Topologie en Meetkunde 2011

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CHAPTER 1

The Classification Problem for Compact Surfaces

1. Introduction

In this Chapter we will introduce and start dealing with the classification problem for compact surfaces. Giving a complete solution to this problem is one of the main goals of the course. It will serve as motivation for many of the concepts that will be introduce.

Our approach to the classification problem will be the following:

(1) We will give a list of compact connected surfaces, all of which will be constructed from a polygonal region in the plane by identifying its edges in pairs.
(2) We will show that any compact connected surface is homeomorphic to one in the list.
(3) We will show that any two surfaces in the list are not homeomorphic to each other.

Parts (1) and (2) will be dealt with in this chapter, while part (3) will be done only after we introduce the fundamental group and learn how to calculate it (via the Seifert - Van Kampen Theorem). To be a bit more precise about part (2) in the plan above, what we will show is that any triangulable compact surface is homeomorphic to one in the list. It turns out that every compact surface is in fact triangulable, and we hope to come back to this at some point in the course.

2. Topological Manifolds

The main objects that will be studied in this chapter are topological surfaces, which are simply 2-dimensional topological manifolds.

Definition 1.1. *An **n-dimensional topological Manifold** is a topological space $(X, \mathcal{T})$ which satisfies the following properties:*

(1) *X is Hausdorff;*
(2) *X admits a countable open cover $\{U_i\}_{i\in\mathbb{N}}$ such that each U_i is homeomorphic to an open set in $\mathbb{R}^n$.*

*Each open U_i together with a homeomorphism $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$ will be called a **coordinate chart** of X.*

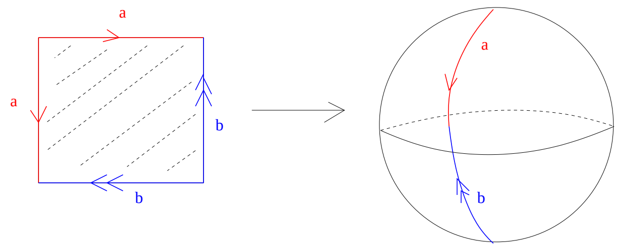
Remark 1.2. *The second condition in the definition above can be restated as: X is second countable and each point of X admits an open neighborhood which is a chart. It the follows that X is locally compact and second countable, and thus metrizable, i.e., the topology of X is induced by a metric.*

Definition 1.3. *A **surface** is a 2-dimensional topological manifold.*

Example 1.4 (The Sphere $\mathbb{S}^2$). *We define the sphere $\mathbb{S}^2$ to be the quotient space obtained from a square by identifying its border according to the Figure 1. Thus, if we denote the unit interval $[0, 1]$ by I, then*

$$\mathbb{S}^2 = \{(x, y) \in I \times I\}/\sim$$

where we identify $(0, y) \sim (1 - y, 1)$, and $(x, 0) \sim (1, 1 - x)$.



The sphere obtained from a square glueing as indicated in the picture

FIGURE 1.

EXERCISE 1.1. (1) Show that $\mathbb{S}^2$ is homeomorphic to the standard sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1.\}$$

(2) Show that it is a surface.

EXAMPLE 1.5 (The Torus $\mathbb{T}^2$). *We define the torus $\mathbb{T}^2$ to be the quotient space obtained from the unit square by identifying its border according to the Figure 2. Thus,*

$$\mathbb{T}^2 = \{(x, y) \in I \times I\}/\sim$$

where we identify $(0, y) \sim (1, y)$*, and* $(x, 0) \sim (x, 1)$*.*

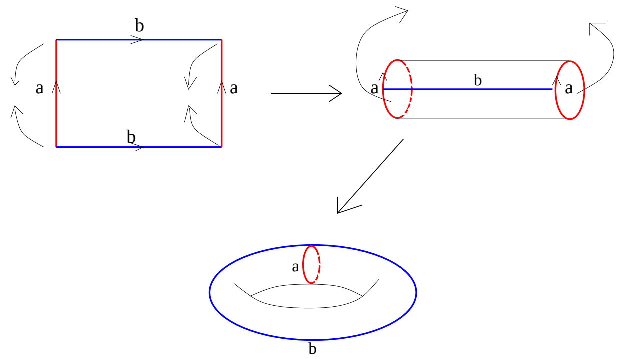


FIGURE 2.

EXERCISE 1.2. Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the standard circle.

(1) Show that the torus $\mathbb{T}^2$ is homeomorphic to a $\mathbb{S}^1 \times \mathbb{S}^1$.
(2) Show that it is a surface.

EXAMPLE 1.6 (The Projective Space $\mathbb{P}^2$). *We define the projective space $\mathbb{P}^2$ to be the quotient space obtained from the unit square by identifying its border according to the Figure 3. Thus,*

$$\mathbb{P}^2 = \{(x, y) \in I \times I\}/\sim$$

where we identify $(0, y) \sim (1, y)$*, and* $(x, 0) \sim (1 - x, 1)$*.*

EXERCISE 1.3. Let $D \subset \mathbb{R}^2$ denote the (closed) disk of radius 1.

(1) Show that $\mathbb{P}^2$ is homeomorphic to the quotient space obtained from D by identifying its border $\mathbb{S}^1$ via the antipodal map (Figure 4)

$$A : \mathbb{S}^1 \to \mathbb{S}^1, \quad A(x, y) = (-x, -y).$$

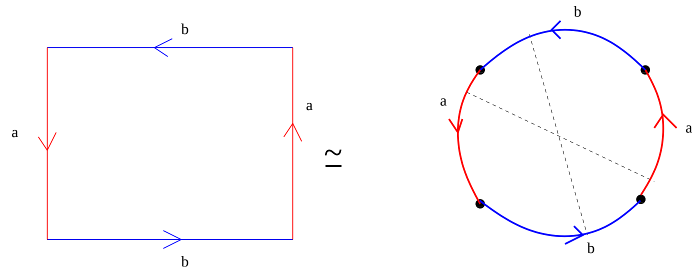


FIGURE 3.

(2) Show that $\mathbb{P}^2$ is homeomorphic to the quotient space obtained from the standard sphere by identifying a point p with its antipodal $-p$.
(3) Show that $\mathbb{P}^2$ is homeomorphic to the space of lines through the origin in $\mathbb{R}^3$.
(4) Show that it is a surface.

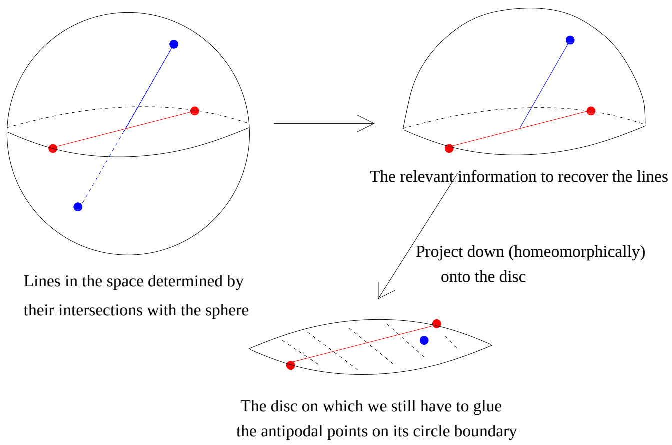


FIGURE 4.

There is a very basic operation which allows us to construct a new manifold out of two given manifolds.

DEFINITION 1.7. *Given two topological manifolds M and N of the same dimension, define their connected sum, denoted $M\#N$ as follows: remove from M and N two "small balls" B_1 and B_2 and glue $M - B_1$ and $N - B_2$ along the sphere $\partial B_1 = \partial B_2$.*

For surfaces, it means that we remove two small disks and we glue the remaininig spaces along the bounday circles (Figure 5). We can describe this operation with more details:

Remove an Open Disk: We remove from M and N an open subset D_1 and D_2 each of which is homeomorphic to an open disk in $\mathbb{R}^2$.

Glue along the Boundary: We fix a homeomorphism $\varphi : \partial D_1 \to \partial D_2$ and we take the quotient space

$$M\#N = (M - D_1)\coprod(N - D_2)/\sim$$

where $x \sim y$ if and only if $x = y$ or $x \in \partial D_1$, $y \in \partial D_2$, and $\varphi(x) = y$.

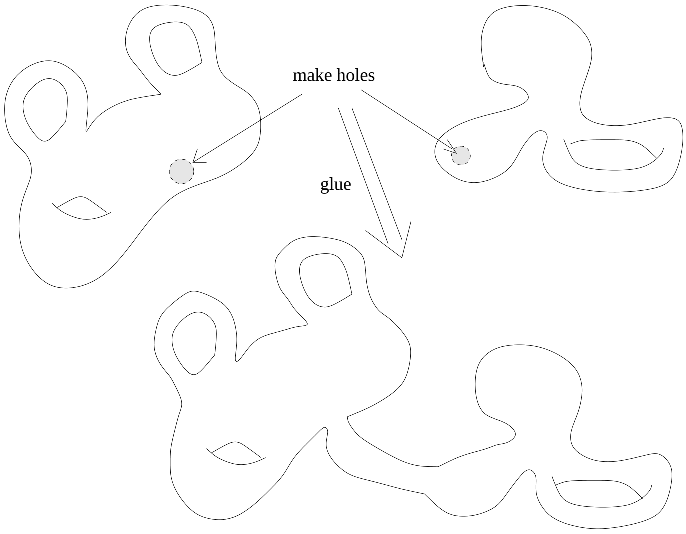


FIGURE 5. Connected Sum

EXAMPLE 1.8. The connected sum of two tori is the double torus $\mathbb{T}_2$. Repeating the operation of connected sum, one obtains all tori with arbitrary number of holes (see Figure 6 for the $g = 2$):

$$\mathbb{T}_g = \underbrace{\mathbb{T}\#\ldots\#\mathbb{T}}_{g \text{ times}}.$$

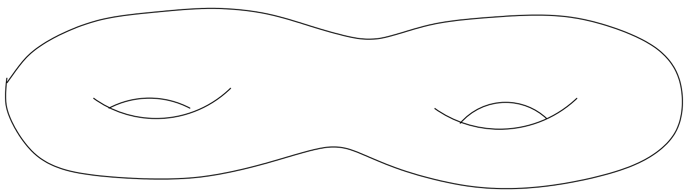


FIGURE 6. Double Torus.

Similarly, one considers the connected sum of h copies of $\mathbb{P}^2$:

$$\mathbb{P}_h = \underbrace{\mathbb{P}^2\#\ldots\#\mathbb{P}^2}_{h \text{ times}}.$$

EXERCISE 1.4. Show that the connected sum $M\#S^2$ of any surface M with the sphere S^2 is homeomorphic to M itself.

We could, in principal, consider more surfaces by considering other examples of connected sums (for example of a torus with a projective space), but as we will soon see, we have already obtained a complete list of all compact connected surfaces:

THEOREM 1.9. *Any compact connected surface is homeomorphic to one of the following:*

(1) *A sphere* $\mathbb{S}^2$*,*
(2) *A connected sum of Tori (plural of Torus)* $\mathbb{T}_g$*, with* $g \in \mathbb{N}$*, or*
(3) *A connected sum of projective spaces* $\mathbb{P}_h$*, with* $h \in \mathbb{N}$*.*

3. The Basic Building Blocks: Polygonal Regions

In this section we will show how to construct surfaces out of polygonal regions of the plane, by identifying its edges in pairs. Intuitively, a polygonal region is a subset of the plane which "looks like" in Figure 7. Let us explain how to make this precise.

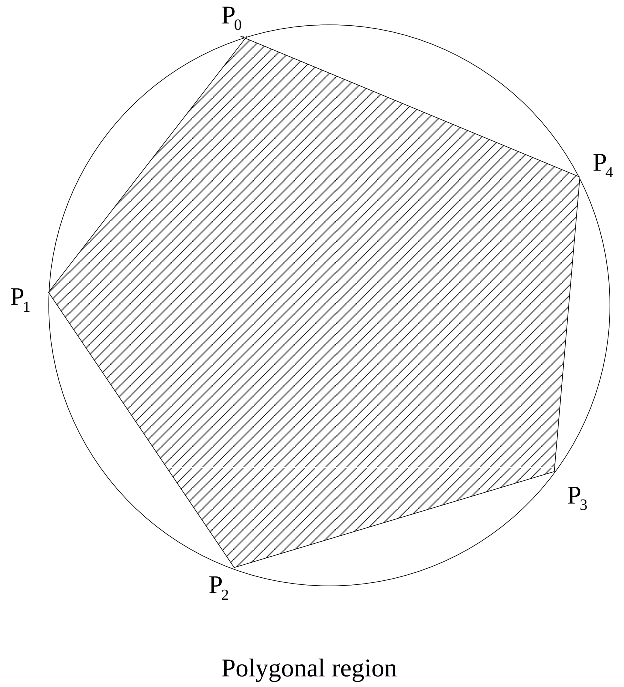


FIGURE 7. Polygonal Region.

- Fix a circle in $\mathbb{R}^2$, pick $n+1$ points on it and order them in counterclockwise direction $\{p_0, \ldots, p_n\}$.
- For each $0 < i \leq n$ consider the line passing through p_{i-1} and p_i. It divides $\mathbb{R}^2$ into two half-planes. Let H_i be the half-plane which contains all the other points p_j.
- Let P be the set
$$P = H_1 \cap H_2 \cap \cdots \cap H_n.$$

DEFINITION 1.10. *An n-sided* ***polygonal region*** *of the plane is any subset of $\mathbb{R}^2$ obtained by the "recipe" above.*

Associated to a polygonal region will will use the following notation:

Vertices: The points p_i will be called vertices of P. The set of all vertices of P will be denoted by $V(P)$.

Edges: The line segment joining p_{i-1} and p_i will be denoted by e_i, and will be called an edge of P. The set of all edges of P will be denoted by $E(P)$

Border: The union of all edges of P will be denoted by ∂P and will be called the border of P.

Interior: The complement of ∂P in P will be denoted by $\mathrm{Int}(P)$ and will be called the interior of P.

It will also be important to introduce orientations on the edges of a polygon, and to specify what a "map" between edges is (this is how we will be able the make precise the notion of "glueing one edge to another").

Definition 1.11.

(1) *Let $L \subset \mathbb{R}^2$ be a line segment. An* ***orientation*** *of L is a choice of ordering of its end points. Such an orientation will be represented by an arrow, and we will say that* *L is a line from a to b (Figure 8).*

(2) *If L is a line from a to b, and L' is a line segment from c to d, then a* ***positive linear map*** *from L to L' is the homeomorphism $h : L \to L'$ which associates to $x = (1-t)a + tb \in L$ the point $h(x) = (1-t)c + td$.*

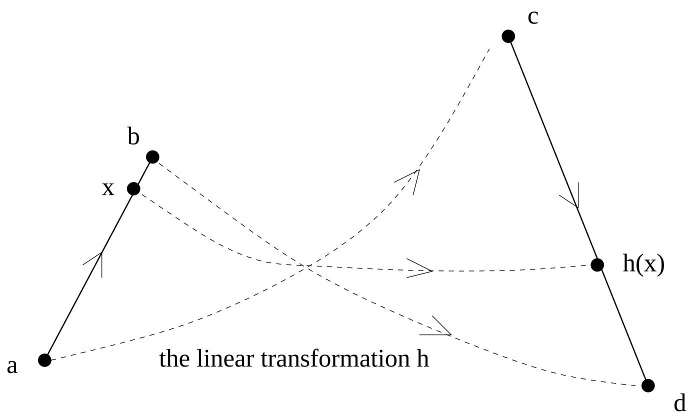


Figure 8. Positive Linear Maps.

4. Glueing the Edges of a Polygonal Region

Since we will be considering (disjoint unions of) polygonal regions with several identifications on the borders, we must find a convenient way of keeping track of such "glueing procedures". For this, we will introduce the concept of labels:

Definition 1.12. *A* ***labeling*** *of a polygonal region P is a map $E(P) \to \Lambda$ from the set of edges of P to a set Λ, whose elements will be called* ***labels****.*

Given a polygonal region along with: (1) a labeling of its edges, and (2) an orientation on edge, we consider the space

$$X = P/\sim$$

where

- If $p \in \mathrm{Int}(P)$, then p is equivalent only to itself, i,e,m $p \sim p$;
- If e_i and e_j are edges with the same label, we let $h : e_i \to e_j$ be a positive linear map and we set
$$x \in e_i \sim h(x) \in e_j.$$

In this case we say that X was obtained from P by glueing its edges together according to the orientation and the labeling.

We remark that we also allow X to be obtained from a finite disjoint unit of polygonal regions with identifications on the edges. Thus X may be either connected or disconnected. As an illustration of spaces obtained in this way, consider the following examples:

Example 1.13. *The disk can be obtained from a triangle with two labels (a and b) and orientations on the edges as shown in the Figure 9 below.*

Example 1.14. *As we have seen in Figure 1 the sphere can be obtained from a square with to labels and orientations on the edges.*

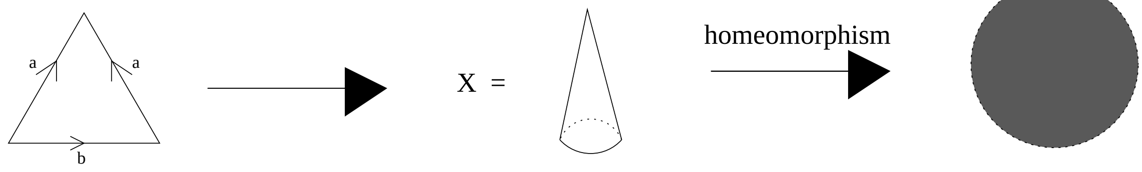


FIGURE 9. The Disk

EXAMPLE 1.15. *In Figure 10 we illustrate the fact that since we allow X to be obtained by glueing the edges of more than one polygonal regions, it follows that X is not necessarily connected.*

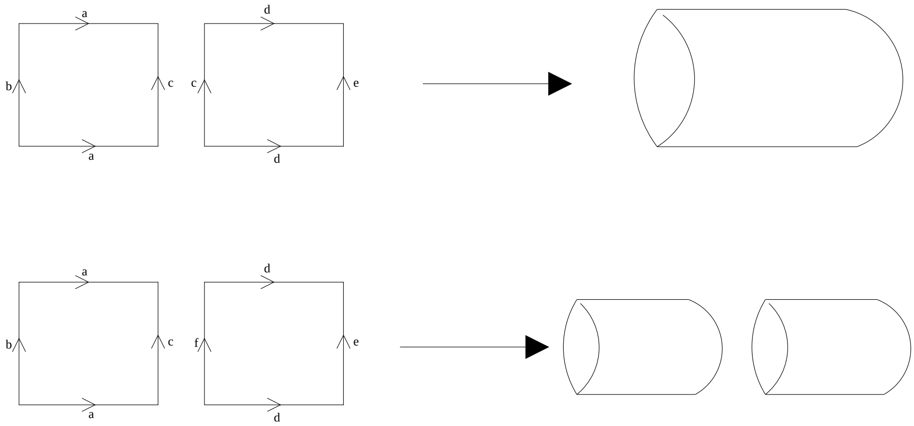


FIGURE 10. X can be connected or disconnected.

Finally, in order to keep track of the orientations of the edges along with the labels, we will now introduce the notion of a labeling scheme. Let e_k is an edge of P with label a_{i_k}. If e_k is oriented from p_{k-1} to p_k, then we put en exponent $+1$ on a_{i_k}. If e_k is oriented from p_k to p_{k-1}, then we put an exponent -1 on a_{i_k}. Then P, its labels, and the orientations on its edges is totally specified up to a homeomorphism which respects the quotient space X by the symbol

$$w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_n}^{\epsilon_n}, \quad \epsilon_i = \pm 1.$$

DEFINITION 1.16. *The symbol $w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_n}^{\epsilon_n}$ will be called a* ***labeling scheme*** *for P with respect to its labels and orientations.*

In Figure 11 are some examples of how to go back and forth from a (disjoint union of) polygonal regions with labels and orientations to labeling schemes.

5. Operations on Labeling Schemes

It is important to note that we are interested in the quotient space X obtained from a polygonal region by gluing its edges and not on the labeling scheme itself. With this in mind, we will now introduce some operations we can perform on the labeling scheme (or equivalently on the polygonal region) which will leave the resulting quotient space unchanged.

I) Cutting: The operation of cutting is described at the level of labeling schemes as follows. Suppose that $w = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}$ is a labeling scheme and let b be a

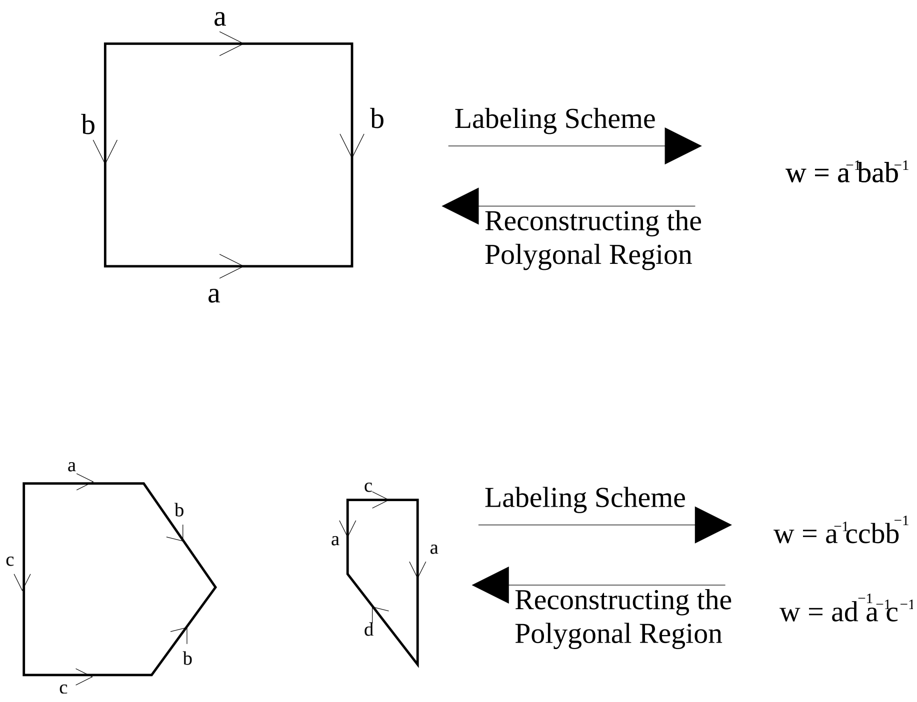


FIGURE 11. Labeling Schemes.

label which does not appear elsewhere in the scheme. Then we may replace w by a pair of labeling schemes

$$w_1 = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} b, \text{ and } w_2 = b^{-1} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}.$$

For a geometric interpretation see the Figure 12.

II) Glueing: The reverse operation of cutting is known as glueing. In terms of the labeling scheme it can be described as follows: If

$$w_1 = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} b, \text{ and } b^{-1} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}$$

are labeling schemes, and the label b only appears where it is indicated above, then we may replace w_1 and w_2 by the labeling scheme $w = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}$.

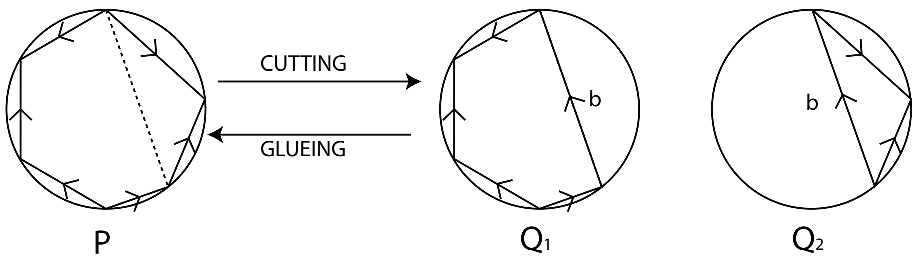


FIGURE 12. Cutting & Glueing

Before we go on with the description of the operations, let us take a small break to write down more formally the result of cutting and glueing:

PROPOSITION 1.17. *Suppose that X is obtained by glueing the edges of n polygonal regions with labeling scheme*

$$w_1 = y_0 y_1, w_2, \ldots, w_n.$$

Let b be a label that does not appear in the scheme. If both y_0 and y_1 have length at least 2, then X can also be obtained by $n+1$ polygonal regions with labeling scheme

$$y_0 b, b^{-1} y_1, w_2, \ldots, w_n.$$

EXERCISE 1.5. The purpose of this exercise is to prove the proposition above. Denote by $P_1, \ldots, P_n$ the original n polygonal regions and by $Q_0, Q_1, P_2, \ldots P_n$ the $n+1$ polygonal regions obtained by cutting P_1. Denote also by $X = (\coprod_{i=1}^n P_i)/\sim$ the space obtained by glueing the edges before cutting, and by $Y = (Q_0 \coprod Q_1 \coprod_{i=2}^n P_i)/\sim$ the space obtained after performing the cutting operation. Consider the obvious map

$$\Phi : Q_0 \coprod Q_1 \coprod_{i=2}^n P_i \longrightarrow \coprod_{i=1}^n P_i.$$

Show that:

(1) Φ induces a well defined map $\varphi : Y \to X$, i.e., if $q \sim q'$, then $\Phi(q) \sim \Phi(q')$.
(2) φ is continuous (use the definition of the quotient topology).
(3) φ is injective, i.e., if $\Phi(q) \sim \Phi(q')$, then $q \sim q'$.
(4) φ is surjective.
(5) X and Y are both compact and Hausdorff.
(6) φ is a homeomorphism.

With only the operations of cutting and gluing we can now easily understand how to construct the connected sums of tori (and projective spaces) out of polygonal regions with identification on the borders:

EXAMPLE 1.18 (The Double Torus $\mathbb{T}^2$). *Let P be the 8-sided polygonal region with labeling scheme $w = aba^{-1}b^{-1}cdc^{-1}d^{-1}$. In order to see that $X = P/\sim$ is homeomorphic to a double torus, we will apply the cutting and glueing operations described above (see Figure 13). Thus we first cut P into two 5-sided polygonal regions Q_1 and Q_2, with labeling schemes $w_1 = aba^{-1}b^{-1}e$ and $w_2 = e^{-1}cdc^{-1}d^{-1}$ respectively. Now, it is clear that after identifying the vertices of Q_1 correspond to the endpoints of e, we obtain the usual representation of the torus as a quotient of the unit square, but with an open disk removed. The edge e then becomes the border of the open disk. The same is obviously true also for Q_2. Thus, if we now apply the glueing operation, what we obtain is the quotient of two copies of the torus, both with an open disk removed by identifying the border of the disk. This is precisely the construction of the connected sum.*

EXERCISE 1.6. Show that $\mathbb{T}_g$ is obtained from a $4g$-sided polygonal region with labeling scheme

$$w = (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}).$$

EXAMPLE 1.19. *A similar argument as the one presented above shows that for $h > 1$, $\mathbb{P}_h$ can be obtained from a $2h$-sided polygonal region with labeling scheme*

$$w = (a_1 a_1) \cdots (a_h a_h).$$

We now continue to describe the rest of the operations that may be performed on the labeling scheme. We suggest that you convince yourself that each of these operations leave the quotient space unchanged.

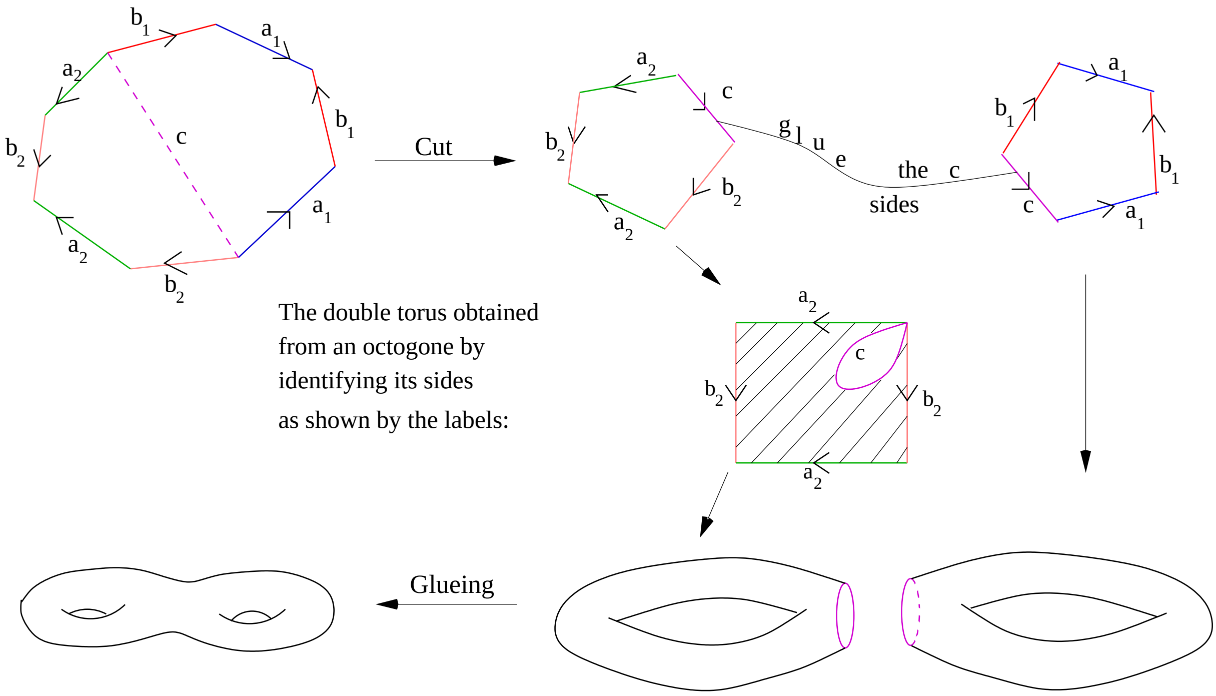


FIGURE 13. Connected Sum of Two Tori.

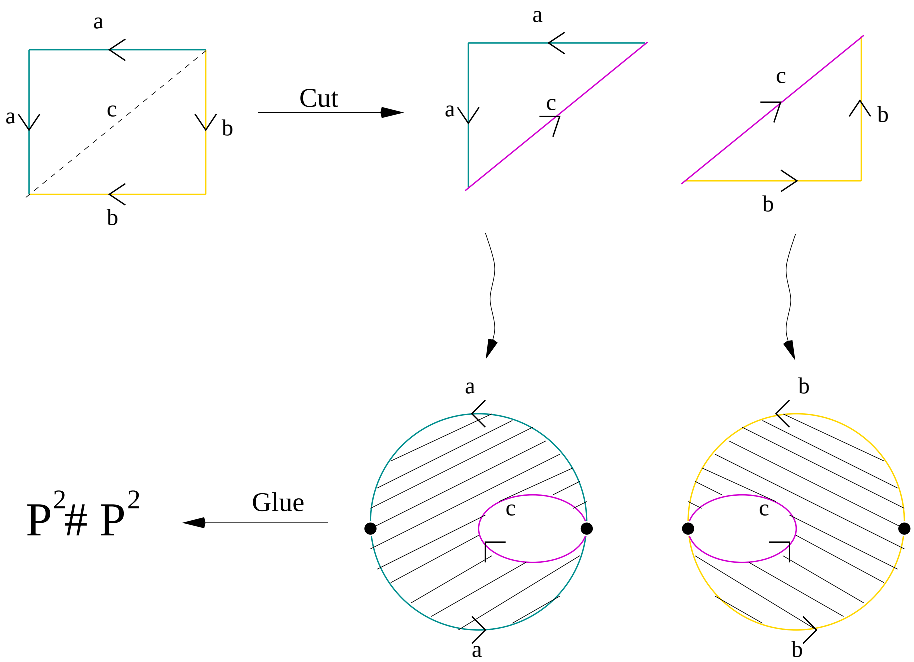


FIGURE 14. Connected Sum of Two Projective Spaces.

DEFINITION 1.20. *Let $w_1, w_2, \ldots, w_n$ be a labeling scheme and let y be a string of labels that appears in the labeling scheme (it may appear in more that one place of the scheme). We will say that y is a* ***removable string*** *if*

(1) *all labels of* y *are distinct, i.e.,* $y = a_{i_1}^{\epsilon_1} \cdots a_{i_k}^{\epsilon_k}$ *with* $a_{i_p} \neq a_{i_q}$ *for all* $i_p \neq i_q$*, and*
(2) *the labels of* y *do not appear elsewhere (outside of* y*) in the labeling scheme, i.e., for all* $1 \leq p \leq k$*, if* a_{i_p} *appears in the labeling scheme, then it belongs to the string* y*.*

III) Unfolding Edges: If y is a removable string of a labeling scheme, then we may replace y by a label that does not appear elsewhere in the scheme. Geometrically, this can be interpreted as replacing a sequence of edges (a "folded line segment"), by a single edge (a line segment) (See Figure 15).

IV) Folding Edges: The reverse operation to unfolding edges is that of folding edges described by: replace all appearances of a single label by a removable string of labels.

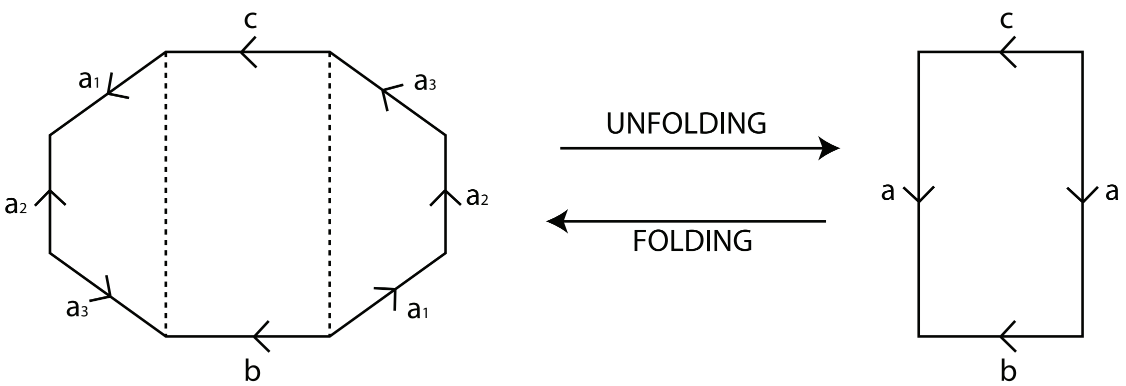


FIGURE 15. Fold/Unfold

V) Reversing Orientations: We may change the the sign of the exponent of all occurrences of a single label in the labeling scheme. In order to understand why the quotient space is left unchanged, recall that we are identifying the points on two oriented edges with the same label by means of a positive linear map. Note that if the orientation on both edges are reversed, the identification remains unchanged.

VI) Cyclic Permutation: It is clear that if instead of writing the labeling scheme of a polygonal region by starting with the label on the edge e_1, we decide to start with the label on a different edge and then continue in the same counterclockwise direction, then the quotient space X is unchanged. We may think of this as performing a rotation on the polygonal region. The effect on the labeling scheme is to take a cyclic permutation of its labels

$$a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n} \longrightarrow a_{i_2}^{\epsilon_2} \cdots a_{i_n}^{\epsilon_n} a_{i_1}^{\epsilon_1}.$$

VII) Flip: We may replace a labeling scheme by its <u>formal inverse</u>:

$$w = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n} \longrightarrow a_{i_n}^{-\epsilon_n} \cdots a_{i_1}^{-\epsilon_1}.$$

Geometrically this corresponds to flipping the polygonal region as in Figure 16 (and then performing a cyclic permutation if necessary) .

REMARK 1.21. The operations of permutation and flipping should be thought of as instances of the same phenomena. If may apply any <u>Euclidean transformation of the plane</u> (i.e., translations, reflections and rotations) to our original polygonal region, the resulting object will be again a polygonal region whose quotient space is homoemorphic to the original one.

Finally, for completeness and also for further reference, we describe two operations that are obtained by composing the operations of cutting/glueing with that of folding/unfolding:

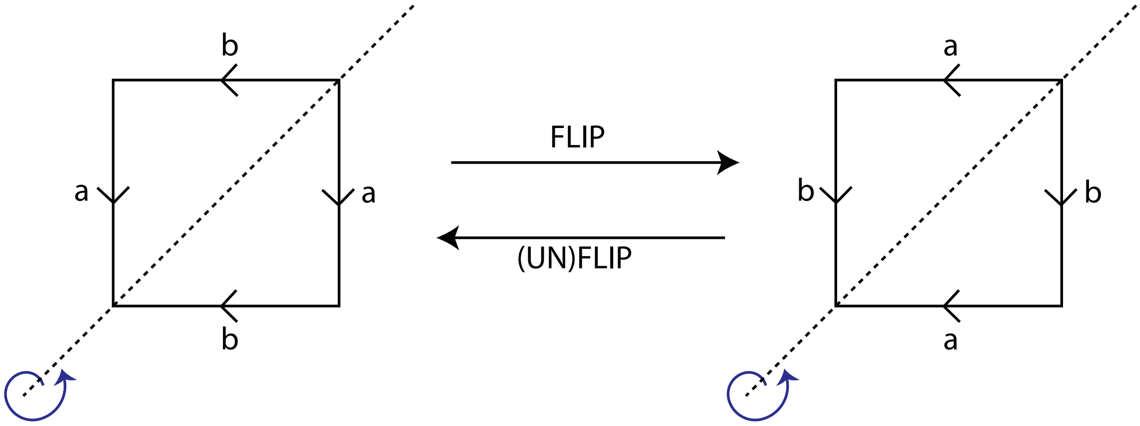


FIGURE 16. Flip/Unflip

VIII) Cancel: We may replace a labeling scheme of the form $y_0aa^{-1}y_1$ by y_0y_1 provided that a does not appear elsewhere in the labeling scheme, and both y_0 and y_1 have length at least 2. Geometrically, this operation is represented by the sequence of diagram in Figure 17.

IX) Uncancel: Under the same conditions as above, we may reverse the operation of canceling by replacing a scheme y_0y_1 by the labeling scheme $y_0aa^{-1}y_1$, as indicated in Figure 17.

It should clear that the operations above leave the quotient space X unchanged. Thus, it is natural to pose the following definition:

DEFINITION 1.22. *Two labeling schemes are* ***equivalent*** *if one can be obtained from the other by applying the operations (I) - (IX) described above.*

EXERCISE 1.7. Show that this defines an equivalence relation on the set of all labeling schemes.

EXAMPLE 1.23. *We have seen that the Klein bottle is the quotient of the unit square by the identification whose labeling scheme is* aba^-1b. *Let us prove that the Klein bottle is homeomorphic to the connected sum of two projective spaces:*

$$\begin{aligned}
aba^{-1}b \; &\longrightarrow abc \quad \& \quad c^{-1}a^{-1}b && \textit{(cutting)}\\
&\longrightarrow cab \quad \& \quad b^{-1}ac && \textit{(permuting and flipping)}\\
&\longrightarrow cabb^{-1}ac && \textit{(glueing)}\\
&\longrightarrow caac && \textit{(canceling)}\\
&\longrightarrow aacc && \textit{(permuting)}.
\end{aligned}$$

6. Geometric Surfaces

In this section we will consider surfaces which are obtained from a polygonal region by identifying it edges in pairs. We wil then show that every such surface is homeomorphic to one in the list given in Theorem 1.9.

DEFINITION 1.24. *A compact and connected topological surface X is called a* ***geometric surface*** *if it can be obtained from a polygonal region by glueing its edges in pairs.*

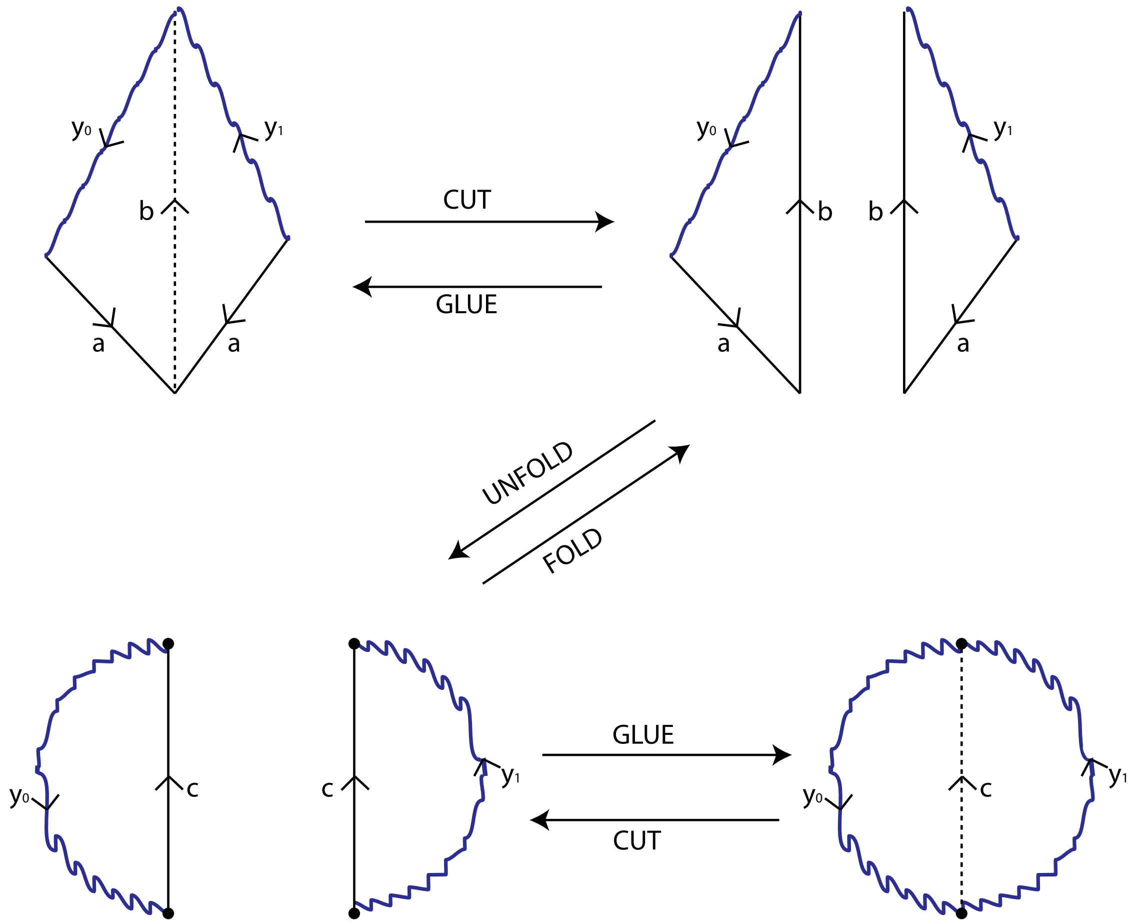


Figure 17. Cancel/Uncancel

The remainder of this section will be dedicated to proving the following theorem:

Theorem 1.25 (Classification of Geometric Surfaces). *Let X be a geometric surface. Then X is homeomorphic to one of the following: $\mathbb{S}^2$, $\mathbb{T}_g$, or $\mathbb{P}_h$ (for some $g, h \in \mathbb{N}$).*

The idea of the proof is to consider labeling schemes which give rise to geometric surfaces (known as proper labeling schemes) and then to show that any proper labeling scheme can be put into a normal form by means of the operations introduced in the last section.

Definition 1.26. *A labeling scheme $w_1, \ldots, w_m$ (for m polygonal regions) is called a* ***proper labeling scheme*** *if each label appears exactly twice in the scheme.*

Remark 1.27. We note that if we start with a proper labeling scheme, then by applying any of the operations introduced in the preceding section gives rise to another proper labeling scheme.

We can now restate Theorem 1.25 into a more algebraic form:

Theorem 1.28 (Normal Forms of Proper Labeling Schemes). *Let w be a proper labeling scheme of length greater or equal to 4 (of a single polygonal region). Then w is equivalent to one of the following labeling schemes:*

(1) $aa^{-1}bb^{-1}$,

(2) $abab$,
(3) $(a_1b_1a_1^{-1}b_1^{-1})\cdots(a_gb_ga_g^{-1}b_g^{-1})$, *or*
(4) $(a_1a_1)(a_2a_2)\cdots(a_ha_h)$.

REMARK 1.29. *Of course, in the list above (1) is a sphere, (2) is a projective space, (3) is a connected sum of tori, and (4) is a connected sum of projective spaces.*

The first step in the proof of Theorem 1.28 is to distinguish between two classes of proper labelings that will then be treated separately:

DEFINITION 1.30. *Let w be a proper labeling scheme for a single polygonal region. If every label of w appears one with an exponent $+1$ and once with exponent -1 we say that w is of* ***torus type****. Otherwise, we say that w is of* ***projective type****.*

We begin by dealing with labeling schemes of projective type:

PROPOSITION 1.31. *Let w be a labeling scheme of projective type. The w is equivalent to a labeling scheme of the following form:*

$$w \sim (a_1a_1)\cdots(a_ka_k)w_1,$$

where w_1 is a labeling scheme of torus type.

The proof of this proposition will follow from the following lemma:

LEMMA 1.32. *If w is a proper labeling scheme of the form $w = [y_0]a[y_1]a[y_2]$, where each $[y_i]$ is a string of labels (which may be empty), the w is equivalent to a labeling scheme of the form*

$$w \sim aa[y_0y_1^{-1}y_2].$$

PROOF. We separate the proof into two cases:
Case 1: $[y_0] = \emptyset$**.** In this case $w = a[y_1]a[y_2]$.

- If $[y_1]$ is empty, then we are done.
- If $[y_2]$ is empty, the we proceed as follows:

$$\begin{aligned} w = a[y_1]a \quad &\longrightarrow a^{-1}[y_1^{-1}]a^{-1} && \text{(flipping)} \\ &\longrightarrow a^{-1}a^{-1}[y_1^{-1}] && \text{(permuting)} \\ &\longrightarrow aa[y_1^{-1}] && \text{(reversing orientation of } a). \end{aligned}$$

- If both $[y_1]$ and $[y_2]$ are not empty, the we apply the operations described in Figure 18.

Case 2: $[y_0] \neq \emptyset$**.** Again we exclude the most trivial case first. If both $[y_1]$ and $[y_2]$ are empty, then $w = [y_0]aa$ and a permutation brings w to the desired form. Assume now that either $[y_1]$

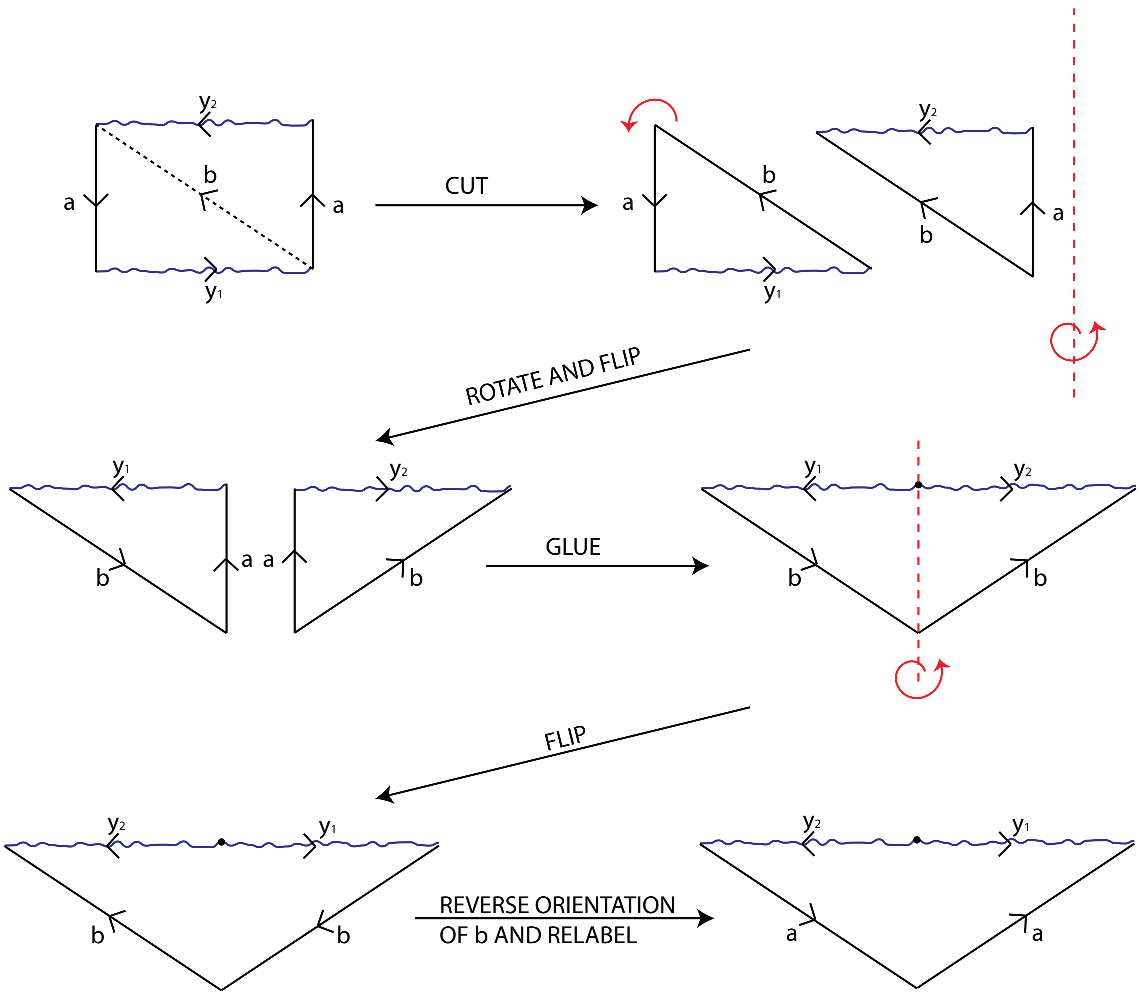


FIGURE 18. Case 1

or $[y_2]$ are non-empty. Then:

$$\begin{aligned}
w = [y_0]a[y_1]a[y_2] &\longrightarrow [y_0]ab \;\&\; b^{-1}[y_1]a[y_2] && \text{(cutting)}\\
&\longrightarrow [y_0^{-1}]b^{-1}a^{-1} \;\&\; a[y_2]b^{-1}[y_1] && \text{(flipping and permuting)}\\
&\longrightarrow [y_0^{-1}]b^{-1}[y_2]b^{-1}[y_1] && \text{(glueing and canceling)}\\
&\longrightarrow b^{-1}[y_2]b^{-1}[y_1y_0^{-1}] && \text{(permuting)}\\
&\longrightarrow b^{-1}b^{-1}[y_2^{-1}y_1y_0^{-1}] && \text{(case 1)}\\
&\longrightarrow [y_0y_1^{-1}y_2]bb && \text{(flipping)}\\
&\longrightarrow aa[y_0y_1^{-1}y_2] && \text{(permuting and relabeling).}
\end{aligned}$$

□

EXERCISE 1.8. Write the algebraic sequence of arguments presented in the proof of case 1, and make diagrams to describe the geometric sequence of arguments presented in case 2 of the proof.

PROOF (OF PROPOSITION 1.31). Let w be a labeling scheme of projective type. Then there is at least one label of w which appears twice with the same sign. Thus,

$$w = [y_0]a[y_1]a[y_2]$$

and by using the lemma, we obtain that w is equivalent to $aa[y_0y_1^{-1}y_2]$. If $[y_0y_1^{-1}y_2]$ is of torus type then we are done. Otherwise, there is a label b is $[y_0y_1^{-1}y_2]$ which appears twice with the same sign, and thus we may assume that w is equivalent to

$$w \sim aa[z_0]b[z_1]b[z_2].$$

We apply the lemma again, this time to the labeling scheme $[aaz_0]b[z_1]b[z_2]$, to obtain that

$$w \sim bbaa[z_0z_1^{-1}z_2].$$

If $[z_0z_1^{-1}z_2]$ is of torus type we are done. Otherwise we continue this process which will end as soon as we have put w into the desired form $w \sim (a_1a_1)\ldots(a_ka_k)w_1$ with w_1 a labeling scheme of torus type. □

REMARK 1.33. *We can conclude from Proposition 1.31 that if w is a proper labeling scheme, then either: (1) w is of torus type, or (2) w is of the form $(a_1a_1)\ldots(a_ka_k)w_1$ with w_1 a labeling scheme of torus type, or (3) w is of the form $(a_1a_1)\ldots(a_ka_k)$ in which case we are done (X is a connected sum of projective spaces).*

We must now examine how to reduce w to a simpler form when w is of the form (1) or (2).

EXERCISE 1.9. Show that if w is a proper labeling scheme of length 4, then w must be equivalent to one of the following labeling schemes:

$$aabb, \quad abab \quad aa^{-1}bb^{-1}, \quad aba^{-1}b^{-1}$$

From now on we assume that w has length greater then 4, and moreover, that it is **irreducible**, i.e., it does not contain any adjacent terms having the same label, but opposite signs (in which case we could perform the operation of canceling to reduce the length of w). In this case we have the following lemma:

LEMMA 1.34. *Suppose that w is a proper labeling scheme of the form $w = w_0w_1$, where w_1 is an irreducible scheme of torus type. Then w is equivalent to a scheme of the form w_0w_2, where w_2 has the same length as w_1, and has the form:*

$$w_2 = aba^{-1}b^{-1}w_3,$$

where w_3 is of torus type or is empty.

PROOF. We will divide the proof of this lemma into several steps.

Step 1: We may assume that w is of the form

$$w = w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5],$$

where some of the strings of labels $[y_i]$ may be empty.

To see this we proceed as follows. Let a be the label in w_1 whose occurrences are as close as possible (with the minimal amount of labels in between them). If a appears first with an exponent -1, then we revert the orientation of both appearances of a. Next, let b be any label in between a and a^{-1}. Then, since a and a^{-1} are the closest labels to each other in w_1, it follows that either b^{-1} appears after a^{-1}, in which case we are done, or b^{-1} appears in front of a, in which case we simply exchange the labels of b and a.

Step 2: (1st Surgery) w is equivalent to

$$w \sim w_0 a[y_2]b[y_3]a^{-1}[y_1y_4]b^{-1}[y_5].$$

We may assume that $[y_1] \neq \emptyset$ (or else there is nothing to prove). Then, we perform the following operations:

$$\begin{aligned}
w &= w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5] \\
&\longrightarrow [y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5]w_0c \ \& \ c^{-1}[y_1]a && \text{(permuting and cutting)} \\
&\longrightarrow [y_4]b^{-1}[y_5]w_0c[y_2]b[y_3]a^{-1} \ \& \ ac^{-1}[y_1] && \text{(permuting)} \\
&\longrightarrow [y_4]b^{-1}[y_5]w_0c[y_2]b[y_3]c^{-1}[y_1] && \text{(glueing)} \\
&\longrightarrow w_0a[y_2]b[y_3]a^{-1}[y_1y_4]b^{-1}[y_5]. && \text{(permuting and relabeling)}
\end{aligned}$$

Step 3: (2nd Surgery) w is equivalent to

$$w \sim w_0a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5].$$

First of all, assume that w_0, y_1, y_4, and y_5 are all empty. Then

$$w \sim a[y_2]b[y_3]a^{-1}b^{-1}$$

and the result follows by permuting and relabeling.

Now assume that at least one of the strings w_0, y_1, y_4, or y_5 is non-empty. Then, we can perform the following sequence of operations:

$$\begin{aligned}
w &\sim w_0a[y_2]b[y_3]a^{-1}[y_1y_4]b^{-1}[y_5] \\
&\longrightarrow a[y_2]b[y_3]a^{-1}c \ \& \ c^{-1}[y_1y_4]b^{-1}[y_5]w_0 && \text{(permuting and cutting)} \\
&\longrightarrow [y_3]a^{-1}ca[y_2]b \ \& \ b^{-1}[y_5]w_0c^{-1}[y_1y_4] && \text{(permuting)} \\
&\longrightarrow [y_3]a^{-1}ca[y_2][y_5]w_0c^{-1}[y_1y_4] && \text{(glueing)} \\
&\longrightarrow w_0c^{-1}[y_1y_4y_3]a^{-1}ca[y_2y_5] && \text{(permuting)} \\
&\longrightarrow w_0a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5]. && \text{(relabeling)}
\end{aligned}$$

Step 4: (3nd Surgery) w is equivalent to

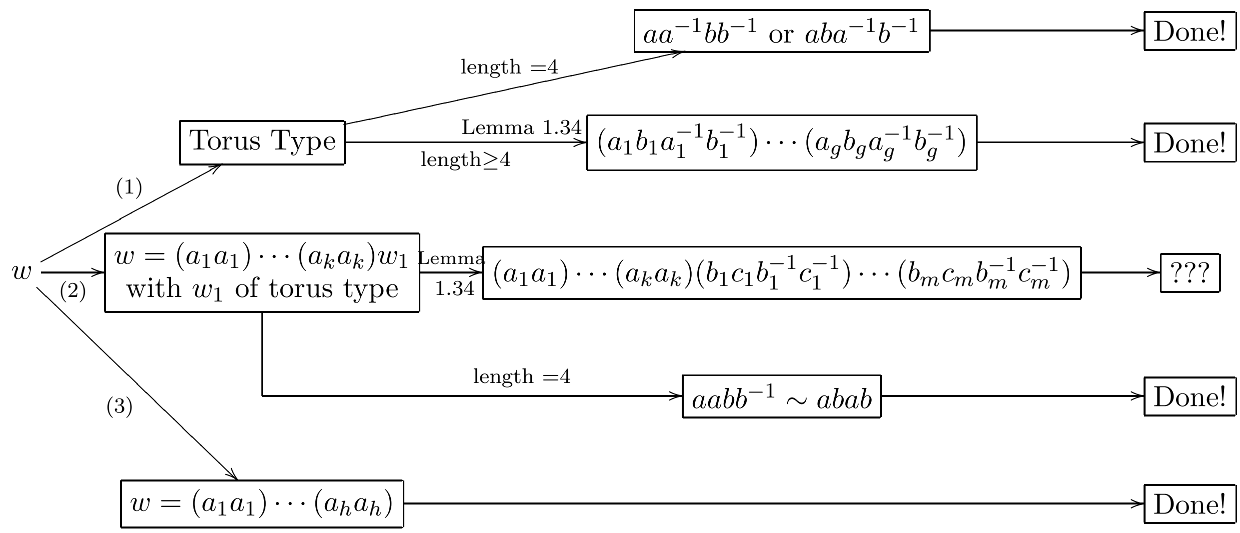
$$w \sim w_0aba^{-1}b^{-1}[y_1y_4y_3y_2y_5].$$

We perform the following sequence of operations:

$$
\begin{aligned}
w &\sim w_0 a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5] \\
&\longrightarrow [y_1y_4y_3]ba^{-1}c \;\&\; c^{-1}b^{-1}[y_2y_5]w_0a && \text{(permuting and cutting)} \\
&\longrightarrow a^{-1}c[y_1y_4y_3]b \;\&\; b^{-1}[y_2y_5]w_0ac^{-1} && \text{(permuting)} \\
&\longrightarrow a^{-1}c[y_1y_4y_3][y_2y_5]w_0ac^{-1} && \text{(glueing)} \\
&\longrightarrow w_0aba^{-1}b^{-1}[y_1y_4y_3y_2y_5]. && \text{(permuting and relabeling)}
\end{aligned}
$$

□

The following graph summarizes the results that we have obtained so far:



Remark 1.35. *The three arrows coming out of w correspond to remark 1.33, while the cases where the length of w is equal to 4 follow from exercise 1.9.*

Thus, in order to conclude the proof of Theorem 1.28 we need to describe what the connected sum of tori and projective spaces correspond to, i.e., to reduce

$$w = (a_1a_1)\cdots(a_ka_k)(b_1c_1b_1^{-1}c_1^{-1})\cdots(b_mc_mb_m^{-1}c_m^{-1})$$

to its normal form. This follows from the following lemma:

Lemma 1.36. *If $w = w_0(aa)(bcb^{-1}c^{-1})w_1$ is a proper scheme, then*

$$w \sim w_0(aabbcc)w_1.$$

Proof. We will make use repeatedly of Lemma 1.32 which states that

$$[y_0]a[y_1]a[y_2] \sim aa[y_0y_1^{-1}y_2].$$

To prove the lemma we consider the following sequence of operations:

$$\begin{aligned} w &= w_0(aa)(bcb^{-1}c^{-1})w_1 \\ &\longrightarrow (aa)[bc][cb]^{-1}[w_1w_0] && \text{(permuting)} \\ &\longrightarrow [bc]a[cb]a[w_1w_0] && \text{(Lemma 1.32)} \\ &\longrightarrow [b]c[a]c[baw_1w_0] && \text{(regrouping the terms)} \\ &\longrightarrow [cc]b[a^{-1}]b[aw_1w_0] && \text{(Lemma 1.32 and regrouping the terms)} \\ &\longrightarrow w_0(bbccaa)w_1. && \text{(Lemma 1.32 and permuting)} \end{aligned}$$

The result then follows by relabeling the terms. □

We can thus conclude from the Lemma, by applying it several times if necessary, and then relabeling, that

$$(a_1a_1)\cdots(a_ka_k)(b_1c_1b_1^{-1}c_1^{-1})\cdots(b_mc_mb_m^{-1}c_m^{-1}) \sim (a_1a_1)\cdots(a_{k+2m}a_{k+2m}).$$

This finishes the proof of Theorem 1.28.

Exercise 1.10. Throughout this section we have implicitly described an algorithm to reduce any proper labeling scheme to one in the normal form of Theorem 1.28. Write down this algorithm explicitly.

Exercise 1.11. Use the algorithm you developed in the exercise above to determine which surface corresponds to the following labeling schemes:

(1) $abacb^{-1}c^{-1}$
(2) $abca^{-1}cb$
(3) $abbca^{-1}ddc^{-1}$
(4) $abcda^{-1}c^{-1}b^{-1}d^{-1}$
(5) $abcdabdc$
(6) $abcda^{-1}b^{-1}c^{-1}d^{-1}$