Fibrations in symplectic and Poisson geometry

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Introduction

Abstract

Connections and fibre bundles A fibre bundle of fibre type F is a smooth mapping $\pi : M \to B$ between manifolds, such that for every $b \in B$ the inverse image $\pi^{-1}(b)$ is isomorphic with F. A fibre bundle is often visualised as a map going downwards:

$$\begin{array}{c} M \\ \big\downarrow{\scriptstyle \pi} \\ B \end{array}$$

One may view the fibres then as hanging vertically above the base manifold B. We call the tangent spaces to the fibre vertical. There is, however, no intrinsic way to determine the horizontal direction. This leads to the notion of a connection. The idea of a connection on a fibre bundle goes back to Ehresmann, who published his article in 1950, [10]. An Ehresmann connection Γ on a fibration $\pi : M \to B$ is a distribution Hor on M, such that

1. for all $x \in M$ the tangent a space T_xM is decomposed in Hor_x and the tangent space to the fibre Vert_x.

2. for every piecewise smooth path $\gamma : [0,1] \to B$ starting from b and for each $x \in \pi^{-1}(b)$ there is a piecewise smooth path $\gamma^{\#} : [0,1] \to M$ starting at x such that $\pi \circ \gamma^{\#} = \gamma$, and $\frac{d}{dt}\gamma^{\#}(t) \in \mathrm{Hor}_{\gamma^{\#}(t)}$, for all $t \in [0,1]$, where it is defined.

$\gamma^{\#}$ is called the horizontal lift of γ starting at x. This construction induces for every path γ in B a diffeomorphism $\phi_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$, called parallel transport. In this thesis we study fibre bundles for which the fibre has a special structure, e.g. it is a symplectic or Poisson manifold, and we study connections on them for which the parallel transport preserves this structures. Parallel transport with respect to a loop γ based at $b \in B$ is a diffeomorphism from $\pi^{-1}(b)$ to itself. All diffeomorphisms which can be constructed in this way form a group $\phi(b)$, called the holonomy group. An Ehresmann connection is a distribution. The obstruction for being an integrable distribution is measured by the so called curvature 2-form. One may see the curvature Ω as a map:

$$\Omega : \mathfrak{X}(B) \times \mathfrak{X}(B) \to \mathfrak{X}_{\mathrm{Vert}}(M).$$

A visualisable description of the curvature is the following: consider the parallel transport along a contractible loop in B. If we contract such loop we get a vector field on the fibre. Vector fields which can be obtained in this way are in the image of the curvature.

Dirac structures A more recent development in mathematics is the introduction of Dirac structures on manifolds by T.J. Courant in 1990 [6]. An almost Dirac structure on a manifold M is a smooth subbundle L of the generalised tangent bundle of M:

$$L \subset \mathcal{T}M = TM \oplus T^*M,$$

such that the rank of L is the dimension of M, and L is isotropic with respect to the natural paring. In particular, graphs of 2-forms, and graphs of bivectors are almost Dirac structures. An almost Dirac structure is called a Dirac structure if, moreover, the sections of L are closed under the so called Courant bracket $[,]_c$. The graphs of 2-forms, and bivectors are Dirac if, and only if they are respectively closed, and Poisson.

An almost Dirac structures of special type on the total space M of a fibration $\pi : M \to B$ induces an Ehresmann connection. Moreover, such an almost Dirac structure induces a horizontal 2-form, and a vertical bivector. If such an almost Dirac structure is really a Dirac structure, then this 2-form is closed, and this vertical bivector is Poisson.

Conversely, all Ehresmann connections give rise to such an almost Dirac structure. It turns out that an Ehresmann connection which can be described by a Dirac structure has a very restrictive holonomy.

Connections in symplectic geometry A symplectic structure ω on a manifold M is a nondegenerate closed 2-form. Not all manifolds admit symplectic structures. For instance, symplectic manifolds must be orientable and even dimensional. Connections on fibrations of fibre type a symplectic manifold, for which the parallel transport preserves this symplectic structure, are called symplectic connections. A symplectic connection can be described by an almost Dirac structure L_τ which is the graph of an compatible 2-form τ on the total space. These are 2-forms which restriction to the fibres coincides with the given symplectic structure on the fibres. A natural question to ask is whether we can find a compatible symplectic structure on the total space. In particular, then L_τ is a Dirac structure. As said before, this restricts the holonomy of the connection, even if we only ask τ to be closed. The main theorem on connections in symplectic geometry comes down to is the following equivalence: there exists a closed 2-form $\tau \in \Omega^2(M)$ such that the induced connection $\Gamma_\tau = \Gamma$ is a symplectic connection, if and only if the holonomy group $\phi(b)$ of Γ around any contractible loop in B is acts on the fibre $\pi^{-1}(b)$ in a Hamiltonian fashion.

Connections in Poisson geometry A Poisson structure on a manifold M is a Lie bracket $\{.,.\}$ on $C^\infty(M)$, which satisfies a Leibniz rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Equivalently, a Poisson structure on M is a bivector π on M, for which the Schouten-Nijenhuis bracket $[\pi, \pi]$ vanishes. In particular, all symplectic manifolds are Poisson manifolds. A remarkable property of Poisson manifolds is that they are foliated into symplectic manifolds.

Meaning that there is a partition of (M, π) of immersed submanifolds $i : L \hookrightarrow M$ all carrying an induced symplectic structure. Connections on fibrations of fibre type a Poisson manifold, for which the parallel transport preserves the Poisson structure, are called Poisson connections. In particular, the total space of a Poisson fibration does always admit a Poisson structure. One simply glues the Poisson structures on the fibres all together and obtains a vertical Poisson bivector π_V. Recall that fibre nondegenerate almost Dirac structures also induce vertical bivectors. A fibre nondegenerate almost Dirac structure for which these bivectors coincide is called a compatible almost Dirac structure. The main theorem on connections in Poisson geometry comes down to is the following equivalence: there exists a compatible Dirac structure for which the induced connection is a Poisson connection if and only if if the holonomy group $\phi(b)$ with respect to this connection acts on the fibre in a Hamiltonian fashion. Clearly, this is a generalisation of the main theorem for symplectic connections.

Principal bundles A special class of fibre bundles is formed by the principal bundles $\pi : P \to M$. The fibre type of these bundles are Lie groups G. Principal bundles are very useful for several reasons. A principal connection is an Ehresmann connection for which the diffeomorphisms obtained by parallel transport are group isomorphisms. They may be described using a G-equivariant $\mathfrak{g}$-valued 1-form on the total space. Also the curvature may be seen here as a $\mathfrak{g}$-valued 2-form on P. By the G-equivariance of the connection one can describe the vector field in the image of the curvature here with a single element in $\mathfrak{g}$. The holonomy group $\phi(u)$ of a connection based a point $u \in P$ is a Lie subgroup of G. A theorem of Ambrose and Singer [1] states that the Lie algebra of $\phi(u)$ is generated by curvature of the connection.

The problem Another nice property of principal G-bundles is the following: if one has a G-space F, i.e. a manifold F on which G acts from the left, one can attach F to P, such that one gets a fibre bundle $P \times_G F$ of fibre type F. The other way round, we can construct a principal G-bundle $\mathcal{P} \to B$ of for any fibre bundle $\pi : M \to B$. G is here a subgroup of the group of diffeomorphisms of F, where F is the fibre type of the fibration. If we attach F to $\mathcal{P} \to B$, then we get our old frame bundle back. Moreover, an Ehresmann connection on $\pi : M \to B$ induces a principal connection on $\mathcal{P} \to B$, and vice versa. The main theorems for the symplectic and the Poisson connections can be proved by using the theorem of Ambrose and Singer for this new $\mathcal{P} \to B$. But here is a problem: in the prove of this theorem it is explicitly used that G is a finite dimensional group, while nothing guarantees us that that is the case here. Fortunately, we can do better. We can reduce $\mathcal{P} \to B$ to a principal $\phi(u)$-bundle. It is not clear whether $\phi(u)$ is finite dimensional, and if we indeed can prove our main theorems in full generality.

The structure of the thesis

This thesis contains four chapters. In the first chapter we give an introduction to symplectic, and Poisson geometry. We give an exposition on Hamiltonian group actions, and we introduce the concept of Lie algebroid. We show that Dirac as well as Poisson structures may be viewed as Lie algebroids. In the second chapter we explain Ehresmann connections and show how (almost) Dirac structures can be used to obtain them. We finish the chapter with a short excursion on Dirac gauging of the Dirac structures. In the third chapter we discuss vector bundles and principal bundles. Connections on vector bundles may be viewed as covariant derivatives. The fourth and the fifth chapter are devoted to the symplectic and the Poisson fibrations.

Conventions

In this thesis all manifolds are second countable and finite dimensional.

Acknowledgments

Contents

Chapter 1

Symplectic and Poisson geometry

Abstract In this thesis many of the concepts and the terminology of differential geometry, symplectic geometry and Poisson geometry are used. The aim of this first chapter is to give a short introduction in these fields, and at the same time to develop the notation. We restrict ourself in this chapter to that material we will really use later. This chapter is divided in four parts. In the first part, about differential geometry, we discuss the concepts of differential forms, exterior derivatives, covariant and contravariant vector fields and Lie derivatives. Most of the statements made here are well known results, and proofs are often omitted. The second part gives a short introduction into symplectic geometry. It includes symplectic manifolds, symplectomorphisms, Hamiltonian vector fields and Hamiltonian group actions. Also various examples of symplectic manifolds are given like the symplectic structure on the cotangent bundle of a manifold, the Fubini-Study structure, the orbit spaces of the coadjoint action of a Lie group on the dual of its Lie algebra, are included. The latter is used in the part on Poisson geometry, as an example of a symplectic foliation. Moreover we discuss the basics of Dirac geometry. Dirac structures of a special type will turn out to be very usefull in the study of connections.

1.1 Basic differential geometry

Fibre bundles

Definition 1.1.1. A fibre bundle $\pi : M \to B$ with fibretype a given manifold F is a smooth mapping between two manifolds with the following property:there is an open covering $\{U_i\}_i$ of B and for every i a diffeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$, such that the composition of ϕ_i with the projection on the first component $U_i \times F \to U_i$ is the mapping $\pi : \pi^{-1}(U_i) \to U_i$. These maps ϕ_i are called local trivialisations of the fibre bundle and the manifold B is called the base space. We summarize this definition in a diagram:

$$\begin{array}{ccc} M \supset \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_1} \\ U_i \subset B & \xrightarrow{\text{id}} & U_i \subset B. \end{array}$$

For every $b \in B$ we denote $\pi^{-1}(\{b\})$ by F_b and we call it the fibre of the fibration above b. For each $b \in U_i$ and all i, we obtain a diffeomorphism, $F_b \to \{b\} \times F$ by restricting ϕ_i to F_b. We conclude that all fibres are diffeomorphic with F. For all i, j we construct the map $\phi_{i,j} : U_i \cap U_j \to \mathrm{Diff}(F)$ by $b \mapsto \phi_j(b) \circ \phi_i^{-1}(b)$, where $\phi_i(b)$, is the restriction of ϕ_i to F_b followed by the projection to F. These maps are called the transition maps of the fibre bundle.

Definition 1.1.2. A vector bundle is a fibre bundle of the fibretype F a vector space, and where the transition maps are linear isomorphisms.

A trivial example of a vector bundle is the product space $M \times \mathbb{R}^n$, with the projection on M. Also the tangent bundle of a manifold is a vector bundle. Here is the fibre F_m the tangent space of M at m. A section of a vector bundle is a map $s : M \to E$ such that $\pi \circ s$ is the identity map on M. All sections together form a vector space which we will denote with $\Gamma(M, E)$, or, if the base is clear from the context over, with $\Gamma(E)$.

Vector fields An intuitive way to describe a vector field is by drawing an arrow on every point on your manifold, such that if one walks along the arrows with a speed proportional to the length of the arrow, one is not required to make impossible moves. An exact definition is that a vector field X is a smooth section of the tangent bundle. Indeed, X assigns to every point m of the manifold M a vector in the tangent space $T_m M$. These vectors are the arrows, and the smoothness takes care that 'the impossible moves' are excluded. The space of vector fields on a manifold M is so important, that it deserves an own notation; $\mathfrak{X}(M)$.

Differential forms There is a dual notion to vector fields, obtained by replacing the tangent bundle with the cotangent bundle T^*M. The fibre of the cotangent bundle over m is the dual T_m^*M of the tangent space $T_m M$. Smooth sections of the cotangent bundle are called differential 1-forms, or shortly 1-forms. The space of 1-forms on the manifold M is denoted with $\Omega^1(M)$. A vector field X and a 1-form α can be combined to a smooth function on M:

$$\alpha(X) : M \to \mathbb{R} : m \mapsto \alpha(m)(X(m)).$$

Therefore a 1-form α can also be seen as a map:

$$\alpha : \mathfrak{X}(M) \to C^\infty(M),$$

where $C^\infty(M)$ denotes the space of smooth functions on M. This map is $C^\infty(M)$-linear, that means that for all $f \in C^\infty(M)$, $f\alpha(X) = \alpha(fX)$. Dually, a vector field X can be seen as a $C^\infty(M)$-linear map:

$$X : \Omega^1(M) \to C^\infty(M).$$

The name 1-form for the sections of the cotangent bundle already suggest that there is a notion of n-forms, where n is a natural number. An n-form is a section of $\bigwedge^n T^*M$ the n-th exterior product of T^*M. For a discussion on the notion in of the exterior product we refer to [12] . The space with n-forms on M is denoted with $\Omega^n(M)$. There dual notion is that of n-vector fields, which are sections of the exterior product $\bigwedge^n TM$. The space of n-vector fields are denoted with $\mathfrak{X}^n(M)$. In particular $\Omega^n(M)$ and $\mathfrak{X}^n(M)$ are empty for all $n > \dim M$. Analogous to the contruction above we may see n-forms ω as $C^\infty(M)$-multilinear antisymmetric maps:

$$\omega : \underbrace{\mathfrak{X}(M) \times ... \times \mathfrak{X}(M)}_{\text{n-times}} \to C^\infty(M),$$

and n-vector fields π as $C^\infty(M)$-multilinear antisymmetric maps:

$$\pi : \underbrace{\Omega^1(M) \times ... \times \Omega^1(M)}_{\text{n-times}} \to C^\infty(M).$$

2-forms and bivectors, will be used frequently in this thesis.

The flow of a vector field Let M be a manifold, let $X \in \mathfrak{X}(M)$, and $m \in M$. We consider the following differential equation:

$$X(\gamma(t)) = \frac{d}{dt}(\gamma(t)), \qquad \gamma(0) = m. \quad (*)$$

γ is, a priori, a path in M. It follows from the general theory of ordinary differential equations that this equation has an unique maximal solution, i.e. there is an open interval $I_m \subset \mathbb{R}$ containing 0, and a path $\gamma : I_m \to \mathbb{R}$, being a solution of $(*)$, and all other solutions are restrictions of γ to an interval $J \subset I_m$. The path γ is called a integral curve of X through m.

Definition 1.1.3. A vector field X on M is complete if for each $m \in M$ the integral curve through m is defined for all of $\mathbb{R}$.

Remark 1.1.4. Vector fields on compact manifolds are always complete.

Definition 1.1.5. Let X be a complete vector field on a manifold M. Then the flow φ_X of X, is a map

$$\varphi_X : \mathbb{R} \times M \to M,$$

which satisfies the following properties:

1. $\varphi_X(t, -) : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$.
2. For all $m \in M$, $\varphi_X(-, m) : \mathbb{R} \to M$ is the solution curve of X through m.
3. $\varphi_X(t, -) \circ \varphi_X(s, -) = \varphi_X(t + s, -) = \varphi_X(s, -) \circ \varphi_X(t, -)$, for any $t, s \in \mathbb{R}$.

Such a maps are also called 1-parameter groups. Sometimes we denote the flow with φ^t_X.

Lie derivatives The idea behind Lie derivatives is to measure how functions, forms, and vector fields change under the flow of a vector field. The Lie derivative of functions, forms and vector field are still functions, forms and vector fields, respectively. We start with functions. For $f \in C^\infty(M)$, the Lie derivative of f with along a vector field X is defined by:

$$X(f)(p) = (L_X f)(p) := \frac{d}{dt}|_{t=0}(f \circ \varphi_X^t(p)) = \lim_{h\to 0} \frac{f(\varphi_X^h(p)) - f(p)}{h}.$$

For differential n-forms $\alpha \in \Omega^n(M)$, we define the Lie derivative along $X \in \mathfrak{X}(M)$ by:

$$L_X\alpha(p) = \lim_{h\to 0} \frac{((\varphi_X^h)^*\alpha)(p) - \alpha(p)}{h}.$$

Here $(\varphi_X^h)^*\alpha$ is the pull-back of α along f_X^h. And, finally, for a vector field $Y \in \mathfrak{X}(M)$ we define the Lie derivative with respect to X by:

$$(L_X Y)(p) = \lim_{h\to 0} \frac{Y(p) - (\varphi_{X_*}^h Y)(p)}{h}.$$

Here $\varphi_{X_*}^h Y$ is the push-forward of the vector field Y along φ_X^h. One often uses the notation $[X, Y]$ for$L_X Y$. $[\cdot, \cdot]$ is a Lie bracket on $\mathfrak{X}(M)$, i.e. $(\mathfrak{X}(M), [\cdot.\cdot])$ is a Lie algebra.

Schouten-Nijenhuis bracket The bracket $[\cdot, \cdot]$ can be extended to multivector fields. It is called Schouten-Nijenhuis bracket. We restrict ourself to the Schouten-Nijenhuis bracket, between elements of $\mathfrak{X}(M)$ and $\mathfrak{X}^2(M)$. Let $\pi \in \mathfrak{X}^2(M)$, then $\pi = \sum_i A_i \wedge B_i$, for some $A_k, B_l \in \mathfrak{X}(M)$. The push forward of π under a homeomorphism $\phi : M \to N$, between 2-manifolds is defined by: $\phi_*\pi = \sum_{ij} \phi_* A_i \wedge \phi_* B_j$. The Lie derivative of π along X is defined by:

$$L_X\pi(p) = [X, \pi](p) := \lim_{h\to 0} \frac{\pi(p) - (\phi_*\pi)(p)}{h}.$$

Definition 1.1.6. For $X, Y \in \mathfrak{X}(M)$, and $\rho, \pi \in \mathfrak{X}^2(M)$, and $A, B \in \mathfrak{X}(M)$, such that $\rho = A \wedge B$, the Schouten-Nijenhuis bracket is defined by

$$\begin{aligned}
[X, Y] &= L_X Y \in \mathfrak{X}(M), \\
[X, \pi] &= [\pi, X] = L_X\pi \in \mathfrak{X}^2(M), \\
[\pi, \rho] &= [\rho, \pi] = A \wedge [B, \pi] - B \wedge [A, \pi] \in \mathfrak{X}^3(M),
\end{aligned}$$

Projectable vector fields

Definition 1.1.7. Let $\pi : M \to N$ be a submersion between manifolds. A vector field $X \in \mathfrak{X}(M)$ is called projectable if there is a vector field $V \in \mathfrak{X}(N)$, such that $T_x\pi(X(x)) = V(\pi(x))$ for all $x \in M$. It is also said that X descends to V.

Lemma 1.1.8. *Let $\pi : M \to N$ be a submersion between manifolds. Let $X, Y \in \mathfrak{X}(M)$ descend to $V, W \in \mathfrak{X}(N)$ respectively, then $[X, Y]$ descends to $[V, W]$.*

Proof.

$$\begin{aligned} T_p\pi([X,Y](p)) &= T_p\pi(\lim_{h\to 0}(\frac{Y(p) - (\varphi_X^{h*}Y(p))}{h})) \\ &= \lim_{h\to 0}\frac{(W(\pi(p)) - (\pi\circ\varphi_X^h)^*(W(\pi(p))}{h} \\ &= L_V W = [V,W]. \end{aligned}$$

□

The exterior derivative An important tool in differential geometry is the exterior derivative. It is a map

$$d : \Omega^k(M) \to \Omega^{k+1}(M),$$

which satisfies the following properties:

1. $d \circ d = 0$
2. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
3. If ω_1 is a k-form then $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$.

This maps exists and the properties above determine d uniquely. We define d here using the Lie derivative.

Definition 1.1.9. Let M be a manifold. The exterior derivative $d\omega$ of a k-form ω is defined by:

$$(d\omega)(X_1, ..., X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1, ..., \hat{X}_i, ..., X_{k+1})) + \sum_{1\le i<j\le k+1}(-1)^{i+j}(\omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{k+1})).$$

where $X_1, ..., X_{k+1} \in \mathfrak{X}(M)$.

Another way to define the exterior derivative can be found in the appendix The explicit formula for the exterior derivative of a one form will be used frequently;

$$(d\omega)(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Foliations

Definition 1.1.10. A (regular) distribution D of rank r on a manifold M, is the assignment to each $p \in M$ of an r-dimensional subspace D_p of T_pM, in a smooth fashion in the following sense: for every point p there is an open neighbourhood U, and there are vector fields $X_1, ...X_r$ on U, such that they form a basis of D_q at every $q \in U$. In other words $D \subset TM$ is a smooth subbundle of rank r.

We look for connected submanifold $P \subset M$ such that for all $p \in P$ the tangent space T_pP coincides with D_p. Such a submanifold is called an integral manifold of the distribution. More exactly, if $f : P \to M$ is an imbedding of P into M, then is P called an integral manifold if $T_pf(T_pP) = D_p$, whenever $p \in P$. P is called a leaf or a maximal integral manifold, if there is no other integral manifold of D which contains M.

Definition 1.1.11. A distribution D on a manifold M is called integrable if through each $p \in M$ there passes an integral manifold of D.

An important way to check if a differentiable distribution is integrable follows from the theorem of Frobenius.

Theorem 1.1.12 (Frobenius). *Let D be a regular k-dimensional distribution on a m-dimensional manifold M, such that for all vector fields $X, Y \in \Gamma(D)$, $[X, Y] \in \Gamma(D)$. Then for every $p \in M$ there is a coordinate system (x, U) around p, such that*

$$x(U) = (-\epsilon, \epsilon) \times ... \times (-\epsilon, \epsilon) \subset \mathbb{R}^m,$$

for some $\epsilon \in \mathbb{R}$, and such that for all $a^{k+1}, ..., a^m$, with all $|a^i| < \epsilon$, the set

$$\{q \in U : x^{k+1}(q) = a^{k+1}, ..., x^m(q) = a^m\}$$

is an integral manifold of D. Moreover any connected integral manifold of the restriction $D|_U$ is contained in one of these sets.

For a discussion and the proof of this theorem we refer to the book of Spivak, [14]. It follows that we can equivalently define a distribution D on M to be integrable if $\Gamma(D)$ is closed under the Lie bracket, i.e. $[X, Y] \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. Note that if we have a integrable distribution on M, we get a partition of M in submanifolds. These submanifolds lie in some sense smooth next to each other. This leads to the notion of a foliation.

Definition 1.1.13. A foliation $\mathcal{F}$ of a manifold M is a partition of M by immersed k-dimensional connected submanifolds $L \hookrightarrow M$, such that around every point $m \in M$ there is a coordinate system (x, U), with

$$x(U) = (-\epsilon, \epsilon) \times ... \times (-\epsilon, \epsilon) \subset \mathbb{R}^m,$$

such that the connected components of $L \cap U$, for $L \in \mathcal{F}$ are sets of the form

$$\{q \in U : x^{k+1}(q) = a^{k+1}, ..., x^m(q) = a^m\}, \quad |a^i| < \epsilon.$$

The elements of $\mathcal{F}$ are called the leaves of the foliation.

Definition 1.1.14. Let $\mathcal{F}$ be a foliation of a manifold M. $T\mathcal{F}$ is the subbundle of TM, which is at the point $m \in M$ the tangent space at m of the leaf containing m. Sections of $T\mathcal{F}$ are vector fields which are tangent to the leaves of $\mathcal{F}$. They are denoted with $\mathfrak{X}(F)$, and they are called foliated vector fields. Similarly are foliated differential n-forms defined; these are sections of $\bigwedge^n T^*\mathcal{F}$, and are denoted with $\Omega^n(\mathcal{F})$.

Singular distributions and Lie algebroids

Definition 1.1.15. A singular distribution D on a manifold M, is the assignment to each $p \in M$ of a subspace D_p of T_pM.

The dimension of the vector spaces D_p in a singular distribution may vary with respect to p. In the sequel we meet quite a few possibly singular distributions. The notions of integral manifolds and leaves for singular distributions are defined as for regular distributions. We call such a singular distribution integrable if through every point $p \in M$, there passes an integral manifold of D. Unfortunately, for singular distributions there is not such a nice theorem as the one of Frobenius, which would make it easy to check if a distribution is integrable. A class of singular foliations which are integrable can be studied using Lie algebroids.

Definition 1.1.16. A Lie algebroid A over a manifold M, is a vector bundle $p : A \to M$, together with a bundle map $\rho_A : A \to TM$, called the anchor map, and a Lie bracket $[.,.]_A$ on the space of sections $\Gamma(M, A)$, which satisfies the following property:

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + (L_{\rho_A \circ \alpha} f)\beta,$$

for all $f \in C^\infty(M)$.

Remark 1.1.17. ρ_A induces a map $\Gamma(A) \to \mathfrak{X}(M)$, also denoted by ρ_A, where $\rho_A(\alpha)(x) = \rho_A(\alpha(x))$. Note that ρ_A is $C^\infty(M)$-linear, i.e. $\rho_A(f \cdot \alpha) = f \cdot \rho_A(\alpha)$, for all $f \in C^\infty(M)$.

If confusion is unlikely, we sometimes omit the subscript A for the anchor map and the Lie bracket.

Lemma 1.1.18. $\rho : \Gamma(A) \to \mathfrak{X}(M)$, *is a Lie algebra homomorphism.*

Proof. We use the Jacobi identity for $[,]_A$ and the Leibniz rule. Take $\alpha, \beta, \gamma \in \Gamma(A)$, and $f \in C^\infty(M)$.

$$\begin{aligned}
0 &= [[\alpha, \beta], f\gamma] + [[f\gamma, \alpha], \beta] + [[\beta, f\gamma], \alpha] \\
[[\alpha, \beta], f\gamma] &= f[[\alpha, \beta], \gamma] + L_{\rho([\alpha,\beta])}(f)\gamma \\
[[f\gamma, \alpha], \beta] &= f[[\gamma, \alpha], \beta] + L_{\rho(\beta)}(f)[\gamma, \alpha] + L_{\rho(\alpha)}(f)[\gamma, \beta] + L_{\rho(\beta)}L_{\rho(\alpha)}(f)\gamma \\
[[\beta, f\gamma], \alpha] &= f[[\beta, \gamma], \alpha] - L_{\rho(\beta)}(f)[\gamma, \alpha] - L_{\rho(\alpha)}(f)[\gamma, \beta] - L_{\rho(\alpha)}L_{\rho(\beta)}(f)\gamma.
\end{aligned}$$

If we add those last three expressions, and use the Jacobi identity another time, we are left with

$$L_{\rho([\alpha,\beta])}(f)\gamma = L_{[\rho(\alpha),\rho(\beta)]}(f)\gamma,$$

for all $\alpha, \beta, \gamma \in \Gamma(A)$, and $f \in C^\infty(M)$. Which implies that ρ preserves the bracket. $\square$

The image of ρ_A is a distribution on M, which is not necessarily regular. We call a Lie algebroid regular if this distribution is regular. For regular Lie algebroids the image of ρ_A is integrable by Frobenius theorem, and the previous lemma. To make sense of integrability for singular Lie algebroids we use the so called A-paths.

Definition 1.1.19. An A-path on a Lie algebroid over a manifold M is a pair (a, γ), where $a : [0,1] \to A$ is a path in A, and $\gamma : [0,1] \to M$ is a path in M, such that

1. a lies over γ, in the sense that $a(t) \in p^{-1}(\gamma(t))$ for all $t \in [0,1]$.
2. $\rho_A(a(t)) = \frac{d}{dt}\gamma(t)$, for all $t \in [0,1]$.

We interpret the a of an A-path as an 'A derivative' of the path γ, and one says that A-derivative is related to the usual derivative by the anchor map. We find a equivalence relation on M: x and y are equivalent if there is an A-path (a, γ), such that γ joins x and y. The equivalence classes are called the orbits of A.

Proposition 1.1.20. *The orbits of a Lie algebroid A over a manifold M are connected immersed submanifolds of M, which integrate* $\operatorname{im}\rho$*, i.e. any orbit $\mathcal{O}$ is an immersed submanifold, and $\rho_x(A) = T_x\mathcal{O}$, for all $x \in M$.*

The proof is based on to the local forms of Lie algebroids.

Remark 1.1.21. By an immersed submanifold we mean that the inclusion map $i : \mathcal{O} \hookrightarrow M$ is an injective immersion, but the image is not necessarily closed in M. What could happen is that an orbit comes arbitrary close to itself.

Example 1.1.22. TM itself is a Lie algebroid over M, where the anchor map is the identity, and the Lie bracket is the usual Lie bracket for vector fields. For any path γ joining two points x and y, $(\frac{d}{dt}\gamma(t), \gamma(t))$ is a TM-path. Hence the orbits are the path-connected components of M.

Integration over fibres Let $\pi : M \to B$ be a fibre bundle of fibre type a compact orientable manifold F. We would like to make sense of integration over the fibre of a form τ living on the total space. In this paragraph we set: $\dim M = m, \dim B = b$. Because F is orientable, there are nondegenerate forms $\omega \in \Omega^{m-b}(F)$. These forms are called volume forms.

Lemma 1.1.23. *Let $\pi : M \to B$ be a fibre bundle and let $U \subset B$ be a neighbourhood over which the bundle is trivial. Differential forms $\tau \in \Omega^l(\pi^{-1}(U))$, with $l \geq m - b$ are linear combinations of forms of the form;*

1. *$\tau = (\pi^*\alpha)f(x, y)\eta$, with η a differential form on F of degree less then $m - b$, and α a differential form on U.*
2. *$\tau = (\pi^*\alpha)f(x, y)\omega$, with ω a volume form on F, and α a differential form on U.*

Now the integration map $\int_F$ is defined by

$$\int_F \tau = (\pi^*\phi)f(x, y)\eta = 0 \int_F (\pi^*\phi)f(x, y)\omega = \phi \int_F f(x, y)\omega.$$

1.2 Symplectic structures

1.2.1 Defintion, examples and obstructions

Symplectic vector spaces Let V be a n-dimensional vector space over $\mathbb{R}$ and let $\Omega : V \times V \to \mathbb{R}$ be a skew-symmetric (or anti-symmetric) bilinear form, i.e. satisfying

$$\Omega(u, v) = -\Omega(v, u) \text{ for all } u, v \in V,$$

We call Ω nondegenerate if $\Omega(u, v) = 0$ for a fixed $u \in V$ and all $v \in V$ implies that $u = 0$.

Definition 1.2.1. A symplectic vector space (V, Ω) is a vector space V equipped with a nondegenerate skew-symmetric bilinear form Ω.

Ω is called the symplectic form of V. In a symplectic vector space one can always find a basis $\{e_1, ..., e_k, f_1, ..., f_k\}$, such that

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$$
$$\Omega(e_i, f_j) = -\Omega(f_j, e_i) = 1$$

Such a basis is called a symplectic basis. The existence of such a basis implies that a symplectic vector space is always even dimensional. Given a symplectic form Ω, we find immediately an isomorphism between V and its dual V^*;

$$\Omega_\# : V \to V^*, v \mapsto \Omega(v, -).$$

The nondegeneracy condition implies that $\Omega_\#$ is an isomorphism.

Definition 1.2.2. Given a symplectic vector space (V, Ω), and $W \subset V$ be a subspace, the symplectic complement of Ω is defined as

$$W^{\Omega} = \{u \in V : \Omega(u, w) = 0 \text{ for all } w \in W\}.$$

One has:

$$\dim V = \dim W + \dim W^{\Omega}.$$

We call a subspace $W \subset V$ isotropic if $\Omega_{|W \times W} = 0$ or equivalently if $W \subset W^{\Omega}$, and we call it symplectic if $\Omega_{|W \times W}$ is nondegenerate or equivalently $W \cap W^{\Omega} = \{0\}$.

Symplectic manifolds

Definition 1.2.3. A symplectic manifold (M, ω) is a manifold equipped with a 2-form ω, such that

- every tangent space $T_x M$ together with ω_x is a symplectic vector space.
- ω is closed, i.e. $d\omega = 0$.

Definition 1.2.4. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A diffeomorphism $\phi : M_1 \to M_2$ is called a symplectomorphism if $\phi^* \omega_2 = \omega_1$.

The most basic example of a symplectic manifold is the even dimensional Euclidean space, $\mathbb{R}^{2n}$, with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$, and symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. A famous theorem of Darboux which says that all symplectic manifolds locally look like this. The nondegeneracy of a symplectic form ω implies that the $2n$-form

$$\omega^n := \underbrace{\omega \wedge ... \wedge \omega}_{n-\text{times}}$$

is nowhere vanishing, i.e. it is a volume form. It follows that manifolds which admits a symplectic structure must be orientable.

Example 1.2.5. An example of a symplectic manifold is the 2-dimensional sphere, $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$, with $\omega = dh \wedge d\theta$. Here h is the height function, and θ is the angle in the xy-plane. One would maybe expect that all even dimensional spheres S^{2n} are symplectic, but that is not the case. If ω is a symplectic form for such a sphere, with $n > 1$, then ω must be exact since $H^2_{\text{de Rham}} = 0$. Hence then there is a 1-form $\omega = d\alpha$, and ω^n is exact as well: $\omega^n = d(\alpha \wedge (d\alpha)^{n-1} := d\theta$. Using Stokes' theorem we get:

$$\text{The area of } S^{2n} = \int_{S^{2n}} \omega^n = \int_{\partial S^{2n}} \theta = 0,$$

which is clearly a contradiction. Therefore such a symplectic form does not exist. More generally we conclude that compact manifolds M, without boundary, and for which $H^2_{\text{de Rham}}(M)$ is trivial, can not have a symplectic structure.

Example 1.2.6. Another example is the torus $\mathbb{T}^2 = S^1 \times S^1$. Let θ_1, θ_2 be the coordinate on the both circles. The symplectic form on the torus is now, $d\theta_1 \wedge d\theta_2$.

Cotangent bundle The cotangent bundle of any manifold X carries a symplectic structure. To see this we first define a 1-form on it.

$$\alpha_{tau} : T^*X \to T^*(T^*X), (x, \xi) \to \xi \circ T_{(x,\xi)}\pi.$$

Here $\pi : T^*X \to X$ is the projection. This 1-form is called the tautological 1-form, and it has the nice property that if $\tau \in \Omega^1(X)$, then $\tau^*(\alpha_{tau}) = \tau$. We define a 2-form on T^*X by taking the exterior derivative.

$$\omega_{can} = -d\alpha_{tau}.$$

It is possible to describe these forms locally in terms of the coordinates. Let U be a coordinate neighbourhood on X, with coordinates $\{x_1, ..., x_n\}$. For every $x \in U$ the differentials $\{(dx_1)_x, ..., (dx_n)_x\}$ form a basis for T_x^*X. Hence, for a $\xi \in T_x^*X$, we find real numbers $\xi_1, ..., \xi_n$ such that $\xi = \sum_1^n \xi_i (dx_i)_x$. Now we have coordinates $\{x_1, ..., x_n, \xi_1, ..., \xi_n\}$, for the $2n$-dimensional manifold T^*X. For the tautological 1-form we find

$$\alpha_{tau} = \sum_{i=1}^n \xi_i dx_i.$$

And hence,

$$\omega_{can} = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

The complex projective space Another example of a symplectic manifold is the complex space $\mathbb{C}^n$. Elements in this space are written as $z = x + iy$, where $x, y \in \mathbb{R}^n$. The symplectic form is then $\sum_{i=1}^n dx_i \wedge dy_i$. There is also a less obvious way to define a symplectic structure on $\mathbb{C}^n$, the so called Fubini-Study structure. It has the nice property that it induces a symplectic structure on the complex projective space $\mathbb{C}P^{n-1}$.

Definition 1.2.7. The projective complex space, $\mathbb{C}P^n$ is de the space of all complex-1-dimensional subspaces of $\mathbb{C}^{n+1}$. It is obtained from $\mathbb{C}^{n+1} - \{0\}$, modulo the equivalence relation $(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n)$ for $\lambda \in \mathbb{C} - \{0\}$. We denote the equivalence class $([z_0, ..., z_n)$ with $[z_0, ..., z_n]$.

Before we can introduce the Fubini-Study structure we have to introduce some more notation.

$$dz_j = dx_j + idy_j \quad \text{and} \quad d\bar{z}_j = dx_j - idy_j$$
$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Note that with these definitions we can write the standard symplectic form on $\mathbb{C}^n$ as $\frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$.

A way to find a symplectic form ω which is compatible with the complex structure i.e. $\omega(u, v) = \omega(iu, iv)$, is by taking a so called strictly plurisubharmonic function ρ. Then

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho,$$

is the form we were looking for.

Definition 1.2.8. Let M be a complex manifold. A function $\rho \in C^\infty(M;\mathbb{R})$ is strictly plurisubharmonic if, on each local complex chart $(U, z_1, ..., z_n)$, where $n = \dim_{\mathbb{C}}(M)$, the matrix $\left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k(p)}\right)$ is positive definite at all $p \in U$.

A discussion about why such functions give rise to compatible symplectic forms can be found on chapter 16 of [5]. To obtain the Fubini-Study metric we choose

$$\rho : \mathbb{C}^n \to \mathbb{R}, z \mapsto \log(|z|^2 + 1)$$

For this function we find

$$\left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k(z)}\right) = \frac{\delta_{jk}(|z|^2+1) - z_k\bar{z}_j}{(|z|^2+1)}$$

Recall that the unitary group $U(n)$ acts transitively on S^{2n-1}, and observe that ρ is invariant under the action $U(n)$; it depends only on the norm of z. Therefore, to check that the matrix is positive definite, it is sufficient to check that the matrix is positive definite in our favorite direction, which is given by the vector $v = (1, 0, ..., 0)^T$. For this vector we have

$$v^* \left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k(z)}\right) v = \frac{|z|^2 + 1 - z_1\bar{z}_1}{(|z|^2+1)^2} > 0$$

So ρ is strictly plurisubharmonic. Let U_i be the open subset in $\mathbb{C}^n$ defined by $z_i \neq 0$. To come to the projective space we study the behaviour of ρ and its induced 2-form under the biholomorphic bijective map

$$\varphi_i : U_i \to U_i, (z_1, ..., z_n) \mapsto \frac{1}{z_i}(z_1, .., z_{i-1}, 1, z_{i+1}..., z_n).$$

We take the pull-back of ρ along φ.

$$\varphi_i^*\rho(z) = \log(\frac{1}{z_i}(|z|^2+1)) = \rho(z) - \log(z_i) - \log(\bar{z}_i).$$

Observe that the last two terms are holomorphic respectively anti-holomorphic. Hence

$$\partial\bar{\partial}(\log(z_i) - \log(\bar{z}_i)) = 0.$$

And we get

$$\begin{aligned} \varphi_1^*(\partial\bar{\partial}\rho(z)) &= \partial\bar{\partial}\varphi_i^*\rho(z) \\ &= \partial\bar{\partial}(\rho(z) - \log(z_i) - \log(\bar{z}_i)) \\ &= \partial\bar{\partial}\rho(z) \end{aligned}$$

Which implies the invariance under φ_i of the Fubini-Study structure on $\mathbb{C}^n$.

Coadjoint orbits. Another important family of symplectic manifolds is given by the orbits of the coadjoint action of a Lie group G, on the dual $\mathfrak{g}^*$ of its Lie algebra. A Lie group G acts on itself by conjugation.

$$C_g : G \to G, h \mapsto ghg^{-1}.$$

The Lie algebra of a Lie group is the tangent space of G at its unit element e. If we take the tangent map of C_g in e we obtain a map $\mathfrak{g} \to \mathfrak{g}$. In this way we find a map

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), g \mapsto T_e C_g = Ad(g).$$

This is called the adjoint action of G on $\mathfrak{g}$. If we take again the tangent in e, but now from Ad, we get a map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g}).$$

If X is an element of $\mathfrak{g}$, we obtain a vector field v_X on G, by

$$v_X(x) = T_e(l_x)X,$$

where l_x is the left multiplication by x.

Denote with α_X the maximal integral curve of v_X, with starting point e. The exponential map is defined as

$$\exp : \mathfrak{g} \to G, X \mapsto \alpha_X(1).$$

We obtain the following commuting diagram.

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{gl}(\mathfrak{g}) \end{array}$$

G acts on the dual of its Lie algebra with the so called coadjoint action Ad^*, which is defined by

$$\left\langle \mathrm{Ad}_g^*(\xi), Y \right\rangle = \left\langle \xi, \mathrm{Ad}_{g^{-1}}(X) \right\rangle, \text{ for all } Y \in \mathfrak{g}.$$

Here $\langle \xi, X \rangle = \xi(X)$ is the natural paring of $\mathfrak{g}^*$ and $\mathfrak{g}$. We denote the orbit of this action through a ξ in $\mathfrak{g}^*$ with $\mathcal{O}_\xi$.

Lemma 1.2.9. *The orbits of the coadjoint action of G on the dual of its Lie algebra $\mathfrak{g}^*$ have a symplectic structure.*

Proof. The coadjoint action gives rise to a vector field $X^\#$ on $\mathfrak{g}^*$, for every $X \in \mathfrak{g}$. Let $\xi \in \mathfrak{g}^*$, then for every $Y \in \mathfrak{g}$, $X^\#$ is defined by

$$\begin{aligned}
\left\langle X_\xi^\#, Y \right\rangle &= \frac{d}{dt}_{|t=0} \mathrm{Ad}^*_{\exp(tX)}(\xi)(Y) \\
&= \frac{d}{dt}_{|t=0} \left\langle \xi, \mathrm{Ad}_{\exp(-tX)} Y \right\rangle \\
&= \langle \xi, [Y, X] \rangle
\end{aligned}$$

The tangent space of the orbit through ξ is generated by the vector fields $X_\xi^\#$. To see this, we define for every $\xi \in \mathfrak{g}^*$ a bilinear form on $\mathfrak{g}$;

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle .$$

The kernel of this bilinear form (i.e. $\{X \in \mathfrak{g} : \omega_\xi(X, Y) = 0 \, \text{for all} \, Y \in \mathfrak{g}\}$) is the Lie algebra of the stabilizer of ξ under the adjoint action. So, if we restrict this bilinear form to the orbit of ξ, we get a nondegenerate 2-form. It remains to show that this form is closed. Let, X, Y, Z be in $\mathfrak{g}$ tangent to the orbits.

$$\begin{aligned}
d\omega_\xi(X, Y, Z) &= -(\omega_\xi([X, Y], Z) + \omega_\xi([Z, X], Y) + \omega_\xi([Y, Z], X)) \\
&+ L_Z \omega_\xi(X, Y) + L_Y \omega_\xi(Z, X) + L_X \omega_\xi(Y, Z) \\
&= -\xi([[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X]) + 0 = 0
\end{aligned}$$

Here we used the Jacobi identity in $\mathfrak{g}$. □

Moser's stability theorem Often, one can find more then one symplectic form on a compact manifold M. An interesting question is then: if M equipped with two different symplectic forms ω_0 and ω_1, is there a diffeomorphism $\phi : M \to M$, such that $\phi^* \omega_0 = \omega_1$? If they are symplectomorphic, it would be interesting to know if we can deform one smoothly into another, i.e. if there is a smooth family ω_t of symplectic forms joining ω_0 to ω_1. It would also be interesting if there was a family of symplectomorphisms, such that $\phi_t^* \omega_t = \omega_0$. A necessary condition is that ω_0 and ω_1 live in the same De Rham-cohomology class. Moser proved that such a family exists, provided one extra condition. The exact formulation of this theorem, taken from [5], is as follows.

Theorem 1.2.10 (Moser). *Let M be a compact manifold. Suppose that $[\omega_0] = [\omega_1]$ and that the 2-form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for each $t \in [0,1]$. Then there exists an isotopy $\phi : M \times \mathbb{R} \to M$, such that $\phi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$.*

Proof. This theorem is proved by a trick called the Moser trick. The idea behind it is to show that the existence of the isotopy depends on the solution of a differential equation, and that this equation can be solved if $[\omega_0] = [\omega_1]$. Here is the exact trick. Suppose that there is a isotopy $\phi : M \times \mathbb{R} \to M$ such that $\phi_t^*\omega_t = \omega_0$, for $0 \leq t \leq 1$. Let

$$v_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1}, \quad t \in \mathbb{R}.$$

Then we find the following equivalence.

$$\begin{array}{rcl} 0 = \dfrac{d}{dt}(\phi_t^*\omega_t) & = & \phi_t^*(L_{v_t}\omega_t + \dfrac{d\omega_t}{dt}) \\ & \Leftrightarrow & \\ L_{v_t}\omega_t + \dfrac{d\omega_t}{dt} & = & 0(*) \end{array}$$

Conversely suppose that we can find a smooth time-dependent vector field v_t, $t \in \mathbb{R}$, which satisfies this last differential equation (*). M is compact, so we can integrate v_t to an isotopy $\phi_t : M \times \mathbb{R} \to \mathbb{R}$, with

$$\frac{d}{dt}(\phi_t^*\omega_t) = 0.$$

Which implies that $\phi_t\omega_t = \phi_0\omega_0 = \omega_0$. So all we have to do is to find a solution of the differential equation (*). For this we use the assumptions on ω_0 and ω_1. Because $\omega_t = (1-t)\omega_0 + t\omega_1$, we have

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0$$

ω_1 and ω_0 live in the same de Rham cohomology class, so there is a 1-form μ such that

$$\omega_1 - \omega_0 = d\mu.$$

Next we observe with Cartan's formula and because ω_t is closed by assumption that

$$L_{v_t}\omega_t = d\iota(v_t)\omega_t + \iota(v_t)d\omega_t = d\iota(v_t)\omega_t.$$

Hence the original differential equation is reduced to

$$d\iota(v_t)\omega_t + d\mu = 0.$$

By the nondegeneracy of ω_t this can be solved pointwise. The result is an unique smooth vector field v_t. $\square$

Theorem 1.2.11 (Darboux). *Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(U, x_1, ...x_n, y_1, ..., y_n)$ centered at p such that on U*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof. Choose a symplectic basis for T_pM, and use this to contruct coordinates $U', x_1', ..., x_n', y_1', ..., y_n'$ around p such that

$$\omega_p = \sum_{i=1}^{n} dx_i \wedge dy_i|_p.$$

On U' we now have 2 symplectic forms. $\omega_0 = \omega$, and $\omega_1 = \sum_{i=1}^{n} dx_i \wedge dy_i$. Let $\omega_t = t\omega_1 + (1-t)\omega_0$. Then $\omega_t(p) = \omega_0(p) = \omega_1(p)$. Being symplectic is an open condition, hence by the compactness of $[0, 1]$, there is an open neighbourhood U of p, such that ω_t is symplectic on U_1. By Moser's theorem there is a symplectomorphism $\varphi : (U, \omega_0) \to (U, \omega_1)$ with $\varphi(p) = p$. We obtain the desired coordinate system by setting the new coordinates $x_i = x_i' \circ \varphi, y_i = y_i' \circ \varphi$. $\square$

1.2.2 Hamiltonian actions and the Hamiltonian group

Hamiltonian vector fields Let (M, ω) be a symplectic manifold. A vector field X on M is called a symplectic vector field if its flow preserves the symplectic form ω, i.e.

$$\varphi_X^{t*}\omega = \omega$$

Naturally, we find in that case, using that ω is closed,

$$0 = L_X\omega = \iota(X)d\omega + d\iota(X)\omega = d\iota(X)\omega.$$

The first De Rham cohomology group of a coordinate neighborhood is trivial so, at least locally, we find a function f such that

$$\iota(X)\omega = -df$$

If we find such a function globally, then we call X a Hamiltonian vector field.

Definition 1.2.12. A vector field X on on a symplectic manifold (M, ω) is called Hamiltonian if there is a function $H : M \to \mathbb{R}$, such that

$$\iota(X)\omega = -dH$$

The function H, is then called the Hamiltonian function of X.

Conversely, starting from a $f \in C^\infty(M)$, we find a Hamiltonian vector field X_f, for which f is the Hamiltonian function. To see this, we observe that there is at each $m \in M$ an isomorphism

$$T_m M \to T_m^* M, v \to \omega(v, -).$$

Such a vector field is a symplectic vector field as we see by the following computation:

$$L_{X_f}\omega = d\iota(X_f)\omega + \iota(X_f)d\omega = -ddf = 0$$

The flow of a Hamiltonian vector field preserves not only the symplectic form, but it also preserves its Hamiltonian function.

$$L_{X_f} f = d\iota(X_f)f + \iota(X_f)df = \iota(X_f)\iota(X_f)\omega = 0.$$

The cotangent bundle of a manifold M has a canonical symplectic form ω_{can} and a canonical 1-form α_{tau}, which are related by $\omega_{can} = -d\alpha_{tau}$. If $f : M \to M$ is a diffeomorphism, then it can be lifted to a diffeomorphism

$$F : T^*M \to T^*M, (x, v^*) \mapsto (f(x), ((T_x f)^*)^{-1} v^*).$$

In particular, F is a symplectomorphism: $F^*\omega_{can} = \omega_{can}$. But, even stronger, it preserves α_{tau}. We have a look to a certain set of diffeomorphisms from M to itself: the 1-parameter group of diffeomorphisms, $\psi_t : M \to M$, generated by a vector field Y. Let $X : T^*M \to TT^*M$ be the vector field which generates the lifts $\Psi_t : T^*M \to T^*M$. Then X is a Hamiltonian vector field with Hamiltonian function

$$H : T^*M \to \mathbb{R} : (m, v^*) \mapsto v^*(Y(m))$$

The Lie derivative of α_{tau} with respect to X is zero. Therefore

$$\iota(X)\omega_{can} = -d\iota(X)\alpha_{tau},$$

and hence, it is sufficient to prove that $H = \iota(X)\alpha_{tau}$. By the construction of X, we have

$$d\pi \circ X = Y \circ \pi : T^*M \to TM$$

Here $\pi : T^*M \to M$ is the projection. We have:

$$\begin{aligned} \iota(X)\alpha_{tau}(m, v^*) &= \alpha_{tau}(m, v^*)(X(m, v^*)) \\ &= v^* \circ d\pi \circ X(m, v^*) \\ &= v^* \circ Y \circ \pi(m, v^*) \\ &= H(m, v^*) \end{aligned}$$

Here we used that $\alpha_{tau}(m, v^*) = v^* \circ d_m\pi$.

Hamiltonian group actions Let (M, ω) be a symplectic manifold, and let G be a group acting on it. This means that we have a group homomorphism $\psi : G \to \mathrm{Diff}(M)$. For a complete vector field on M, then its flow φ_X^t is an 1-parameter group of diffeomorphisms. Hence, every complete vector field gives rise to an action of $\mathbb{R}$ on M.

Definition 1.2.13. An action ψ is called a symplectic action, if for all $g \in G$, $\psi_g \in \mathrm{Symp}(M, \omega)$.

Clearly, a symplectic action gives rise to a symplectic vector field and vice versa.

Definition 1.2.14. An action of $\mathbb{R}$ is called Hamiltonian if it is generated by a Hamiltonian vector field.

If G is not $\mathbb{R}$, but another Lie group, then we can act with $\mathbb{R}$ on M via G. Let $X \in \mathfrak{g}$. Then we obtain a 1-parameter group of diffeomorphisms of M, by $\psi_{\exp(tX)}$. Denote with $X^\bullet$ the corresponding vector field.

Definition 1.2.15. An action $\psi : G \times M \to M$ of a Lie group G on a symplectic manifold (M, ω) is Hamiltonian if there is a map

$$\mu : M \to \mathfrak{g}^*$$

such that,

1. $d\mu^X = \iota(X^\bullet)\omega$, where $\mu^X : M \to \mathbb{R}, p \mapsto \langle \mu(p), X\rangle$.
2. μ is equivariant with respect to the given action ψ of G, and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:
$$\mu \circ \psi_g = Ad_g^* \circ \mu, \text{ for all } g \in G.$$

The map μ is called the moment map of the action.

We call $(M,\, G, \mu)$ a Hamiltonian G-space. Hamiltonian actions appear in a natural way, as will see in the next two examples

Example 1.2.16 (The action of G on T^*G). G acts on its own cotangent bundle T^*G by

$$G \times T^*G \to T^*G, (g, (h, v^*)) \mapsto (hgHgnv, r_{g*}^T v^*)$$

where r_{g*}^T is the transpose of r_{g*}. For every $g \in G$ we define a map

$$L_g : \mathfrak{g} \to TG, \xi \mapsto (g, l_{g*}(\xi)).$$

For all $\xi \in \mathfrak{g}$ the 1-parameter group $t \mapsto r_{\exp(-t\xi)}$ is generated by the vector field

$$G \to TG : g \mapsto L_g\xi.$$

The lift of the action by $r_{g^{-1}}$, is exactly the action G on T^*G. Hence G acts Hamiltonian on T^*G, with moment map

$$\mu : T^*G \to \mathfrak{g}^*, (h, v^*) \mapsto -L_g^* v^*$$

Example 1.2.17. This example is similar to the previous one. If P is a principal G-bundle, then G-acts on T^*P:

$$G \times T^*P, (g, (p, \eta)) \mapsto (pg^{-1}, r_{g^{-1}*}\eta).$$

This a Hamiltonian action with moment map

$$\mu_P : T^*P \to \mathfrak{g}^*, (p, \eta) \mapsto -L_p^*\eta.$$

Here is $L_p : \mathfrak{g} \to T_P, \xi \mapsto p \cdot \xi = \frac{d}{dt}|_{t=0} p \exp(t\xi)$.

The Marsden-Weinstein quotient The goal of this section is to prove the following theorem.

Theorem 1.2.18 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)/G$ is a manifold, and there is a symplectic form ω_{red} on $\mu^{-1}(0)/G$, satisfying $i^*\omega = \pi^*\omega_{red}$.*

Definition 1.2.19. The action of a group G on a set M is called free if for all elements $m \in M$ the stabilizers the unity element e.

Denote with $\mathfrak{g}_p$ the Lie algebra of the stabilizer of $p \in M$, let $\mathfrak{g}_p^\circ \subset \mathfrak{g}_p^*$ be its annihilator. Denote with $\mathrm{orb}(p)$ the the orbit of p in M.

Lemma 1.2.20. *The map $d\mu_p : T_pM \to \mathfrak{g}^*$ has as kernel $(T_p\,\mathrm{orb}(p))^{\omega_p}$, and it has as image $\mathfrak{g}_p^\circ$.*

Proof. μ is the moment map of a Hamiltonian action. So for all $v \in T_pM$ and $X \in \mathfrak{g}$, we have

$$\omega_p(X_p^\bullet, v) = \langle d\mu_p(v), X\rangle \,.$$

Moreover, observe that $T_p \operatorname{orb}(p) = \{X_p^\bullet : X \in \mathfrak{g}\}$. If $X \in \mathfrak{g}_p$, then

$$X_p^\bullet = \frac{d}{dt}|_{t=0} p \exp(tX) = \frac{d}{dt}|_{t=0} p = 0.$$

Now we are ready to compute the kernel of $d\mu_p$. Let $v \neq 0$ be in this kernel, then

$$0 = \langle d\mu_p(v), X\rangle = \omega_p(X_p^\bullet, v).$$

Which is equivalent with $v \in T_p \operatorname{orb}(p)^{\omega_p}$. In order to prove that $\operatorname{im}(d\mu_p) \subset \mathfrak{g}_p^\circ$ we take $X \in \mathfrak{g}_p$ and $v \in T_pM$.

$$\langle d\mu_p(v), X\rangle = \omega_p(X_p^\bullet) = 0.$$

So $d\mu_p(v)$ is indeed in the annihilator of $\mathfrak{g}_p$. To finish the prove we count the dimensions of the spaces involved.

$$\begin{aligned}
\dim orb(p) + \dim \mathfrak{g}_p &= \dim \mathfrak{g} \\
\dim \mathfrak{g}_p^\circ + \dim \mathfrak{g}_p &= \dim \mathfrak{g}, \text{ hence} \\
\dim \mathfrak{g}_p^\circ &= \dim \operatorname{orb}(p). \\
\dim M &= \dim \ker d\mu_p + \dim \operatorname{im} d\mu_p \\
&= \dim M - \dim \operatorname{orb}(p) + \dim \operatorname{im} d\mu_p
\end{aligned}$$

Finally we find

$$\dim \operatorname{im} d\mu_p = \dim \mathfrak{g}_p^\circ$$

and we may conclude that these spaces are the same. □

If an action is locally free in a point $p \in M$, then is the stabilizer trivial and hence is μ a submersion in p. Now G acts freely on $\mu^{-1}(0)$, so 0 is a regular value of μ and $\mu^{-1}(0)$ is a closed submanifold of M.

Lemma 1.2.21. *Let (V, ω) be a symplectic vector space. Suppose that W is an isotropic subspace, i.e., $W \subset W^\omega$. Then ω induces a canonical symplectic form Ω on W^ω/W.*

Proof. Let $u, v \in W$, and denote $[u], [v]$ for their respective classes in W^ω/W. Define

$$\Omega([u], [v]) = \omega(u, v)$$

It has to be checked that Ω is welldefined and nondegenerate. For the welldefinedness we compute for all $i, j \in W$

$$\omega(u+i, v+j) = \omega(u, v) + \omega(u, j) + \omega(i, v) + \omega(i, j) = \omega(u, v).$$

To check the nondegeneracy suppose that $u \in W^\omega$, such that $\omega(u, v) = 0$ for all $v \in W^\omega$, then $u \in (W^\omega)^\omega = W$, and then $[u] = 0$. □

Theorem 1.2.22. *Let G be a compact Lie group which acts freely on a manifold M, then M/G is a manifold.*

Proof. See[5] page 143. □

Proof of the Marsden-Weinstein theorem. G act freely on $\mu^{-1}(0)$, hence then $\mu^{-1}(0)$ is a closed submanifold of M. By the previous theorem $\mu^{-1}(0)/G$ is a manifold. Using lemma 1.2.21 we see that all what is left to prove is that the orbits of G in $\mu^{-1}(0)$ are isotropic. For $p \in \mu^{-1}(0)$ we have

$$T_p \operatorname{orb}(p) \subset T_p\mu^{-1}(0) = \operatorname{Ker}(d\mu_p) = (T_p \operatorname{orb}(p))^{\omega_p}.$$

□

The flux homomorphism We study some properties of the group of symplectomorphisms $\operatorname{Symp}(M, \omega)$. We assume, throughout this section, that M is compact and without boundary. $\operatorname{Symp}(M, \omega)$ is a locally path connected Lie group. Denote with $\operatorname{Symp}_0(M, \omega)$ the identity component of $\operatorname{Symp}_0(M, \omega)$. That means that for all $\psi \in \operatorname{Symp}_0(M, \omega)$ there is an isotopy ψ_t, such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$. Such a isotopy induces a family of symplectic vector fields, by the following relation.

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

If, moreover, for all t $\iota(X_t)\omega$ is exact, then the ψ_t are Hamiltonian symplectomorphisms. $\operatorname{Ham}(M, \omega)$ denotes the set of Hamiltonian symplectomorphisms.

Lemma 1.2.23. $\operatorname{Ham}(M, \omega) \subset \operatorname{Symp}(M, \omega)$ *is a normal and pathconnected subgroup.*

Proof. Let ψ_t and ϕ_t be Hamiltonian isotopies which are generated by H_t and G_t respectively and let X_t and Y_t be their respective families of vector fields. Then $\phi_t \circ \psi_t$ is a symplectomorphism for all t. Define the vector fields:

$$Z_t = \phi_{t*} \circ X_t \circ \phi_t^{-1} + Y_t.$$

This family of vector fields satisfies:

$$\frac{d}{dt}(\phi_t \circ \psi_t) = \phi_{t*} \circ X_t + Y_t \circ \phi_t \circ \psi_t = (\phi_{t*} \circ X_t \circ \phi_y^{-1} + Y_t) \circ \phi_t \circ \psi_t,$$

and

$$\iota(Z_t)\omega = \iota((\phi_t^{-1})^* X_t + Y_t)\omega = d(H_t \circ \phi_t^{-1} + G_t).$$

So $\phi_t \circ \psi_t$ is a Hamiltonian isotopy with Hamiltonian function $H_t \circ \phi_t^{-1} + G_t$. With a similar computation we see that

$$Z_t = -\psi_{t*}^{-1} \circ X_t \circ \psi_t,$$

is the family of vector fields corresponding to ψ_t^{-1}, with Hamiltonian function $-H_t \circ \psi_t$. So, $\mathrm{Ham}(M, \omega)$ is a subgroup of $\mathrm{Symp}(M, \omega)$. Moreover, for any $\phi \in \mathrm{Symp}(M\omega)$ $H_t \circ \phi$ generates the isotopy $\phi^{-1} \circ \psi_t \circ \phi$, hence $\mathrm{Ham}(M, \omega)$ is a normal subgroup. $\mathrm{Ham}(M, \omega)$ is path-connected by its very definition. □

Soon we will define the flux homomorphism. The flux homomorphism is a homomorphism between the universal cover, $\widetilde{\mathrm{Symp}}_0(M, \omega)$, of $\mathrm{Symp}_0(M, \omega)$, which exists out of homotopy classes of smooth paths $\{\psi_t\}$ in $\mathrm{Symp}_0(M, \omega)$, and the cohomology group, $H^1(M, \mathbb{R})$. The flux homomorphism is of our interest, since $\mathrm{Ham}(M, \omega)$ appears exactly as its kernel. First we recall that there is a natural identification:

$$\begin{aligned} H^1(M, \mathbb{R}) &\cong \mathrm{Hom}(\pi_1(M), \mathbb{R}) \\ \alpha &\mapsto ([\gamma] \mapsto \int_\gamma \alpha). \end{aligned}$$

Definition 1.2.24. The flux homomorphism, $\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0(M, \omega) \to H^1(M, \mathbb{R})$ is defined by:

$$\mathrm{Flux}(\{\psi_t\}) = \int_0^1 [\iota(X_t)\omega] dt.$$

Here is X_t induced by ψ_t as above.

We have to prove that this is a welldefined homomorphism, i.e it depends only on homotopy classes. By the identification given above, the $\mathrm{Flux}(\{\psi_t\})$ corresponds to the homomorphism:

$$\gamma \mapsto \int_0^1 \int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma}(s)) ds dt.$$

We define the map $\beta : S^1 \times [0, 1] \to M, (s, t) \mapsto (\psi_t^{-1}(\gamma(s)))$, and we get $\psi_t(\beta(s, t)) = \gamma(s)$. If we differentiate this with respect to s and t we get, since $\frac{\partial}{\partial t}\psi_t = X_t \circ \psi_t$,

$$\frac{\partial}{\partial s}[\psi_t(\beta(s,t))] = d\psi_t(\beta(s,t))\frac{\partial \beta}{\partial s} = \dot{\gamma}(s)$$
$$\frac{\partial}{\partial t}[\psi_t(\beta(s,t))] = (X_t \circ \psi_t(\beta(s,t)) + d\psi_t(\beta(s,t))\frac{\partial \beta}{\partial t} = 0.$$

Therefore, we have:

$$\begin{aligned}\text{Flux}(\{\psi_t\})(\gamma) &= \int_0^1 \int_0^1 -\psi_t^*\omega(\frac{\partial \beta}{\partial t}, \frac{\partial \beta}{\partial s})dsdt \\ &= \int_0^1 \int_0^1 \omega(\frac{\partial \beta}{\partial s}, \frac{\partial \beta}{\partial t})dsdt,\end{aligned}$$

since $\psi_t^*\omega = \omega$. This last expression depends on the homotopy class of β only. β is dependent on ψ_t with fixed endpoint. Therefore the Flux depends only on the homotopy class of ψ_t with $\psi_0 = \text{id}$, and $\psi_1 = \psi$,and hence it is well-defined. It is left to prove that the Flux is a homomorphism. To see that, recall that if ψ_t generates X_t and ϕ_t generates Y_t, then $\phi_t \circ \psi_t$ generates $X_t + (\psi_t^{-1})^*Y_t$. Therefore;

$$\begin{aligned}\text{Flux}(\{\phi_t\}\{\psi_t\}) &= \int_0^1 [\iota(X_t + (\psi_t^{-1})^*Y_t)]\omega dt \\ &= \int_0^1 [\iota(X_t)]\omega dt + \int_0^1 [\iota((\psi_t^{-1})^*Y_t)]\omega dt \\ &= \text{Flux}(\{\phi_t\}) + \text{Flux}(\{\psi_t\}),\end{aligned}$$

and the Flux is a homomorphism. Now we come to our main theorem of this section.

Theorem 1.2.25. *Let* $\psi \in \text{Symp}_0(M,\omega)$*. Then* ψ *is a Hamiltonian symplectomorphism if and only if there exists a symplectic isotopy* ψ_t *with* $\psi_0 = \text{id}$*, and* $\psi_1 = \psi$*, such that* $\text{Flux}(\{\psi_t\}) = 0$*. Moreover, if the* $\text{Flux}(\{\psi_t\}) = 0$*, then* $\{\psi_t\}$ *is isotopic with fixed endpoints to a Hamiltonian isotopy.*

In the proof of this theorem we will integrate families of vector fields. With $\int_0^1 X_t dt$ we mean the vector field:

$$\int_0^1 X_t dt : M \to TM, m \mapsto \lim_{n\to\infty} \sum_1^n X_{t_i}(m) - X_{t_{i-1}}(m).$$

Here the t_i make a partition of $[0,1]$, such that the mesh of this partitions goes to zero, if $n \to \infty$.

Proof. Let ψ be Hamiltonian, then it is the endpoint of a Hamiltonian isotopy ψ_t, corresponding to a family of Hamiltonian functions. We get

$$\mathrm{Flux}(\{\psi_t\}) = \int_0^1 [\iota(X_t)\omega] = \int_0^1 [dH_t]dt.$$

Elements in this last class are clearly exact, so indeed

$$\mathrm{Flux}(\{\psi_t\}) = 0$$

For the converse we start with a isotopy class $\{\psi_t\}$, for which the flux is zero. We want to change this isotopy such that $\iota(X_t)w$ is exact. We make use of the following lemma:

Lemma 1.2.26. *$\iota(X_t)\omega$ is exact for all $t \in [0,1]$, precisely if $\int_0^T \iota(X_t)\omega dt$ is exact for all $T \in [0,1]$.*

Proof. Proof of the lemma. Let there be for all $t \in [0,1]$, functions f_t such that, $\int_0^T [\iota(X_t)\omega]dt = df_t$. Then

$$\int_0^T \iota(X_t)\omega dt = \int_0^T df_t dt = d\int_0^T f_t dt,$$

i.e. $\int_0^T \iota(X_t)\omega dt$ is exact. The other way round, if $\int_0^T \iota(X_t)\omega dt$ is exact for all $T \in [0,1]$, then it is in particular closed. We get, using lemma **??**, that $\iota(X_t)\omega$ is exact for all $t \in [0,1]$.

□

$\mathrm{Flux}(\{\psi_t\}) = 0$, and hence there is a function $F : M \to \mathbb{R}$, such that

$$\int_0^1 \iota(X_t)\omega dt = dF.$$

Let ϕ_s be the Hamiltonian flow of F. Then it is the flow of a vector field Y, such that $\iota(Y) = dF$. Therefore

$$\iota(Y)\omega = \int_0^1 \iota(X_t)\omega dt,$$

hence $Y = \int_0^1 X_t dt$. Now we change the ψ_t by juxtaposition with ϕ_s. It we get an isotopy

$$\psi'_t = \left\{ \begin{array}{ll} \psi_{2t} & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \phi_{1-2t} & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{array} \right.$$

This isotopy generates a vector field X'_t, which we integrate:

$$\int_0^1 X'_t dt = \int_0^1 X_t dt - \int_0^1 Y dt = Y - Y = 0.$$

Without loss of generality, we may assume in that ψ_t is an isotopy such that $\int_0^1 \iota(X_t)\omega dt = 0$, and $\int_0^1 X_t dt = 0$. For every $t \in R$, define the symplectic vector field.

$$Z_t = \int_0^t X_s ds,$$

which flow we denote with ζ_r^t. Now we consider the isotopy $\phi_t = \zeta_r^t \circ \psi_t$. Observe that $\zeta_r^0 = \zeta_r^1 = \text{id}$, and therefore we have

$$\phi_0 = \text{id}, \phi_1 = \psi.$$

ϕ_t is our desired Hamiltonian isotopy. We check this:

$$\begin{aligned} \text{Flux}(\{\phi_t\}_{0\le t\le T}) &= \text{Flux}(\{\zeta_1^t\}_{0\le t\le T}) + \text{Flux}(\{\psi_t\}_{0\le t\le T}) \\ &= \text{Flux}(\{\zeta_r^T\}_{0\le r\le 1}) + \int_0^T [\iota(X_t)\omega]dt \\ &= [\iota(Y_T)\omega] + \int_0^T [\iota(X_t)\omega]dt = 0. \end{aligned}$$

□

1.3 Poisson geometry

The Poisson bracket The idea of Poisson geometry is to look to the properties of the space of smooth functions on a manifold: $C^\infty(M)$. A Poisson structure on a manifold is defined in terms of a bracket on this space.

Definition 1.3.1. A Poisson bracket on a manifold M is a map

$$\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

such that the following properties are satisfied:

1. $C^\infty(M)$ equipped with a $\{,\}$ is a Lie algebra, i.e. for all $f, g \in C^\infty(M)$:

$$\begin{aligned} &\{f,g\} = -\{g,f\} && \text{(skew-symmetry)}, \\ &\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0 && \text{(Jacobi-identity)}, \end{aligned}$$

2. $\{.,.\}$ satisfies a Leibniz rule:

$$\{f, gh\} = \{f,g\}h + g\{f,h\}$$

Definition 1.3.2. A manifold M with a Poisson bracket $\{.,.\}$ is a Poisson manifold $(M, \{.,.\})$.

Example 1.3.3. Any symplectic manifold (M, ω) is a Poisson manifold. Recall that for every function $f \in C^\infty(M)$ there is Hamiltonian vector field X_f on M, such that $\iota(X_f)\omega = -df$. The Poisson bracket on M is defined by;

$$\{f, g\} = \omega(X_f, X_g).$$

This bracket is clearly skew-symmetric, and since $\omega(X_f, X_g) = -df(X_g) = dg(X_f) = L_{X_f}g = -L_{X_g}f$, the Leibniz rule follows directly from the Leibniz rule for the exterior derivative. We have to check the Jacobi identity. First we compute the exterior derivative of $\{g, f\}$:

$$d\{f, g\} = dL_{X_f}g = L_{X_f}dg = -L_{X_f}(\iota(X_g)\omega) = -\iota([X_f, X_g])\omega.$$

Here we used that $L_{X_f}\omega = 0$ in the formula:

$$\iota([X_f, X_g])\omega = L_{X_f}(\iota(X_g)\omega) - \iota(X_g)L_{X_f}\omega.$$

This expression for $d\{f, g\}$ implies:

$$X_{\{f,g\}} = [X_f, X_g].$$

We take an arbitrary $h \in C^\infty(M)$, and let this work on both sides:

$$\begin{aligned} \{\{f, g\}, h\} &= X_{\{f,g\}}(h) = [X_f, X_g](h) \\ &= X_f(X_g(h)) - X_g(X_f(h)) \\ &= X_f(\{g, h\}) - X_g(\{f, h\}) \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\}. \end{aligned}$$

This equation is equivalent to the Jacobi identity, which one can see by rearranging components using the skew symmetry of the Poisson bracket.

Poisson bivector There is an equivalent way to define a Poisson manifold. Let π be a bivector on M. Recall that we may consider π as a map:

$$\pi : \Omega^1(M) \times \Omega^1(M) \to C^\infty(M).$$

A bracket is now defined by:

$$\{f, g\} = \pi(df, dg).$$

This bracket may fail to satisfy the Jacobi identity. It satisfies this identity exactly if π yields $[\pi, \pi] = 0$ for the Schouten-Nijenhuis bracket. In that case we call π a Poisson bivector. This construction gives rise to a bundle map:

$$\pi^{\#} : T^*M \to TM, \mu \mapsto \pi^{\#}(\mu),$$

with $\nu(\pi^{\#}(\mu)) = \pi(\mu, \nu)$ for all $\nu \in T^*M$.

Poisson maps and submanifolds

Definition 1.3.4. A map $\varphi : (M, \pi_N) \to (N, \pi_N)$ between two Poisson manifolds is called a Poisson map if for all $f, g \in C^\infty(N)$

$$\{f \circ \varphi, f \circ \varphi\}_M = \{f, g\}_N \circ \varphi.$$

Definition 1.3.5. A Poisson map $\varphi : (M, \pi_N) \to (N, \pi_N)$ which is a diffeomorphism is called a Poisson diffeomorphism.

There is not a well defined way to pull-back a Poisson structure along a map, unless this map is a diffeomorphism. Therefore the definition of a Poisson submanifold looks a bit ugly.

Definition 1.3.6. A Poisson submanifold $(M, \pi_M) \subset (N, \pi_N)$ is a submanifold such that the immersion $i : M \hookrightarrow N$ is a Poisson map.

Note that not on all submanifolds of N, such a Poisson structure exists. If it exists, then it is unique. Later we will derive some conditions for submanifolds, which guarantee the existence of the Poisson structure.

Hamiltonian vector fields Given a function $f \in C^\infty(M)$, we get a derivation $\{f, .\}$ on $C^\infty(M)$. There is an one-to-one correspondence between derivations on $C^\infty(M)$ and vector fields $\mathfrak{X}(M)$: for every $f \in C^\infty(M)$, there is a $X_f \in \mathfrak{X}(M)$, such that for all $g \in C^\infty(M)$

$$\{f, g\} = L_{X_f} g.$$

Definition 1.3.7. A vector field X on a Poisson manifold $(M, \{., .\})$ is called a Hamiltonian vector field if there is a $f \in C^\infty(M)$, such that for all $g \in C^\infty(M)$

$$\{f, g\} = L_X g.$$

The function f is then called the Hamiltonian function of X.

Hamiltonian vector fields are not dependent on the function, but on the exterior derivative of the function. In relation with the bundle map $\pi^{\#}$ it yields $X_f = \pi^{\#} \circ df$. In the case that M is a symplectic manifold we found that $\{f, g\} = L_{X_f} g$ for the Hamiltonian vector field we defined above. So, the two definitions of Hamiltonian vector fields are compatible.

Group actions Like in symplectic geometry we distinguish some types of group actions of Lie groups G. Let $\rho : G \times M \to M$ be such an action. Denote with $\rho_g : M \to M$ the map one gets if one fixes an element $g \in G$.

Definition 1.3.8. We call an action on a Poisson manifold (M, π) a Poisson action if ρ_g is a Poisson map for all $g \in G$.

Definition 1.3.9. Let $(M, \{.,.\})$ be a Poisson manifold, and let G act on it in a Poisson fashion. The action is called Hamiltonian if there is a moment map:

$$\mu : M \to \mathfrak{g}^*,$$

such that for all $X \in \mathfrak{g}$, $\tilde{X} = \pi^{\#}(d\mu^X)$, where $\mu^X : X \to \mathbb{R} : p \mapsto \langle \mu(p), X \rangle$.

Note that in this situation $\tilde{X}$ is the Hamiltonian vector field of the function μ^X. The definition seems therefore suitable.

Symplectic foliation and the splitting theorem Let (M, π) be a Poisson manifold. We have seen that this means that we have a bundle map

$$\pi^{\#} : T^*M \to TM.$$

The image $\pi^{\#}(T^*M) \subset TM$ is a distribution, which we denote with D, and which is not necessarily regular.

Lemma 1.3.10. *Let (M, π) be a Poisson manifold. $D_m = \pi^{\#}_m(T^*_m M) \subset T_m M$ is a symplectic vector space, for every $m \in M$. On D_m the symplectic form ω_m is defined by:*

$$\omega_m(u, v) = \pi(\alpha, \beta).$$

Here is $\pi^{\#}(\alpha) = u$, and $\pi^{\#}(\beta) = v$.

Proof. First we prove that ω_m is well-defined. Suppose that $\alpha' \in T^*_m M$, such that $\pi^{\#}(\alpha') = u$. Then

$$\pi(\alpha - \alpha', \beta) = \beta(\pi^{\#}(\alpha)) - \beta(\pi^{\#}(\alpha')) = 0.$$

Therefore $\pi(\alpha, \beta) = \pi(\alpha', \beta)$, and hence ω_m is well-defined. That ω_m is skew-symmetric follows directly from the skew-symmetry of π. Suppose that $\omega_m(u, v) = 0$ for all $v \in D_m$, then it yields for all $\beta \in T^*_m M$, that $\beta(u) = 0$. Which implies that $u = 0$. We conclude that ω_m is nondegenerate, and really defines a symplectic structure on D_m. □

Definition 1.3.11. Let (M, π) be a Poisson manifold. The distribution $D = \operatorname{im} \pi^{\#}(T^*M)$ is called the symplectic distribution of π.

It is natural to wonder if this distribution is integrable to a foliation, and if that is the case if the symplectic forms on the tangent spaces can be extended to symplectic forms on the leaves. The answer is yes, as we will later see using Lie algebroids. The following theorem describes the local structure of Poisson manifolds.

Theorem 1.3.12 (The splitting theorem, Weinstein). *Let (M, π) be a Poisson manifold and choose $x \in M$ arbitrary. Denote $\operatorname{rank} \pi_x = 2k$, and let $\dim M = 2k + l = m$. There exists a neighbourhood centered as x with coordinates $(q_1, ..., q_k, p_1, ..., p_k, y_1, ...y_l)$, such that*

$$\pi = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j},$$

with $\phi_{ij}(0) = 0$.

Proof. In case $k = 0$ we are done. If $k > 0$, then there are two functions $f, g \in C^\infty(M)$, such that $\{f, g\}(x) \neq 0$. Label $g = q_1$. Because $X_{q_1}(f)(x) = \{f, g\}(x) \neq 0$, we have $X_{q_1} \neq 0$. One can find local coordinates $x_1, ..., x_m$ around x, such that $\frac{\partial}{\partial x_1} = X_{q_1}$. Then $\{q_1, x_1\} = \frac{\partial}{\partial x_1}(x_1) = 1$. Now choose p_1 to be x_1. Note that X_{p_1} and X_{q_1} are linear independent at x, because otherwise $1 = X_{p_1}(q_1)(x) = \{q_1, \lambda q_1\}(x) = 0$, which is clearly a contradiction. Linear independence of vector fields is an open condition, so X_{q_1} and X_{p_1} are linear independent on a neighbourhood of x. Since $[X_{q_1}, X_{p_1}] = X_{\{q_1, p_1\}} = 0$, we can use Frobenius theorem to find a two dimensional foliation, i.e. we find local coordinates $p_1, q_1, y_3, ..., y_m$ round x, such that $\{p_1, q_1\} = 1, \{p_1, y_i\} = \{q_1, y_i\} = 0$. The Jacobi identity implies that $\{p_1, \{y_i, y_j\}\} = \{q_1, \{y_i, y_j\}\} = 0$. Which means that $\frac{\partial}{\partial q_1}\{y_i, y_j\} = \frac{\partial}{\partial p_1}\{y_1, y_j\} = 0$. So, in this coordinates the bivector is of the form:

$$\pi = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \sum_{ij} \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.$$

If $k = 1$ we are done, otherwise we can find two functions $h, k \in C^\infty(M)$, such that $\{h, k\}(x) \neq 0$, and, moreover, such that $X_{p_1}(x), X_{q_1}(x), X_h(x), X_k(x)$, are linear independent. With this functions we repeat the construction. □

Remark 1.3.13. The splitting theorem states that all Poisson manifolds look locally like the product of a transversal N and some symplectic plaques. These transversal Poisson structures are unique up to Poisson diffeomorphisms, as we will see later using Dirac geometry.

Proposition 1.3.14. *The space of 1-forms $\Omega^1(M)$ on a Poisson manifold (M, π) is a Lie algebroid over M with anchor map $\pi^\#$. The bracket on the sections is defined by:*

$$\begin{aligned} [.,.]_\pi &: \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M) \\ [\alpha, \beta]_\pi &= L_{\pi^\#(\alpha)}\beta - L_{\pi^\#(\beta)\alpha} - d(\pi(\alpha, \beta)) \\ &= \iota(\pi^\#(\alpha))d\beta - \iota(\pi^\#(\beta))d\alpha + d(\pi(\alpha, \beta)). \end{aligned}$$

Proof. We have to prove that $[,]_\pi$ is a Lie bracket, and that it satisfies the Leibniz rule:

$$[\alpha, f\beta]_\pi = f[\alpha,\beta]_\pi + (L_{\pi^\#(\alpha)}f)\beta.$$

We start with the Leibniz rule.

$$\begin{aligned}
[\alpha, f\beta]_\pi &= \iota(\pi^\#(\alpha))d(f\beta) - f\iota(\pi^\#(\beta))d\alpha + d(f\pi(\alpha,\beta)) \\
&= f[\alpha,\beta]_\pi + \iota(\pi^\#(\alpha))df \wedge \beta + \pi(\alpha,\beta)df \\
&= f[\alpha,\beta]_\pi + (\iota(\pi^\#(\alpha)))\beta \\
&= f\,[\alpha,\beta]_\pi + (L_{\pi^\#(\alpha)})\beta.
\end{aligned}$$

It is left prove that $[,]_\pi$ is a Lie bracket. $[,]_\pi$ is clearly skew symmetric. First we check the Jacobi identity on two exact forms; df dg. If we put those in our second expression for the bracket we get:

$$[df, dg]_\pi = d(\pi(df, dg)) = d\{f, g\}.$$

So the Jacobi identity for the exact forms is equivalent with the Jacobi identity for the Poisson bracket. Because all 1-forms are locally linear combinations of functions times exact 1-forms, it is sufficient to check the Jacobi identity for (fdx, dy, dz), where $f, x, y, z \in C^\infty(M)$.

$$\begin{aligned}
[fdx, [dy,dz]_\pi]_\pi &= f[dx[dy,dz]_\pi]_\pi + L_{\pi^\#([dy,dz]_\pi)}(f)dx \\
[dy, [dz, fdx]_\pi]_\pi &= \left[dy, f\,[dz,dx]_\pi + L_{\pi^\#(dz)}(f)dx\right]_\pi \\
&= f\,[dy,[dz,dx]_\pi]_\pi + L_{\pi^\#(dy)}(f)[dz,dx]_\pi \\
&+ L_{\pi^\#(dz)}(f)[dy,dx]_\pi + L_{\pi^\#(dy)}L_{\pi^\#(dz)}(f)dx \\
[dz\,[fdx,dy]_\pi]_\pi &= f\,[dz,[dx,dy]_\pi]_\pi + L_{\pi^\#(dz)}(f)[dx,dy]_\pi \\
&- L_{\pi^\#(dy)}(f)[dz,dx]_\pi - L_{\pi^\#(dz)}L_{\pi^\#(dy)}(f)dx.
\end{aligned}$$

When we add those three expression then everything cancels by the Jacobi identity for exact 1-forms, and the following identity for Lie derivatives:

$$L_{\pi^\#([dy,dz]_\pi)}(f)dx = L_{\pi^\#(dy)}L_{\pi^\#(dz)}(f)dx - L_{\pi^\#(dz)}L_{\pi^\#(dy)}(f)dx.$$

□

Definition 1.3.15. The orbits of the Lie algebroid described above are called the symplectic leaves of the Poisson π structure on M. The foliation is called the symplectic foliation on of the Poisson structure.

All Hamiltonian vector fields are tangent to the symplectic leaves. We find the symplectic leaves in a set theoretical way by an equivalence relation; two points x, y are equivalent if there are Hamiltonian vector fields $X_1, .., X_n$, with respective flows $\varphi^1, ..., \varphi^n$, and a $t \in \mathbb{R}^n$, such that

$$y = \varphi^n_{t_n} \circ \cdots \circ \varphi^1_{t_1}(x).$$

Linear Poisson structures

Definition 1.3.16. A Poisson structure on a vector space V is called linear if the bracket of two linear functions on V is again linear.

The space of the linear functions on V is the dual V^* of V, and hence a linear Poisson structure gives a Lie algebra structure on the dual space. We take V to be finite dimensional, such that V is canonically isomorphic to V^{**}. So V is the dual of a Lie algebra $\mathfrak{g}$, and we denote V with $\mathfrak{g}^*$.

A way to define a Poisson structure on $\mathfrak{g}^*$ is by:

$$\{f, g\}(\mu) = \langle \mu, [d_\mu f, d_\mu g] \rangle .$$

Again we made use of the fact that $\mathfrak{g}^{**} = \mathfrak{g}$. That this bracket satisfies the Leibniz condition follows directly from the Leibniz rule for the exterior derivative. To check the Jacobi identity we make use of structure of the Lie algebra.

Definition 1.3.17. Let $x_1, ..., x_n$ be a basis of a Lie algebra $\mathfrak{g}$. There are constants c^k_{ij} such that:

$$[x_i, x_j] = \sum_k c^k_{ij} x_k.$$

These constants are called structure constants.

So, chosen a basis for $\mathfrak{g}$, we find the following expression:

$$\{f, g\} = \sum_{ijk} c^k_{ij} x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \; f, g \in C^\infty(\mathfrak{g}^*).$$

Therefore the structure constant of $\mathfrak{g}$ are exactly the structure constants of $(C^\infty(M), \{.,.\})$, and the Jacobi identity is satisfied. The Poisson structure described here is called the Konstant-Kirilov-Souriau Poisson structure.

We have a look at the symplectic leaves of $\mathfrak{g}^*$. Let $u, v \in \mathfrak{g}^*$. We may consider them as linear functions on $\mathfrak{g}$, and hence they have Hamiltonian vector fields:

$$X_u v(\mu) = \{u, v\}(\mu) = \langle \mu, [u, v] \rangle = \langle ad^*_u(\mu), v \rangle .$$

So, X_u is tangent to the coadjoint orbits, and the symplectic leaves are these orbits. We did meet this orbits already as examples of symplectic manifolds in the previous section.

Example 1.3.18. We compute an explicit example. Take $\mathfrak{g}$ to be $\mathbb{R}^2$ and denote the (e_1, e_2) basis. The Lie product is defined by $[e_1, e_2] = e_1$. Denote with (f_1, f_2) the dual basis, and with (h_1, h_2) the dual of that basis of $\mathfrak{g}^{**}$. The isomorphism, $\mathfrak{g}^{**} \to \mathfrak{g}$, which is in fact independent of all these bases, is given by $ah_1 + bh_2 \mapsto ae_1 + be_2$. Take $u_1 = a_1h_1 + b_1h_2, u_2 = a_2h_1 + b_2h_2 \in \mathfrak{g}^{**}$ and $\mu = cf_1 + df_2 \in \mathfrak{g}^*$, then:

$$\{u_1, u_2\}(\mu) = \mu([a_1h_1 + b_1h_2, a_2h_1 + b_2h_2]) = \mu((a_1b_2 - b_1a_2)e_1) = c(a_1b_2 - b_1a_2).$$

1.4 Dirac geometry

1.4.1 Linear Dirac structures

Let W be a finite dimensional vector space. On the direct sum $W \oplus W^*$ there is a natural paring:

$$\langle (v, \nu), (v', \nu') \rangle = \nu'(v) + \nu(v').$$

Definition 1.4.1. A linear Dirac structure on W is a subspace $L \subset W \oplus W^*$, such that L is maximal isotropic with respect to the paring, $\langle , \rangle$, or equivalently L is isotropic and $\dim(L) = \dim(W)$.

The image of the projection $pr_1 : L \to W$ is called the range R of L. We claim that the annihilator R° is the set $\{\alpha \in W^* : (0, \alpha) \in L\} = W^* \cap L$.

Proof. Let $\alpha \in R^\circ$, then $\alpha(v) = 0$ for all $v \in R$. So for all $(v, \xi) \in L$ it yields $\langle (0, \alpha), (v, \xi) \rangle = 0$, and hence $L + \mathbb{R}(0, \alpha)$ is isotropic. By the maximality of L, this means that $(0, \alpha) \in L$. For the other inclusion, we just take $(0, \alpha), (v, \xi) \in L$. Then $\alpha(v) = \langle (0, \alpha), (v, \xi) \rangle = 0$. So $\alpha \in R^\circ$. □

Denote with $pr_2 : L \to W^*$ the projection of L on W^*. The annihilator of $pr_2(L)$ is called the kernel K of the Dirac structure. In a very similar way as the prove above one can prove that $K = \{v \in W : (v, 0) \in L\} = W \cap L$.

Given a Dirac structure L, we define a natural 2-form ω on the range of L. Let $(v, \xi), (u, \eta) \in L$, then $\omega(u, v) = \xi(u) = -\eta(v)$. Conversely, given a subset R of W and a 2-form ω, we obtain a Dirac structure $L_{R,\omega}$ in the following way.

$$L_{R,\omega} = \{(v, \xi) \in W \oplus W^* : v \in R, \xi|_R = i_v\omega\}.$$

This set is isotropic because: $\langle (v, \xi), (u, \eta) \rangle = \xi(u) + \eta(v) = \iota(v)\omega(u) + \iota(u)\omega(v) = 0$. Counting the dimensions gives the maximality property. Note that if we reconstruct the range and the natural 2-form from $L_{R,\omega}$, we indeed get R and ω back. Introducing an R and a ω is an equivalent way of defining a Dirac structure.

With the kernel we can do more or less the same trick. A Dirac structure L implies a bivector on W/K. Let again $(v,\xi),(u,\eta) \in L$, and define $\pi(\xi,\eta) = \xi(u)$. This is a antisymmetric form which has K as kernel. Such a 2-form can also be seen as a map $\pi : pr_2(L) \to pr_2(L)^*$. And using the canonical isomorphism $W \simeq W^{**}$, we find a bivector $\pi : (W/K)^* \to W/K$.

Definition 1.4.2. Let $V \subset W$ be a subspace, we call a Dirac structure $L_{R,\omega}$ V-nondegenerate if $\omega|_{V\cap R}$ is nondegenerate.

Proposition 1.4.3. *$L_{R,\omega}$ is V-nondegenerate if and only if $(V \times V^\circ) \cap L_{R,\omega} = \{0\}$.*

We postpone the proof of this proposition a little. First we explore some nice properties of linear Dirac structures.

Proposition 1.4.4. *Let L be a Dirac structure on a vector space W, and subsets V of W, such that $(V \times V^\circ) \cap L = \{0\}$. Then the space*

$$H = \{v \in W : \text{there is a } \xi \in V^\circ, \text{ such that } (v,\xi) \in L\}.$$

is an orthogonal complement of V in W.

Proof. We check that $H \cap V = \{0\}$. Take $v \in H \cap V$, then, since $v \in H$, there is a $\xi \in V^0$ such that $(v,\xi) \in L$. Hence we obtain

$$(v,\xi) \in (V \times V^\circ) \cap L = \{0\},$$

and v has to be 0. Next we check that $W = V + H$. Counting dimensions we find $\dim L = \dim W = \dim(V \times V^\circ)$, and since L and $V \times V^\circ$ intersect trivially, we must have

$$W \times W^* = (V \times V^\circ) \oplus L.$$

Therefore, for all $w \in W$ there is a decomposition

$$(w,0) = (v,\xi) + (h,\eta),$$

with $v \in V, \xi \in V^\circ$ and $(h,\eta) \in L$. It follows that $\eta = -\xi$, and hence in V°, so $h \in L$. We found that $w = v + h$, and we conclude

$$W = H \oplus V.$$

□

Corollary 1.4.5. *Dually there is the decomposition $W^* = H^\circ \oplus V^\circ$.*

Example 1.4.6. We consider the case that $V = R$. Then $L_{R,\omega}$ is V-nondegenerate exactly if ω is nondegenerate. $H = \{0\}$ and $V = R = W$ is a symplectic vector space.

Example 1.4.7. The other extreme case is $V \cap R = \{0\}$. Then all $L_{R,\omega}$ are V-nondegenerate. And for all $X \in R$ there is $\alpha \in V^\circ$, such that $\alpha|_R = \iota(X)\omega$. Hence $R = H$.

Lemma 1.4.8.

$$H^\circ = \{\alpha \in W^* : \text{ there is a } X \in V, \text{ such that } (X, \alpha) \in L\}$$

$$V^\circ = \{\beta \in W^* : \text{ there is a } Y \in H, \text{such that } (Y, \beta) \in L\}$$

Proof. We only prove the first equation, the second can be proved in a similar way. Take $X \in V$, such that there is an $\alpha \in W^*$ for which $(X, \alpha) \in L$. By the definition of H, there is for every $Y \in H$ a $\beta \in V^\circ$, with $(Y, \beta) \in L$. Because L is isotropic we find

$$\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X) = \alpha(Y) = 0.$$

Therefore, such α indeed annihilates H. Moreover, the decomposition $W \times W^* = (V \times V^\circ) \oplus L$, induces that we can write every $\xi \in W^*$, as

$$(0, \xi) = (X, \eta) + (Y, \alpha) \in (V \times V^\circ) \oplus L.$$

It follows that $Y = -X \in V$. The decomposition $W^* = H^\circ \oplus V^\circ$ induces that if $\xi \in H^\circ$, then $\eta = 0$. Hence for any $\alpha \in H^*$ there is a $Y \in V$, such that $(Y, \alpha) \in L$. □

Proof of proposition 1.4.3. If $V \cap R = \{0\}$, then V is clearly nondegenerate and the intersection $(V \times V^\circ) \cap L_{R,\omega} = \{0\}$. Assume that there is a $X \neq 0$ in $V \cap R$. Suppose first that L is V-nondegenerate, then there is an $Y \in R \cap V$, such that

$$\iota(Y)\xi|_{R \cap V} = \iota(Y)\iota(X)\omega|_{V \cap R} \neq 0,$$

which implies that $\xi \notin V^\circ$, i.e. $(V \times V^\circ) \cap L_{R,\omega} = \{0\}$. For the converse we use the decomposition $W = H \oplus V$. Assume that there is a $X \in V \cap R$ such that $\omega(X, Y) = 0$ for all $Y \in V \cap R$, $\omega(X, Y) = 0$. Then there is a $\xi \in W^*$ such that $(X, \xi) \in L$, i.e. $\xi|_R = \iota(X)\omega$. Moreover $\xi \in H^\circ$, and $H \subset R$. Therefore this implies that $\iota(X)\omega = 0$, which on its turn implies that $X = 0$ or $\omega = 0$. In the first case we are done, in the second we get

$$\begin{aligned} L_{R,\omega} &= \{(X, \xi) \in W \oplus W^* : X \in R, \xi|_R = \iota(X)\omega\} \\ &= \{(X, \xi) \in W \oplus W^* : X \in R, \xi \in R^\circ\}. \end{aligned}$$

$(V \times V^\circ) \cap (R \times R^\circ) = \{0\}$ implies that $V \cap R = \{0\}$, which we assumed not to be the case. Hence ω is V-nondegenerate. □

1.4.2 Dirac manifolds

To go from the study of linear Dirac structures to the study of Dirac manifolds we first generalise the tangent bundle of a manifold M:

$$\mathcal{T}M = TM \oplus T^*M,$$

on which we have again a natural paring.

$$\langle (x,\xi),(x',\xi')\rangle = \xi'(x) + \xi(x').$$

Note that sections of $\mathcal{T}M$ are described by elements of $\mathfrak{X}(M) \times \Omega^1(M)$. On them we define the so called Courant bracket:

$$[(x,\xi),(x',\xi')]_c = ([x,x'], L_x\xi' - L_{x'}\xi + d(\xi(x'))).$$

Definition 1.4.9. An almost Dirac structure on a manifold M is a subbundle $L \subset \mathcal{T}M$ of rank $\dim M$ such that $\langle \alpha, \beta\rangle = 0$ for all $\alpha, \beta \in \Gamma(L)$.

Definition 1.4.10. A Dirac structure on a manifold M is an almost Dirac structure which is closed under the Courant bracket, i.e. $[\alpha,\beta]_c \in \Gamma(L)$ for all $\alpha, \beta \in \Gamma(L)$.

Note that on an almost Dirac structure the Courant bracket is anti-symmetric.

Example 1.4.11 (Dirac structures of closed 2-form type). The graph $\{(X, \iota(X)\omega)\}$ of a 2-form on M defines an almost Dirac structure L_ω. This almost Dirac structure is a Dirac structure if for all vector fields $X, Y \in \mathfrak{X}(X)$, it yields

$$\iota([X,Y])\omega = L_X(\iota(Y)\omega) - L_Y(\iota(X)\omega) + d((\iota(X)\omega)(Y)).$$

Because,

$$d((\iota(X)\omega)(Y)) = L_Y(\iota(X)\omega) - \iota(Y)d(\iota(X)\omega) = L_Y(\iota(X)\omega) - \iota(Y)L_X\omega + \iota(Y)\iota(X)d\omega,$$

we get that L_ω is Dirac exactly if

$$\iota([X,Y])\omega = L_X(\iota(Y)\omega) - \iota(Y)L_X\omega + \iota(Y)\iota(X)d\omega.$$

Which is equivalent to $d\omega = 0$, i.e. to ω is closed. Dirac structures of this form are called of closed 2-form type or presymplectic Dirac structures. Note that for such Dirac structures the map $pr_1 : L \to TM$ is surjective. The other way round, given a Dirac structure L for which this projection is surjective, one can find a 2-form for which this structure is the graph; for any vector field $X \in \mathfrak{X}(M)$ there is an unique 1-form α_X, such that $(X, \alpha_X) \in \Gamma(M)$. We find ω to be

$$\omega(X,Y) = \alpha_X(Y) = -\alpha_Y(X).$$

Example 1.4.12 (Dirac structures of Poisson bivector type). The graph of a bivector π on M, $\{(\pi^{\#}(\xi),\xi) : \xi \in T^*M\}$ is an almost Dirac structure, as is easily seen by this computation:

$$\left\langle(\pi^{\#}(\xi),\xi),(\pi^{\#}(\xi'),\xi')\right\rangle = \pi(\xi,\xi') + \pi(\xi,\xi').$$

This structure is Dirac if for all $\xi,\xi' \in T^*M$, $[\pi^{\#}(\xi),\pi^{\#}(\xi')] = \pi^{\#}(\eta)$, where

$$\eta = L_{\pi^{\#}(\xi)}\xi' - L_{\pi^{\#}(\xi')}\xi + d(\xi(\pi^{\#}(\xi'))) = [\xi,\xi']_{\pi}.$$

Which is equivalent to requiring that $\pi^{\#}([\xi,\xi']_{\pi}) = [\pi^{\#}(\xi),\pi^{\#}(\xi')]$, i.e the bivector preserves bracket or equivalently it is a Poisson bivector. Note that for such Dirac structures the map $pr_2 : L \to T^*M$ is surjective. The other way round, given a Dirac structure L for which this projection is surjective, one can find a bivector for which this structure is the graph. For every $\xi \in T^*M$ there is an unique $X_\xi \in TM$ such that $(X_\xi,\xi) \in L$. Define the bivector by:

$$\pi^{\#} : T^*M \to TM : \xi \mapsto X_\xi.$$

A way to measure if L is Dirac There is a 3-form T_L on an almost Dirac structure L, which vanishes identically if and only if L is a Dirac structure. Let $(s_1,s_2,s_3) \in \Gamma(L)$, then

$$T_L(s_1,s_2,s_3) = \langle[s_1,s_2]_c,s_3\rangle\,.$$

For later computational purposes we give a more explicit expression for this 3-form. Denote $s_1 = (X,\alpha)$, $s_2 = (Y,\beta)$, $s_3 = (Z,\gamma)$.

$$\begin{aligned}T_L(s_1,s_2,s_3) = \frac{1}{2}(&L_X(\beta(Z)) + L_Y(\gamma(X)) + L_Z(\alpha(Y)) + \\ &\alpha([Y,Z]) + \beta([Z,X]) + \gamma([X,Y]))\end{aligned}$$

Proof.

$$\begin{aligned}T_L(s_1,s_2,s_3) &= \frac{1}{2}((L_X\beta - L_Y\alpha + d(\alpha(Y)))(Z) + \gamma([X,Y]))\\ &= \frac{1}{2}(L_X(\iota(Z)\beta) - \iota([X,Z])\beta\\ &\quad -L_Y(\iota(Z)\alpha) + \iota([Y,Z])\alpha + \iota(Z)d(\alpha(Y)) + \gamma([X,Y]))\\ &= \frac{1}{2}(L_X(\beta(Z)) + L_Y(\gamma(X)) + L_Z(\alpha(Y))\\ &\quad +\alpha([Y,Z]) + \beta([Z,X]) + \gamma([X,Y])).\end{aligned}$$

□

The Dirac structure as Lie algebroid

Theorem 1.4.13. *A Dirac structure L on a manifold M is a Lie algebroid over M with anchor map $pr : L \to TM$. The bracket on the sections is the Courant bracket.*

Proof. We have to check the Jacobi identity and the Leibniz rule, which both can be done by straight forward computations. We only do the Leibniz rule:

$$\begin{aligned}
[(x,\xi), f(x',\xi')]_c &= ([x, fx'], L_x f\xi' - L_{fx'}\xi + d(\xi(fx'))) \\
&= (f[x,x'] + (L_x f)\xi', (L_x f)\xi' + fL_x\xi' - fL_{x'}\xi - \xi(x')df + f(\xi(x')) + \xi' df) \\
&= f[(x,\xi),(x',\xi')]_c + L_x f\xi'
\end{aligned}$$

□

Definition 1.4.14. Let (M, L) be a Dirac manifold. The orbits of the Dirac structure are called the presymplectic leaves of the Dirac structure. They form the presymplectic foliation $\mathcal{F}_L$ of the Dirac structure.

Lemma 1.4.15. *Let (M, L) be a a Dirac manifold. All presymplectic leaves carry a natural closed 2-form.*

Proof. For every $X \in \mathfrak{X}(\mathcal{F}_L)$ there is a $\alpha_X \in \Omega^2(\mathcal{F})$, such that $(X, \alpha_X) \in \Gamma(L)$. Let S be a leaf of $\mathcal{F}_L$, and define ω_S as follows:

$$\omega_S(X,Y) = \alpha_X(Y)|_S = -\alpha_Y(X)|_S.$$

We check that this form is closed.

$$\begin{aligned}
d\omega_S(X,Y,Z) &= (\alpha_X([Y,Z]) + \alpha_Y([Z,X]) + \alpha_Z([X,Y]))|_S \\
&\quad +L_Z(\omega_S(X,Y)) + L_X(\omega_S(Y,Z) + L_Y\omega_S(Z,X)) \\
&= T_L((X,\alpha_X),(Y,\alpha_Y),(Z,\alpha_Z))|_S.
\end{aligned}$$

T_L vansihes, because L is a Dirac structure. □

Theorem 1.4.16. *A Dirac structure L on a manifold M such that the characteristic distribution is regular, is determined by a regular foliation $\mathcal{F}$ together with a foliated 2-form $\omega \in \Omega^2(\mathcal{F})$, which is leafwise closed, i.e. the restriction of ω to any leaf S is closed.*

Proof. Let $\mathcal{F}$ and ω be as in the theorem. The Dirac structure is given by:

$$L = \{(X, \iota(X)\omega) : X \in \mathfrak{X}(F)\}.$$

□

Corollary 1.4.17. *Let M be a manifold. There is an one-to-one correspondence between Poisson structures on M for which the symplectic distribution is regular, and regular symplectic foliations of M, i.e. foliations $\mathcal{F}$ and a nondegenerate leafwise closed 2-form $\omega \in \Omega^2(M)$.*

Proof. The graph of a Poisson bivector determines a Dirac structure on the manifold. The presymplectic leaves of this Dirac structure are precisely the symplectic leaves induced by the Poisson structure. The other way round, a symplectic foliation on a manifold induces in a unique way a Dirac structure. By the nondegeneracy of the 2-form is the projection of this Dirac structure to the cotangent bundle bijective. Therefore, it is the graph of a Poisson bivector. □

In the other extreme example, where the Dirac structure is the graph of a closed 2-form, there is only one presymplectic leaf; the manifold itself.

Remark 1.4.18. The theorem above can be extended to the situation where the characteristic distribution is singular.

Dirac gauging Given a Dirac structure L on a manifold M, one could wonder how to change it a little such that it is still Dirac. This can be done using so called Dirac gauging. The idea is to change the foliated 2-form on M by adding a 2-form $B \in \Omega^2(M)$, such that for all leaves S the restrictions

$$\omega_S + B|_S \in \Omega^2(S)$$

are closed. In particular all closed 2-forms on M would work. Doing this one obtains a new Dirac structure which is the image of the map

$$\tau_B : \mathcal{T}M \to \mathcal{T}M : (X, \xi) \to (X, \xi - \iota(X)B).$$

Such a 2-form is called a Dirac gauge 2-form, the map τ_B is called a Dirac gauge transformation, and the new Dirac structure is called the gauge transform.

Remark 1.4.19. Set theoretically, the leaves of the presymplectic foliation of a Dirac structure don't change.

Poisson gauging If we gauge a Poisson Dirac manifold (M, L_π) structure then we wonder if the gauge transform is again the graph of a Poisson bivector. In this paragraph we discuss the conditions this imposes on the gauge 2-form. The natural condition is that for all the leaves S the restrictions

$$\omega_S + B\mid_S \in \Omega^2(S)$$

are symplectic. A calculation yields that

$$\ker((\omega_S + B|_S)_x) = \ker((\mathrm{id}_{TM} - \pi^\# \circ B_\#)_x),$$

for all $x \in S$ and all symplectic leaves (S, ω_S). This proves the following lemma.

Lemma 1.4.20. *$\omega_S + B|_S$ is symplectic for every leaf S of the foliation if and only if $\mathrm{id}_{TM} - \pi^\# \circ B_\# : TM \to TM$ is an isomorphism.*

Which is the motivation for the coming definition.

Definition 1.4.21. A closed 2-form B on a Poisson manifold (M, π) for which all symplectic leaves $(S, \omega_S + B|_S)$ are again a symplectic manifolds, is called a Poisson gauging form for (M, π).

Proposition 1.4.22. *Let B be a is a Poisson gauging form for (M, π), then*

$$\pi_B^\# = (id_{TM} - \pi^\# \circ B_\#)^{-1} \circ \pi^\# : T^*M \to TM$$

defines a Poisson structure on M.

Proof. We have to check that $[\pi_B, \pi_B] = 0$ for the Schouten-Nijenhuis bracket. Note that for all 1-forms on M we have $\pi_(\xi, \eta) = \pi(\xi + \pi_B^\#(\xi), \eta)$. So $[\pi, \pi] = 0$ implies that $[\pi_B, \pi_B] = 0$. □

$\pi_B^\#$ is called the gauge transformation of π, with respect to B. We use this theory of Poisson gauging to prove for some special manifolds that they are Poisson diffeomorphic. We do this using a result very similar to the Moser theorem in the symplectic case.

Theorem 1.4.23. *Let M be a compact manifold and be π_0, π_1 be Poisson structures on M. Assume that there is a smooth family $\{B_t\}_{t\in[0,1]}$ of gauging forms for (M, π_0) such that*

1. *$B_0 = 0$,*
2. *π_1 is a gauge transformation of π_0 with respect to B_1,*
3. *$\frac{d[B_t]}{dt} = [0]$ in $H^2_{de\ Rham}(M)$,*

then (M, π_0) and (M, π_1) are Poisson diffeomorphic.

Proof. We consider the problem leavewise, and use the Moser theorem from symplectic geometry. Denote the symplectic leaves of (M, π_0) and (M, π_1), with (S, ω_S^0) and (S, ω_S^1) respective. On each leaf S, $\omega_{S,t} = \omega_S^0 + B_t|_S$ is a smooth family of symplectic forms, in the same cohomology class, such that $\omega_{S,0} = \omega_S^0$ and $\omega_{S,1} = \omega_S^1$. So, all leaves are symplectomorphic under the gauge transformation. This means that there is a diffeomorphism

$$\Psi : M \to M,$$

such that for all leaves S, $\Psi|_S : (S, \omega_S^0) \to (S, \omega_S^1)$ is a symplectomorphism. Ψ is the desires Poison diffeomorphism. □

Maps between Dirac manifolds In general one can say that a nice feature of vector fields is that one can push them forward, and a nice feature of forms is that one can pull them back. For Dirac structures, being a combination of those two, we define concepts in both directions.

Definition 1.4.24. Let $\phi : (M, L_M) \to (N, L_N)$ be a smooth map between Dirac manifolds.

1. ϕ is called a forward Dirac map or shortly f-Dirac, if:
   $L_N = \phi_* L_M = \{(Y, \beta) \in TN \oplus T^*N : \text{ there is a } X \in TM, \text{ with } d\phi(X) = Y \text{ and } (X, d\phi^*\beta) \in L_M\}$
2. ϕ is called a backward Dirac map or shortly b-Dirac, if:
   $L_M = \phi^* L_N = \{(X, \alpha) \in TM \oplus T^*M : \text{ there is a } \beta \in T^*N, \text{ with } d\phi^*(\beta) = \alpha \text{ and } (d\phi(X), \beta) \in L_N\}$

In case ϕ is a diffeomorphism, these notions coincide.

Reduction Dirac reduction looks quite similar to the reduction we have seen in the Marsden-Meyer-Weinstein quotient. First we translate the notion of a Hamiltonian group action to the Dirac language.

Definition 1.4.25. Let $G \times M \to M$ be a proper action of a Lie group G on a Dirac manifold, (M, L). We call the action Hamiltonian if there is a moment map:

$$\mu : M \to \mathfrak{g}^* \text{ such that: } (X_\xi, d\mu^\xi) \in \Gamma(L) \text{ for all } \xi \in \mathfrak{g},$$

which is G-equivariant:

$$\mu(g \cdot x) = (\mathrm{Ad}_g^*) \circ \mu(x)$$

$X_\xi \in \mathfrak{X}(M)$ is defined by:

$$X_\xi(m) = \frac{d}{dt}|_{t=0} \exp(t\xi) \cdot m,$$

and $\mu^\xi : M \to \mathbb{R} : m \mapsto \langle \mu(m), \xi \rangle$.

Lemma 1.4.26. *$\mu : M \to \mathfrak{g}^*$ is a f-Dirac, where $\mathfrak{g}^*$ is equipped with the Dirac structure induced by the Konstant-Kirilov-Souriau Poisson structure.*

Proof. For every $\eta \in \mathfrak{g}^*$, we have $T^*_\mu \mathfrak{g}^* \simeq \mathfrak{g}$. We use this identification to give an explicit description of the Dirac structure on $\mathfrak{g}^*$:

$$L_{\mathfrak{g}^*} = \{(\langle \eta, [\cdot, \xi] \rangle, \xi) \in T\mathfrak{g}^* \oplus T^*\mathfrak{g}^*\}.$$

So μ is a f-Dirac means that $d_m\mu(X_\xi(m)) = \langle \eta, [\cdot, \xi] \rangle$, for $\mu(m) = \eta$, and all $\xi \in \mathfrak{g}$, and moreover $d\mu^*(\xi) = d\mu^\xi$. The last requirement is true by the definition of the dual of a linear map. For the first requirement we compute:

$$d_m\mu(X_\xi(m)) = d_m\mu(\frac{d}{dt}|_{t=0} \exp(t\xi) \cdot m) = ad^*_\xi(\mu(m)) = \langle \mu(m), [\cdot, \xi] \rangle .$$

Where we used the G-equivariance of the moment map.

□

Proposition 1.4.27. *Let $G \times M \to M$ be a Hamiltonian action of a compact Lie group G on a Dirac manifold (M, L), with momentum map $\mu : M \to \mathfrak{g}^*$. Let 0 be a regular value of μ, and let the action of G on $\mu^{-1}(0)$ be free, then*

1. *$i : \mu^{-1}(0) \hookrightarrow M$, is a Dirac submanifold so it inherits the pull-back Dirac structure i^*L.*
2. *The orbit space $M_{red} = \mu^{-1}(0)/G$ inherits a Dirac structure L_{red}, which is uniquely defined by the requirement that the quotient map, $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$, is a f-Dirac.*

Proof. 1. 0 is a regular value of μ, so $\mu^{-1}(0)$ is a submanifold of codimension $\dim G = g$ of M. The only thing to check is that the pull-back Dirac structure i^*L is of constant rank $\dim M - \dim G$. First we give the explicit expression for i^*L.

$$i^*L = \{(X, \alpha_{|T\mu^{-1}(0)}) : X \in T\mu^{-1}(0), (X, \alpha) \in L\}$$

Let X_ξ be a vector field induced by the action of G, and let $x \in \mu^{-1}(0)$. Then, by the G-equivariance of μ,

$$\mu(\exp(t\xi) \cdot x) = \mathrm{Ad}^*_{\exp(t\xi)} \cdot \mu(x) = 0.$$

So all vector fields X_ξ are tangent to $\mu^{-1}(0)$. i^*L contains the pairs $(X_\xi, d\mu|_{T\mu^{-1}(0)}) = (X_\xi, 0)$, for all $\xi \in \mathfrak{g}$, because $T_p\mu^{-1}(0) = \ker d_p\mu$, for all $p \in \mu^{-1}(0)$. Define for all $m \in \mu^{-1}(0)$, $H_m = \mathrm{span}\{X_\xi(m) : \xi \in \mathfrak{g}\}$. Then L contains all pairs $(0, \alpha)$, such that α annihilates H_m, and so does i^*L. Because G acts freely on $\mu^{-1}(0)$, all X_ξ are nonzero and moreover rank of H is $\dim G$. So the rank of i^*L is at least, $g+(m-g)-g = m-g$. Because i^*L is isotropic it rank is also at most $m - g$, hence i^*L is of constant rank.

2. G is a compact Lie group which acts freely on $\mu^{-1}(0)$. Hence $\mu^{-1}(0)$ is a manifold. L_{red} is desribed by:

$$\begin{aligned} L_{red} = \{(d\pi X, \beta) \in T(\mu^{-1}(0)/G) \oplus T^*(\mu^{-1}(0)/G) : \\ X \in T\mu^{-1}(0), d\pi^*\beta = \alpha|_{T\mu^{-1}(0)}\ (X, \alpha) \in L\}. \end{aligned}$$

Note that $d\pi(X_\xi) = 0$ for all $\xi \in \mathfrak{g}$.

□

1.5 Poisson geometry continued

Recall that it is not always possible to find a compatible Poisson structure on a submanifold of a Poisson manifold. In this section we discuss a type of submanifolds on which such a Poisson structure always can be found.

Transversal and cosymplectic submanifolds

Definition 1.5.1. A transversal of a Poisson manifold (M, π) through x is a submanifold $N \subset M$ such that

$$T_xM = T_xN \oplus T_xS,$$

where S is the symplectic leaf containing x.

In fact transversal submanifolds are Poisson manifolds, as we will see later.

Definition 1.5.2. Let $N \subset M$ be a submanifold. Define

$$(T_xN)^\perp = \pi^\#((T_xM/T_xN)^*) \subset T_xM$$

N is called of constant rank if $\dim(T_xN)^\perp$ is independent of x.

The notation $(T_xN)^\perp$ suggests that this has to do with some kind of orthogonality. In fact if M is a symplectic manifold, then we $(T_xN)^\perp = (T_xN)^{\omega_x}$, the symplectic orthogonal. In general $(T_xN)^\perp$ is the symplectic orthogonal of $T_xN \cap T_xS$ inside T_xS.

Definition 1.5.3. A submanifold $N \subset M$ is called cosymplectic in (M, π), if for all $x \in N$

$$T_xM = T_xN + (T_xN)^\perp$$

Proposition 1.5.4. *If N is cosymplectic in (M, π), then $T_xM = T_xN \oplus (T_xN)^\perp$ for all $x \in N$. In particular is N of constant rank.*

Proposition 1.5.5. *For $x \in M$, there is at least one cosymplectic transversal that passes through x. Moreover, for each transversal , $\tilde{N}$, that passes through x, there exists an open subspace $N \subset \tilde{N}$, containing x, such that N is a cosymplectic transversal.*

Proof. $\tilde{N}$ is transversal in x, so $T_x\tilde{N} \cap T_xS = \{0\}$. Because $(T_xN)^\perp$ is the symplectic orthogonal of $T_xN \cap T_xS$ inside T_xS, this implies that $(T_x\tilde{N})^\perp = T_xS$. So at x we have indeed

$$T_xM = T_x\tilde{N} \oplus (T_x\tilde{N})^\perp.$$

This is an open condition, so there is an open neighbourhood of x for which this is true. □

Theorem 1.5.6. *A transversal cosymplectic submanifold N of a Poisson manifold (M,π) inherits a Poisson structure, such that it is a Poisson submanifold.*

Proof. Let $L = L_\pi$ be the Dirac structure on M induced by π. N is a Poisson submanifold exactly if there is a Poisson bivector π^N on N, such that the pull-back Dirac structure,

$$L_N = \{(X, \alpha_{|TN}) : (X,\alpha) \in L\}$$

is the graph of a π^N. L_N is a graph of a Poisson bivector precisely $pr_2 : L_N \to T^*N$ is a bijection, or here equivalently an injection. The kernel of this projection is

$$\ker(pr_2) = \{(X, \alpha_{|TN}) : (X,\alpha) \in L, \alpha_{|TN} = 0\}.$$

We conclude that L_N is the graph of a bivector exactly as for all $\alpha \mathbb{T}^*M$, $\pi^\#(\alpha) = 0$ if and only if $\alpha \mid_{TN}= 0$, i.e. L_N is the graph of a Poisson bivector π^N if and only if N is a cosymplectic transversal. □

Poisson diffeomorphic transversals

Theorem 1.5.7. *Let x_0, x_1 be elements of one symplectic leaf of a Poisson manifold (M,π). Let N_0 and N_1 be transversals through respective x_0 and x_1. Then (N_0, x_0) and (N_1, x_1) are Poisson diffeomorphic.*

Proof. Symplectic leaves of a Poisson manifold are path connected, so it is sufficient to prove this for x_0 and x_1 in one coordinate neighbourhood U of M. By the splitting theorem we may assume that there are coordinates $x_0, ..., x_k, y_1, ..., y_k, z_1, ...z_l$ around x_0, such that $z_1, ...z_l$ are the coordinates in N_0. U looks like $N_0 \times \mathbb{R}^{2k}$. N_0 is cosymplectic in M, hence there is an induced Poisson structure π_0 on N_0, such that $\pi_0(x) = 0$. On the product $N_0 \times \mathbb{R}^{2k}$, we have a Poisson structure $\pi_0 \times \pi_{can}$. Where $\pi_{can} = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$. N_1 is also transversal, therefore in this product we may write N_1 as the graph of a map $\phi : N_0 \to \mathbb{R}^{2k}$. Because N_1 is cosymplectic as well, we also have an induced Poisson structure π_1 on N_1. The first projection $pr_1 : (x, \phi(x)) \to N_0$ gives a map from $N_1 \to N_0$. This induces a second Poisson structure π_ϕ on N_0, such that pr_1 is a Poisson diffeomorphism. Now we have to prove that (N_0, π_0) and (N_0, π_ϕ) are Poisson diffeomorphic. We do this using the theory of Poisson gauging and the Moser theorem, discussed above.

On $\mathbb{R}^{2k}$ we have the canonical symplectic 2-form $\omega_{can} = \sum_i^k dx_i \wedge dy_i$. Let $B = \phi^*(\omega_{can}) \in \Omega^2(N_0)$.

Lemma 1.5.8. *B is a gauging form for (N_0, π_0) if and only if N_ϕ is cosymplectic in M.*

We postpone the proof of the lemma a little. We finish the proof of the theorem with noting that $B_t = tB$ is a family of exact gauging for (N_0, π_0), with $B_0 = 0$, and π_ϕ is the gauge transformation of π_0 with respect to B_1. So, by the Moser theorem, we have that (N_0, π_0) and (N_ϕ, π_ϕ) are Poisson diffeomorphic. □

Proof of the lemma. In this prove we omit the 0 from (N_0, π_0). It suffices to prove the following equality for any $x \in N$:

$$\ker(\mathrm{id}_{TN} - \pi^{\#} \circ B_{\#})_x.$$

We have:

$$\begin{aligned}
T_{(x,\phi(x))}N_\phi &= \{(X, T_x\phi(X)) : X \in T_xN\} \\
(T_{(x,\phi(x))}M/T_{(x,\phi(x))}N_\phi)^* &= \{(\phi^*(\xi), \xi) \in T_x^*N \times T_{\phi(x)}^*\mathbb{R}^{2k}\} \\
(T_{(x,\phi(x))}N_\phi)^\perp &= \{(\pi^{\#}(\phi^*\xi), \pi_{can}^{\#}(\xi)) : \xi \in T_{\phi(x)}^*\mathbb{R}^{2k}\} \\
&= \{(\pi^{\#} \circ (\phi^*\omega_{can})_{\#}(V), V) : V \in T_{\phi(x)}\mathbb{R}^{2k}\}.
\end{aligned}$$

In the last equality we used $(\pi_{can}^{\#})^{-1} = (\omega_{can})_{\#}$. We compute the intersection:

$$\begin{aligned}
T_{(x,\phi(x))}N_\phi \cap (T_{(x,\phi(x))}N_\phi)^\perp &= \{((X, T_x\phi(X))) : X \in T_xN, X = \pi^{\#} \circ B_{\#}(T_x\phi(X)))\} \\
&\simeq \ker(\mathrm{id}_{TN} - \pi^{\#} \circ B_{\#})_x
\end{aligned}$$

□

Chapter 2

Connections on fibre bundles

2.1 Ehresmann connections

Abstract In this chapter we discuss the notion of connections on a fibre bundle. These can be described with the help of a special kind of almost Dirac structures on the total space of the fibration.

Fibre bundles

Theorem 2.1.1. *Let M be a compact connected manifold. A smooth mapping $\pi : M \to B$ is a locally trivial fibration if and only if π is a surjective submersion.*

Proof. Let π be a locally trivial fibration of fibretype F. Then for every $b \in B$, there is a neighbourhood U_i, and a diffeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$, such that $\pi|_{\pi^{-1}(U_i)} = pr_1 \circ \phi_i$. pr_1 is a surjective submersion, so π is a surjective submersion as well. For the converse we use the submersion theorem. HAS TO BE REPAIRED. $\square$

Let $\pi : M \to B$ be a fibre bundle. The vertical subspace Vert_x of T_xM at $x \in M$ is:

$$\mathrm{Vert}_x = \mathrm{Ker}\, T_x\pi.$$

The vertical space Vert_x at $x \in M$ is the tangent space at x of the fibre containing x. All vertical spaces together for the the vertical bundle $\mathrm{Vert} \subset TM$. Sections of this bundle are called vertical vector fields. The space of vertical vector fields on M is denoted with $\mathfrak{X}_{\mathrm{Vert}}(M)$.

Ehresmann connections

Definition 2.1.2. An Ehresmann connection Γ on a fibration $\pi : M \to B$ is a distribution Hor on M, such that

1. for all $x \in M$, $T_xM = \mathrm{Hor}_x \oplus \mathrm{Vert}_x$.
2. for every piecewise smooth path $\gamma : [0,1] \to B$ starting from b and for each $x \in F_b$ there is a piecewise smooth path $\gamma^\# : [0,1] \to M$ starting x such that $\pi \circ \gamma^\# = \gamma$, and $\frac{d}{dt}\gamma^\#(t) \in \mathrm{Hor}_{\gamma^\#(t)}$, for all $t \in [0,1]$, where it is defined.

Remark 2.1.3. Note that $\gamma^\#$ will be unique. It is called the horizontal lift of γ with respect to Γ.

A connection Γ on a fibration $\pi : M \to B$ allows us to decompose every $X \in TM$ into a horizontal and a vertical part, which we denote with $X = X^{\mathrm{hor}} + X^{\mathrm{vert}}$. Moreover, for every $x \in M$ one gets an isomorphism:

$$T_x\pi|_{\mathrm{Hor}_x} : \mathrm{Hor}_x \to T_{\pi(x)}B.$$

Definition 2.1.4. Let $\pi : M \to B$ be a fibre bundle, and let Γ be a connection on it. The horizontal lift of a vector field $X \in \mathfrak{X}(B)$ with respect Γ is the vector field $X^\# \in \mathfrak{X}(M)$, which is defined by:

$$X^\#(x) = (T_x\pi|_{\mathrm{Hor}_x})^{-1}(X(\pi(x))).$$

In other words, $X^\#$ is the unique horizontal vector field on M which is projectable to X. See also the paragraph on projectable vector field in chapter 1.

Lemma 2.1.5. *Let $\pi : M \to B$ be a fibre bundle with a connection Γ. For vector fields $v_1, v_2 \in \mathfrak{X}(B)$ the following equality holds:*

$$[v_1^\#, v_2^\#]^{hor} = [v_1, v_2]^\#.$$

Proof. The horizontal lift $v_i^\#$ is defined exactly such that it descends to v_i. It follows by lemma 1.1.8 that $[v_1^\#, v_2^\#]$ descends to $[v_1, v_2]$. Also, $[v_1, v_2]^\#$ is horizontal and descends to $[v_1, v_2]$. Hence, $[v_1^\#, v_2^\#]^{\mathrm{hor}} = [v_1, v_2]^\#$. □

Definition 2.1.6. Let $\pi : M \to B$ be a fibre bundle, let Γ be a connection, and let γ be a path in B. Parallel transport along γ with respect to Γ is a map:

$$\phi_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1)),$$

which sends a $p \in \pi^{-1}(\gamma(0))$ to the point $\gamma_p^\#(1)$. Here $\gamma_p^\#$ is the horizontal lift of γ which starts at p.

Proposition 2.1.7. *A fibre bundle $\pi : M \to B$ of fibre type a compact manifold F admits connections.*

Proof. First we consider the local problem. Let γ be at path in B contained in a coordinate neighbourhood (U, x) such that the bundle is trivial over that neighbourhood. There is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$, such that $pr_1 \circ \phi = \pi$. The horizontal lift of the local vector field $\frac{\partial}{\partial x_i}$, is given by $(\frac{\partial}{\partial x_i}, v_i)$ with $v_i : F \to TF$ a vector field. We describe the path γ as the solution curve of $\sum_i^n \alpha_i(t) \frac{\partial}{\partial x_i}|_{\gamma(t)}$. The horizontal lift of γ is of the form $(\gamma, \zeta)(t)$, such that

$$(\dot{\gamma}, \dot{\zeta}) = \sum_i^n \alpha_i(t)(\frac{\partial}{\partial x_i}|_{\gamma(t)}, v_i(t)).$$

So, in fact all we have to do is to find a path ζ in F, such that

$$\dot{\zeta}(t) = \sum_i^n \alpha_i(t) v_i(t).$$

Since F is compact, this differential equation has an unique solution. Next, we have to prove that a path which is not contained in such a neighbourhood is still uniquely liftable. Since B is covered by open coordinate neigbourhoods U_α as above on which the bundle is trivial, and since the interval $[0, 1]$ is compact, we find finitely many t_i with $t_0 = 0 < t_1 < ... < t_n = 1$, such that $\gamma([t_i, t_{i+1}])$ is contained in such a neighbourhood. On all this neighbourhoods γ has horizontal lift. On the overlap of to the lifts conicide, hence the lifts can be glued together to one lift. □

Curvature

Definition 2.1.8. Let Γ be a connection on a fibration $\pi : M \to B$. The curvature 2-form $\Omega \in \Omega^2(M, \text{Vert})$ of Γ is a defined by:

$$\Omega : (X, Y) \mapsto [X^{\text{hor}}, Y^{\text{hor}}]^{\text{vert}}.$$

The curvature 2-form is a horizontal form in the sense that $\iota(X)\Omega = 0$ for any vertical vector X. Therefore one can also view Ω as a form on the base space, having values in the vertical vector fields, i.e. as a map:

$$\Omega : \mathfrak{X}(B) \to \mathfrak{X}_{\text{Vert}}(M) : \Omega(v_1, v_2) = [v_1^\#, v_2^\#]^{\text{vert}} = [v_1^\#, v_2^\#] - [v_1, v_2]^\#.$$

Remark 2.1.9. Th Lie bracket of two horizontal vector fields is not necessarily horizontal, and this failure is measured precisely by Ω. In particular, the horizontal distribution in integrable if and only if the curvature vanishes identically.

2.2 Connections defined by almost Dirac structures

In this section we discuss connections induced by almost Dirac structures. This discussion is inspired by [4], and we use our discussion on linear Dirac structures 1.4.1.

Definition 2.2.1. Let $p : M \to B$ be a fibration. An almost Dirac structure L on the total space of the fibration is called fibre nondegenerate if

$$(\mathrm{Vert} \oplus \mathrm{Vert}^\circ) \cap L = \{0\}.$$

Proposition 2.2.2. *A fibre nondegenerate almost Dirac structure defines a connection Γ_L on the fibration. The horizontal spaces are:*

$$\mathrm{Hor} = \{X \in TM : \textit{there is a } \alpha \in \mathrm{Vert}^\circ, (X, \alpha) \in L\}.$$

Moreover, any connection on $p : M \to B$ can be obtained in this way.

Proof. Note that for each $x \in M$, L_x is Vert_x-nondegenerate in T_xM. We saw in the section about linear Dirac structures that this induces the existence of a natural orthogonal complement H_x of Vert_x in T_xM. Note that this H_x coincides with the Hor_x described above. Indeed we find a splitting:

$$TM = \mathrm{Hor} \oplus \mathrm{Vert}\,.$$

This proves the first part of the proposition. Conversely, given a connection Γ on M, then we find a fibre nondegenerate almost Dirac structure by

$$L = \mathrm{Hor} \oplus \mathrm{Hor}^\circ\,.$$

The associated connection to this fibre nondegenerate almost Dirac structure is the old Γ. □

Remark 2.2.3. Two different fibre nondegenerate almost Dirac structures may lead to the same connection. In fact, a fibre nondegenrate Dirac structure induces much more then a connection.

The horizontal 2-form and the vertical bivector Note that the splitting $TM = \mathrm{Hor} \oplus \mathrm{Vert}$ induces a splitting on the dual bundle $T^*M = \mathrm{Hor}^* \oplus \mathrm{Vert}^*$, and we have the following identifications:

$$\mathrm{Hor}^\circ \simeq \mathrm{Vert}^*, \text{ and } \mathrm{Vert}^\circ \simeq \mathrm{Hor}^*\,.$$

We use these identifications several times in the sequal.

Proposition 2.2.4. *Let $p : M \to B$ be a fibration. Any fibre nondegenerate almost Dirac structure L on M induces a smooth horizontal 2-form*

$$\omega_L : \mathrm{Hor} \times \mathrm{Hor} \to \mathbb{R},$$

which is the restriction of the natural 2-form to the horizontal subspaces.

Proof. For every $X \in \mathrm{Hor}$ there is an unique $\alpha \in \mathrm{Vert}^\circ$, such that $(X, \alpha) \in L$. Define the 2-form ω_L by:

$$\omega_L(X_1, X_2) = \frac{1}{2}(\alpha_1(X_2) - \alpha_2(X_1)) = \alpha_1(X_2) = -\alpha_2(X_1),$$

where $\alpha_1, \alpha_2 \in \mathrm{Vert}^\circ$, such that $(\alpha_1, X_1), (\alpha_2, X_2)$ are in L. □

Proposition 2.2.5. *Let $p : M \to B$ be a fibration. For any fibre nondegenerate almost Dirac structure and any fibre $i : F_b \hookrightarrow M$ the pull-back almost Dirac structure is well defined and coincides with the graph of a vertical bivector field $\pi_L \in \mathfrak{X}^2_{\mathrm{Vert}}(M)$.*

Proof. We define a 2-form $\pi_L : \mathrm{Hor}^\circ \times \mathrm{Hor}^\circ \to \mathbb{R}$, and then use the identification $\mathrm{Hor}^\circ \simeq \mathrm{Vert}^*$ to obtain a vertical bivector field. For every $\alpha \in \mathrm{Hor}^\circ$ there is a unique $X \in \mathrm{Vert}$, such that $(X, \alpha) \in L$. Define:

$$\pi_L(\alpha_1, \alpha_2) = \frac{1}{2}(\alpha_1(X_2) - \alpha_2(X_1)) = \alpha_1(X_2) = -\alpha_2(X_1),$$

with $X_1, X_2 \in \mathrm{Vert}$ such that $(X_1, \alpha_1)(X_2, \alpha_2) \in L$. By the identification mentioned before this defines a bivector field on the fibres of $p : M \to B$. Fix a fibre $i : F_b \hookrightarrow M$, and consider the pull-back of the Dirac structure i^*L:

$$\begin{aligned} i^*L &= \{(X, \alpha|_{\mathrm{Vert}}) \in \mathrm{Vert} \oplus \mathrm{Vert}^* : (X, \alpha) \in L\} \\ &= \{(X, \alpha) \in \mathrm{Vert} \oplus \mathrm{Hor}^\circ : (X, \alpha) \in L\} \\ &= \{(X, \alpha) \in \mathrm{Vert} \oplus \mathrm{Hor}^\circ : X = \pi_L(\alpha, \cdot) \in L\} \\ &= \mathrm{graph}(\pi_L). \end{aligned}$$

□

Summarizing this section: a fibre nondegenerate almost Dirac structure L on a fibration $p : M \to B$ induces:

1. a connection Γ_L,
2. a horizontal 2-form $\omega_L \in \Omega^2_{\mathrm{Hor}}(M)$,
3. a vertical bivector field $\pi_L \in \mathfrak{X}^2_{\mathrm{Vert}}(M)$.

Conversely, given such a triple (Γ, ω, π) there exists a unique fiber nondegenerate almost Dirac structure L which induces it. It is given by:

$$L = \text{graph}(\pi_L) \oplus \text{graph}(\omega_L).$$

2.3 Connections defined by Dirac structures

Conditions on (Γ, ω, π)**.** Some connections can be described by fibre nondegenerate Dirac structures. In the next proposition we see which conditions this requirement imply on (Γ, ω, π). In the last chapter about Poisson fibrations we investigate the influence on the so called holonomy group. In fact, to ask that the almost Dirac structure is Dirac turns out be a quite strong requirement.

Proposition 2.3.1. *Let $(\Gamma_L, \omega_L, \pi_L)$ be the triple which is determined by a fibre nondegenerate almost Dirac structure L on a fibration $p : P \to M$. Then L is a Dirac structure if and only if the following conditions hold:*

1. *The vertical bivector π_L is a Poisson bivector.*
2. *Parallel transport with respect to Γ_L preserves the vertical Poisson structure π_L:*

$$L_{\tilde{v}}\pi_L = 0, \text{ for all } v \in \mathfrak{X}(M).$$

3. *The horizontal 2-form ω_L is closed.*
4. *The following identity for the curvature Ω_Γ of Γ is satisfied:*

$$\Omega_\Gamma(v_1, v_2) = \pi_L^\#(d\iota(\tilde{v}_1)\iota(\tilde{v}_2)\omega_L), \text{ for all } v_1, v_2 \in \mathfrak{X}(M).$$

Proof. Recall that L is a Dirac structure if and only if the 3-form T_L on L vanishes identically.

1. Let $\alpha, \beta, \gamma \in \Gamma(\text{Hor}^\circ)$, then $X = \pi_L^\#(\alpha), Y = \pi_L^\#(\beta), Z = \pi_L^\#(\gamma)$ are the unique vector fields in $\mathfrak{X}_{\text{Vert}}(M)$ such that $(X, \alpha), (Y, \beta), (Z, \gamma) \in \Gamma(L)$. The explicit formula for T_L gives:

$$\begin{aligned} T_L((X,\alpha),(Y,\beta),(Z,\gamma)) &= \frac{1}{2}(\alpha([X,Y]) + \beta([Z,X]) + \gamma([X,Y]) + \\ &\quad L_X(\beta(Z) + L_Y(\gamma(X)) + L_Z(\alpha(Y))) \\ &= \frac{1}{2}((L_Z\alpha)(Y) + (L_X\beta)(Z) + (L_Y\gamma)(X)) \\ &= \frac{1}{2}(\pi_L(L_{\pi_L^\#(\gamma)}\alpha, \beta) + \pi_L(L_{\pi_L^\#(\alpha)}\beta, \gamma) + \pi_L(L_{\pi_L^\#(\beta)}\gamma, \alpha)) \\ &= -\frac{1}{2}[\pi_L, \pi_L](\alpha, \beta, \gamma). \end{aligned}$$

We used the following formula for a 1-form σ and vector fields V, W;

$$\sigma([V,W]) = -L_W\sigma(V) + (L_W\sigma)(V).$$

2. Let $v \in \mathfrak{X}(M)$, and let $\beta, \gamma \in \Gamma(Hor^\circ)$. There exists an unique 1-form $\alpha \in \Gamma(\text{Vert}^\circ)$ and unique vector fields $Y, Z \in \Gamma(\text{Vert})$ such that $(v^\#, \alpha), (\beta, Y), (\gamma, Z) \in \Gamma(L)$. The explicit expression for T_L gives:

$$\begin{aligned}
& T_L((v^\#,\alpha),(\beta,Y),(\gamma,Z)) \\
= \; & \frac{1}{2}(\alpha([Y,Z]) + \beta([Z,v^\#]) + \gamma([v^\#,Y]) + \; v^\# \cdot \beta(Z) + Y \cdot \gamma(v^\#) + Z \cdot \alpha(Y)) \\
= \; & \frac{1}{2}(\beta([Z,v^\#]) + \gamma([v^\#,Y]) + v^\# \cdot \beta(Z)) \\
= \; & \frac{1}{2}(\iota(v^\#)L_Z\beta - L_{v^\#}(\beta(Z)) - \iota(v^\#)L_Y\gamma + L_{v^\#}(\gamma(Y)) + v^\# \cdot \beta(Z)) \\
= \; & \frac{1}{2}(L_{v^\#}\pi_L(\beta,\gamma) + L_{v^\#}\pi_L(\beta,\gamma) - L_{v^\#}\pi_L(\beta,\gamma)) \\
= \; & \frac{1}{2}L_{v^\#}\pi_L(\beta,\gamma).
\end{aligned}$$

We used that the Lie bracket of vertical vector fields is vertical, and that $L_Z\beta, L_Y\gamma \in \Gamma(Hor^\circ)$.

3. Let $v_1, v_2, v_3 \in \mathfrak{X}(M)$. There exists unique 1-forms $\alpha, \beta, \gamma \in \Gamma(\text{Vert}^\circ)$ such that $(v_1^\#, \alpha), (v_2^\#, \beta), (v_3^\#, \gamma) \in \Gamma(L)$. The explicit expression for T_L gives:

$$\begin{aligned}
& T_L((v_1^\#,\alpha),(v_2^\#,\beta),(v_3^\#,\gamma)) \\
= \; & \frac{1}{2}(\alpha([v_2^\#,v_3^\#]) + \beta([v_3^\#,v_1^\#]) + \gamma([v_1^\#,v_2^\#]) + v_1^\# \cdot \beta(v_3^\#) + v_2^\# \cdot \gamma(v_1^\#) + v_3^\# \cdot \alpha(v_2^\#)) \\
= \; & \frac{1}{2}(\alpha([v_2^\#,v_3^\#]^{\text{hor}}) + \beta([v_3^\#,v_1^\#]^{\text{hor}}) + \gamma([v_1^\#,v_2^\#]^{\text{hor}}) \\
& \quad + v_1^\# \cdot \beta(v_3^\#) + v_2^\# \cdot \gamma(v_1^\#) + v_3^\# \cdot \alpha(v_2^\#)) \\
= \; & \frac{1}{2}(\omega_L([v_1^\#,[v_2^\#,v_3^\#]^{\text{hor}}) + \omega_L([v_2^\#,[v_3^\#,v_1^\#]^{\text{hor}}) + \omega_L([v_3^\#,[v_1^\#,v_2^\#]^{\text{hor}}) \\
& \quad + v_1^\# \cdot \omega_L(v_2^\#,v_3^\#) + v_2^\# \cdot \omega_L(v_3^\#,v_1^\#) + v_3^\# \cdot \omega_L(v_1^\#,v_2^\#)) \\
= \; & \frac{1}{2}d\omega_L(v_1^\#,v_2^\#,v_3^\#).
\end{aligned}$$

4. Let $v_1, v_2 \in \mathfrak{X}(M)$, and let $\gamma \in \Gamma(\text{Hor}^\circ)$. There are unique $\alpha, \beta \in \Gamma(\text{Vert}^\circ)$ and $Z \in \Gamma(\text{Vert})$ such that $(v_1^\#, \alpha), (v_2^\#, \beta), (Z, \gamma) \in \Gamma(L)$. The explicit expression for T_L gives:

$$
\begin{aligned}
T_L((v_1^\#,\alpha),(v_2^\#,\beta),(Z,\gamma)) &= \frac{1}{2}(v_1^\# \cdot \beta(Z) + v_2^\# \cdot \gamma(v_1^\#) + Z\cdot\alpha(v_2^\#) \\
&\quad +\alpha([v_2^\#, Z]) + \beta([Z, v_1^\#]) + \gamma([v_1^\#, v_2^\#])) \\
&= \frac{1}{2}(Z\cdot\alpha(v_2^\#) + \gamma([v_1^\#, v_2^\#])).
\end{aligned}
$$

We used that the Lie bracket between a horizontal and a vertical vector field is vertical. We continue the computation after making the observation that $d(\omega_L(v_1^\#, v_2^\#)) \in \Gamma(\mathrm{Hor}^\circ)$.

$$
\begin{aligned}
\frac{1}{2}(Z\cdot\alpha(v_2^\#) + \gamma([v_1^\#, v_2^\#])) &= \frac{1}{2}(d(\alpha(v_2^\#))(Z) + \gamma([v_1^\#, v_2^\#])^{\mathrm{vert}}) \\
&= \frac{1}{2}(d(\omega_L(v_1, v_2))(Z) + \gamma(\Omega_\Gamma(v_1, v_2))) \\
&= \frac{1}{2}(\pi_L(d(\omega_L(v_1, v_2)), \gamma) + \gamma(\Omega_\Gamma(v_1, v_2))).
\end{aligned}
$$

Therefore if L is Dirac structure we get that

$$\Omega_\Gamma(v_1, v_2) = \pi_L^\#(d(\iota(v_1^\#)\iota(v_2^\#)\omega_L)).$$

□

A natural question to ask is how fibre nondegenerate symplectic Dirac structures and Poisson bivector Dirac structures look like.

Lemma 2.3.2. *Let ω be a closed 2-form on the total space of a fibration $p : P \to M$. Then the associated Dirac structure is fibre nondegenerate if and only if the pull-back of ω to each fibre is nondegenerate.*

Proof. The Dirac structure is fibre nondegenerate if and only if at every $x \in P$, ω_x is Vert_x-nondegenerate, which is equivalent to the requirement that the pull-back of ω to all fibres is nondegenerate. □

Fibre bundles with a nondegenerate closed 2-form on the fibres are called symplectic fibrations. To them we have devoted a whole chapter.

Lemma 2.3.3. *Let π be a Poisson structure on the total space of a fibration $p : P \to M$. Then the associated Dirac structure L_π is fibre nondegenerate if and only if the bilinear form*

$$\pi\mid_{\mathrm{Vert}^\circ} : \mathrm{Vert}^\circ \times \mathrm{Vert}^\circ \to \mathbb{R}$$

is nondegenerate.

Proof. L_π is fibre nondegenerate means that $\xi \in \mathrm{Vert}^\circ$ if and only if $\pi^\#(\xi) \notin \mathrm{Vert}$. So in that case $\xi \in \mathrm{Vert}^\circ$ and $\pi(\xi, \eta) = 0$ for all $\eta \in \mathrm{Vert}^\circ$ indeed implies $\pi^\#(\xi) = 0$, i.e. $\pi|_{\mathrm{Vert}^\circ}$ is nondegenerate.

For the converse note that $\pi\,|_{\mathrm{Vert}^\circ}$ is nondegenerate implies that for $\xi \in \mathrm{Vert}^\circ$, $\pi^\#(\xi) = 0$ if and only if $\xi = 0$.Therefore, if we assume that $\xi \neq 0$, we get that there is a $\eta \in \mathrm{Vert}^\circ$ such that $\pi(\xi, \eta) \neq 0$, hence $\pi^\#(\xi) \notin \mathrm{Vert}$. □

In this case, the horizontal form ω_L is nondegenerate. Hor is constructed such that $\pi^\#|_{\mathrm{Vert}^\circ} : \mathrm{Vert}^\circ \to \mathrm{Hor}$ is an isomorphism.

2.4 Dirac gauging of fibre bundles

We wonder when the gauge transforms of fibre nondegenerate Dirac structures on M are again fibre nondegenerate. We start with the extreme case.

Example 2.4.1. Assume $pr_1(L) \cap \mathrm{Vert} = \{0\}$. Then the horizontal distribution is given by $pr_1(L)$. This implies in fact that Hor is integrable and that the foliation it induces coincides with the presymplectic leaves of the Dirac structure. Moreover, since gauge transformations leave the range of the Dirac structure as well as the vertical bundle invariant, all gauge transformations leave the Dirac structure fibre nondegenerate.

Things get less obvious if we assume that $pr_1(L) \cap \mathrm{Vert}$ is nontrivial. Analogue to the linear case we get:

Lemma 2.4.2. *A Dirac gauging transformation τ_B preserves the fibre nondegeneracy condition if and only if for all presymplectic leaves S,*

$$\omega_S + B|_{pr_1(L)\cap \mathrm{Vert}},$$

is still nondegenerate.

Corollary 2.4.3. *Any horizontal Dirac gauge 2-form preserves the fibre nondegeneracy condition of a Dirac structure.*

Proof. Since horizontal 2-forms vanish on vertical vectors we get for a horizontal gauging 2-form B:

$$\omega_S + B|_{pr_1(L)\cap \mathrm{Vert}} = \omega_S|_{pr_1(L)\cap \mathrm{Vert}}.$$

□

Chapter 3

Connections on vector bundles and principal G-bundles

Abstract A broad and important family of fibre bundles are the so called principal G-bundles. Here G is a Lie group which acts on the total space. The fibre type of a principal bundle is this Lie group G. Connections on a principal bundle can be defined in terms of a Lie algebra valued 1-form on the total space of the fibration. One of the nice properties of principal bundles is that if one has a manifold F on which G acts, one can construct a new fibration which has as fibre type F. This is called the associated bundle. We will see that associated bundles are very useful to find a relation between a fibre bundle and its so called linear frame bundle. We use this relation to show that connections on linear frame bundles are in one-to-one correspondence with covariant derivatives ∇.

3.1 Connections on vector bundles

Vector bundle valued differential forms We extend the notion of differential forms to the notion vector bundle valued differential forms.

Definition 3.1.1. Let (E, π) and (E', π') be two vector bundles over a manifold M. The homomorphism bundle $\mathrm{Hom}(E, E')$ consists of bundle maps $\varphi : E \to E'$, such that for every $m \in M$, $\varphi_m : E_m \to E'_m$ is linear, i.e. it has the fibre $\mathrm{Hom}(E, E')_m = \mathrm{Lin}(E_m, E'_m)$.

Definition 3.1.2. Let M be a manifold and let (E, π) be a vector bundle over M. A E-valued 1-form on M is a smooth section of $\mathrm{Hom}(TM, E)$. The space of E-valued 1-forms on M is denoted with $\Omega^1(M, E)$.

Similar as with the 1-forms discussed in chapter 1, we may consider $\sigma \in \Omega^1(M, E)$ as a $C^\infty(M)$-linear map:

$$\sigma : \mathfrak{X}(M) \to C^\infty(M, E) : \sigma(X)(m) = \sigma(m)(X(m)).$$

Where $C^\infty(M, E)$ is the space of smooth sections of E.

Definition 3.1.3. Let M be a manifold and let (E, π) be a vector bundle over M. A E-valued k-form on M is a smooth section of $\mathrm{Hom}(\bigwedge^k TM, E)$. The space of E-valued k-forms on M is denoted with $\Omega^k(M, E)$.

We may consider $\sigma \in \Omega^k(M, E)$ as a $C^\infty(M)$-multilinear skew-symmetric map:

$$\sigma : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k-\text{times}} \to C^\infty(M, E).$$

Denote with S_m the group of permutations of $\{1, ..., m\}$. An n, k-shuffle with $n + k = m$ is a permutation s such that $s(\{1, ..., n\}) = \{1, ..., n\}$. We use shuffles to define a wedge product for $\omega \in \Omega^n(M)$ and $\sigma \in \Omega^k(M, E)$:

$$(\omega \wedge \sigma)(X_1, ..., X_n, X_{n+1}, ..., X_{n+k}) := \sum_{s \text{ is a n,k-shuffle}} \mathrm{sign}(s)\omega(X_{s(1)}, ..., X_{s(n)})\sigma(X_{s(n+1)}, ..., X_{s(n+k)}).$$

Note that $\omega \wedge \sigma \in \Omega^{n+k}(M, E)$.

The covariant derivative Let $\pi : E \to M$ be a vector bundle over a manifold M. To generalise the exterior derivative to vector bundle valued forms, we need the notion of a connection.

Definition 3.1.4. Let (E, π) be a vector bundle over M. A connection on E is a differential operator

$$\nabla : C^\infty(M, E) \to \Omega^1(M, E),$$

which satisfies a Leibniz rule:

$$\nabla(f\sigma) = df \wedge \sigma + f\nabla\sigma.$$

for all $\sigma \in C^\infty(M, E)$ and $f \in C^\infty(M)$.

Example 3.1.5. Let $E = M \times \mathbb{R}$ be the trivial line bundle over M. Then $C^\infty(M, E) = C^\infty(M)$, $\Omega^1(M, E) = \Omega^1(M)$, and the exterior derivative is a connection.

Connections ∇ are used to differentiate sections of E with respect to any vector field X on M. More precisely one defines:

$$\begin{aligned} \nabla_X : C^\infty(M, E) &\to C^\infty(M, E) \\ \sigma &\to (\nabla\sigma)(X). \end{aligned}$$

Example 3.1.6. Let E be again the trivial line bundle over M, and let d be the connection. Then for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we get:

$$d_X f = \iota(X)df = df(X) = L_X f,$$

the Lie derivative of f along X.

Definition 3.1.7. Let E be a vector bundle over a manifold M. A covariant derivative is a map

$$\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E),$$

for all nonnegative intergers k. Moreover, it has to satisfy:

$$\nabla(\omega \wedge \mu) = -\omega \wedge \nabla\mu + (-1)^k d\omega \wedge \mu,$$

for $\mu \in \Omega^k(M, E)$ and $\omega \in \Omega^n(M)$.

The covariant derivative ∇ is characterised in a very similar way as the exterior derivative. Namely by the relation

$$\begin{aligned}(\nabla\mu)(X_1, ..., X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\nabla_{X_i}(\mu(X_1, ..., \hat{X}_i, ..., X_{k+1})) + \\ \sum_{1\leq i<j\leq k+1}(-1)^{i+j}(\mu([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{k+1}).\end{aligned} \tag{3.1}$$

In particular, for a $\sigma \in \Omega^1(M, E)$, and $X, Y \in \mathfrak{X}(M)$ we have:

$$\nabla\sigma(X, Y) = \nabla_X(\sigma(Y)) - \nabla_Y(\sigma(X)) - \sigma([X, Y]).$$

The curvature Note that in the definition of a covariant derivative we do not have the analogue of the property $d \circ d = 0$ for the exterior derivative. In fact $\nabla \circ \nabla$ does not need to vanish. It gives the so called curvature of the connection.

Definition 3.1.8. Let E be a vector bundle over M, and let ∇ be a connection. The curvature Ω of ∇ is $\nabla \circ \nabla : C^\infty(M, E) \to \Omega^2(M, E)$.

Proposition 3.1.9. *Let E be a vector bundle over M, and let ∇ be a connection. For all $X, Y \in \mathfrak{X}(M)$, and $\sigma \in C^\infty(M, E)$:*

$$\Omega(\sigma)(X, Y) = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma.$$

Moreover, this expression is $C^\infty(M)$-linear in all arguments.

Proof.

$$\begin{aligned}\Omega(\sigma)(X,Y) = \nabla(\nabla\sigma)(X,Y) &= \nabla_X(\nabla\sigma(Y)) - \nabla_Y(\nabla\sigma(X)) - \nabla\sigma([X,Y]) \\ &= \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma.\end{aligned}$$

Which proves the first the first part of the proposition. For the second part, take $f \in C^\infty(M)$:

$$\begin{aligned}\nabla(\nabla f\sigma) &= \nabla(df \wedge \sigma + f\nabla\sigma) \\ &= -df \wedge \nabla\sigma - ddf \wedge \sigma + df \wedge \nabla\sigma + f\nabla\sigma \\ &= f\nabla\sigma,\end{aligned}$$

and

$$\begin{aligned}\Omega(\sigma)(fX,Y) &= f\nabla_X\nabla_Y\sigma - \nabla_Y(f\nabla_X\sigma) - \nabla_{f[X,Y]} - \nabla_{(L_Xf)Y}\sigma \\ &= f\nabla_X\nabla_Y\sigma - f\nabla_Y\nabla_X\sigma - L_Yf\nabla_X\sigma - f\nabla_{[X,Y]}\sigma - L_Xf\nabla_Y\sigma \\ &= f\Omega(\sigma)(X,Y).\end{aligned}$$

□

Corollary 3.1.10. *The curvature* $\Omega \in \Omega^2(M, \mathrm{End}(E))$.

Definition 3.1.11. Let E be a vector bundle over a manifold M, and let ∇ be a connection. ∇ is called a flat connection if the curvature vanishes identically.

Remark 3.1.12. For any trivial vector bundle $E = M \times V$ there is an unique flat covariant derivative. This covariant exterior derivative is denoted with d, and it coincides with the exterior derivative when $V = \mathbb{R}$.

Pull-back vector bundle Let $\phi : N \to M$ be a differentiable map between manifolds. Let (E, π) be a vector bundle over M. The pull-back vector bundle ϕ^*E is the bundle over N defined by:

$$\phi^*E = \{(e, a) \in E \times N : \pi(e) = \phi(a)\}.$$

As general notation, for each $s \in \Gamma(M, E)$, we denote by $\phi^*(s)\Gamma(N, \phi^*E)$ the section given by $\phi^*(s)(x) = s(\phi(x))$. In general, every section of ϕ^*E can be written as a sum of sections of type $f\phi^*(s)$, with $f \in C^\infty(N), s \in \Gamma(M, E)$. Let with a covariant derivative ∇ on E. The pull-back of the covariant derivative along ϕ is the covariant derivative on ϕ^*E is characterised by:

$$\nabla^{\phi*E}(f\phi^*s) = df \wedge \phi^*s + f\phi^*(\nabla s).$$

for all $s \in \Gamma(M, E)$, $f \in C^\infty(N)$.

Parallel sections Let E be a vector bundle over a manifold M. For a path $\gamma : \mathbb{R} \to M$ we are looking for lifts in the total space E, i.e. paths $a : \mathbb{R} \to E$, such that $a(t) \in E_{\gamma(t)}$ for all t. Such lifts can be viewed as sections of the pull-back vector bundle $\gamma^* E$. In the presence of a connection ∇ on E, it is interesting to look at the following special class of lifts. Namely lifts a such that:

$$\nabla^{\gamma^* E}_{\dot{\gamma}(t)} a(t) = 0, \quad \text{for all } t \in \mathbb{R}.$$

Such lifts are called parallel with respect to ∇.

Proposition 3.1.13. *Let E be a vector bundle over a manifold M, and let ∇ be a connection. Let $\gamma : [0,1] \to M$ be a path joining the points $x, y \in M$. For any $p \in E_{\gamma(0)}$, there is a unique parallel lift $a(t) \in C^\infty(\mathbb{R}, \gamma^* E)$ starting at p.*

This proposition follows directly from the theory of ordinary differential equations.

3.2 Principal G-bundles

Definition 3.2.1. A principal G-bundle is a manifold P together with a free and proper right action from G.

The orbit space $M := P/G$ is called the base space of the principal G-bundle, and $\pi : P \to M$ is the natural projection. We denote a principal G-bundle with total space P with $P(M, G)$.

Proposition 3.2.2. *$P(M, G)$ is a fibre bundle of fibre type G.*

Proof. □

Remark 3.2.3. The fibres of a principal G-bundle are isomorphic with G, but not canonically. Once one has chosen a $u \in \pi^{-1}(x)$, then all other elements of $\pi^{-1}(x)$ are of the form ug for $g \in G$, and one obtains a isomorphism $G \to \pi^{-1}(x) : g \mapsto ug$.

Example 3.2.4 (Hopf fibration). Let $P = S^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and let G be the circle group S^1. Then G acts on P by

$$G \times P \to P : (e^{i\varphi}, (z_1, z_2)) \mapsto (e^{i\varphi} z_1, e^{i\varphi} z_2).$$

For a point $(z_1, z_2) \in P$ we find an element $e^{i\varphi}$ of S^1, such that $e^{i\varphi} z_1$ is a real number. In this way we find the quotient S^3/S^1, and the projection π.

$$\begin{array}{rcl} \pi : S^3 & \to & S^3/S^1 \simeq S^2 = \{(x, z) \in \mathbb{R} \times \mathbb{C} : x^2 + |z|^2 = 1\} \\ (r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) & \mapsto & (r_1, r_2 e^{i(\varphi_2 - \varphi_1)}). \end{array}$$

Example 3.2.5 (Frame bundle). Let M be a m-dimensional manifold. A frame u at $p \in M$ is a ordered basis of $T_p M$, which we see as a linear map $u : \mathbb{R}^m \to T_p M$. Let $L(M)$ be the set of all linear frames above all $p \in M$, and let the projection π map a frame u at p to p. The linear group $\mathrm{GL}(m)$ acts on $L(M)$ by precomposition, and $L(M)$ becomes a principal $\mathrm{GL}(m)$ bundle. We call this bundle the linear frame bundle of M.

The vertical bundle Since any principal G bundle $P(M,G)$ is a fibre bundle we can talk about vertical subspaces and the vertical bundle $VP \subset TP$. In this case however, there is a nicer description of vertical vectors:

Lemma 3.2.6. *For all $p \in P$ there is a linear isomorphism between $\varphi_p : \mathfrak{g} \to V_pP$.*

Proof. For $p \in P$ define

$$\varphi_p : \mathfrak{g} \to T_pP, \xi \mapsto \frac{d}{dt}|_{t=0} \exp(t\xi) \cdot p.$$

We check that $\varphi_p(\xi)$ is vertical for all $\xi \in \mathfrak{g}$. Let $f \in C^\infty(M)$, then

$$\frac{d}{dt}|_{t=0}(\pi^* f)(p \exp(t\xi)) = \frac{d}{dt}|_{t=0} f(\pi(p)) = 0.$$

φ_p is linear. The fibres of the fibre bundle are diffeomorphic to G, hence $\mathfrak{g}$ and V_pP have the same dimension. It is left to show that the map is injective. G acts freely on P hence, $\frac{d}{dt}|_{t=0}\, p\exp(t\xi) = 0$ if and only if $\exp(t\xi) = e$. □

Definition 3.2.7. Let $P(M,G)$ be a principal bundle. The fundamental vector field on P induced by $\xi \in \mathfrak{g}$ is the vector field constructed by

$$\tilde{\xi} : P \to TP, p \mapsto \varphi_p(\xi).$$

Note that all fundamental vector fields are vertical. They lead to a map $\varphi : \mathfrak{g} \to \mathfrak{X}_{\text{Vert}}(P) : \xi \to \tilde{\xi}$.

Example 3.2.8 (The Hopf fibration). We consider the infinitesimal action of S^1 of the Hopf fibration. The Lie algebra of S^1, is $i\mathbb{R}$, and the exponential map is the usual exponential map: $i\varphi \mapsto e^{i\varphi}$. Hence, the fundamental vector field for $i\varphi$ is given by $\widetilde{i\varphi}(z_1, z_2) = i\varphi(z_1, z_2)$.

Reduced subbundles

Definition 3.2.9. A homomorphism f between two principal bundles, $P'(M', G')$ and $P(M,G)$ is a map $f : P' \to P$ and a homomorphism $f' : G' \to G$, such that $f(u'g') = f(u')f'(g)$.

Remark 3.2.10. A homomorphism between two principal bundles sends fibres to fibres, hence it induces a map from $M' \to M$.

Definition 3.2.11. We call $P'(M', G')$ a reduced subbundle of $P(M,G)$ if $M' = M$, $f : P' \to P$ is an imbedding, $f' : G' \to G$ is injective, and the induced map $M' \to M$ is the identity mapping.

Example 3.2.12. Let M be an orientable manifold. A frame u above $p \in M$ is a orthogonal ordered basis of T_pM, which we see as a orthogonal linear map $u : \mathbb{R}^m \to T_pM$. Let $OL(M)$ be the set of all orthogonal linear frames above all $p \in M$, and let the projection π map a frame u at p to p. The orthogonal linear group $O(m)$ acts on $OL(M)$ by precomposition. $OL(M)(M, O(m))$ is a principal $O(m)$ bundle. We call this bundle the orthogonal linear frame bundle of M. The orthogonal linear frame bundle is a reduced bundle of the linear frame bundle.

Lemma 3.2.13. *Let $P(M, G)$ be a principal G-bundle over M and let H be a Lie subgroup of G. Let $Q \subset P$. Assume that:*

1. *the projection $\pi : P \to M$ maps Q onto M.*
2. *$r_h(Q) = Q$ for all $h \in H$.*
3. *for all $u, v \in Q$ such that $\pi(u) = \pi(v)$, there is a $h \in H$, such that $uh = v$.*
4. *every point $x \in M$ has a neighbourhood U and a section $\sigma : U \to P$ such that $\sigma(U) \subset Q$.*

Then $Q(M, H)$ is a reduced subbundle of $P(M, G)$.

Proof. Let $x \in M$ and U a neighbourhood of x as in assumption (4). For $u \in \pi^{-1}(x)$ there is a $g \in G$ such that $u = \sigma(x)g$. Define a map $\psi : \pi^{-1}(U) \to U \times G, u \mapsto (x, g)$. Restrict ψ to Q and get a bijection $\psi : Q \cap \pi^{-1}(U) \to U \times H$. Endow Q with the following topology; $V \subset Q$ is open if $\psi_i : \pi^{-1}(U_i) \to U_i \times H$ is open. Here $\{U_i\}$ is a suitable open cover M, and ψ_i is defined as ψ above. For this topology the $\psi_i|_{\pi^{-1}(U_i) \cap Q}$ are homeomorphisms. Let (V, κ) be a coordinate neighbourhood in $U_i \times H$, then $(\psi_i^{-1}(V), \kappa \circ \psi_i|_{\psi_i^{-1}(V)})$ a coordinate neighbourhood in Q. This induces a smooth structure on Q for which all $\psi_i|_{\pi^{-1}(U_i) \cap Q}$ are diffeomorphisms. It follows easily that $Q(M, H)$ is a reduced subbundle of $P(M, G)$. □

Associated bundles Given a principal G-bundle $P(M, G)$ and a manifold F on which G acts from the left, we construct a new fibre bundle over M of fibre type F.

Definition 3.2.14. Let $P(M, G)$ be a principal bundle and let F be a G-manifold. Then the associated bundle of F to $P(M, G)$ is the following fibre bundle over the space M. The total space $P \times_G F$ is the orbit space of the action of G on $P \times F$ given by $g \cdot (p, f) = (pg, g^{-1}f)$. The projection map $\pi_F : P \times_G F \to M$ sends the equivalence class $[p, f]$ to $\pi(p)$, where π is the projection of the original principal bundle.

We need to prove that $P \times_G F$ is a fibre bundle over M.

Proof. Note tha π_F is well defined because $\pi(pg) = \pi(p)$. Note that the fibres of $P \times_G F$ are diffeomorphic with F. Let U be an open neighbourhood M such that $\pi^{-1}(U) \cong U \times G$ say by isomorphism ϕ. We find an isomorphism

$$\pi_F^{-1}(U) \to U \times F : [p, \xi] \mapsto (\pi(p), pr_2 \circ \phi(p)\xi).$$

Here is pr_2 the projection to the second component. The transition functions of this associated fibre bundle are by means of the group action on F.

$$\psi_{\alpha\beta} : F \to F, \xi \mapsto (pr_2 \circ \phi_\alpha(b))(pr_2 \circ \phi_\beta(b)^{-1})\xi.$$

Hence the associated fibre bundle is a local trivial bundle. □

3.3 Connections on principal G-bundles

Like for general fibre bundles a connection on a principal bundle is a horizontal distribution on P. In the definition of a connection on a principal G-bundle the action of G is taken into account.

Definition 3.3.1. Let $P(M, G)$ be a principal G-bundle. A connection Γ on P is a distribution H, such that for all $p \in P$ and $g \in G$:

1. $T_pP = H_pP \oplus V_pP$
2. $H_{pg}P = (r_g)_*(H_p)$.

Vectors in H_p are called horizontal vectors, and sections of H are horizontal vector fields. The space of horizontal vector fields on P is denoted with $\mathfrak{X}_{\mathrm{Hor}}(P)$.

Remark 3.3.2. A connection Γ on $P(M, G)$ determines a decomposition $X = X^{\mathrm{hor}} \oplus X^{\mathrm{vert}}$ for all $X \in TP$.

The connection 1-form. An equivalent way to define a connection on a principal G-bundle is by using a $\mathfrak{g}$-valued 1-form.

Definition 3.3.3. A connection 1-form on a principal G-bundle is a $\mathfrak{g}$-valued differential 1-form ω on P that satisfies the following conditions:

1. for all $\xi \in \mathfrak{g}$, $\omega(\varphi(\xi)) = \xi$.
2. for all $g \in G$ and $p \in P$, $\omega_{pg}(T_pr_g(X)) = \mathrm{Ad}(g^{-1})\omega_p(X)$.

Proposition 3.3.4. *Let ω be a connection 1-form on a principal bundle $P(M, G)$. Then*

$$H_pP = \{X \in T_pP | \omega(X) = 0\},$$

is a connection on $P(M, G)$.

Proof. Choose $p \in P$. First we check that $T_pP = H_pP \oplus V_pP$. Take $X \in H_pP \cap V_pP$, then there is a $\xi \in \mathfrak{g}$ such that $X = \varphi_p(\xi)$, and $\omega(X) = \xi$, which implies that $\xi = 0$. $H_p \subset T_pP$ is a subspace of codimension $\dim \mathfrak{g}$. Dimension count gives $T_pP = H_pP \oplus V_pP$. Take $X \in H_p$ and $g \in G$, then $\omega(T_p(r_g)(X)) = Ad(g^{-1})\omega(X) = 0$, and hence we have $T_pr_g(H_pP) = H_{pg}P$. □

Conversely, for a connection Γ on a principal bundle we find a connection 1-form, for which the kernel is the horizontal distribution.

Lemma 3.3.5. *Let $P(M,G)$ be a principal bundle with a connection Γ. Define $\omega \in \Omega^1(P,\mathfrak{g})$: for $X \in T_pP$, $\omega_p(X) = \xi$, such that $\varphi_p(\xi) = X^{vert}$. ω is a connection 1- form, and the connection it induces is Γ.*

Proof. The first condition for connection 1-forms is clear by the contruction of ω. All elements X of T_pP can be written as a unique sum $X^{\text{hor}} + X^{\text{vert}}$. Therefore it suffices to check the equivariance condition for horizontal X and vertical X. Let X be horizontal then $\mathrm{Ad}(g^{-1})(\omega(X)) = 0$. $T_p(r_g)(X) \in H_{pg}P$, and therefore $T_p(r_g)^*(\omega(X)) = 0$ as well. For a vertical vector X, there is a $\xi \in \mathfrak{g}$ such that $X = \varphi_p(\xi)$. For the fundamental vector field $\tilde{\xi}$, we find

$$\widetilde{\mathrm{Ad}(g^{-1})}\xi(pg) = \frac{d}{dt}|_{t=0} pg\exp(t\,\mathrm{Ad}(g^{-1})\xi) = \frac{d}{dt}|_{t=0} pgg^{-1}\exp(tA)g = (T_pr_g)(\tilde{\xi}(p)).$$

So for the connection 1-form we have indeed $\omega_{pg}(T_pr_g(X)) = Ad(g^{-1})(\omega_p(X))$. ☐

It is now a natural question to ask if connections exists. To prove that this is the case we use the locally trivial nature of principal bundles.

Theorem 3.3.6. *Any principal G-bundle admits a connection.*

Proof. By the very definition of a principal fibre bundle there is a covering $\{U_i\}_I$ of M such that $\pi^{-1}(U_i) \cong U_i \times G$. Note that $T_{p,g}(U \times G) \simeq T_pU \oplus \mathfrak{g}$. Subordinate to the covering we find a partition of unity; that is for every $i \in I$ we find a function χ_i which is zero outside a compact subset of U_i, and moreover, for every $m \in M$, the sum $\sum_{i\in I}\chi_i(m) = 1$. By the trivial structure we find for all $i \in I$ a connection 1-form ω^i; $\omega^i_p(X \oplus \xi) = \xi$ for all $X \oplus \xi \in T_{p,g}(U \times G)$. A connection 1-form for the whole principal fibre bundle is now given by $\omega = \sum_{i\in I}\pi^*(\chi_i)\omega_i$. ☐

Proposition 3.3.7. *Any connection on a principal G-bundle is a Ehresmann connection on $\pi : P \to M$ viewed as a fibre bundle*

To prove this theorem we make use of the following lemma.

Lemma 3.3.8. *Let G be a Lie group. For any path $A : [0,1] \to \mathfrak{g}$, there is a unique path $\gamma : [0,1] \to G$, with $\gamma(0) = e$ and $\frac{d}{dt}\gamma(t)(\gamma(t))^{-1} = A(t)$, for $t \in [0,1]$.*

Proof. We may assume that $A(t)$ is defined for all $t \in \mathbb{R}$. We define a vector field X on $G \times \mathbb{R}$:

$$X(g,s) = (a(s)\cdot g, \frac{d}{dt}|_s).$$

The integral curve of this vector field starting at $(e,0)$ is of the form $(\gamma(t),t)$, with γ our desired path. ☐

Proof of the propostion. We have to prove that there exists horizontal lifts for paths γ in M. Let $\gamma : [0,1] \to M$ be a path in M and let $x \in \pi^{-1}(\gamma(0))$. There are finitely many open neighbourhoods U_i such that $\pi^{-1}(U_i) \simeq U_i \times G$, and, moreover, $\gamma(t)$ is contained in $\cup_i U_i$. Using this isomorphisms we find a path u in P such that $\pi(u(t)) = \gamma(t)$ and $u(0) = x$. The horizontal lift $\gamma^\#$ of γ a starting at x is of the form $u(t)a(t)$, where $a(t)$ is a path in G starting at e. We have a look at the tangent vectors:

$$\dot{\gamma^\#}(t) = \dot{u}(t)a(t) + u(t)\dot{a}(t).$$

Note that $\gamma^\#(t)$ is horizontal precisely when $\omega(\dot{\gamma}^\#(t)) = 0$, i.e. when:

$$\mathrm{Ad}(a(t)^{-1})\omega(\dot{u}(t)) + a(t)^{-1}\dot{a}(t) = 0.$$

Now $\mathrm{Ad}(a(t)^{-1})\omega(\dot{u}(t))$ plays the role of $A(t)$ in the lemma above. Therefore, we conclude that such a path a in G exists. □

Connections on associated bundles Let F be a manifold on which G acts, then a connection on$P(M,G)$ can carried over to the associated bundle $P \times_G F$, which has fibres diffeomorphic to F. As we see in the following proposition.

Proposition 3.3.9. *Let G be a Lie group which acts on a manifold F from the left, and let $P(M,G)$ be any principal G-bundle. Any connection Γ on $P(M,G)$ determines a connection on the associated fibre bundle $P \times_G F$.*

Proof. Define the map $\tilde{\pi} : P \times F \to M$ by $(p,f) \mapsto \pi(p)$. At (p,f) the tangent space of $P \times F$ has the following property.

$$\begin{aligned} T_{(p,f)}(P \times F) &= T_pP \oplus T_fF \\ &= H_pP \oplus V_pP \oplus T_fF. \end{aligned}$$

Moreover, $\mathrm{Ker}\, T_{(p,f)}\tilde{\pi} = V_p \oplus T_fF$. Consider natural projection $\pi_{P\times F} : P \times F \to P \times_G F$, and observe that $\tilde{\pi} = \pi_F \circ pr$ and that $T_{(p,f)}\pi_{P\times F}(V_p) = \{0\}$. Therefore, we conclude that $\mathrm{Ker}\, T_{[p,f]}\pi_F = T_{(p,f)}pr(T_fF)$. It is now natural to require that

$$H_{[p,f]}(P \times_G F) = T_{(p,f)}\pi_{P\times F}(H_p)$$

This defines a connection on $P \times_G F$. □

Remark 3.3.10. Let $X \in T_bM$, and denote with $\bar{X}(p)$ the horizontal lift of X to T_pP, for $p \in \pi^{-1}(b)$. Then the horizontal lift $[p,f]$ in $P \times_G F$ is given by $X^\#([p,f]) = T_{(p,f)}\pi_{P\times F}(\bar{X}(p), 0_f)$. Here 0_f is the zero vector in T_fF.

3.4 Curvature

Analogue to the fibre bundles we define the curvature for connections on principal bundles.

Definition 3.4.1. Let $P(M,G)$ be a principal G-bundle and let Γ be a connection on it with connection 1-form ω. The curvature 2-form $\Omega \in \Omega^2(p, \mathfrak{g})$, is defined by

$$\Omega(X,Y) = \omega([X^{\text{hor}}, Y^{\text{hor}}]).$$

The curvature form can also be defined in terms of the connection 1-form ω.

Proposition 3.4.2. *Let ω be the connection 1-form for a connection Γ on a principal G-bundle, then $\Omega(X,Y) = (d\omega)(X^{hor}, Y^{hor})$.*

Proof. The explicit formula for the exterior derivative gives:

$$(d\omega)(X^{\text{hor}}, Y^{\text{hor}}) = (X^{\text{hor}})(\omega(Y^{\text{hor}})) - (Y^{\text{hor}})(\omega(X^{\text{hor}})) - \omega([X^{\text{hor}}, Y^{\text{hor}}]) = \omega([X^{\text{hor}}, Y^{\text{hor}}]).$$

Here we used that ω vanishes on horizontal vectors. □

There is another relation between the connection form and the curvature form. This relation is given by the so called structure equation.

Theorem 3.4.3. $\Omega_p(X,Y) = (d\omega)_p(X,Y) + [\omega_p(X), \omega_p(Y)]$, *for* $X, Y \in T_pP, p \in P$.

In the proof of this theorem the following lemma is used.

Lemma 3.4.4. *Let $P(M,G)$ be a principal bundle and let Γ be a connection on it. The Lie bracket $[X, \tilde{\xi}]$ between a horizontal vector field and a fundamental vector field on P is again horizontal.*

Proof. The flow of $\tilde{\xi}$ is $\exp(t\xi)$. Hence, for all $p \in P$

$$[X, \tilde{\xi}](p) = -[\tilde{\xi}, X](p) = \lim_{t \to 0} \frac{X(p) - (r_{\exp(t\xi)*}X)(p)}{t}.$$

the horizontal spaces are constructed such that $r_{\exp(t\xi)*}X(p)$ is horizontal. Therefore, $[X, \tilde{\xi}]$ is horizontal. □

Proof of the theorem. A vector in T_pP can be decomposed in a horizontal and a vertical component. Therefore it is sufficient to consider the cases that X, Y are both horizontal, both vertical or that X is horizontal and Y is vertical. If X and Y are both horizontal then $\omega(X) = \omega(Y) = 0$, and then the equation is the same as in the previous proposition. Note that $\Omega(X,Y) = 0$, whenever X or Y is vertical. If X and Y are both vertical, then there are elements $\xi, \eta \in \mathfrak{g}$ such that $\varphi_p(\xi) = X$ and $\varphi_p(\eta) = Y$. Hence:

$$d\omega(\tilde{\xi},\tilde{\eta}) = \tilde{\xi}(\omega(\tilde{\eta})) - \tilde{\eta}(\omega(\tilde{\xi})) - \omega([\tilde{\xi},\tilde{\eta}]) = -[\omega(\tilde{\xi}),\omega(\tilde{\eta})].$$

Because $\omega_p(\tilde{\xi}(p)) = \xi$ for all $p \in P$, $\tilde{\eta}(\omega(\xi))$ vanishes, and $\tilde{\xi}(\omega(\tilde{\eta}))$ vanishes similarly. We are left with

$$d\omega(\tilde{\xi},\tilde{\eta}) = -[\omega(\tilde{\xi}),\omega(\tilde{\eta})],$$

and hence the equation is satisfied. The last case is that X is horizontal and Y is vertical. Since $\omega(X) = 0$ it is sufficient to prove that $d\omega(X,Y) = 0$. Extend X to a horizontal vector field on P, which we denote again with X. Let $\xi \in \mathfrak{g}$ such that $\varphi_p(\xi) = Y$. We find

$$d\omega(X,\tilde{\xi}) = X(\omega(\tilde{\xi})) - \tilde{\xi}(\omega(X)) - \omega([X,\tilde{\xi}]) = X(\omega(\tilde{\xi})) - \omega([X,\tilde{\xi}])$$

$\omega([X,\tilde{\xi}])$ disappears because $[X,\tilde{\xi}]$ is horizontal, as is proved in the lemma above. □

Definition 3.4.5. A connection Γ on a principal bundle $P(M,G)$ is called flat if the curvature 2-form vanishes identically.

Curvature on associated fibre bundles

Proposition 3.4.6. *Let $P(M,G)$ be a principal G-bundle, and let G act on a manifold F. Denote with $\rho_F : \mathfrak{g} \to \mathfrak{X}(F)$ the infinitesimal action of G on F. Let Γ be a connection on $P(M,G)$, and let θ be the connection induced on the associated bundle $P \times_G F$. Let $\bar{X}(p)$ denote the horizontal lift of $X \in \mathfrak{X}(M)$ at p in $P(M,G)$. Denote with $\pi_{P\times F}$ the projection $P \times F \to P \times_G F$. The curvatures Ω_Γ and Ω_θ are related in the following way:*

$$\Omega_\theta(X_p,Y_p)([p,f]) = T_{(p,f)}\pi_{P\times F}(0_p,(\rho_F \circ \Omega_\Gamma(\bar{X}_p,\bar{Y}_p^{\#})))$$

Proof. Let X,Y be to vector fields on M. The vector fields $X' = (\bar{X},0), Y(\bar{Y},0)' \in \mathfrak{X}(P\times F)$ descend under $\pi_{P\times F}$ to $X^{\#},Y^{\#} \in \mathfrak{X}(P\times_G F)$.

Define a 2-form on $P\times F$:

$$\Omega'(X',Y') = [X',Y'] - [X,Y]'$$

$\Omega'(X',Y')$ descends to $[X^{\#},Y^{\#}] - [X,Y]^{\#} = \Omega_\theta(X,Y)$. On the other hand we have $\Omega'(X',Y') = ([\bar{X},\bar{Y}],0) - (\overline{[X,Y]},0)$. Hence,

$$\Omega'(X',Y') = (\varphi(\Omega_\Gamma(X,Y)),0).$$

We finish the prove by showing that $T\pi_{P\times F}(\varphi_p(\xi),0) = T\pi_{P\times F}(0,\rho_F(\xi))$. This last equality is is equivalent to $T\pi_{P\times F}(\varphi_p(\xi),-\rho_F(\xi)) = 0$. Consider the path $\gamma(t) = (p\exp(t\xi), exp(-t\xi)f)$ in $P\times F$. $\pi_{P\times F}(\gamma(t))$ is a constant path in $P\times_G F$. Hence we have:

$$0 = \frac{d}{dt}|_{t=0}T\pi_{P\times F}(\gamma(t)) = T\pi_{P\times F}\frac{d}{dt}|_{t=0}\gamma(t) = T\pi_{P\times F}(\varphi_p(\xi),-\rho_F(\xi)).$$

□

3.5 Connections and the covariant derivative

The linear frame bundle Above we defined the linear frame bundle $L(M)$ on a manifold M in terms of bases of the tangent spaces. We extend the notion of the linear frame bundle to a general vector bundle.

Definition 3.5.1. Let $\pi : E \to M$ be a vector bundle of rank n over manifold M. A frame u above $p \in M$ is a ordered basis of E_p, which we see as a linear map $u : \mathbb{R}^n \to E_p$. Let $\mathrm{GL}(E)$ be the set of all linear frames above all $p \in M$, and let the projection π map a frame u at p to p. The linear group $\mathrm{GL}(n)$ acts on $\mathrm{GL}(E)$ by precomposition. $\mathrm{GL}(E)(M, \mathrm{GL}(m))$ is a principal $\mathrm{GL}(m)$ bundle. We call this bundle the linear frame bundle of $\pi : E \to M$.

$\mathrm{GL}(n)$ acts on $\mathbb{R}^n$ in the usual way. The associated bundle $\mathrm{GL}(E) \times_{\mathrm{GL}(n)} \mathbb{R}^n$ is isomorphic with the original vector bundle E. The isomorphism is given by $[u, v] \mapsto u(v)$.

On vector bundles we have defined the covariant derivative and in principal bundle we have defined the connections. The aim of this section is to prove the following proposition.

Proposition 3.5.2. *There is an one-to-one correspondence between connections on the frame bundle* $\mathrm{GL}(E)$ *and covariant derivatives on the vector bundle* E.

From a covariant derivative to a connection We start with a covariant derivative, and construct with that a connection on the frame bundle. We defined the covariant derivative to be the linear map:

$$\nabla : C^\infty(M, E) \to \Omega^1(M, E),$$

which satisfies the Leibniz rule: for $f \in C^\infty(M)$, $s \in C^\infty(M, E)$,

$$\nabla(fs) = f\nabla s + df \wedge s.$$

A local section σ of the frame bundle on a neighbourhood U is of the following form.

$$\sigma : m \mapsto \{\sigma_1(m), ..., \sigma_n(m)\},$$

where $\{\sigma_1(m), ..., \sigma_n(m)\}$ forms a basis of E_m. So in fact σ exists of n sections $M \to E$ such that the $\sigma_i(m)$ are linear independent for $m \in U$. We call a section of a frame bundle horizontal if $\nabla\sigma_i(m) = 0$ for all i. Now we define the horizontal subspace of $T_p \mathrm{GL}(E)$, to be the tangent spaces of the horizontal local sections i.e. $H_p \mathrm{GL}(E) = T_{\sigma(m)=p}\sigma(m)$, where $\nabla\sigma_i(m) = 0$. We have to check that $H_p \mathrm{GL}(E)$ is invariant under the action of $\mathrm{GL}(n)$. If $A \in \mathrm{GL}(n)$, then forms $A \circ \sigma(m)$ a new basis of E_m; $\{\tilde{\sigma}_j(m)\}$. The covariant derivative is linear so $\nabla\tilde{\sigma}_j(m) = 0$, exactly if all $\nabla\sigma_i(m) = 0$. Therefore, the in this way defined horizontal spaces, are invariant under the action.

From a connection to a covariant derivative Next we start with a principal G-bundle, $P(M, G)$. Let E be a vector space, and let $\rho : G \to \mathrm{GL}(E)$ be a linear representation of G. $P \times_G E$ be the associated bundle. Denote with $C^\infty(P, E)^G$ be the space of equivariant maps. Those are maps $f : P \to E$ satisfying $f(pg) = \rho(g^{-1})f(p)$. If $X \in \mathfrak{g}$, then $\exp(tX) \in G$, for $t \in \mathbb{R}$, hence we have

$$f(p \exp(tX)) = \rho(\exp(-tX))f(p).$$

Therefore we have on the infinitesimal level that

$$(\tilde{X}(f))(p) + \rho_*(X)f(p) = 0, \text{ for } X \in \mathfrak{g},$$

Definition 3.5.3. A form α on a principal bundle $P(M, G)$ is called horizontal if $\iota(X)\alpha = 0$ for all vertical vector fields $X \in \mathfrak{X}(P)$

Definition 3.5.4. A form α on a principal G-bundle, taking values in a representation (E, ρ) of G, i.e $\alpha \in \Omega^1(P, E)$ is called basic if it is horizontal and invariant, i.e. $g\alpha = \alpha$ for all $g \in G$. The space of basic forms with values in E is denoted with $\Omega_{bas}(P, E)$.

Next two proposition come from [3] on page 19 and 20.

Proposition 3.5.5. *There is a natural isomorphism between* $\Gamma(M, P \times_G E)$ *and* $C^\infty(P, E)^G$*, given by sending* $s \in C^\infty(P, E)^G$ *to* s_M *defined by*

$$s_M(x) = [p, s(p)];$$

here p *is any element of* $\pi^{-1}(x)$ *and* $[p, s(p)]$ *is the element of* $P \times_G E$ *corresponding to* $(p, s(p)) \in P \times E$*.*

Proposition 3.5.6. *If* $\alpha \in \Omega^q_{bas}(P, E)$*, define* $\alpha_M \in \Omega^q(M, P \times_G E)$ *by*

$$(\alpha_M)(\pi_* X_1, ..., \pi_* X_q)(x) = [p, \alpha(X_1, ..., X_q)(p)].$$

Here p *is any point in* $\pi^{-1}(x)$*, and* $X_i \in T_r P$*. Then the map* $\alpha \mapsto \alpha_M$ *from* $\Omega^q_{bas}(P, E)$ *to* $\Omega^q(M, P \times_G E)$ *is an isomorphism.*

Proof. First we have to check that this map is well-defined. For $p' \in \pi^{-1}(x)$ another, there is a $g \in G$ such that $p' = pg$. We observe $[p', \alpha(X_1, ..., X_q)(p')] = [pg, \alpha(X_1, ..., X_q)(pg)] = [p, \alpha(X_1, ..., X_q)(p)]$. The last equality follows from the fact that α is basic. The map $\alpha \mapsto \alpha_M$ is linear, so to prove injectivity let $\alpha \in \Omega^q_{bas}(P, E)$ be in the kernel. Then $\alpha_M(\pi_* X_1, ..., \pi_* X_q)(x) = 0$ for all $X_i \in \mathfrak{X}(P)$. This only happens if $\alpha = 0$. For surjectivity, let $\beta \in \Omega^q(P \times_G E)$, and define $\alpha \in \Omega^q_{bas}(P, E)$ by $\alpha(X_1, ..., X_q)(p) = (\beta)(\pi_* X_1, ..., \pi_* X_q)(\pi(p))$. □

Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form. Let $\rho : G \to \mathrm{GL}(E)$ the representation of G. Then the differential $\rho_* := T\rho$ is a mapping $\mathfrak{g}g \to \mathrm{gl}(E)$. We define the map:

$$\rho_*(\omega) : C^\infty(P, E)^G \to \Omega^1(P, E)^G,$$

by

$$\langle \rho_*(\omega)s, X \rangle_p = \rho_*(\omega(X(p)))s(p).$$

And in a similar way we define:

$$\rho_*(d\omega) : \Omega^1(P, E)^G \to \Omega^2(P, E)^G,$$

by

$$\langle \rho_*(d\omega)\alpha, (X, Y) \rangle_p = \rho_*(d\omega(X(p), Y(p)))\alpha(X(p), Y(p)).$$

We find a covariant derivative of an associated vector bundle by means of the following diagram.

$$\begin{array}{ccc}
C^\infty(P, E)^G & \xrightarrow{\ d + \rho_*(\omega)\ } & \Omega^1_{bas}(P, E) \\
\cong \downarrow & & \downarrow \cong \\
\Gamma(M, E) & \xrightarrow{\ \nabla\ } & \Omega^1(M, E)
\end{array}$$

Let $s \in C^\infty(P, F)^G$ end let $X \in \mathfrak{g}$, then

$$\iota(\tilde{X})(ds + \rho_*(\omega)(s)) = \tilde{X}(s) + \rho(X)(s) = 0$$

Moreover $ds + \rho_*(\omega(s))$ is G-equivariant, if s equivariant. Therefore $ds + \rho_*(\omega(s)) \in \Omega^1_{bas}(P, E)$. To check that ∇ is a covariant derivative, we choose a $s_M \in \Gamma(M, P \times_G E)$ and a C^∞-function f on M, and we observe what $\nabla f s_M$ is. The isomorphism of 3.5.5 sends $f s_M$ to $(f \circ \pi)s \in C^\infty(P, E)^G$. Since $d(f \circ \pi)s = d(f \circ \pi) \cdot s + (f \circ \pi)ds$, so it is sent to $d(f \circ \pi) \cdot s + (f \circ \pi)ds + (f \circ \pi)\rho(\omega)s \in \Omega^1_{bas}(P, E)$. On its turn the isomorphism of 3.5.6 sends it to $df \cdot s + f \nabla s_M \in \Omega^1(M, E)$.

The curvature For both, the vector bundles and the principal bundles we discussed the notion of curvature. These are related by the following extension of the diagram above.

$$\begin{array}{ccccc} C^\infty(P,E)^G & \xrightarrow{d+\rho_*(\omega)} & \Omega^1_{bas}(P,E) & \xrightarrow{d+\rho_*(d\omega)} & \Omega^2_{bas}(P,E) \\ \cong\big\downarrow & & \big\downarrow\cong & & \big\downarrow\cong \\ \Gamma(M,E) & \xrightarrow{\quad\nabla\quad} & \Omega^1(M,E) & \xrightarrow{\quad\nabla\quad} & \Omega^2(M,E) \end{array}$$

Note that $d\omega$ is the curvature of the principal bundle here, because it maps from the basic forms.

3.6 Parallel transport and the holonomy group

Parallel transport The concepts of a horizontal lifts and parallel transport are similar as for fibre bundles. Nice is that parallel transport in principal bundle preserves the additional date; the group structure of the fibre.

Lemma 3.6.1. *Let $P(M,G)$ be principal bundle, and let Γ be a connection on it. Parallel transport $\phi_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ along a path γ with respect to Γ is an isomorphism.*

Proof. ϕ_γ is bijective because the different horizontal lifts of γ don't intersect each other. Recall that one can obtain all elements from a fibre $\pi^{-1}(x)$, by multiplying one chosen element by the elements of G. Because multiplying with elements of G, sends horizontal paths to horizontal paths, we observe that ϕ_γ is really a homomorphism and therefore a isomorphism between the fibres. □

The holonomy group.

Definition 3.6.2. Let $P(M,G)$ be a principal fibre bundle with a connection Γ. Choose $x \in M$. The holonomy group $\Phi(x)$ in x with respect to Γ is the following subgroup of $\mathrm{Aut}(\pi^{-1}(x))$: $\Phi(x)$ exists out of those automorphisms of π^{-1} which can be obtained via parallel transport ϕ_γ with respect to Γ along a loop in M around x.

This definition requires a prove that $\Phi(x)$ is really a subgroup of $\mathrm{Aut}(\pi^{-1})$.

Proof. Take two elements $\phi_\gamma, \phi_{\gamma'} \in \Phi(x)$. Then $\phi_\gamma \circ (\phi_{\gamma'})^{-1}$ is obtained by parallel transport along $\gamma * (\gamma')^{-1}$, with

$$\gamma * (\gamma')^{-1}(t) = \begin{cases} \gamma(t), & 0 \le t \le \frac{1}{2} \\ \gamma'(t), & \frac{1}{2} \le t \le 1 \end{cases}$$

Hence, $\Phi(x)$ is a subgroup. □

Lemma 3.6.3. *Let $P(M, G)$ be a principal fibre bundle with a connection Γ. For two points x and x' in the same path connected component of the base space M the holonomy groups $\Phi(x)$ and $\Phi(x')$ with respect to Γ are isomorphic.*

Proof. Let α be an path joining x and x'. Then

$$\Phi(x) \to \Phi(x') : \phi_\gamma \mapsto \phi_\alpha \circ \phi_\gamma,$$

is an isomorphism. □

There is another equivalent way to define the holonomy group. It uses that the fibres of a principal G-bundle are isomorphic with G.

Definition 3.6.4. Let $P(M, G)$ be a principal fibre bundle with a connection Γ, and let $u \in P$. The holonomy group $\Phi(u)$ in u with respect to Γ is the subgroup of G which exists out of those $g \in G$ such that there is a horizontal path from g to ug.

Again, this definition requires a prove that $\Phi(u)$ is a subgroup of G.

Proof. For $g, h \in \Phi(u)$, there are paths γ, α in M such that $\phi_\gamma(u) = ug$ and $\phi_\alpha(u) = uh$. By combining these paths we get $\phi_{\alpha*\gamma}(u) = \phi_\alpha(ug) = \phi_\alpha(u)g = uhg$. Hence $hg \in \Phi(u)$. □

Lemma 3.6.5. *Let $P(M, G)$ be a principal fibre bundle with a connection Γ, and let $u \in P$ and $x = \pi(u)$. The holonomy groups $\Phi(x)$ and $\Phi(u)$ are isomorphic.*

Proof. The map which sends $\phi_\gamma \in \Phi(x)$ to that $g \in \Phi(u)$ such that $\phi_\gamma(u) = ug$, is the isomorphism. □

Proposition 3.6.6. $\Phi(u)$ *is a Lie subgroup of a G*

A proof of this proposition can be found in [11].

Definition 3.6.7. Let $P(M, G)$ be a principal G-bundle, and let Γ be a connection on it. The restriction of the holonomy group $\Phi(u)$ in u to elements generated by the loops in M which are contractible is called the restricted holonomy group in u, and denoted with $\Phi^0(M)$.

Lemma 3.6.8. *Let $P(M, G)$ be a principal G-bundle whose base manifold is connected, and let Γ be a connection on it. Choose $u \in P$, and let $x = \pi(u)$. There is a surjective group homomorphism from the fundamental group $\pi_1(M, x)$ of M to the quotient $\Phi(u)/\Phi^0(u)$.*

Proof. Let γ and γ' be loops in around x, such that $[\gamma] = [\gamma'] \in \pi_1(M, x)$. γ and γ' both define an element in $\Phi(u)$. $\gamma \cdot \gamma'$ is null-homotopic, so they define the same element in $\Phi(u)/\Phi^0(u)$. This defines the homomorphism from $f : \pi_1(M, x) \to \Phi(u)/\Phi^0(u)$. □

Corollary 3.6.9. *For principal G-bundles for which the base manifold is connected is $\Phi(u)/\Phi^0(u)$ countable.*

Proof. Manifolds which are connected have countable fundamental groups. □

A theorem on holonomy There is an important result due to Ambrose and Singer which gives a relation between the holonomy group of a connection and its curvature. First we introduce the notion of the holonomy bundle.

Definition 3.6.10. Let $P(M, G)$ be a principal fibre bundle with a connection Γ. Let $u \in P$. The set of all elements v in P such that there exists a horizontal path joining u and v is called the holonomy bundle with respect to connection Γ of u, and denoted with $P(u)$.

That $P(u)$ is a fibre bundle is hidden in the proof of the following theorem.

Theorem 3.6.11 (Reduction). *Let $P(M, G)$ be a principal fibre bundle with connected base M, and let Γ be a connection on it. Let $u \in P$, then*

1. *$P(u)$ is a reduced bundle with structure group $\Phi(u)$.*

2. *the connection Γ is reducible to a connection in $P(u)$.*

Proof. To prove this theorem we use lemma 3.2.13. For the first statement it is sufficient to check the assumptions of this lemma. Let $x = \pi(u)$. Since M is connected there is a path to any $y \in M$. Therefore $\pi(P(u)) = M$. If $v \in P(u)$, then $va \in P(u)$ if $a \in \Phi(u)$, by the very definition of $\Phi(u)$. To verify assumption (4) we take a local coordinate system $x_1, ..., x_n$ around x, where x is mapped to the origin. Let $\delta > 0$ and define U by $U = \{|x_i| < \delta\}$. For a $y \in P$ such the $\pi(y) \in U$, let γ be a path in U joining x and $\pi(y)$. Define $\sigma(y)$ to be obtained by parallel transport along this γ. Then σ is a section and $\sigma(y) \in P(u)$. This proves the first statement of the theorem. To prove the second statement, we observe that $H_pP \subset T_p(P(u))$, for $p \in P(u)$. We now define $H_pP(u)$ to be H_pP. □

Now we are ready to state and prove the holonomy theorem.

Theorem 3.6.12 (Holonomy). *Let $P(M, G)$ be principal bundle. Let Γ be a connection on P, and denote with ω its connection 1-form and with Ω its curvature. Let $u \in P$ be an arbitrary point, and denote with $P(u)$ the holonomy bundle through u. Then the Lie algebra of the holonomy group $\Phi(u)$ is equal to the following subspace $\mathfrak{g}'$ of $\mathfrak{g}$.*

$$\mathfrak{g}' = \{\Omega_v(X, Y) : v \in P(u),\ X, Y \in \mathrm{Hor}_v\}.$$

Proof. By the reduction theorem we may assume that $P(u) = P$, and hence $G = \Phi(u)$. Because for all horizontal X, Y, we have $\Omega(X, Y) = -\omega([X, Y])$, and because of the G equivariance of ω, we find that $\mathfrak{g}'$ is $\mathrm{Ad}\, G$ invariant. Therefore $\mathfrak{g}'$ is a Lie algebra itself. We will prove that in fact $\mathfrak{g}' = \mathfrak{g}$. Thereto we define a distribution S in P. Let

$$\begin{aligned} S_v &= \mathrm{Hor}_v + \mathfrak{g}'_v \text{ for } v \in P, \text{ here} \\ \mathfrak{g}'_v &= \{\tilde{A}(v) : A \in \mathfrak{g}'\}. \end{aligned}$$

We claim that S is differentiable and integrable. Let $v \in P$, and let U be an open neighbourhood of $y = \pi(v)$ in M such that $\pi^{-1}(U) \simeq U \times G$. Let $X_1, ..., X_m \in \mathfrak{X}(M)$ be differentiable vector fields such that for all $y' \in U$, $X_1(y'), ..., X_m(y')$ forms a basis for $T_{y'}M$. Let $A_1, ..., A_r$ be a basis for $\mathfrak{g}'$. Then we obtain a basis for S_v: $X_1^\#(v), ...X_m^\#(v), \tilde{A}_1(v), ..., \tilde{A}_r(v)$. By the differentiability of the vector fields involved, this implies that S in differentiable. We use this basis to prove the integrability. Because $\mathfrak{g}'$ is an algebra and $[\tilde{A}_i, \tilde{A}_j] = \widetilde{[A_i, A_j]}$, we have that $[\tilde{A}_i, \tilde{A}_j] \in S$. We proved before that $[\tilde{A}_i, X_j^\#]$ is horizontal, so, vectors of that form are also contained in S. For horizontal vectors we have $\omega([X_i^\#, X_j^\#]) = -\Omega(X_i^\#, X_j^\#) \in \mathfrak{g}'$, hence $[X_i^\#(v), X_j^\#(v)] = \tilde{A}(v)$ for some $A \in \mathfrak{g}'$. We conclude that S is integrable. Let P_0 be the maximal integral manifold in of S through u. Because S contains the horizontal distribution, $P = P(u) \subset P_0 = P$. Therefore:

$$\dim \mathfrak{g}' = \dim P_0 - r = \dim P - r = \dim \mathfrak{g},$$

and hence $\mathfrak{g}' = \mathfrak{g}$. □

3.7 Connections on the frame bundle

In this section we take a fibration $\pi : M \to B$ with fibre F. The structure group G of the fibration is a subgroup of $\mathrm{Diff}(F)$, and is not necessarily finite dimensional. We introduce a more general concept of frame bundle, and we compare the holonomy groups of the frame bundle with the holonomy group of the original bundle.

Frame bundles

Definition 3.7.1. The frame bundle, $p : \mathcal{P} \to B$ of the fibration $\pi : M \to B$ is a principal G-bundle whose fibre over $b \in B$ is

$$\mathcal{P}_b = \{u : F \to F_b : u \text{ is a diffeomorphism}\}.$$

G acts on $\mathcal{P}$ from the right by precomposition:

$$\mathcal{P} \times G \to \mathcal{P} : (u, g) \mapsto u \circ g.$$

The original fibre bundle is isomorphic with the associated bundle $\mathcal{P} \times_G F$, and a connection on the original bundle induces a connection on the frame bundle. In order to see that we need the notion of the tangent space $T_u\mathcal{P}$ of $\mathcal{P}$, which we will describe without specifying the differential structure on $\mathcal{P}$. Take a path u_t, $t \in [0, 1]$ in $\mathcal{P}$, with $u_0 = u$. The choice of this path involves the specification of a path $b(t), t \in [0, 1]$ in B, such that $u_t : F \to F_{b(t)}$. Note that for all $x \in F$, $u_t(x), t \in [0, 1]$, is a path in M. Therefore we get a map

$$\frac{\partial}{\partial t} u_t|_{t=s} : F \to T_{\pi^{-1}(b(s))}M.$$

Let X be the map we get if we take $s = 0$. We compute

$$T_{u(x)}\pi(X(x)) = \frac{d}{dt}|_{t=0}\pi(u_t(x)) = \frac{d}{dt}|_{t=0}b(t),$$

which is independent of x. Finally we have computed that

$$T_u\mathcal{P} = \{X : F \to TM : X(x)T_{u(x)}\pi \cdot X(x) \in T_{p(u)}B \text{ is constant}\}.$$

The vertical spaces of $p : \mathcal{P} \to B$ are the kernels of $T_u p$. These are those vector fields in $T_u\mathcal{P}$ which image in vertical in the original fibre bundle. Therefore the vertical spaces are given by

$$\mathcal{V}_u = \{du \circ X : X \in \mathfrak{X}(F)\}$$

Now let Γ be a connection on the original fibration. It induces a connection on $p : \mathcal{P} \to B$ in the following way.

$$\mathcal{H}_u = \{v^{\#} \circ u : v \in T_b B\}.$$

Moreover its induced connection on $\mathcal{P} \times_G F$ is the original connection Γ. $p : \mathcal{P} \to B$ is a principal G-bundle, hence the connection can also be described with the help of a $\mathfrak{g}$ valued connection 1-form. The Lie algebra of $\mathrm{Diff}(F)$ is $\mathfrak{X}(F)$. The fundamental vector field on $\mathcal{P}$ induced by $X \in \mathfrak{X}(F)$ is

$$\tilde{X}(u) = du \circ X.$$

Curvature We describe how the curvature in the original bundle Ω_Γ is related with the curvature in the frame bundle F_Γ. For every $x \in F$ and $\xi \in T_u\mathcal{P}$ we have $\xi(x) = \xi(x)^{\text{hor}} + \xi(x)^{\text{vert}}$. And the connection 1-form ω at ξ has as value at u that vector field $X \in \mathfrak{X}(F)$, such that

$$\tilde{X}(x)_u = u \circ \frac{d}{dt}|_{t=0} \exp(tX)(x) = \xi(x)^{\text{vert}}.$$

The curvature is given by

$$F_\Gamma(\xi, \eta) = \omega([\xi^{\text{hor}}, \eta^{\text{hor}}]) \in \mathfrak{X}(F),$$

and $du \circ \omega([\xi^{\text{hor}}, \eta^{\text{hor}}]) \in \mathcal{V}_u$. The curvature of the original bundle is given by

$$\Omega_\Gamma(v_1, v_2)_{u(x)} = [v_1^{\#}, v_2^{\#}]^{\text{vert}} \in \mathfrak{X}(F_b),$$

where $b = p(u)$. Because for all $\xi \in T_u\mathcal{P}$, $T_{u(x)}p \cdot \xi(x) = \text{constant}$, there are $v_1, v_2 \in T_bB$ such that $\xi(x)^{\text{hor}} = v_1(x)^{\text{hor}}$, and $\eta(x)^{\text{hor}} = v_2(x)^{\text{hor}}$. Now it is not hard to see that the relation between the two curvatures is given by:

$$\Omega_{\Gamma b} = du \circ F_\Gamma \circ u^{-1},$$

where $u \in \mathcal{P}_b$.

The holonomy groups The following lemma gives a relation between the holonomy group of the original bundle and the frame bundle.

Lemma 3.7.2. *Let $b \in B$, and $\Phi(b)$ the holonomy group for a connection Γ on the bundle $\pi : M \to B$. Let $u : F \to F_b$ be in $\mathcal{P}$. Then the $\Phi(b)$ is isomorphic to the holonomy group $\Phi(u)$ of the induced connection on $p : \mathcal{P} \to B$.*

Proof. $\Phi(u)$ exists out of those elements $g \in \text{Diff}(F) = G$, such that u and ug can be joined by a horizontal path f_t in $\mathcal{P}$. f_t is the horizontal lift of a path $\gamma(t)$ in B. u and ug live in the same fibre, so $\gamma(t)$ is a loop in B, and $u \circ g \circ u^{-1} \in \Phi(b)$. The map

$$\Phi(u) \to \Phi(b) : g \mapsto u^{-1} \circ g \circ u,$$

is our desired isomorphism. □

Chapter 4

Symplectic fibre bundles

Abstract In this chapter we concentrate on symplectic fibre bundles. These are fibre bundles of type a symplectic manifold (F, σ) and the transition functions are symplectomorphisms. The total space of a symplectic fibration doesn't always admit a compatible symplectic structure itself. We derive some conditions on the fibration, for which the total space is symplectic. We discuss the notions of a symplectic connection. These are connections for which the parallel transport preserves the symplectic structure. They can be described by vertically closed 2-forms. If such a 2-form turns out to be closed and nondegenerate then the total space is in fact a symplectic space, and the form is called a coupling form. We finish with a theorem, which shows that the connection induced by a coupling form has to have quite restricted holonomy groups. The theory is illustrated with examples. The outline of this chapter is mostly based on chapter 6 of monograph of McDuff and Salomon, [13].

4.1 Symplectic fibre bundles

A fibre bundle $\pi : M \to B$ has a structure group $G \subset \mathrm{Diff}(F)$ if all transition functions take values in G. In this chapter the fibre is a compact symplectic manifold (F, σ).

Definition 4.1.1. A fibration $\pi : M \to B$ with local trivialisation (ϕ_i, U_i) is a symplectic fibration if all transition functions take values in $\mathrm{Symp}(F, \sigma)$, i.e. $\phi_{i,j}(b)^*\sigma = \sigma$, for all i, j and $b \in U_i \cap U_j$.

The fibres F_b of a symplectic fibration are equipped with a natural symplectic structure $\sigma_b \in \Omega^2(F_b)$ defined by $\sigma_b = \phi_i(b)^*\sigma$. We could define a symplectic fibration equivalently, by requiring that every fibre has a symplectic structure σ_b such that $\sigma_b = \phi_i(b)^*\sigma$ yields. Then for the transition functions it yields,

$$\phi_{i,j}(b)^*\sigma \quad = (\phi_j(b) \circ \phi_i(b)^{-1})^*\sigma = \phi_i(b)^{-1*}\sigma_b = \sigma.$$

Hence, they are in $\mathrm{Symp}(F, \sigma)$. A symplectic form ω on the total space M is called compatible with the symplectic fibration if each fibre (F_b, σ_b) is a symplectic submanifold of (M, ω),i.e.

for all $b \in B$ $\iota_b^*\omega = \sigma_b$, where $\iota_b : F_b \to M$ denotes the inclusion. Given a fibration $\pi : M \to B$ with symplectic fibre (F, σ), and a structure group $G \in \text{Symp}(F, \sigma)$, we would like to understand under which conditions M admits a compatible symplectic form ω. The other way round, if we have a fibration $\pi : M \to B$, and a symplectic form $\omega \in \Omega^2(M)$, such that the fibres are all symplectic manifolds of M, we would like to understand if the fibration admits the structure of a symplectic fibration which is compatible with ω. The last question is answered by the following theorem.

Theorem 4.1.2. *Let $\pi : M \to B$ be a fibre bundle with connected base and let $\omega \in \Omega^2(M)$ be a symplectic form such that the fibres are all symplectic submanifolds of M. Then $\pi : M \to B$ admits the structure of a symplectic fibration which is compatible with ω.*

Proof. For every $b \in M$ we have $\sigma_b = \iota_b^*\omega \in \Omega^2(F_b)$ and a diffeomorphism $\phi_i(b) : F_b \to F$, hence $\phi_i(b)^{-1*}\sigma_b \in \Omega^2(F)$. We will show that for all b these represent the same cohomology class in $H^2_{\text{de Rham}}(F)$. Using lemma **??** hAS TO BE REPAIRED.we note that the difference

$$\phi_i(b)^{-1*}\sigma_b - \phi_j(b')^{-1*}\sigma_{b'}$$

is an exact form. M is a connected manifold, so in particular for two points b and b' there is a path γ, such that $\gamma(0) = b$ and $\gamma(1) = b'$. First take b and b' both containded in a neighborhood U_i, over which the bundle is trivial. Let $\gamma(t)$ be contained in U_i, for all $t \in I$. Define

$$\phi = \iota_{\gamma(t)} \circ \phi_i(\gamma(t))^{-1} : I \times F \to M$$

This is a smooth map, and using the lemma we conclude that $(\iota_b \circ \phi_i(b)^{-1})^*\omega - (\iota_{b'} \circ \phi_i(b')^{-1})^*\omega$, is an exact form for such b and b'. If two point b, b' don't lie in such a neighbourhood, but in neighbourhoods U_i, U_j such that $U_i \cap U_j \neq \emptyset$, then choose $c \in U_i \cap U_j$, and conclude that $(\iota_b \circ \phi_i(b)^{-1})^*\omega$ and $(\iota_b' \circ \phi_j(b')^{-1})^*\omega$ are both cohomologous with $(\iota_c \circ \phi_i(c)^{-1})^*\omega$. For points which are further away from each other we construct a path γ joining them. Then $\gamma^{-1}(U_i)_i$ gives a cover of $[0, 1]$. $[0, 1]$ is compact, so we get after only finitely many steps that $(\iota_b \circ \phi_i(b)^{-1})^*\omega$ and $(\iota_b' \circ \phi_j(b')^{-1})^*\omega$ are cohomologous in $H^2_{dR}(F)$.

To finish the proof we take again a path γ joining two points $b, b' \in M$. Now define $\sigma_t = (1 - t)\sigma + t\sigma_{\gamma(t)} \in \Omega^2(F)$. They are also all cohomologous. Now Moser's stability theorem gives that there is a family of diffeomorphisms $\psi_t : F \to F$, such that

$$\psi_0 = \text{id}$$
$$\psi_t^*\sigma_t = \sigma_0 = \sigma$$

Therefore all fibres are symplectomorphic to the standard fibre (F, σ). □

The converse is not obvious.

The next result, due to Thurston, shows that for M compact and for B a symplectic manifold, there is a necessary and sufficient condition, which guarantees the existence of a compatible symplectic form on M.

Theorem 4.1.3 (Thurston). *Let $\pi : M \to B$ be a symplectic manifold with compact total space M, symplectic fibre (F, σ), and connected symplectic base (B, β). Suppose, there is a class $a \in H^2_{de\ Rham}(M)$ such that*

$$\iota_b^* a = [\sigma_b]$$

for some $b \in B$. Then for every sufficiently large real $K > 0$, there exists a symplectic form $\omega_K \in \Omega^2(M)$ which is compatible with the fibration π and represents the class $a + K[\pi^*\beta]$.

Proof. Let τ_0 be any closed 2-form which represents the class $a \in H^2_{\text{de Rham}}(M)$. For any α denote by $\sigma_\alpha \in \omega^2(U_\alpha \times F)$ the 2-form obtained from σ via the pull-back under the projection $U_\alpha \times F \to F$. The open sets U_α are chosen to be contractible, so it follows that the forms $\phi_\alpha^* \sigma_\alpha - \tau_0 \in \Omega^2(\pi^{-1}(U_\alpha))$ are exact. Choose a collection of 1-forms $\lambda_\alpha \in \Omega^1(\pi^{-1}(U_\alpha))$ such that

$$\phi_\alpha^* \sigma_\alpha - \tau_0 = d\lambda_\alpha.$$

Let $\{\rho_\alpha : B \to [0, 1]\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$, and define $\tau \in \Omega^2(M)$ by:

$$\tau = \tau_0 + \sum_\alpha d((\rho_\alpha \circ \pi)\lambda_\alpha).$$

All terms are closed, so τ is closed as well, and represents the class a. Moreover, the 1-form $d(\rho_\alpha \circ \pi)$ vanishes on vertical vectors, and hence

$$\begin{aligned}
\iota_b^* \tau &= \iota_b^* \tau_0 + \sum_\alpha (\rho_\alpha \circ \pi) \iota_b^* d\lambda_\alpha \\
&= \sum_\alpha (\rho_\alpha \circ \pi) \iota_b^* (\tau_0 + d\lambda_\alpha) \\
&= \sum_\alpha (\rho_\alpha \circ \pi) \iota_b^* \phi_\alpha^* \sigma_\alpha \\
&= \sum_\alpha (\rho_\alpha \circ \pi) \sigma_b = \sigma_b.
\end{aligned}$$

The form τ need not to be nondegenerate. However, it is nondegenerate on the vertical subspaces, and hence it determines a field of horizontal subspaces $\mathrm{Hor}_x = (\mathrm{Vert}_x)^\tau$. The form

$$\omega_K = \tau + K\pi^*\beta$$

agrees with τ on Vert_x, is nondegenerate on Hor_x for $K > 0$ sufficiently large, and satisfies $\omega_K(\xi, \eta) = 0$ for $\xi \in \mathrm{Hor}_x$ and $\eta \in \mathrm{Vert}_x$. This shows that ω_K is nondegenerate for K sufficiently large. □

Constructing a symplectic fibre bundle. There is a natural way to construct a symplectic fibration from a principal G-bundle and a symplectic manifold.

Theorem 4.1.4. *If (P, B) is a principal G-bundle, and if G acts in a symplectic fashion on a symplectic manifold (F, σ), then $P \times_G F$ is a symplectic fibration.*

Proof. As we have seen in the chapter about principal G-bundles are the transition functions of the associated fibre bundle, really by means of the group action on F. This group action is here symplectic, and therefore the transition functions as well. □

Example 4.1.5. Let $P(G, B)$ be a principal G-bundle. The symplectic form on the orbits of the coadjoint action on $\mathfrak{g}^*$ is invariant under this action. So G acts in particular in a symplectic way on $\mathcal{O}_\xi$. Therefore, for all orbits $P \times_G \mathcal{O}_\xi$ is a symplectic fibration over B.

If we allow G to be infinite dimensional, like $\mathrm{Symp}(F, \sigma)$ is, then every symplectic fibration gives rise to a principal G-bundle, namely the symplectic frame bundle.

Definition 4.1.6. Let $\pi : M \to B$ be a symplectic bundle with fibre F. The symplectic frame bundle, $\mathcal{P} \to B$, has as fibres

$$\mathcal{P}_b = \{f : F \to F_b : f^*\sigma_b = \sigma\}.$$

The group of symplectomorphisms of (F, σ) acts on $\mathcal{P}$ by precomposition.

4.2 Symplectic connections

Definition 4.2.1. An Ehresmann connection Γ on a symplectic fibration $\pi : M \to B$ is called symplectic if the diffeomorphisms, constructed by parallel transport along paths γ, preserve the symplectic structures in the fibres.

Ehresmann connections may be described by fibre nondegenerate almost Dirac structures on the total space. We consider those which are the graph of a compatible 2-form $\tau \in \Omega^(M)$, i.e. $\iota_b^*\tau = \sigma_b$ for all $b \in B$. Note that the compatability condition assures that the graph of τ is indeed fibre nondegenerate. Not all compatible 2-forms on M lead to a symplectic connection. The following lemma indicates some obstructions on τ to induce a symplectic connection.

Lemma 4.2.2. *Assume that $\tau \in \Omega^2(M)$ is compatible with the fibration. Then the connection Γ_τ is symplectic if and only if τ is vertically closed in the sense that*

$$d\tau(\eta_1, \eta_2, \cdot) = 0$$

for all $x \in M$ and all vertical tangent vectors $\eta_1, \eta_2 \in \mathrm{Vert}_x$.

Proof. The connection Γ_τ is symplectic if the flow of all horizontal lifts $v^\#$, of vector fields on B, preserve the restriction of τ to the fibres. Equivalently,

$$L_{v^\#}\tau(\eta_1, \eta_2) = 0,$$

for all $v \in \mathfrak{X}(B)$, and vertical vectors η_1, η_2. By Cartans formula we get

$$d(\iota(v^\#)\tau)(\eta_1, \eta_2) + (\iota(v^\#)d\tau)(\eta_1, \eta_2) = 0.$$

By the definition of Hor, the 1-form $\iota(v^\#)\tau$, vanishes on vertical vectors. Since the pull-back and the exterior derivative commute it yields,

$$\iota_b^* d(\iota(v^\#)\tau) = d\iota_b^*(\iota(v^\#)\tau) = 0.$$

Therefore, the first term is zero. That the last term is zero is equivalent to the statement that

$$d\tau(\eta_1, \eta_2, \xi) = 0$$

for all horizontal ξ. The restriction to the fibres of τ is closed by assumption so we obtain:

$$d\tau(\eta_1, \eta_2, \cdot) = 0.$$

□

Definition 4.2.3. Let $\pi : M \to B$ be a symplectic fibration. A connection 2-form on M is a form $\tau \in \Omega^2(M)$ which is compatible with the symplectic fibre structure and which is vertically closed.

It is clear that all connection 2-forms give rise to symplectic connections. Next, we prove that they really exist.

Lemma 4.2.4. *Any symplectic fibrations $\pi : M \to B$ of fibre type (F, σ) admits a connection 2-form τ.*

Proof. Let $\{\phi_\alpha : U_\alpha\}_\alpha$ be the local trivialization. Denote with $\sigma_\alpha \in \Omega(U_\alpha \times F)$ the pull-back of σ under the projection $U_\alpha \times F$. Let $\rho_\alpha : B \to \mathbb{R}$ be a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$. Define τ by:

$$\tau = \sum_\alpha (\rho_\alpha \circ \pi)\phi_\alpha^* \sigma_\alpha.$$

We claim that this τ is a connection 2-form. To check the compatibility of τ we take an arbitrary $b \in B$, such that $b \in U_\beta$, and compute:

$$
\begin{aligned}
\iota_b^*\tau &= \iota_b^* \sum_\alpha (\rho_\alpha \circ \pi)\phi_\alpha^*\sigma_\alpha \\
&= \sum_\alpha (\rho_\alpha \circ \pi)\iota_b^*\phi_\alpha^* pr_\alpha^*\sigma \\
&= \sum_\alpha (\rho_\alpha \circ \pi)(pr_\alpha \circ \phi_\alpha \circ \iota_b)^*\sigma \\
&= \phi_\beta(b)^*\sigma = \sigma_b.
\end{aligned}
$$

Furthermore, we compute $d\tau$:

$$
d\tau = \sum_\alpha d(\rho_\alpha \circ \pi) \wedge \phi_\alpha^*\sigma_\alpha - \sum_\alpha (\rho_\alpha \circ \pi)\phi_\alpha^* pr_\alpha^* d\sigma.
$$

σ is closed, so the second term vanishes. Because Vert $=$ Ker $d\pi$, the first term vanishes on vertical vectors, i.e. τ is vertically closed. □

Example 4.2.5. In this example we study a fibration over the torus $\mathbb{T}^2$. The total space M, is the quotient space $\mathbb{R}^4/\sim$, where $\sim$ is the equivalence relation

$$
(s,t,x,y) \sim (s+i, t+j, x+it+k, y+l),
$$

with $i,j,k,l \in \mathbb{Z}$. The projection is given by $\pi : M \to \mathbb{T}^2 : [s,t,x,y] \mapsto [s,t]$. The fibres are

$$
\pi^{-1}([s,t]) = \{[s,t,x+it+k, y+l] \in M : k,l \in \mathbb{Z}\}
$$

again the torus. The symplectic form on the fibre $dx \wedge dy$. The following form is a connection 2-form on the total space;

$$
\tau = dx \wedge dy - s dt \wedge dy.
$$

The second term of this form is zero on vertical vectors, so τ is really compatible with the symplectic structure on the fibres. To check that it is really a connection 2-form we take the exterior derivative.

$$
d\tau = -ds \wedge dt \wedge dy.
$$

Since $ds \wedge dt$ vanishes on vertical vectors, τ is vertically closed. Next we compute how the connection induced by this 2-form looks like. Denote with $m = [s,t,x,y]$ a point in M. Elements of Vert_m are of the form $(0,0,u,v)$, with $u,v \in \mathbb{R}$. Then

$$
\begin{aligned}
\text{Hor}_m = \text{Vert}_m^\tau &= \{(a,b,c,d) \in \mathbb{R}^4 : cv - du - sbv = 0, \text{ for all } u,v \in \mathbb{R}\} \\
&= \{(a,b,sb,0) \in \mathbb{R}^4\}.
\end{aligned}
$$

Hence, the horizontal subspaces are dependent on m.

Example 4.2.6.

Definition 4.2.7. The quaternions form a ring with elements of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$, and with the following commutation relations:

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j \end{aligned}$$

The quaternions are denoted with $\mathbb{H}$.

It is possible to identify the quaternions with $\mathbb{C}^2$. To do that we define two maps:

$$\varphi_1 : \mathbb{C} \to \mathbb{H}, a + bi \mapsto a + bi$$
$$\varphi_2 : \mathbb{C} \to \mathbb{H}, a + bi \mapsto aj + bk,$$

and combine this map to:

$$\varphi : \mathbb{C}^2 \to \mathbb{H} : (z_1, z_2) \mapsto \varphi_1(z_1) + \varphi_2(z_2).$$

This map has the nice property that $\varphi(\lambda(z_1, z_2)) = \varphi_1(\lambda)(\varphi_1(z_1) + \varphi_2(z_2))$, for all $\lambda \in \mathbb{C}$. Therefore, it induces a map of projective spaces:

$$\tilde{\varphi} : \mathbb{C}P^3 \to \mathbb{H}P^1.$$

The fibres of this fibration can be described as follows:

$$\tilde{\varphi}^{-1}([1, h_2]) = \{[z_1, z_2, z_3, z_4] \in \mathbb{C}P^3 : \frac{\varphi_1(z_3) + \varphi_2(z_4)}{\varphi_1(z_1) + \varphi_2(z_2)} = h_2\}$$
$$\tilde{\varphi}^{-1}([h_1, 1]) = \{[z_1, z_2, z_3, z_4] \in \mathbb{C}P^3 : \frac{\varphi_1(z_1) + \varphi_2(z_2)}{\varphi_1(z_3) + \varphi_2(z_4)} = h_1\}.$$

Parallel transport in terms of the connection 2-form. Let τ be a connection 2-form on a symplectic fibration $\pi : M \to B$. We give an explicit description of the parallel transport in terms of τ. Because the horizontal lift is a local concept, it suffices to look to a trivial bundle $M = B \times F$. Where $B \subset \mathbb{R}^m$ is an open set. Denote with $x_1, ..., x_m$ the standard coordinates on $\mathbb{R}^m$. A connection 2-form $\tau \in \Omega^2(M)$ is of the form:

$$\tau = \sigma + \sum_i \alpha_i \wedge dx_i + \sum_{i<j} f_{ij} dx_i \wedge dx_j.$$

Where for all $x \in B$, $\sigma(x) \in \Omega^2(F), \alpha_i(x) \in \Omega^1(B)$, and $f_{ij} \in C^\infty(F)$. Abbreviate $\frac{\partial}{\partial x_i} = \partial_i$. The exterior derivative of τ is:

$$\begin{aligned}
d^{B\times F}\tau &= d\sigma + \sum_i \partial_i\sigma \wedge dx_i \\
&+ \sum_i d\alpha_i \wedge dx_i + \sum_{i,j} \partial_j\alpha_i \wedge dx_j \wedge dx_i \\
&+ \sum_{i<j} df_{ij} dx_i \wedge dx_j + \sum_{i<j<k} (\partial_k f_{ij} + \partial_j f_{ki} + \partial_i f_{ik}) dx_i \wedge dx_j \wedge dx_k.
\end{aligned}$$

So, τ is vertically closed if and only if

$$d\sigma(x) = 0, \qquad \text{and } \; \partial_i\sigma(x) + d\alpha_i(x) = 0.$$

The horizontal lift of a tangent vector $\xi \in T_bB$, at a point $(b,q) \in B \times F$ is a vector $\xi^\# = (\xi, X) \in \mathbb{R}^m \times T_qF$, such that for any vector $(0,Y) \in \mathbb{R}^m \times T_qF$,

$$\tau((\xi, X), (0, Y)) = 0.$$

Insert this in the expression for τ. We find that this implies:

$$\iota(X)\sigma_b = \sum_{i=1}^m \xi_i\alpha_i(b).$$

Hence, the holonomy along a path γ in B is given by the diffeomorphisms $\Phi_t : F \to F$ defined by

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t, \qquad \text{and } \iota(X_t)\sigma_t = \sum_{i=1}^m \alpha_i(t)(x(t))\dot{x}_i(t).$$

Construction of a symplectic fibre bundle with a closed connection 2-form A connection 2-form doesn't have to be closed. However in the following special case we can always find a closed connection 2-form.

Theorem 4.2.8 (Weinstein). *Let $G \to \mathrm{Symp}(F, \sigma)$ be a Hamiltonian action. Then every connection on a principal G-bundle $P \to B$ gives rise to a closed connection 2-form on $P \times_G F$.*

Proof. First we consider the case that F is T^*G. We have seen that G acts on T^*G in a Hamiltonian fashion. Since the fibres of a principal G-bundle are diffeomorphic with G, $P \times_G T^*G$ can be identified with the vertical cotangent bundle of P. A connection 1-form $A \in \Omega^2(P, \mathfrak{g})$ determines a projection:

$$TP \to VP$$
$$(p, v) \mapsto (p, \varphi_p A(v)).$$

Which, on its turn, determines an injection:

$$\iota_A : V^*P \to T^*P.$$

A cotangent bundle has a canonical symplectic form, and so does T^*P. By pulling it back along the injection above we obtain a 2-form on V^*P:

$$\omega_A = \iota_A^* \omega_{can}.$$

The equivariant behaviour of the connecting 1-form induces that ι_A is equivariant, i.e.

$$\iota_A(g \cdot (p, v^*)) = g\iota(p, v^*) \text{ for all } (p, v^*) \in T^*P \text{ and } g \in G.$$

Therefore, we get invariance of ω_A under the action of G. ω_A restricts to the canonical symplectic forms on the fibres on V^*P. Next we take (F, σ) to be a symplectic manifold with Hamiltonian G-action and moment map $\mu_F :\to \mathfrak{g}^*$. The form

$$\tau_A = \omega_A \oplus \sigma \text{ on } W = V^*P \times F$$

is closed, since both ω_A and σ are. G acts on W in a Hamiltonian fashion with moment map:

$$\begin{aligned} \mu_W &= \mu_P \circ \iota_A \oplus \mu_F : W \to \mathfrak{g}^* \\ ((p, v^*), f) &= \mu_P \circ \iota_A(p, v^*) + \mu_F(f). \end{aligned}$$

To come to our result we observe that μ_P is a submersion, therefore 0 is a regular value of μ_W. Furthermore we have that

$$\mu_W^{-1}(0) = \{((p, v^*), f) : \mu_P \circ \iota_A(p, v^*) = -\mu_F(f)\}.$$

Recall that for every $p \in P$, $v \mapsto \omega_p(v)$, is an isomorphism between V_pP and $\mathfrak{g}$. Therefore, for every $f \in F$ and every $p \in P$ there is an unique $v^* \in V_p^*P$ such that $((p, v^*), f) \in \mu_W^{-1}(0)$. We find a natural identification

$$M = \mu_W^{-1}(0)/G \cong P \times_G F.$$

Since τ_A is invariant under the action of G it is projectable to a form on M which we call again τ_A. Symplectic reduction on the fibres shows that this form restricts to the canonical symplectic forms on the fibres. □

Connections on the symplectic frame bundle Recall from section 3.7, that an Ehresmann connection on a fibre bundle induces a connection on the frame bundle. Similarly, a symplectic connection Γ on a symplectic fibre bundle $\pi : M \to B$ induces a connection on the symplectic frame bundle $\mathcal{P}$. Note that the symplectic frame bundle of $\pi : M \to B$ is a subbundle of the frame bundle, and hence, the tangent space $T_f\mathcal{P}$ at a symplectomorphism $f : F \to F_b$ is a subspace of:

$$\{X : F \to TM : T_{f(x)}\pi \cdot X(x) \in T_{p(f)}B \text{ is constant}\}.$$

Let τ be the connection 2-form on M inducing Γ. Let f_t be a path in $\mathcal{P}$, with $f_0 = f$. f_t is a symplectomorphism for all t, hence:

$$f_t^*(\iota_{b_t}^*\tau) = (\iota_{b_t} \circ f_t)^*\tau = \sigma.$$

Therefore, in terms of the Lie derivative, for $\xi \in T_f\mathcal{P}$ has to satisfy:

$$L_\xi \tau \overset{fibre}{=} 0,$$

where $\overset{fibre}{=}$ means that the restriction to the fibres is on both side the same. By the formula of Cartan, we get

$$0 = L_\xi\tau \overset{fibre}{=} d\iota(\xi)\tau + \iota(\xi)d\tau \overset{fibre}{=} d\iota(\xi)\tau.$$

Note that a general vertical vector in $T_{f(x)}M$ can be expressed as $df(x)w$, where $w \in T_xF$. Thus the condition for f_t to be symplectic translates to the condition that the 1-form on F

$$T_xF \to \mathbb{R} : w \to \tau(\xi(x), df(x)w),$$

is closed. Hence,

$$\begin{aligned} T_f\mathcal{P} = \{X : F \to TM \quad | \quad & T_{f(x))}\pi \cdot X(x) = \text{constant}, \\ & \text{and the 1-form } T_xF \to \mathbb{R} : w \to \tau(\xi(x), df(x)w) \text{is closed}\}. \end{aligned}$$

Now we find the vertical and thehorizontal spaces:

$$\begin{aligned} \mathcal{V}_f &= \{\xi \in T_f\mathcal{P} : \xi = df \circ X \text{ for } X \in \mathfrak{X}(F, \sigma)\}, \\ \mathcal{H}_f &= \{\xi \in T_f\mathcal{P} : \xi = v^\# \circ f, v \in T_{p(f)}B\}. \end{aligned}$$

4.3 Symplectic gauge theory and the curvature form

Let $\pi : M \to B$ be a symplectic fibration, and let τ be a connection 2-form. A projectable diffeomorphism $\phi : M \to M$, which preserves the symplectic structure of the fibre, i.e.

$$\phi^*\tau \overset{fibre}{=} \tau,$$

is called a fibrewise symplectomorphism of M. The Lie algebra of the group of fibrewise symplectomorphisms are the fibrewise symplectic vector fields. Those are vector fields for which

$$d(\iota(X)\tau) \overset{fibre}{=} 0.$$

Since on vertical vectors the expression $\iota(X)d\tau$ vanishes, an equivalent condition is that

$$L_X\tau \overset{fibre}{=} 0.$$

Which looks like the original condition for a vector field to be symplectic. A special kind of fibrewise symplectomorphisms of M, are those who descend to the identity on B. i.e.

$$\pi \circ \phi = \pi.$$

These are called symplectic gauge transformations, and they form a subgroup $\text{Diff}(M, \tau)$ of the fibrewise symplectomorphisms. From the following commuting diagram it follows directly that it is in fact a normal subgroup.

$$\begin{array}{ccccccc}
M & \xrightarrow{\phi} & M & \xrightarrow{\psi} & M & \xrightarrow{\phi^{-1}} & M \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
B & \xrightarrow{f} & B & \xrightarrow{\text{id}} & B & \xrightarrow{f^{-1}} & B
\end{array}$$

The Lie algebra of $\text{Diff}(M, \sigma)$ must be a subalgebra of the space of fibrewise symplectic vector fields. The symplectic gauge transformations only interchange elements inside the fibres, so the Lie algebra of $\text{Diff}(M, \sigma)$ are the vertical fibrewise symplectic vector fields, which we denote with $\mathfrak{X}_{\text{Vert}}(M, \sigma)$. Given a $Y \in \mathfrak{X}(M, \sigma)$, and a fibrewise symplectomorphism ϕ, we find:

$$\exp(\text{Ad}(\phi)Y) = \phi^{-1} \exp(Y)\phi \in \text{Diff}(M, \tau).$$

Which induces that $\text{Ad}(\phi)Y \in \mathfrak{X}_{\text{Vert}}(M, \sigma)$. If we differentiate another time we get that for any fibrewise symplectic vector field X,

$$\text{ad}(X)(Y) = [X, Y] \in \mathfrak{X}^v(M, \sigma).$$

So in particular, for a horizontal lift $v^{\#}$, and a vertical fibrewise symplectic vector field Y, we have $[v^{\#}, X]$ is again a vertical fibrewise symplectic vector field.

Lemma 4.3.1. *Let Y be a vertical Hamiltonian vector field, with Hamiltonian function H in the sense that*

$$\iota(Y)\tau \overset{fibre}{=} dH.$$

Then, for every fibrewise symplectic vector field X, the Lie bracket $[X,Y]$ is a vertical Hamiltonian vector field, with Hamiltonian function $L_X H$.

Proof. Let $\phi : M \to M$ be a fibrewise symplectomorphism, and let Y be as above. Then for a vector field X we have,

$$\begin{aligned} \phi^{-1*}(\iota(\phi^* Y)\tau)(X) &\overset{fibre}{=} \tau(\phi_*^{-1} \circ Y \circ \phi, \phi_*^{-1} X) \\ &\overset{fibre}{=} \tau(Y \circ \phi, X) \overset{fibre}{=} d(H \circ \phi)(X) \end{aligned}$$

So $\phi^* Y$ is a vertical Hamiltonian vector field.

Now we compute $\iota([X,Y])\tau$, where Y is as above and X is a fibrewise symplectomorphism.

$$\begin{aligned} \iota([X,Y])\tau &\overset{fibre}{=} \iota(\lim_{h\to 0} \frac{Y - \phi_h^* Y}{h}) \\ &\overset{fibre}{=} \lim_{h\to 0} \iota(\frac{Y - \phi_h^* Y}{h}) \\ &\overset{fibre}{=} d(\lim_{h\to 0} \frac{H - H \circ \phi_h}{h}) \overset{fibre}{=} d(L_X H). \end{aligned}$$

□

Lemma 4.3.2. *Let τ be connection 2-form, then the curvature of the connection Γ_τ, satisfies*

$$\iota([v_1^{\#}, v_2^{\#}]^{vert})\tau \overset{fibre}{=} d\iota(v_2^{\#})\iota(v_1^{\#})\tau - \iota(v_2^{\#})\iota(v_1^{\#})d\tau.$$

Proof. Let Y be any vertical vector field, then we express:

$$\begin{aligned} d\tau(v_1^{\#}, v_2^{\#}, Y) = \quad & L_{v_1^{\#}}(\tau(v_2^{\#}, Y)) + L_{v_2^{\#}}(\tau(Y, v_1^{\#})) + L_Y(\tau(v_1^{\#}, v_2^{\#})) \\ & -\tau([v_1^{\#}, v_2^{\#}], Y) - \tau([Y, v_1^{\#}], v_2^{\#}) - \tau([v_2^{\#}, Y], v_1^{\#}). \end{aligned}$$

We have seen above that the bracket $[v^{\#}, Y]$ is vertical. By definition of the horizontal subspaces $\tau(v^{\#}, Y) = 0$. We observe that most terms vanish and we are left with:

$$d\tau(v_1^{\#}, v_2^{\#}, Y) = L_Y(\tau(v_1^{\#}, v_2^{\#})) - \tau([v_1^{\#}, v_2^{\#}], Y) = \iota(Y)d\iota(v_2^{\#})\iota(v_1^{\#}) - \tau([v_1^{\#}, v_2^{\#}]^{\mathrm{vert}}, Y).$$

□

4.4 The coupling form

We have seen that a connection Γ_τ is symplectic if the connection 2-form τ is vertically closed. τ can not always be taken to be closed.

Example 4.4.1. Above we encountered the example of a symplectic fibration over the torus. The total space M is $\mathbb{R}^4/\sim$, where $\sim$ is the equivalence relation

$$(s,t,x,y) \sim (s+i, t+j, x+it+k, y+l),$$

with $i,j,k,l \in \mathbb{Z}$. As connection 2-form we found

$$\tau = dx \wedge dy - sdt \wedge dy,$$

which is clearly not closed. In fact, M doesn't admit a closed connection 2-form. If it would, then M would be symplectic itself, but M is a nonorientable manifold. To see that, we restrict the tangent bundle $TM \to M$, to N, where $N \subset M$ is $\{[s+i, t+j, x+it+k, y+l] \in M : s, y \in \mathbb{R} \text{ fixed}\}$. N is isomorphic with the Möbius band, which is the most classical example of a nonorientable surface. Hence, M is nonorientable.

The next important theorem gives when a closed τ can be found.

Theorem 4.4.2. *Let $\pi : M \to B$ be a symplectic fibration and let Γ be a symplectic connection on M. Then the following are equivalent:*

1. *There exists a closed connection 2-form $\tau \in \Omega^2(M)$ such that $\Gamma_\tau = \Gamma$.*
2. *The holonomy of Γ around any contractible loop in B is Hamiltonian.*

Moreover, in this case, there exists a unique closed connection 2-form $\tau_\Gamma \in \Omega^2(M)$ which generates Γ and satisfies the normalization condition:

$$\int_F \tau_\Gamma^{k+1} = 0 \in \Omega^2(B),$$

where $\int_F$ denotes integration over the fibre. Any other closed connection 2-form τ which generates Γ is related to τ_Γ by:

$$\tau - \tau_\Gamma = \pi^*\beta$$

for some closed form $\beta \in \Omega^2(B)$ that is uniquely determined by τ.

A 2-form which satisfies the condition above is called the coupling-form.

Corollary 4.4.3. *If $\pi : M \to B$ is a symplectic fibration, such that the base is a compact symplectic manifold and the equivalent statements in the previous theorem are fulfilled, then M carries a symplectic structure.*

Proof. By assumption there is a compatible closed 2-form $\tau \in \Omega^2(M)$. Let β be the symplectic form on B. Then the theorem 4.1.3 of Thurston shows that $\omega_k = \tau + k\pi^*\beta$ is symplectic for k sufficiently large. □

Proof of theorem 4.4.2. First we prove, '1 implies 2'. The other direction will follow later. Let $D \subset \mathbb{C}$ be the unit disc, and let $u : D \to B$ be a smooth map. Any contractible loop in B can be seen as the boundary of a the image of D under such map. We define the pull back of the bundle u^*M, which has a total space

$$\{(m, z) \in M \times D : \pi(m) = u(z)\},$$

and the projection is the projection on the second coordinate. u lifts to a bundle map

$$\phi : D \times F \to M,$$

such that for each $z \in D$ the map $(F, \sigma) \to (F_{u(z)}, \sigma_{u(z)}) : q \mapsto \phi(z, q)$ is a symplectomorphism. Hence the pull-back of the connection 2-form is of the form:

$$\phi^*\tau = \sigma + \alpha \wedge dx + \beta \wedge dy + f dx \wedge dy,$$

where $z = x + iy$ is the coordinate on D, $\sigma(z) = \sigma$, $\alpha(z) = \beta(z) \in \Omega^1(F)$ and $f(z) \in C^\infty(F)$. The connection 2-form τ is closed, hence for all $z \in D$:

$$d\alpha = d\beta = 0, \text{ and } df = \partial_x\beta - \partial_y\alpha.$$

As in the paragraph about the relation between parallel transport and coupling form we see that the holonomy of ϕ^* around the $z(t) = e^{2\pi it} = x(t) + iy(t)$ is the path of symplectomorphisms $\Phi_t : F \to F$ given by

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t, \text{ and } \iota(X_t)\sigma = \alpha_t,$$

$$\text{where } \alpha_t = \alpha(z(t))\dot{x}(t) + \beta(z(t))\dot{y}(t).$$

The formula $df = \partial_x\beta - \partial_y\alpha$ leads to the relation

$$\int_0^1 \alpha_t dt = d\int_D f(x, y) dx dy,$$

which implies that the flux of the path $\{\Phi_t\}_{0\leq t\leq 1}$ is zero, and hence that $\Phi_1 : F \to F$ is a Hamiltonian symplectomorphism.

□

The natural way to find the relationship between the holonomy and the connection 2-form is the theorem, 3.6.12, of Ambrose and Springer. To do that we need a principal bundle; the symplectic frame bundle. We restate the holonomy theorem of Ambrose and Springer for this particular case.

Theorem 4.4.4. *Given a symplectic fibre bundle, $\pi : M \to B$,whose base manifold is B is connected and paracompact. Then the symplectic frame bundle, $\mathcal{P} \to B$, is a principal* $\mathrm{Symp}(F,\sigma)$*-bundle. Let τ, be a connection 2-form on M, and $\Gamma_{\mathcal{P},\tau}$ be the induced connection on $\mathcal{P}$. Let $\mathcal{P}(f)$ be the holonomy group with reference point $f \in \mathcal{P}$. Let $\mathcal{P}(f)$ be the holonomy bundle through f of $\Gamma_{\mathcal{P},\tau}$. Denote with $\mathfrak{g}' \subset \mathfrak{X}(F,\sigma)$, the subspace generated by elements of the form $\Omega_\Gamma(v,w)$, where w, v are tangent vectors to $\mathcal{P}$ at the points in $\mathcal{P}(f)$. Then the subgroup of* $\mathrm{Symp}(F,\sigma)$*, generated by $\mathfrak{g}'$, is $\mathcal{P}^0(f)$; the holonomy group, generated by contractible paths in B.*

Proof. Has to be done still. □

Corollary 4.4.5. *The symplectomorphisms from $F_b \to F_b$ obtained by parallel transport along contractible loops in B are all Hamiltonian if and only if the curvature form takes values in the space of Hamiltonian vector fields.*

Proof. We denote the parallel transport along a contractible curve γ in B, in $\mathcal{P}$ with Φ_γ and in M with ϕ_γ, an the fibre above b in $\mathcal{P}$ with $\mathcal{P}_b$. $\mathcal{P}$ is a principal $\mathrm{Symp}(F,\sigma)$ bundle, so for a $f \in \mathcal{P}_b$, we have a $\psi \in \mathrm{Symp}(F,\sigma)$ such that

$$\Phi_\gamma(f) = f \circ \psi$$

Note that if $f \in \mathcal{P}_b$ then is $\phi_\gamma \circ f \in \mathcal{P}_b$ as well. The horizontal vectors in $T\mathcal{P}$ are of the form $v^\# \circ f$, and the solution curve of $v^\#$ gives the parallel transport. So we find

$\Phi_\gamma(f) = \phi_\gamma \circ f$.

If we combine this results we get

$$f \circ \psi = \phi_\gamma \circ f$$

Now if the curvature form ends in the space of Hamiltonian vector fields if and only if the holonomy around the contractible loops in B is Hamiltonian, in the notation here this means that ψ is Hamiltonian. We observed earlier that $\mathrm{Ham}(F,\sigma)$ is a normal subgroup of $\mathrm{Symp}(F,\sigma)$. Hence we get that ψ is Hamiltonian if and only if ϕ_γ is Hamiltonian. □

Continuation of the proof of 4.4.2. Let Γ be a symplectic connection on $\pi : M \to B$, such that the holonomy around every contractible loop in b is Hamiltonian. We search for a closed connection 2-form τ, such that $\Gamma = \Gamma_\tau$. By the compatibility of τ, we know the value of $\tau(X,Y)$ on vertical vector spaces. Moreover, we know that $\tau(X, v^\#) = 0$, where X is vertical and $v \in \mathfrak{X}(B)$. So we are left with defining τ on horizontal vector fields, for which we use the condition on the holonomy. The curvature form takes values in the space of Hamiltonian vector fields on F. So, for two vector fields $v_1, v_2 \in \mathfrak{X}(B)$, there is a function $H_{v_1,v_2} : M \to \mathbb{R}$ such that

$$\iota([v_1^\#, v_2^\#]\,\mathrm{Vert})\sigma_b = H_{v_1,v_2}|_{F_b},$$

which is unique up to a constant. Therefore, we normalise it such that

$$\int_{F_b} H_{v_1,v_2}\sigma_b^k = 0.$$

Since for all $x \in M$, with $\pi(x) = b$, the form $T_bB \times T_bB \to \mathbb{R} : (v_1, v_2) \mapsto H_{v_1,v_2}$ is bilinear and skew-symmetric, there is a unique connection 2-form, τ_Γ, which generates Γ and which satisfies $\tau_\Gamma(v_1^\#, v_2^\#)$. We will prove that this form is closed. By the compatibility of τ_Γ, $d\tau_\Gamma$ vanishes on all triples of vertical vectors, and, since Y is vertically closed, $d\tau_\Gamma$ vanishes already when two vertical vectors are involved. We have a closer look to the curvature identity:

$$\iota([v_1^\#, v_2^\#]^{\mathrm{vert}})\tau_\Gamma \overset{fibre}{=} d\iota(v_2^\#)\iota(v_1^\#)\tau_\Gamma - \iota(v_2^\#)\iota(v_1^\#)d\tau_\Gamma,$$

and insert the definition of τ_Γ, and get:

$$dH_{v_1,v_2} \overset{fibre}{=} dH_{v_1,v_2} - \iota(v_2^\#)\iota(v_1^\#)d\tau_\Gamma.$$

So $d\tau_\Gamma(v_1, v_2, X) = 0$ for vertical vector fields X. It remains to prove that $d\tau_\Gamma$ vanishes on triples of horizontal vector fields. As we saw before we have:

$$\begin{aligned} d\tau_\Gamma(v_1^\#, v_2^\#, v_3^\#) = \quad & L_{v_1^\#}(\tau_\Gamma(v_2^\#, v_3^\#)) + L_{v_2^\#}(\tau_\Gamma(v_3^\#, v_1^\#)) + L_{v_3^\#}(\tau_\Gamma(v_1^\#, v_2^\#)) \\ & -\tau_\Gamma([v_1^\#, v_2^\#], v_3^\#) - \tau_\Gamma([v_3^\#, v_1^\#], v_2^\#) - \tau_\Gamma([v_2^\#, v_3^\#], v_1^\#). \end{aligned}$$

We write this as the sum of three fibrewise Hamiltonian functions:

$$\begin{aligned} H_1 &= L_{v_3^\#}(\tau_\Gamma(v_1^\#, v_2^\#)) - \tau_\Gamma([v_1^\#, v_2^\#], v_3^\#) \\ H_2 &= L_{v_1^\#}(\tau_\Gamma(v_2^\#, v_3^\#)) - \tau_\Gamma([v_2^\#, v_3^\#], v_1^\#) \\ H_3 &= L_{v_2^\#}(\tau_\Gamma(v_3^\#, v_1^\#)) - \tau_\Gamma([v_3^\#, v_1^\#], v_2^\#). \end{aligned}$$

With the help of lemma 4.3.1 we see that Hamiltonian vector field of H_1 is

$$Y_1 = \Omega_\Gamma([v_1, v_2], v_3) - [v_3^\#, \Omega(v_1, v_2)].$$

We write Y_1 in another way:

$$
\begin{aligned}
Y_1 &= \Omega_\Gamma([v_1, v_2], v_3) - [v_3^\#, \Omega(v_1, v_2)] \\
&= [[v_1, v_2]^\#, v_3^\#]^{\text{vert}} + [\Omega(v_1, v_2), v_3^\#] \\
&= [[v_1^\#, v_2^\#], v_3^\#]^{\text{vert}}.
\end{aligned}
$$

Where we have used that $[v_1, v_2]^\# + \Omega(v_1, v_2) = [v_1^\#, v_2^\#]$, and that $[\Omega(v_1, v_2), v_3^\#]$ is vertical. The other Hamiltonian vector fields are found by cyclic permuting the indices in the last expression for Y_1. So, by the Jacobi-identity, the Hamiltonian vector field of $d\tau_\Gamma(v_1^\#, v_2^\#, v_3^\#)$ is 0, and hence $d\tau_\Gamma(v_1^\#, v_2^\#, v_3^\#)$ is zero itself. Therefore, τ_Γ is closed. Thus '2 implies 1' is proven.

We prove that this τ_Γ satisfies $\int_{F_b} \tau_\Gamma^{k+1} = 0$, which is equivalent to $\int_{F_b} \iota(v_2^\#)\iota(v_1^\#)\tau_\Gamma^{k+1} = 0$ for all $b \in B$ and $v_1, v_2 \in T_bB$. This, in its turn is equivalent to

$$
\int_{F_b} \tau_\Gamma(v_1^\#, v_2^\#)\tau_\Gamma^k = 0.
$$

This follows from the beginning of the proof where we took $\int_{F_b} H_{v_1,v_2}\sigma_b^k = 0$.

Next, we have to prove that if τ is another closed compatible connection 2-form which generates Γ, then there is a closed $\beta \in \Omega^2(B)$, such that $\tau - \tau_\Gamma = \pi^*\beta$. First we concentrate on a U_α on which the fibration is locally trivial. Let $(x_1, ...x_n)$ be the coordinates on U_α. We write

$$
\tau - \tau_\Gamma = \omega + \sum_i \alpha_i \wedge dx_i + \sum_{i<j} f_{ij} dx_i \wedge dx_j,
$$

where $\omega \in \Omega^2(F), \alpha_i(x) \in \Omega^1(F)$, and $f_{ij}(x) \in C^\infty(F)$. The vertical subbundle is contained in the kernel of $\tau - \tau_\Gamma$, because τ and τ_Γ generate the same connection. So, $\omega(x) = 0$ and $\alpha_i(x) = 0$. Since τ and τ_Γ are closed we get $df_{ij} = 0$, hence the f_{ij} are constant in the fibres. Therefore, there is a $\beta \in \Omega^2(U_\alpha)$ such that $\tau - \tau_\Gamma = \pi^*\beta$. β is uniquely determined by τ, so there exist a global $\beta \in \Omega^2(B)$. β is closed, because $\tau - \tau_\Gamma$ is closed.

To finish we prove that this τ_Γ is unique. Suppose that there is a compatible closed connection 2-form which generates Γ, such that

$$
\int_F \tau^{k=1} = 0.
$$

Then the curvature identity gives

$$
\iota([v_1^\#, v_2^\#]^{\text{vert}})\tau \overset{fibre}{=} d(\tau(v_1^\#, v_2^\#)).
$$

for $v_1, v_2 \in \mathfrak{X}(B)$. So $[v_1^\#, v_2^\#]^{\text{vert}}$ is Hamiltonian on each fibre, with Hamiltonian function $\tau(v_1^\#, v_2^\#)$. The normalisation condition above gives that the mean value of $\tau(v_1^\#, v_2^\#)$ is zero.

Hence $\tau(v_1^\#, v_2^\#) = H_{v_1,v_2}$, and by the observation made in the beginning of the prove we may conclude that $\tau = \tau_\Gamma$. $\square$

Example 4.4.6. We have seen already that for any principal G-bundle we get a symplectic fibration, if we take the associated bundle with a orbit of the coadjoint action on $\mathfrak{g}^*$, $E = P \times_G \mathcal{O}_\xi$. Now let ω be a connection 1-form on P, let Γ and Ω be the corresponding connection respectively the curvature 2-form. Such a connection induces a connection on the associated bundle. Recall that Ω is $\mathfrak{g}$ valued 2-form, so for a $\mu \in \mathfrak{g}^*$, we get that $\mu \circ \Omega \in \Omega^2(P, \mathfrak{g})$. Now we define a 2-form on E.

$$\Omega_{E[p,\mu]}(X, Y) = \mu \circ \Omega_p(hX, hY) + \langle \mu, [vX, hY] \rangle .$$

Chapter 5

Poisson fibre bundles

Abstract In this chapter we discuss fibrations with a Poisson manifold (F, π) as fibre, and the structure group to be a subgroup of the Poisson diffeomorphisms of F. Different as in the previous chapter, does the Poisson structure of the fibre immediately imply a Poisson structure on the total space. This Poisson bivector will turn out to be vertical and is denoted with π_V. In this chapter we will find as an analogy to the coupling form the coupling Dirac structure. Also will we discuss the effect of Poisson gauging on the total space.

5.1 Poisson fibre bundles

Definition 5.1.1. A Poisson fibration is a fibre bundle $p : P \to M$, with as fibre a Poisson manifold (F, π), and as structure group of the fibration a subgroup of the group of Poisson diffeomorphisms, $\mathrm{Diff}(F, \pi)$.

All fibres of a Poisson fibration carry a natural Poisson structure

$$\pi_b = (\phi_i(b)^{-1})_* \pi$$

Remark 5.1.2. All symplectic fibre bundles are Poisson fibre bundles. The Poisson bivector π, is related to the symplectic 2-form ω, by $\pi^{\#} = -(\omega_{\#})^{-1}$.

For symplectic fibre bundles we found some conditions for which the total space itself would be a symplectic manifold. For a Poisson fibre bundles that is not necessary anymore: we find immediately a Poisson structure on the total space. In fact we can just glue the Poisson structure π_b on the fibres defined above together to

$$\pi_V(x) = \pi_{p(x)}(x), \text{ for all } x \in M.$$

This is a vertical 2-vector field in the sense that it takes values in $\bigwedge^2 \mathrm{Vert}$.

As we have seen before a fibre nondegenerate almost Dirac structure L, give rise to a connection, a horizontal 2-form ω_L and a vertical bivector π_L, such that

$$L = \mathrm{graph}(\pi_L) \oplus \mathrm{graph}(\omega_L)$$

Definition 5.1.3. A fibre nondegenerate Dirac structure L on a Poisson fibration $p : P \to M$, is called compatible with the fibration if $\pi_L = \pi_V$. In this case we call L a coupling Dirac structure.

In general we can say the following about the existence of a Poisson structure on a fibration.

Proposition 5.1.4. *Let $p : M \to B$ be a fibration with a connected base manifold and compact fibres. If L is a fibre nondegenerate Dirac structure, then $p : M \to B$ admits the structure of a Poisson fibration, such that $\pi_L = \pi_V$.*

Proof. If L is a fibre nondegenerate Dirac structure then π_L is a Poisson bivector. L defines a connection and the parallel transport along the along paths define to a chosen fibre F_b, gives the local trivialisation. The compactness of the fibre ensures the completeness of the horizontal lifts. □

5.2 Poisson connections

Definition 5.2.1. A connection Γ on a Poisson fibration $p : P \to M$ is called a Poisson connection, if for all loops γ in B, the map defined by parallel transport

$$F_\gamma : F_b \to F_b$$

is a Poisson diffeomorphism.

Example 5.2.2. We look to the symplectic fibration $M \to \mathbb{T}^2$, with fibre $\mathbb{T}^2$, where The total space M, is the quotient space $\mathbb{R}^4/\sim$, where $\sim$ is the equivalence relation

$$(s, t, x, y) \sim (s + i, t + j, x + it + k, y + l),$$

with $i, j, k, l \in \mathbb{Z}$. The projection is given by $\pi : M \to \mathbb{T}^2 : [s, t, x, y] \mapsto [s, t]$. The symplectic form on the fibre is $\omega = dx \wedge dy$. A symplectic, and hence a Poisson connection is given in terms of the connection 2-form $\tau = dy \wedge dy - sdt \wedge dy$. The graph of τ is the almost Dirac structure L. Note that Hor° is given by $\iota(X)\tau$, where X is vertical. Take X, Y vertical, then

$$\pi_L(\iota(X)\tau, \iota(Y)\tau) = \iota(X)\tau(Y) = \tau(Y, X) = -\omega(X, Y).$$

The horizontal 2-form $\omega_L(X, Y) = \tau(X, Y)$, where X, Y are horizontal. We compute ω_L explicitelly, where we use the expression for the horizontal vectors we found before,

$$\tau_m((a, b, sb, 0), (a', b', sb', 0)) = 0.$$

The horizontal 2-form vanishes. Because $L = \mathrm{graph}(\omega_L) \oplus \mathrm{graph}(\pi_L) = \mathrm{graph}(\tau)$.

A connection is a Poisson connection if and only if the flow of every horizontal vector field preserves the vertical Poisson structure, i.e.

$$L_{v^\#}\pi_V = 0, \text{ for all } v \in \mathfrak{X}(B).$$

Poisson connections do exists as follows from the following proposition.

Proposition 5.2.3. *Let $p : P \to M$ be a Poisson fibration, with fibre (F, π), then there exist a fibre nondegenerate almost Dirac structure L on $p : P \to M$ such that:*

1. *The vertical Poisson structure π_L coincides with π_V*
2. *The connection Γ_L is a Poisson connection.*

Proof. We prove this proposition with the help of the local trivial structure of the fibration. Let $\{U_i, \phi_i\}$ be a trivialisation of the fibre bundle. L_i is the pull-back Dirac structure of the graph of π.

$$L_i = \{((v, w), (0, \eta)) \in T(U_i \times F) \oplus T^*(U_i \times F) : w = \pi^\#(\eta)\}.$$

The vertical vectors in $T(U_i \times F)$ are vectors of the form $(0, w)$, $\mathrm{Vert}^\circ = \{(\xi, 0) \in T^*(U_i \times F)\}$. So elements of $(\mathrm{Vert} \oplus \mathrm{Vert}^\circ) \cap L_i$ are of the form $((0, w), (0, 0))$, where $w = \pi^\#(0) = 0$. So L_i is fibre nondegenerate. Horizontal vectors are of the form $(v, 0)$, and the horizontal 2-form $\omega_{L_i} = 0$. The vertical bivector $\pi_L = \pi$, and horizontal flow doesn't effect this structure. So Γ_{L_i} is a Poisson connection. We glue this Dirac structures together with the help of a partition of unity ρ_i subordinate to the open cover $\{U_i\}$ of M. Define

$$L = \sum_i (\rho_i \circ p)\phi_i^* L_i.$$

The associated data to this new Dirac structure are also obtained by gluing the old data together:

- $\mathrm{Hor}_L = \sum_i (\rho_i \circ p)\phi_i^* \,\mathrm{Hor}_i$
- $\omega_L = \sum_i (\rho_i \circ p)\phi_i^* \omega_{L_i} = 0$
- $\pi_L = \sum_i (\rho_i \circ p)\phi_i^* \pi_{L_i}$

L is fibre nondegenerate and π_L coincides with π_V. All we have to check is that Γ_L is a Poisson connection. To do this we use the following property of the Schouten-Neijenhuis bracket:

$$[fX, Y] = f[X, Y] + (L_X f)Y, \quad X \in \mathfrak{X}(P), Y \in \mathfrak{X}^2(P), f \in C^\infty(P).$$

The horizontal lift of a vector field $v \in \mathfrak{X}(M)$, is $\sum_i (\rho_i \circ p) v_i^{\#}$, where $v_i^{\#}$ is the horizontal lift with respect to Γ_{L_i}. Note that by the reasoning above $L_{v_i^{\#}} \pi_V = 0$ We compute:

$$\begin{aligned}
L_{v^{\#}} \pi_V &= [v^{\#}, \pi_V] \\
&= \sum_i [(\rho_i \circ p) v_i^{\#}, \pi_V] \\
&= \sum_i (\rho_i \circ p)[v_i^{\#}, \pi_V] + (L_{v_i^{\#}}(\rho_i \circ p))\pi_V \\
&= \sum_i (\rho_i \circ p) L_{v_i^{\#}} \pi_V = 0.
\end{aligned}$$

Where we used that $L_{v_i^{\#}}(\rho_i \circ p) = 0$. □

Construction of a Poisson fibration equipped with a coupling Dirac structure There is a way to construct a Poisson fibration with fibre (F, π) if there is a Hamiltonian group action of a Lie-group G on F, and a principal G-bundle. This construction is analogue to theorem 4.2.8.

Theorem 5.2.4. *Let $G \times F \to F$ be a Hamiltonian group action of a compact Lie group G on a Poisson manifold (F, π), with momentum map $\mu_F : F \to \mathfrak{g}^*$. Then every connection on a principal G bundle $P(M, G)$ determines a coupling Dirac structure L on the associated Poisson fibration $P \times_G F \to M$.*

Proof. Note that the fibres of a principal G bundle are isomorphic to G, therefore there is a canonical isomorphism between the vertical bundle and the and $P \times \mathfrak{g}$. Let ω be a connection 1-form on P, this induces a projection:

$$p_\omega : TP \to P \times \mathfrak{g}, (p, X) \mapsto (p, \omega_p(X)),$$

and hence an injection

$$i_\omega : P \times \mathfrak{g}^* \to T^*P,$$

such that $i_\omega(p, \xi^*)$ sends $X \in T_pP$ to $\xi^*(\omega_p(X)) \in \mathbb{R}$. The fibres of the fibre bundle $P \times \mathfrak{g}^*$ are isomorphic with $G \times \mathfrak{g}^*$, which in its turn are isomorphic with T^*G. The cotangent bundle of P has a canonical symplectic structure, ω_{can}. We obtain a closed form on $P \times \mathfrak{g}^*$ if we pull-back ω_{can}:

$$\sigma_\omega = i_\omega^* \omega_{can}.$$

σ_ω coincides on the fibres with σ_G the canonical symplectic structure on T^*G. The graph(σ_ω) is a Dirac structure and G acts Hamiltonian on $P \times \mathfrak{g}^*$:

$$G \times P \times \mathfrak{g}^* \to P \times \mathfrak{g}^* : (g, (p, \xi)) \mapsto (pg^{-1}, \mathrm{Ad}_g^* \xi).$$

The moment map $\mu = \mu_P \circ i_\omega$. Now we define a new space

$$W = P \times \mathfrak{g}^* \times F,$$

on which we have a Dirac structure

$$L = \mathrm{graph}(\sigma_\omega) \oplus \mathrm{graph}(\pi).$$

G acts Hamiltonian on W, with moment map

$$\mu_W = \mu \oplus \mu_F$$

For every $(p, f) \in P \times F$, there is a unique $\xi \in \mathfrak{g}^*$, such that $((p, \xi), f) \in \mu^{-1}(0)$. Hence $\mu^{-1}(0) \simeq P \times F$, and this isomorphism leads to a imbedding:

$$m : P \to P \times \mathfrak{g}^*.$$

$m^*\sigma_\omega$ is a closed 2-form on P. The Dirac structure on obtained on $\mu_W^{-1}(0)$, because 0 is a regular value of μ_W, is described by

$$\mathrm{graph}(m^*\sigma_\omega) \oplus \mathrm{graph}(\pi)$$

G acts freely on P and therefore also on $\mu_W^{-1}(0)$, so with the help of Dirac reduction we obtain a Dirac structure L_{red} on the reduced space $\mu_W^{-1}(0)/G \simeq P \times_G F$. Denote the projection $P \times F \to P \times_G F$, with pr. An explicit description of L_{red} is given by:

$$\begin{aligned} L_{red} = \{(d\,\mathrm{pr}(X, \pi^\#(\xi)), \alpha) \in T(P \times_G F) \oplus T^*(P \times_G F); \\ X \in TP, \xi \in T^*F \text{ and } d\,\mathrm{pr}^*\,\alpha = (\iota(X)(m^*\sigma_\omega), \xi)\} \end{aligned}$$

L_{red} is fibre nondegenerate. Hence L is the desired Dirac structure. □

The Poisson frame bundle Let $p : M \to B$ be a Poisson fibration with (F, π). Like before we construct a principal G-bundle over B, such that the associated bundle with F, is original fibration.

Definition 5.2.5. The Poisson frame bundle, $p : \mathcal{P} \to B$ of the Poisson fibration $p : M \to B$ is a principal G-bundle, where G is a subgroup of $\mathrm{Diff}(F, \pi)$, the group of Poisson diffeomorphisms. The fibre of $p : \mathcal{P} \to B$ over $b \in B$ is

$$\mathcal{P}_b = \{u : (F, \pi) \to (F_b, \pi_V) : u \text{ is a Poisson diffeomorphism}\}.$$

G acts on $\mathcal{P}$ from the right by precomposition:

$$\mathcal{P} \times G \to \mathcal{P} : (u, g) \mapsto u \circ g.$$

The tangent space of $\mathcal{P}$ at u is given by

$$T_u\mathcal{P} = \{X : F \to TM : T_{u(x)}p \cdot X(x) = \text{constant}, L_X\pi_V = 0\}.$$

The vertical spaces are given by

$$\mathcal{V}_u = \{du \circ X : X \in \mathfrak{X}(F, \pi)\}$$

Now let Γ be a connection on the original fibration. It induces a connection on $p : \mathcal{P} \to B$ in the following way.

$$\mathcal{H}_u = \{v^{\#} \circ u : v \in \mathfrak{X}(B)\}.$$

Moreover its induced connection on $\mathcal{P} \times_G F$ is the original connection Γ.

The holonomy theorem for a Poisson frame bundle

Theorem 5.2.6. *Let $p : M \to B$ be a Poisson fibre bundle with a paracompact base manifold B and fibre (F, π). Denote its frame bundle with $p : \mathcal{P} \to B$. Let $\Gamma_{\mathcal{P}}$ be the connection on $\mathcal{P}$ induced by a connection Γ on M. Denote with F and Ω their respective curvatures. Let $u \in \mathcal{P}$ and let $\mathcal{P}(u)$ be the holonomy bundle through u. Let $\Phi(u)$ be the holonomy group of u with respect to $\Gamma_{\mathcal{P}}$. Then the Lie algebra of $\Phi(u)$ is given by:*

$$\{F_v(X, Y) : v \in \mathcal{P}(u),\ X, Y \in \mathcal{H}_v\}$$

Coupling Dirac structures In the chapter about symplectic fibre bundles we have seen a beautiful theorem about the coupling form. It stated that a closed connection 2-form exists if and only if the holonomy around all contractible loops is Hamiltonian. Here we find a similar result. In fact the theorem about the symplectic fibre bundles turns out to be a particular case of the following theorem.

Theorem 5.2.7. *Let $p : P \to M$ be a Poisson fibration and let L be a fibre nondegenerate almost Dirac structure such that the induced connection Γ_L is Poisson. Then the following statements are equivalent:*

1. *L is a Dirac structure.*
2. *for every base point $b \in B$ the action of the holonomy group $\Phi(b)$ of Γ_L on the fibre F_b is Hamiltonian.*

We use the Poisson frame bundle to prove this theorem.

Proof. (2) implies (1). The outline of te prove is as follows. The connection Γ on $p : P \to M$ induces a connection $\Gamma_{\mathcal{P}}$ on the frame bundle. With the reduction theorem we may reduce $\mathcal{P}$ to a $\Phi(u)$ bundle , where $u \in \mathcal{P}$. We denote this new bundle again wtih $\mathcal{P}$. This connection induces a fibre nondegenerate Dirac structure on $\mathcal{P} \times_G F$, where G is the holomomy group on $\mathcal{P} \to B$, with respect to $\Gamma_{\mathcal{P}}$. Because $\mathcal{P} \times_G F \to B$ is isomorphic to the bundle $p : P \to M$, and the induced connection is the original connection, we find that L has to be Dirac.

Recall that the holonomy group $\Phi(b)$ on $p : P \to M$ with respect to Γ is isomorphic to the holonomy group $\Phi(u)$ in the frame bundle with respect to $\Gamma_{\mathcal{P}}$, where $u \in \mathcal{P}_b$.

□

5.3 Poisson gauging on fibrations

A Poisson gauge 2-form for the fibre (F, π) of a Poisson fibration gives rise to a Poisson gauge 2-form for (P, π_V). All Poisson gauge transformations of a fibre nondegenerate Poisson-Dirac structure give rise to a new fibre nondegenerate Poisson structure. In this section we discuss the Poisson gauging of a the total spaces and their relation to Poisson gauging of the fibre. We start with the last.

Lemma 5.3.1. *Let $p : P \to M$ be a Poisson fibre bundle of fibre type (F, π). Let $\{U_i\}_i$ be the open cover of M used for its trivialisation, and let $\{\rho_i\}_i$ be a partition of unity subordinate to it. For any Poisson gauge 2-form $B \in \Omega^2(F)$ for (F, π), is the 2-form*

$$\tilde{B} = \sum_i (\rho_i \circ p)\phi_i^*(B),$$

Poisson gauging for (P, π_V).

Proof. $\tilde{B}$ is clearly a closed form. Recall that a 2-form is Poisson gauging for (P, π_V) if and only if the map

$$(\mathrm{id}_{TP} - \pi_V^{\#} \circ \tilde{B}_{\#}) : TP \to TP$$

is an isomorphism. Equivalently, we proof for all $p \in P$ that the restriction of this map to T_pP is injective. Note that an element in the kernel has to be vertical. So a vector X is in the kernel if and only if $d\phi_i X$ is in the kernel of B for all suitable i. But this kernel is $\{0\}$ by assumption. Therefore $\tilde{B}$ is a Poisson gauging for (P, π_V). □

5.4 Future speculations

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