

①

Integration on manifolds

Def: A differential k-form is something (informal) you can integrate over a k dimensional manifold.

An n-form on an n-mfd provides a volume measuring device

Integrals in $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be open. Then note that $\dim \Omega^n(U) = 1$
so any $\omega \in \Omega^n(U)$ can be written as

$$\omega = f(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n$$

②

Def: We define $\int_U \omega = \int_U f(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n = \int_U f dx^1 \dots dx^n$
(In $\mathbb{R}^n$, just erase the wedges $\wedge$)

How to "upgrade" to a general manifold?

Let $\omega \in \Omega^n(M)$ be an n-form. Suppose that ω has compact support contained in a chart

$$\chi: U \subseteq M \rightarrow \chi(U) \subseteq \mathbb{R}^n$$

Then the pullback $(\chi^{-1})^* \omega$ is an n-form on $\chi(U) \subseteq \mathbb{R}^n$
So we can apply integration on $\mathbb{R}^n$ again $\int$

Def. Let ω be as before ($\omega \in \Omega^n(M)$ with compact support) Then we define:

$$\int_U \omega := \int_{\chi(U)} (\chi^{-1})^* \omega$$

Rk. In our applications, this definition will suffice.

③

w. dont do part f, g, h ∇
 $= (yZ, xZ)$ (part e) 0

Exercises

1) Let $M = \{(x,y,z) \mid z = x^2 + y^2, x^2 + y^2 \leq 1\}$

Let $\omega = z \, dx \wedge dy$
Compute $\int_M \omega$



2) Let $\omega = x \, dy \wedge dz$

Compute $\int_S \omega$



$x = \cos \phi$
 $y = \sin \phi$
 $z = \cos \phi$
 $\chi^{-1}(\phi) \mapsto (x, y, z)$

Example

Consider S' with $\omega = \frac{1}{2}(x \, dy - y \, dx)$

Consider $\chi: S' \setminus \{(-1, 0)\} \rightarrow]-\pi, \pi[$
 $(\cos t, \sin t) \mapsto t$

S' is covered by one chart (up to 1 point but that does not matter)

$$\int_{S'} \omega = \int_{-\pi}^{\pi} (\chi^{-1})^* \omega$$

write $\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow \begin{cases} dx = -\sin t \, dt \\ dy = \cos t \, dt \end{cases}$

$$\Rightarrow (\chi^{-1})^* \omega = \frac{1}{2} dt$$

$$\text{So } \int_{S'} \omega = \frac{1}{2} \int_{-\pi}^{\pi} dt = \pi$$

Def. Let ω be as before (support) Then we define:

$$\int_U \omega :=$$

Rk. In our applications, this

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HW: don't do part f, c

$$F(x, y, z) = (yz, xz) \quad (\text{part})$$

Exercises for today

4.54, 4.56, 4.66, 4.71

About the H

Look at

1

Def: A differential

Integration on manifolds

Exercises

1) Let $M = \{(x,y,z) \mid z = x^2 + y^2, x^2 + y^2 \leq 1\}$

Let $\omega = z \, dx \wedge dy$

Compute $\int_M \omega$



2) Let $\omega = x \, dy \wedge dz$

Compute $\int_S \omega$



$x = \cos \theta \sin \phi$
 $y = \sin \theta \sin \phi$
 $z = \cos \phi$

$\chi^{-1}(\theta, \phi) \mapsto (x, y, z)$

Example

Consider S^1 with $\omega = \frac{1}{2}(x \, dy - y \, dx)$

Consider $\chi: S^1 \setminus \{(-1,0)\} \rightarrow]-\pi, \pi[$
 $(\cos t, \sin t) \mapsto t$

S^1 is covered by one chart (up to 1 point but that does not matter)

$$\int_{S^1} \omega = \int_{-\pi}^{\pi} (\chi^{-1})^* \omega$$

$$\text{write } \begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow \begin{cases} dx = -\sin t \, dt \\ dy = \cos t \, dt \end{cases} \Rightarrow (\chi^{-1})^* \omega = \frac{1}{2} dt$$

$$\text{so } \int_{S^1} \omega = \frac{1}{2} \int_{-\pi}^{\pi} dt = \pi$$

Def: Let ω be as before ($\omega \in \text{support}$) Then we define:

$$\int_U \omega := \int_{\chi(U)} (\chi^{-1})^* \omega$$

Rk: In our applications, this definition

3

HW. dont do part f, g, h
 $F(x, y, z) = (yz, xz)$ (part e)

Exercises for today

- 4.54, 4.56, 4.66, 4.71.

Look at previous exams.

About the Homework

You may choose to do the HWs today at 10:00, or Wednesday at 10:00.

1

Def: A differential k -form is something (informal) you can integrate over a k -dimensional manifold.

An n -form on an n -manifold provides a volume measuring device

Integration on manifolds

2

Def: We define $\int_U \omega = \int_U f(x_1, \dots, x_n) \, dx_1 \wedge \dots \wedge dx_n$
(In $\mathbb{R}^n$, just erase the wedges)



Exercises

*) Let $M = \{(u,v,z) \mid z = \sqrt{u^2 + v^2}, x^2 + y^2 \leq 4\}$

Exercises for today

- 4.54, 4.56, 4.66, 4.71.
- Look at previous exams.

About the Homework 7:

☹ You may choose to do the HW today at 10:00, or Wednesday at 16:00.

If you choose to do it today, At around 9:50 - 9:55 please move (& spread out) to the back of the room.

Example

Consider S^1 with $\omega = \frac{1}{2}(x dy - y dx)$

①
Def: A differential k -form (informal) you can integrate over manifold.

An n -form on an n -manifold is a measuring device

Integrals in $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be open. Then for any $\omega \in \Omega^n(U)$ we can

$$\omega = f(x_1, \dots, x_n)$$