

HOMEWORK 6 (OCTOBER 12, 2026)

Exercise 1. On the 2-sphere S^2 we consider the 1-form and the 2-form given by the following formulas (restricted to S^2):

$$\omega = z^2(ydx - xdy), \quad \sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

Please do the following:

- (a) show that $\omega \wedge \sigma = 0$ and $d\sigma = 0$;
- (b) show that $\omega = F^*(xdy - ydx)$ where $F : S^2 \rightarrow \mathbb{R}^2$, $F(x, y, z) = (yz, xz)$;
- (c) show that, over $U = \{(x, y, z) \in S^2 : z \neq 0\}$, σ can be written as $\sigma = gdx \wedge dy$ for some smooth function g on U that you have to compute;
- (d) compute $d\omega$ and show that it can be written as $d\omega = f\sigma$ for some smooth function f on S^2 that you have to compute;
- (e) show that, for any $p \in S^2$, one has $\sigma_p \neq 0$ (i.e., as a skew-symmetric bilinear map $\sigma_p : T_p S^2 \times T_p S^2 \rightarrow \mathbb{R}$, it is not the zero map). Here you may use item (a) of Homework 5;

Bonus question:

- (f) show that σ cannot be written as the pull-back of a 2-form on $\mathbb{R}^2$ by any smooth function $G : S^2 \rightarrow \mathbb{R}^2$.

Points: 2 pts for a, 2 for b, 1 for c, 2 for d, 3 for e and 2 for f.