To prove the lemma we consider the following sequence of operations:

$$
\begin{aligned}
w &= w_0(aa)(bcb^{-1}c^{-1})w_1 \\
&\longrightarrow (aa)[bc][cb]^{-1}[w_1w_0] && \text{(permuting)} \\
&\longrightarrow [bc]a[cb]a[w_1w_0] && \text{(Lemma 1.32)} \\
&\longrightarrow [b]c[a]c[baw_1w_0] && \text{(regrouping the terms)} \\
&\longrightarrow [cc]b[a^{-1}]b[aw_1w_0] && \text{(Lemma 1.32 and regrouping the terms)} \\
&\longrightarrow w_0(bbccaa)w_1. && \text{(Lemma 1.32 and permuting)}
\end{aligned}
$$

The result then follows by relabeling the terms. □

We can thus conclude from the Lemma, by applying it several times if necessary, and then relabeling, that

$$(a_1a_1)\cdots(a_ka_k)(b_1c_1b_1^{-1}c_1^{-1})\cdots(b_mc_mb_m^{-1}c_m^{-1}) \sim (a_1a_1)\cdots(a_{k+2m}a_{k+2m}).$$

This finishes the proof of Theorem 1.28.

Exercise 1.10. Throughout this section we have implicitly described an algorithm to reduce any proper labeling scheme to one in the normal form of Theorem 1.28. Write down this algorithm explicitly.

Exercise 1.11. Use the algorithm you developed in the exercise above to determine which surface corresponds to the following labeling schemes:

(1) $abacb^{-1}c^{-1}$
(2) $abca^{-1}cb$
(3) $abbca^{-1}ddc^{-1}$
(4) $abcda^{-1}c^{-1}b^{-1}d^{-1}$
(5) $abcdabdc$
(6) $abcda^{-1}b^{-1}c^{-1}d^{-1}$

7. Triangulated Surfaces

In this section we will introduce the notion of a triangulation on a compact Hausdorff topological space. We will then show that:

Theorem 1.37. *Any triangulated compact connected surface can be obtained from a single polygonal region by identifying its edges with respect to a proper labeling scheme.*

Theorem 1.9 will then follow from Theorem 1.28 and the following result which we will not prove now (but we intend to come back to it if time permits).

Theorem 1.38. *Every compact connected surface is triangulable (i.e., can be triangulated).*

We begin with the definition of a triangulation:

Definition 1.39. Let X be a compact Hausdorff topological space.

- A **curved triangle** in X is a subspace A together with a homeomorphism $h : T \to A$, where T is a triangular region in the plane. If $v \in T$ is a vertex, then $h(v)$ is called a **vertex** of A. Similarly, if $e \in T$ is an edge, then $h(e)$ is called an **edge** of A.

- A **triangulation** of X is a collection $A_1, \ldots, A_n$ of curved triangles of X which cover X,

$$\cup_i A_i = X,$$

  and such that if $A_i \cap A_j \neq \emptyset$, then either
  (1) $A_i \cap A_j = \{v\}$ is a vertex, or
  (2) $A_i \cap A_j = e$ is an edge, and furthermore, the map $h_j^{-1} \circ h_i$ which maps the edge $h_i^{-1}(e)$ of T_i to the edge $h_j^{-1}(e)$ of T_j is a linear homeomorphism.

Remark 1.40. *By condition (2) in the definition above we mean the that if we pick an orientation on $h_i^{-1}(e)$ then $h_j^{-1} \circ h_i$ induces a choice of orientation on $h_j^{-1}(e)$ for which $h_j^{-1} \cap h_i : h_i^{-1}(e) \to h_j^{-1}(e)$ becomes a positive linear map.*

For an example of a triangulation of the sphere, see Figure 19.

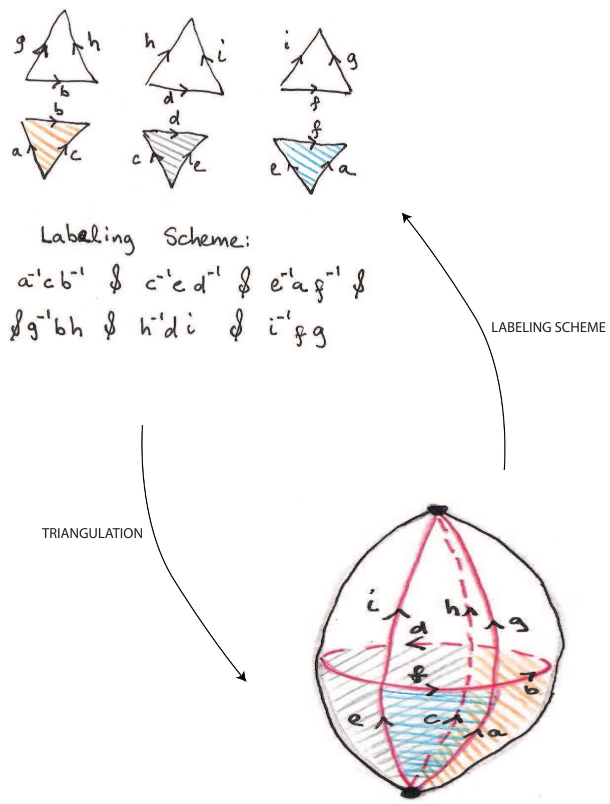

Figure 19.

Exercise 1.12. For each of the following spaces exhibit an explicit triangulation.

(1) A torus
(2) a cylinder
(3) a cone
(4) a projective space
(5) a Möbius band,
(6) a Klein bottle

A triangulation $\{A_1, \dots, A_n\}$ of a compact Hausdorff space X induces a labeling scheme, which will be called the **labeling scheme of the triangulation**, as follows:

Polygonal Regions: For each curved triangle A_i, let $h_i : T_i \to A_i$ be the corresponding homeomorphism defined on the triangular region T_i. The polygonal region we will consider is the disjoint union of the triangles T_i's.

Orientation on Edges: Let $e \subset X$ be an edge appearing in the triangulation (i.e., it is the edge of at least one of the curved triangles). Let v and w be the vertices at the endpoints of e. Choose an orientation on e by declaring it to go from v to w. Then if $h_i^{-1}(e)$ is an edge of T_i, we orient it from $h_i^{-1}(v)$ to $h_i^{-1}(w)$.

Labels: Let

$$\Lambda = \{e \subset X : e \text{ is the edge of (at least) one of the curved triangles}\}$$

be the set of edges of the triangulation. Then if $h_i^{-1}(e)$ is an edge of T_i, we associate to it the label $e \in \Lambda$.

EXAMPLE 1.41. Figure 19 exhibits a triangulation on the sphere, and also the labeling scheme of the triangulation.

EXERCISE 1.13. For each of the spaces in exercise 1.12, determine the labeling scheme of the triangulation.

PROPOSITION 1.42. *If X is a compact triangulated surface, then the labeling scheme of the triangulation is proper.*

PROOF. We need to show that each label appears exactly twice in the labeling scheme. The arguments needed to do this are intuitively clear. However, the easiest way to make them precise is by using the notion of fundamental group. Thus, we will sketch the proof now, but leave the details as an exercise that should be done after the fundamental group is introduced.

The first step is to show that each label appear at least twice in the labeling scheme. Thus, assume that a label appears only once. This means that there is an edge e which is the edge of only one curved triangle. Exercise 1.14 bellow shows that this cannot happen. The intuitive idea is that if $x \in e$, then by removing x we will not "create a hole" in X, but on the other hand, if we remove any point from an open set in $\mathbb{R}^2$, then we do "create a hole"

The next step is to show that there is at most two appearances of each label. Again this will follow by a "removing one point trick". If a label appears more than twice, then there are more than two triangles which intersect in a single edge. Intuitively, this will mean that there is a "multiple corner" which cannot be smoothened into an open subset of $\mathbb{R}^2$ (see figure 20). The precise argument is given in exercise 1.15 bellow.

$\square$

EXERCISE 1.14 (To be done after the definition of homotopy). Let T be a triangle and $x \in T$ be a point in one of the edges of T and let U be any neighborhood of x. Show that any loop in $U - x$ is homotopic to a constant path. Conclude that x does not have any neighborhood which is homeomorphic to an open set of $\mathbb{R}^2$.

EXERCISE 1.15 (To be done after the Seifert van Kampen Theorem). Consider the space obtained by glueing together k triangles along a common edge e, with $k > 2$ (Figure 20 shows the case when $k = 3$). Let $x \in e$ be a point in this common edge. Show that any neighborhood U of x contains a possibly smaller neighborhood $V \subset U$ such that $V - x$ is homotopy equivalent to a bouquet of $k - 1$ circles. Conclude by computing the fundamental group of $V - x$ that x does not have any neighborhood which is homeomorphic to an open subset of $\mathbb{R}^2$.

PROPOSITION 1.43. *If X is a compact triangulated surface, then X is homeomorphic to the space obtained from $\coprod T_i$ by glueing its edges according to the labeling scheme of the triangulation.*

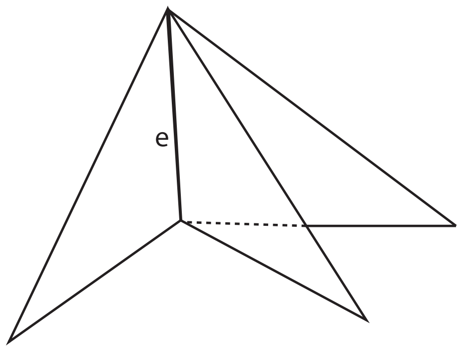

FIGURE 20.

EXERCISE 1.16. Consider the map $h : \coprod T_i \to X$ obtained by putting together all of the maps $h_i : T_i \to A_i \subset X$. Consider the space X' obtained by identifying two points p and q of $\coprod T_i$ if and only if $h(p) = h(q)$. Show that X' is homeomorphic to X.

PROOF. Let us denote Y the quotient space obtained from $\coprod T_i$ by identifying it edges with respect to the labeling scheme of the triangulation. It is an immediate consequence of the exercise above, that h factors through a continuous map $f : Y \to X$, i.e.,

$$\begin{tikzcd} \coprod_i T_i \arrow[d] \arrow[dr, "h"] & \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Moreover, since h is surjective, it follows that f is also surjective. Thus, in order to prove the proposition, it suffices to show that f is injective (because Y is compact and X is Hausdorff).

Let us denote by $[p] \in Y$ the equivalence class – *with respect to the labeling scheme of the triangulation* – of a point p in $\coprod T_i$. Assume that $f([p]) = f([q])$, for some $p \neq q$. Then, by definition, it follows that $h(p) = x = h(q)$. Thus, either x belongs to some edge e of the triangulation on X, in which case it is clear that $[p] = [q]$, or x is a vertex. In this case, in order to show that $[p] = [q]$ (so that f is injective) we must verify that the identification of p with q is "forced" as a consequence of the identification of the edges of the triangles T_i's (see also exercise 1.20).

Suppose that A_i and A_j intersect at a vertex v. What we need to show is that we can find a sequence

$$A_i = A_{i_1}, A_{i_2}, \dots A_{i_m} = A_j,$$

such that A_{i_k} intersects $A_{i_{k+1}}$ on a common edge which contains v as its endpoint (as illustrated in Figure 21). This is the content of the following exercise.

□

EXERCISE 1.17. Given v, define two curved triangles A_i and A_j with vertex v to be equivalent if we can find a sequence A_{i_k} as above. Use the "remove the one point trick" to show that if there is more that one equivalence class of curved triangles with vertex v, then v does not have any neighborhood in X which is homeomorphic to an open set of $\mathbb{R}^2$, and thus X is not a surface.

We now are ready to finish the proof of Theorem 1.37. What we will show is that we may glue the triangles T_i together in order to obtain the desired polygonal region. In fact, start by

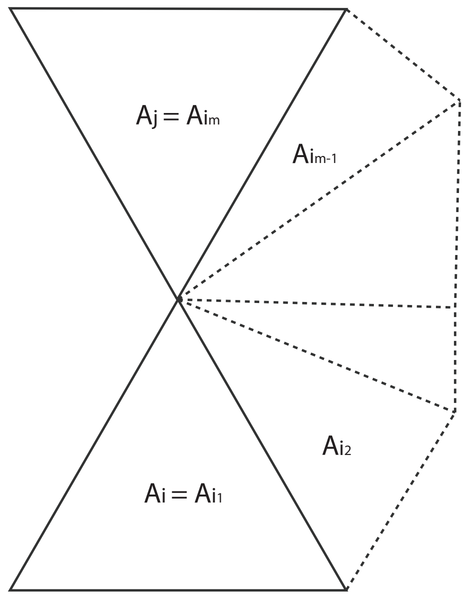

FIGURE 21.

choosing one of the triangles, say T_1. If T_i is another triangle which has a label on one of its edges which is equal to a label of T_1, then (after possibly flipping T_i) we may glue both triangles together. The effect of this is to reduce the original number of triangles by two, at the expense of adding one polygonal region (which in this case has 4 sides) which we denote by P_1. Next, we look at the edges of P_1. If one of the triangles T_j, with $j \neq 1, i$, has a label equal to one of the labels of P_1, then, after flipping T_j if necessary, we may glue it to P_1 obtaining in this way a new polygonal region P_2. We continue this process as long as we have two polygonal regions containing edges that have a common label.

At some point we will reach a situation where either we obtain the polygonal region P_{n+1} that we were looking for, or we obtain more then one disconnected polygonal region in which none of the labels appearing in one of them appear also in the other region. However, it is easy to see that this cannot happen, for in this case the quotient space X will necessarily be disconnected.

EXERCISE 1.18. Determine the space obtained from the following labeling schemes:

(1) abc, dae, bef, cdf.
(2) abc, cba, def, dfe^{-1}.

EXERCISE 1.19. Show that the projective space $\mathbb{P}^2$ can be obtained from two Mobius bands by glueing them along there boundary.

EXERCISE 1.20. Let X be the space obtained from a sphere by identifying its north and south poles (X is not a surface). Find a triangulation on X such that the labeling scheme of the triangulation determines a sphere (i.e., the surface obtained by glueing the edges of the triangles with respect to the labeling scheme of the triangulation is homeomorphic to a sphere). Conclude that two non-homeomorphic compact Hausdorff spaces can have triangulations which induce the same labeling scheme. (We remark that this exercise gives an example of a triangulated space for which the map f from the proof of Proposition 1.43 is not injective.)