

**HOMEWORK 5; GIVEN ON SEPT 28, 2024,
TO BE HANDED IN BY OCTOBER 7TH**

If you are looking at the exercise before you learned about the Lie bracket of vector fields, please be aware that you could just ignore (b) for a while. More precisely, items (c), (d), (e) (skipping (b) from the list) and (f) do not depend on (b).

Exercise 1. We work on the manifold $M = S^2$ (the 2-sphere).

- (a) Defining, for $q = (x, y, z) \in S^2$,

$$X_q^1 := z \left(\frac{\partial}{\partial y} \right)_q - y \left(\frac{\partial}{\partial z} \right)_q,$$

$$X_q^2 := x \left(\frac{\partial}{\partial z} \right)_q - z \left(\frac{\partial}{\partial x} \right)_q,$$

$$X_q^3 := y \left(\frac{\partial}{\partial x} \right)_q - x \left(\frac{\partial}{\partial y} \right)_q,$$

show that X^1 , X^2 and X^3 are vectors fields on S^2 .

- (b) Compute $[X^2, X^3]$.
- (c) Show that, for each $q \in S^2$, the tangent vectors written above span $T_q S^2$.
- (d) Prove that $f : S^2 \rightarrow \mathbb{R}$, $f(x, y, z) = y + z$ is not a submersion.
- (e) Look back at your answers to the previous questions and check whether your arguments/proofs/computations were, indeed, worked out on the given manifold $M = S^2$, or did you secretly move to $\mathbb{R}^3$ (e.g. in the computation in (b), or perhaps in computing df in (d)). Whenever you did, please fix your argument, hopefully by explaining why what you did is correct, or by changing the proof/computation.
- (f) *If you have time, but/and just for fun (as this is not a "routine" question):* X^1 , X^2 and X^3 are three vector fields on S^2 which, at each $q \in S^2$, span $T_q S^2$. Can you actually find two vector fields Y^1 and Y^2 that do the same (hence, since $\dim S^2 = 2$, that would mean that, at each $q \in S^2$, they form a basis of $T_q S^2$). Ok, ok, if you do not find them in 10 min, can you at least find one single vector field Z on S^2 with the property that $Z_q \neq 0$ for all $q \in S^2$? Give it another 10min (at least) and, if that turns out to be too difficult as well, try to find a similar Z on S^3 (think about formulas similar to the ones of X^1 , X^2 and X^3 , but on S^3 now). Hmmm ... things are easier on S^3 ... can you guess/detect why?

The last item can be seen as a "bonus question".