

HOMEWORK 3 (GIVEN ON SEPTEMBER 21, 2026)

Explanation: This "exercise" is longer since I am trying to guide you carefully and including some reminders, in a way that can be used in many other similar exercises, towards the more interesting parts. Think for instance that the aim is to answer item (vii) in a simple, elegant way. Please don't pannick ... in the class you will have to explain only some of these items (which ones ... you will find out in the class).

In the lecture, a tangent vector to a manifold M at a point $p \in M$ was, by definition, an equivalence class of a curve γ in M starting at p :

$$\gamma : I \rightarrow M, \gamma(0) = p \quad (I \subset \mathbb{R} \text{ interval containing } 0).$$

For the definition of the equivalence relation please see the notes. Recall also that the equivalence class of γ was denoted

$$(1) \quad \frac{d\gamma}{dt}(0) \in T_p M \quad (p = \gamma(0)).$$

Sometimes is more useful/suggestive to use the notation

$$\left. \frac{d}{dt} \right|_{t=0} \gamma(t) := \frac{d\gamma}{dt}(0).$$

Hopefully this notation will also help you in answering the following:

Question (i): For $\gamma : I \rightarrow M$ defined on some interval $I \subset \mathbb{R}$, make sense of/define the speed at arbitrary times $t_0 \in I$:

$$\frac{d\gamma}{dt}(t_0) \in T_{\gamma(t_0)} M.$$

The rest of the exercise is meant to help you work with tangent vectors, as defined in the lectures, when looking inside $\mathbb{R}^n$ and embedded submanifolds. The point is to make use, as much as possible, of the of the canonical basis

$$(2) \quad \left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_n} \right)_p \in T_p \mathbb{R}^n$$

You can choose one of the definitions/descriptions we gave:

- (1) Via the standard identification

$$\text{standard}_p : T_p \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad v \mapsto v^{\text{Id}}$$

sending v to its value at the identity chart, the canonical basis (2) corresponds to the standard basis $e_1, \dots, e_n$ of the vector space $\mathbb{R}^n$. Recall also that the abstract speed (1) corresponds to the usual derivative.

- (2) As equivalence classes of curves, the i -th member of the basis corresponds to the linear curve along the i -th axis or, using the terminology of speeds,

$$\left(\frac{\partial}{\partial x_i} \right)_p = \left. \frac{d}{dt} \right|_{t=0} (p_1, \dots, p_{i-1}, p_i + t, p_{i+1}, \dots, p_n).$$

With the help of the basis (2), we can write represent/write an arbitrary tangent vector vector $v \in T_p \mathbb{R}^n$ (uniquely) as

$$(3) \quad v = \lambda_1 \cdot \left(\frac{\partial}{\partial x_1} \right)_p + \dots + \lambda_n \left(\frac{\partial}{\partial x_n} \right)_p, \quad \text{with } \lambda_i \in \mathbb{R}.$$

Please also keep in mind that, given two tangent vectors $v, w \in T_p M$, at some point $p \in M$, to check that $v = w$ it suffices to check that, for some chart χ around p , $v^\chi = w^\chi$. Of course, for $M = \mathbb{R}^n$ one has the identity chart. With these in mind ...

Question (ii): Prove that, for (smooth) curves $\gamma : I \rightarrow \mathbb{R}^n$,

$$\frac{d\gamma}{dt}(t) = \gamma'_1(t) \left(\frac{\partial}{\partial x_1} \right)_{\gamma(t)} + \dots + \gamma'_n(t) \left(\frac{\partial}{\partial x_n} \right)_{\gamma(t)},$$

where $\gamma_i : I \rightarrow \mathbb{R}$ are the components of γ and $\gamma'_i(t) \in \mathbb{R}$ are their usual derivatives.

Ok, let's now move to embedded submanifolds $M \subset \mathbb{R}^n$. Since any (smooth) curve in M is also one in $\mathbb{R}^n$, it follows that we have inclusions

$$T_p M \subset T_p \mathbb{R}^n \quad (\text{for each } p \in M)$$

hence also the tangent vectors to M can be written in the form (3). The question is: given an expression (3), when does it actually belong to $T_p M$? Here are two possible tools to use for answering such questions:

- (1) using speed of curves: to show that $v \in T_p \mathbb{R}^n$ belongs to $T_p M$ you have to find a curve γ in M such that $v = \frac{d\gamma}{dt}(0)$;

Question (iii): For the sphere $S^2 \subset \mathbb{R}^3$, and $p = (1, 0, 0) \in S^2$, show that

$$\left(\frac{\partial}{\partial y} \right)_p, \left(\frac{\partial}{\partial z} \right)_p \in T_p \mathbb{R}^3$$

both belong to $T_p S^2$, using speeds of curves.

- (2) using the RVT: if it happens that $M = \text{Zero}(F)$ as in the regular value theorem (hence F defined from $\mathbb{R}^n$, or an open inside it, to $\mathbb{R}^k$, with 0 regular value), then we know that

$$T_p M = \ker \left((dF)_p : T_p \mathbb{R}^n \rightarrow T_{f(p)} \mathbb{R}^k \right).$$

To work with the differential and the canonical basis, the following is useful:

Question (iv): Show that, for any smooth $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$ its differential at $p \in \mathbb{R}^n$, $(dF)_p : T_p \mathbb{R}^n \rightarrow T_{F(p)} \mathbb{R}^k$, is still given by "the usual formula's", i.e. it sends

$$\left(\frac{\partial}{\partial x_i} \right)_p \mapsto \sum_{j=1}^k \frac{\partial F_j}{\partial x_i}(p) \cdot \left(\frac{\partial}{\partial x_j} \right)_p,$$

where F_j is the j -th component of F .

Question (v): Prove again (iii), but using now the RVT.

And now something handy when having to compute the differential of smooth functions $f : M \rightarrow \mathbb{R}^k$ with $M \subset \mathbb{R}^n$ and embedded submanifold. The best (but very common) scenario is when f extends to a smooth function on the entire $\mathbb{R}^n$ (or an open inside it):

$$\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k.$$

Since $T_p M \subset T_p \mathbb{R}^n$ and, by the discussions above, we know how to handle $d\tilde{f}$, the following is very useful:

Question (vi): If $f = \tilde{f}|_M$ as before, then show that the differential

$$(df)_p : T_p M \rightarrow T_{f(p)} \mathbb{R}^k$$

is the restriction of the differential

$$(d\tilde{f})_p : T_p \mathbb{R}^n \rightarrow T_{\tilde{f}(p)} \mathbb{R}^k$$

to the subspace $T_p M \subset T_p \mathbb{R}^n$. (Hints: inclusion map; the chain rule).

As an illustration:

Question (vii): Compute the differential $(df)_p$ of the function

$$f : S^2 \rightarrow \mathbb{R}^2, \quad f(x, y, z) = (xy, xz),$$

at the point $p = (1, 0, 0)$, applied to the two tangent vectors to S^2 from item (iii).

Then deduce that f is a submersion at $p = (1, 0, 0)$.

Question (viii): Deduce that f is a submersion at $p = (1, 0, 0)$.