ADDENDUM TO CHAPTER 7

1. Action of the Fundamental Group

Throughout these notes, $p : E \to B$ will denote a covering map, and E and B are path connected topological spaces.

We have seen that for any path $\gamma : [0,1] \to B$ and $e \in p^{-1}(\gamma(0))$ there exists a unique lift $\tilde{\gamma}_e : [0,1] \to E$ of γ starting at e (Proposition 7.27). Moreover, it was shown that if γ_1 and γ_2 are path homotopic, then their lifts (starting at the same point) have the same end points and are also path homotopic (Lemma 7.32). What we will now explain is that from both of these properties we obtain a (right) action of $\pi(B, b)$ on the fiber $p^{-1}(b)$.

Definition 1.1. Let X be a topological space and G a topological group (i.e., a topological space endowed with a group structure such that the multiplication $m : G \times G \to G$ and the inversion $\iota : G \to G$ are continuous). An (left) **action of** G **on** X is a continuous map

$$\Psi : G \times X \to X, \quad (g, x) \mapsto \Psi(g, x) = g \cdot x$$

which satisfies:

- $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$, and
- $1 \cdot x = x$ for all $x \in X$.

Similarly, a *right* action of G on X is a continuous map $X \times G \to X$ which satisfies:

- $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and $x \in X$, and
- $x \cdot 1 = x$ for all $x \in X$.

For an action of G on X we define for each x the **orbit of** G **through** x to be the set

$$\mathcal{O}_x = \{g \cdot x : g \in G\} \subset X,$$

and the **isotropy of** G **at** x to be

$$G_x = \{g \in G : g \cdot x = x\} \subset G.$$

Exercise 1. *Show that G_x is a subgroup of G.*

Exercise 2. *Show that for any $x \in X$ there is a bijection between $\mathcal{O}_x$ and G/G_x, where G/G_x denotes the quotient of G by the equivalence relation $g \sim h$ if and only if $gh^{-1} \in G_x$.*

We can now explain the action of $\pi(B, b)$ on $p^{-1}(b)$. It is defined as follows: for each pair $(e, [\gamma]) \in p^{-1}(b) \times \pi(B, b)$ we take $e \cdot [\gamma] = \tilde{\gamma}_e(1)$ where $\tilde{\gamma}_e$ is the unique lift of γ which starts at e.

Exercise 3. *Let E and B be path connected topological spaces and $p : E \to B$ a covering map:*

1. *Show that the map $(e, [\gamma]) \mapsto \tilde{\gamma}_e(1)$ is well defined and determines a right action of $\pi(B, b)$ on $p^{-1}(b)$.*
2. *Show that the action is transitive, i.e., for every $e \in p^{-1}(b)$ we have that $\mathcal{O}_e = p^{-1}(b)$.*
3. *Show that the isotropy of $\pi(B, b)$ at e is isomorphic to $\pi(E, e)$.*
4. *Show that there is a bijection between $p^{-1}(b)$ and $\pi(B, b)/p_*\pi(E, e)$.*
5. *Conclude that if E is simply connected then there is a bijection between $p^{-1}(b)$ and $\pi(B, b)$.*

Exercise 4. *Consider the map $f : \mathbb{S}^n \to \mathbb{P}^n$ which associates to each x in the sphere $\mathbb{S}^n$ the line in $\mathbb{R}^n$ which passes through x and the origin.*

1. *Show that f is a covering map.*
2. *Assuming that $\mathbb{S}^n$ is simply connected, for $n \geq 2$, compute the fundamental group of $\mathbb{P}^n$.*

2. Properly Discontinuous Actions

Definition 2.1. A (continuous) action of G on X is said to be **properly discontinuous** if for every $x \in X$ there exists a neighborhood U_x of x in X such that $g \cdot U_x \cap U_x \neq \emptyset$ implies that $g = 1$.

The importance of properly discontinuous actions for us is given by the following proposition:

Proposition 2.2. *If E is a path connected and simply connected topological space, and G acts properly discontinuously on E, then the quotient map $p : E \to E/G$ is a covering map, and moreover, $\pi(E/G, x) \cong G$, for any $x \in E/G$.*

Exercise 5. *Prove the proposition above by following these steps:*

1. *For $e \in E$, let U_e be a neighborhood such that $g \cdot U_e \cap U_e \neq \emptyset$ implies that $g = 1$. Show that $g \cdot U_e$ is a neighborhood of $g \cdot e$ which satisfies the same property.*
2. *Show that $V_{[e]} = p(U_e)$ is an open neighborhood of $[e]$ in E/G.*
3. *Show that $V_{[e]}$ is evenly covered (i.e., that $p^{-1}(V_{[e]})$ admits a partition into slices). Conclude that p is a covering map.*
4. *Fix $e \in E$ and show that the map $G \to \pi(B, b)$ which associates to any $g \in G$ the homotopy class of $p \circ \tilde{\gamma}$ (where $\tilde{\gamma}$ is any path joining e to $g \cdot e$), is an isomorphism of groups.*

Exercise 6. *Show that the action of $\mathbb{Z}$ on $\mathbb{R}$ given by $(n, x) \mapsto n + x$ is properly discontinuous. Conclude that $\pi(\mathbb{S}^1, p) = \mathbb{Z}$.*

Exercise 7. *Show that the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by $(n, (x, y)) \mapsto (n + x, y)$ is properly discontinuous. Conclude that $\pi(\text{Cylinder}, p) = \mathbb{Z}$.*

Exercise 8. *Show that the action of* $\mathbb{Z}^2$ *on* $\mathbb{R}^2$ *given by* $((n,m),(x,y)) \mapsto (n+x, m+y)$ *is properly discontinuous. Conclude that* $\pi(\mathbb{T}^2, p) = \mathbb{Z}^2$.

Exercise 9. *Show that* $\mathbb{Z}_2 = \{1, -1\}$ *acts properly discontinuously on the sphere* $\mathbb{S}^n$. *What is the quotient space?*

Exercise 10. *Show that a covering of a simply connected space is a homeomorphism.*

Exercise 11. *Can* $\mathbb{S}^2$ *be obtained from a properly discontinuous action of a group on* $\mathbb{R}^2$*? What about* $\mathbb{P}^2$*?*

Exercise 12. *Let* K *be the Klein bottle.*

(1) *Show that there is a covering map* $\mathbb{T}^2 \to K$.
(2) *Show that* K *can be obtained from* $\mathbb{R}^2$ *as the quotient by a properly discontinuous action.*
(3) *Compute the fundamental group of* K.

Exercise 13. *Show that the quotient of a topological manifold by a properly discontinuous action of a group is also a topological manifold.*

Exercise 14. *Show that if* $p : E \to B$ *is a covering, and* B *is a topological manifold of dimension* n, *then* E *is a topological manifold of dimension* n.