

HOMework 7 (OCTOBER 19, 2026)

Exercise 1. As in the previous homework, we consider the 1- and 2-forms on the sphere S^2 given by the following formulas (restricted to S^2):

$$\omega = z^2(ydx - xdy), \quad \sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

We also consider the vector field

$$V = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

Please do the following:

- (a) compute $i_V(\omega)$ and show that $i_V(\sigma) = -dz$;
- (b) compute $L_V(\omega)$ and $L_V(\sigma)$ using the main properties of the Lie derivatives (as discussed in the class);
- (c) compute $L_V(\omega)$ and $L_V(\sigma)$ using the Cartan magic formula;
- (d) find the points $p = (x, y, z) \in S^2$ at which $i_V(\sigma)_p = 0$;
- (e) show that if a form ζ on S^2 can be written as the pull-back of a form on $\mathbb{R}^2$ by the function F from the previous homework, then $i_V(\zeta)$ must vanish at the point $q = (1, 0, 0)$. Deduce that σ cannot be written as such a pull-back.
- (f) prove that, over $U := \{(x, y, z) \in S^2 : z \neq 0\}$, σ can be written as $\omega = f \cdot dx \wedge dy$ for some $f \in C^\infty(U)$ that you have to compute;
- (g) prove that σ is a volume form on S^2 ;
- (h) show that σ cannot be written as the pull-back of a 2-form on $\mathbb{R}^2$ by any smooth function $G : S^2 \rightarrow \mathbb{R}^2$.

Points: 2 points for (c) and (f), and 1 point for each of the rest.