On Hamiltonian actions of symplectic groupoids

A classification of toric $\mathcal{T}_\Lambda$-spaces

Maarten Mol

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supervisor: prof. dr. M.N. Crainic

second reader: prof. dr. E.P. van den Ban

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Introduction

In 1982, both Atiyah [Ati82] and Guillemin and Sternberg [GS82] independently proved the celebrated convexity theorem. This describes the moment map image of a compact and connected Hamiltonian torus-space as the convex hull of a certain finite set of points. Two years later, Kirwan [Kir84] built upon their work and generalized it to Hamiltonian actions of more general Lie groups. Throughout the years, these theorems have been a great driving force for further research into moment maps, which has led to many sorts of generalizations and analogues. In 1988, Mikami and Weinstein [MW88] developed the more general framework of Hamiltonian actions by symplectic groupoids, which set the stage for possible extensions and Poisson geometric explanations of the convexity theorems. In a letter in 2000 [Wei00], Weinstein conjectured such extensions to hold for the actions of *proper symplectic groupoids*. Six years later, Zung [Zun06] showed that moment maps of such actions enjoy certain affinity properties and was hereby able to deduce many existing convexity theorems. In particular, Zung showed the existence of an *integral affine structure* on the orbit space of a proper symplectic groupoid. Crainic, Fernandes and Martínez Torres further researched and established the connection between proper symplectic groupoids and the integral affine structure on their orbit space in [CFT16].

It is not the convexity property of the moment map, but rather its interplay with this integral affine structure that is of main interest to us. More precisely, this thesis is concerned with the classification of certain classes of moment maps in terms of this integral affine structure. Such classifications already appear in the work of Duistermaat [Dui80] and later in the work of Delzant [Del88]. Our main result is a generalization of these two classification results.

Theorem (Main result)**.** *The toric $\mathcal{T}_\Lambda$-spaces over a fixed integral affine manifold (B, Λ) are classified (up to isomorphism) by pairs (Δ, c), consisting of a Delzant submanifold $\Delta \subset (B, \Lambda)$ and a class $c \in \check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$ in the first degree Čech cohomology of the sheaf of Lagrangian sections of $\mathcal{T}_\Lambda$ over Δ.*

Let us provide a bit more insight into this statement. An integral affine structure Λ on B is a smooth Lagrangian lattice in (T^*B, Ω_{can}). Given such an integral affine structure Λ on B, there is a canonical torus bundle

$$\mathcal{T}_\Lambda = T^*B/\Lambda$$

over B and the canonical symplectic form on T^*B descends to a symplectic form Ω_Λ on $\mathcal{T}_\Lambda$. This turns $\mathcal{T}_\Lambda$ into a type of proper symplectic groupoid, called a symplectic torus bundle. Conversely, every symplectic torus bundle over B induces an integral affine structure Λ on B and must be isomorphic to $\mathcal{T}_\Lambda$. The theorem is concerned with a certain class of Hamiltonian $\mathcal{T}_\Lambda$-actions, which we call *toric*. Every such toric action comes with a moment map $(S, \omega) \to B$, the image of which is what we call a *Delzant submanifold* of (B, Λ). Moreover, to every toric $\mathcal{T}_\Lambda$-space we can associate its *Lagrangian Chern class*

$$c \in \check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda)),$$

where Δ is the image of the moment map. Given a fixed Delzant submanifold $\Delta \subset (B, \Lambda)$, this association induces a bijection between the set of isomorphism classes of toric $\mathcal{T}_\Lambda$-spaces with moment image equal to Δ, and the set $\check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$.

Duistermaat studied a class of moment maps called *Lagrangian fibrations*. They correspond to the class of toric $\mathcal{T}_\Lambda$-spaces for which the moment map is a submersion and its image is all of B, as follows. Under certain conditions on the fibers, such a fibration $(S, \omega) \to B$ induces an integral affine structure Λ on B and comes with a free Hamiltonian $\mathcal{T}_\Lambda$-action, with the Lagrangian fibration as moment map. In this way, such Lagrangian fibrations can be interpreted as principal Hamiltonian $\mathcal{T}_\Lambda$-spaces, which are in particular toric. For this class, the moment map image Δ is always the entire base B. So according to our main result, the Lagrangian fibrations over B that induce a fixed integral affine structure Λ on B should be classified by their Lagrangian Chern class. This was shown by Duistermaat.

Delzant studied toric T-spaces, where T is a torus. This is a class of compact Hamiltonian T-spaces for which the moment image is not just a convex polytope, but a *Delzant polytope*. This means that the edges that meet in each vertex are spanned by a $\mathbb{Z}$-basis of the weight lattice Λ_T^* in $\mathfrak{t}^*$. Since the weight lattice is the integral affine structure on the base $\mathfrak{t}^*$, this could be interpreted as saying that the Delzant polytopes are those polytopes which are optimally adapted to the integral affine structure on $\mathfrak{t}^*$. The compact and connected Delzant submanifolds of $(\mathfrak{t}^*, \Lambda_T^*)$ coincide with the Delzant polytopes. In contrast to the Lagrangian fibrations, the moment image of a toric T-space will never be the entire base $\mathfrak{t}^*$ since it has to be compact. It is however always convex and therefore contractible. This leads to the vanishing of the cohomology in our theorem and so, according to our theorem, the toric T-spaces should be classified just by the Delzant polytopes in $(\mathfrak{t}^*, \Lambda_T^*)$. This is precisely Delzant's classification of toric T-spaces.

Our main theorem could therefore be seen as a unifying result, two extremes of which are Duistermaat's and Delzant's classification theorems.

There are two papers which contain results that are very similar to our own. It should be noted that we became aware of their existence only after proving this. First of all, our main result has a great resemblance with that of the paper [Bou89] by Boucetta. Although their approach is slightly different, it seems that the class of moment maps that are classified there is very similar to ours. We however have yet to establish the precise relationship. Our work shares even more similarity with that of Karshon and Lerman [KL15]. Although their main result is strikingly similar to ours, there are some clear differences. On one hand, they define and classify a more general notion of toric T-spaces than we do. On the other hand, we work in the more general setting of Hamiltonian torus bundle actions, whereas Karshon and Lerman only consider torus actions. Therefore, none of the two results implies the other and it would be interesting to see if their more general notion of toric T-space can be extended to the setting of Hamiltonian torus bundle actions. Although their paper was not known to us when discovering the main result, there were two previous papers of which Lerman was author and co-author ([Ler95], [LT97]) that did serve as a great source of inspiration.

Organisation

The organisation of this thesis is as follows.

- In Chapter 1 we give an overview of the relevant theory on symplectic groupoids and their Hamiltonian actions. In this chapter, many proofs are omitted.
- In Chapter 2 we discuss the relationship between symplectic torus bundles and integral affine structures.
- In Chapter 3 we prove a normal form for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-space. We then use this normal form to show that such moment maps are locally polyhedral.
- In Chapter 4 we have a brief intermezzo on stratifications and discuss the examples that we will come across in Chapter 5.
- In Chapter 5 we use the results of Chapter 3 to prove our main result: the classification of toric $\mathcal{T}_\Lambda$-spaces.
- In Chapter 6 we consider proper symplectic groupoids with smooth orbit spaces and extend our results from Chapter 4 to Hamiltonian actions of such groupoids. Moreover, we suggest a notion of toric $\mathcal{G}$-space and show that for this notion the moment image is a Delzant submanifold of the orbit space of the groupoid. We end the chapter with a short outlook on which other results of Chapter 5 we hope to generalize and which new features may play a role.
- In the Appendix we provide a proof of the Marle-Guillemin-Sternberg normal form for Hamiltonian G-spaces. This is used in Chapter 3 to obtain a normal form for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-space.

Chapter 1

Moment maps and symmetries

This chapter serves as an introduction to main objects of study: symplectic groupoids and their Hamiltonian actions. Moreover, it provides an overview of those of their properties that will be relevant for the rest of this thesis. Although we do not give proofs of the results in the first three sections, this overview should give the reader enough feeling for the objects involved to be able to understand the rest of the text. Since we will consider non-zero Poisson structures only in Chapter 6, the Poisson geometric part of this chapter mainly provides a background and is not essential for most of the thesis.

1.1 Moment maps in Poisson geometry

1.1.1 Basic definition and examples in Poisson geometry

There are various ways to characterize a Poisson structure. In this section we will give an overview of these. Proofs and further details can be found in [FM15] or [Vai94].

Poisson brackets and bi-vectors

Definition 1.1.1. A **Poisson structure** on a manifold M is Lie bracket on $\{\cdot,\cdot\}$ on $C^\infty(M)$ that satisfies the Leibniz identity:

$$\{fg,h\} = f\{g,h\} + \{f,h\}g, \quad \forall f,g,h \in C^\infty(M).$$

Alternatively, one can define a Poisson structure as a particular type of bivector

$$\pi \in \Gamma(\Lambda^2 TM).$$

To see this, recall that a k-derivation is a k-linear map

$$\underbrace{C^\infty(M) \times ... \times C^\infty(M)}_{k \text{ times}} \to C^\infty(M)$$

that is a derivation in each argument. Given a skew-symmetric bi-derivation $\{\cdot,\cdot\}$ on M we define its Jacobiator as the skew-symmetric tri-derivation on M, defined by:

$$\mathrm{Jac}(f,g,h) = \{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}, \quad f,g,h \in C^\infty(M).$$

We can now reformulate the definition of a Poisson bracket: it is a skew-symmetric bi-derivation whose Jacobiator vanishes. The vanishing of the Jacobiator is of course just the Jacobi identity. The following allows us to rephrase this in terms of multi-vector fields.

Lemma 1.1.2. *Let M be a manifold. There is a bijective correspondence between k-vector fields on M and skew-symmetric k-derivations on M, which associates to a k-vector field θ the skew-symmetric k-derivation given by*

$$\{f_1, ..., f_k\} = \theta(df_1, ..., df_k), \quad f_i \in C^\infty(M).$$

Now, given a bivector π, we define $\{\cdot,\cdot\}$ to be the corresponding bi-derivation and we define Θ_π to be the tri-vector field on M corresponding to the Jacobiator of $\{\cdot,\cdot\}$. This leads to:

Proposition 1.1.3. *Let M be a manifold. There is a bijective correspondence:*

$$\left\{ \begin{array}{c} \text{Bivectors } \pi \in \Lambda^2 TM \\ \text{satisfying } \Theta_\pi = 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Poisson structures} \\ \{\cdot,\cdot\} \text{ on } M \end{array} \right\}$$

which associates to such a bivector π the bracket:

$$\{f, g\} = \pi(df, dg), \quad f, g \in C^\infty(M).$$

In light of this result, a bivector π on M for which $\Theta_\pi = 0$ is called a **Poisson bivector**.

The Lie algebroid of a Poisson manifold

Definition 1.1.4. A **Lie algebroid** over a manifold M is a triple $(\mathcal{A}, \rho, [\cdot,\cdot])$ consisting of a vector bundle $\mathcal{A}$ over M, a vector bundle map $\rho : \mathcal{A} \to TM$ called the **anchor**, and a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(\mathcal{A})$, that satisfy the Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + (\mathcal{L}_{\rho(\alpha)} f)\beta, \quad \alpha, \beta \in \Gamma(\mathcal{A}), \quad f \in C^\infty(M). \tag{1.1}$$

A Poisson manifold (M, π) gives rise to a Lie algebroid structure on T^*M, as follows.

The anchor: Given a bilinear map $B : V \times V \to \mathbb{R}$ we can define a linear map $B^\flat : V \to V^*$ as $B^\flat(v)(w) = B(v, w)$. Applying this fiberwise to a bivector $\pi \in \Lambda^2 TM$ and identifying $(T^*M)^*$ with TM canonically, we obtain a vector bundle map

$$\pi^\sharp : T^*M \to TM$$

determined by the fact that $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$ for all $\alpha, \beta \in T^*M$. This is the anchor map for T^*M.

The Lie bracket: We define the Lie bracket on T^*M as:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d\pi(\alpha, \beta).$$

This is the unique bilinear, skew-symmetric map

$$\Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$$

satisfying both the Leibniz identity (1.1) and

$$[df, dg]_\pi = d\{f, g\}_\pi, \quad \forall f, g \in C^\infty(M).$$

Proposition 1.1.5. *The triple* $T^*_\pi M := (T^*M, \pi^\sharp, [\cdot,\cdot]_\pi)$ *is a Lie algebroid over* M*.*

Let $(\mathcal{A}, \rho, [\cdot,\cdot])$ be a Lie algebroid. For every $x \in M$ the vector space $\mathfrak{g}_x := \mathrm{Ker}(\rho_x)$ admits a Lie bracket defined by:

$$[\alpha_x, \beta_x]_{\mathfrak{g}_x} = [\alpha, \beta](x)$$

where $\alpha, \beta \in \Gamma(\mathcal{A})$ are arbitrary extensions of $\alpha_x, \beta_x \in \mathrm{Ker}(\rho_x)$.

Proposition 1.1.6. *Let* $(\mathcal{A}, \rho, [\cdot,\cdot])$ *be a Lie algebroid. Then:*

a) The anchor map ρ *induces a morphism of Lie algebras* $\Gamma(\mathcal{A}) \to \mathcal{X}(M)$ *with respect to the Lie bracket of vector fields on* M*.*

b) The bracket $[\cdot,\cdot]_{\mathfrak{g}_x}$ *defined above is a well-defined Lie bracket.*

For a Poisson manifold (M, π), the Lie algebra $(\mathfrak{g}_x, [\cdot,\cdot]_{\mathfrak{g}_x})$ is called the **isotropy Lie algebra** at $x \in M$. The reason for this will become apparent later.

The symplectic leaves of a Poisson manifold

Let (M, π) be a Poisson manifold. To any function $f \in C^\infty(M)$ we associate a vector field

$$X_f := \pi^\sharp(df)$$

on M. This is called the **Hamiltonian vector field** of f (with respect to π). The flow of a Hamiltonian vector field X_f at time t is denoted by Φ^t_f. One can define an equivalence relation on M by:

$$x \sim_\pi y \iff \text{there are } f_1, ..., f_k \in C^\infty(M) \text{ such that } y = \Phi^1_{f_k} \circ ... \circ \Phi^1_{f_1}(x). \tag{1.2}$$

Definition 1.1.7. The **symplectic leaves** of (M, π) are the equivalence classes of the equivalence relation (1.2).

Theorem 1.1.8. *The symplectic leaves of a Poisson manifold* (M, π) *are connected, initial submanifolds of* M*. For every such leaf* L*, the tangent space of* L *at a point* x *is equal to* $Im(\pi^\sharp_x)$*. Furthermore, each leaf* L *admits a symplectic structure* ω_L *determined by:*

$$\omega_L(\pi^\sharp_x(\alpha), \pi^\sharp_x(\beta)) = -\pi_x(\alpha, \beta), \quad \alpha, \beta \in T^*_x M, \quad x \in M.$$

The partition $\{(L, \omega_L)|\ L$ is a symplectic leaf of $(M, \pi)\}$ is called the **symplectic foliation** of (M, π). The Poisson structure is fully determined by its symplectic foliation. The following result provides a way to determine the symplectic leaves of a Poisson manifold, once a sensible guess has been made.

Proposition 1.1.9. *Let* (M, π) *be a Poisson manifold. Suppose that* $\mathcal{P}$ *is a partition of* M *into connected, immersed submanifolds with the property that*

$$T_x P = Im(\pi^\sharp_x)$$

for all $x \in P$ *and* $P \in \mathcal{P}$*. Then* $\mathcal{P}$ *is the partition of* M *into symplectic leaves.*

Some standard examples

We end this section with some standard examples of Poisson manifolds.

Example 1.1.10. Every manifold M can be equipped with the zero-Poisson structure $\pi = 0$. The corresponding anchor map and Lie bracket on T^*M are both the zero ones. The isotropy Lie algebras are the abelian Lie algebras T^*_xM and symplectic leaves of M are just the points of M endowed with the zero-symplectic structure.

Example 1.1.11. Every symplectic manifold (M, ω) admits a Poisson structure. In terms of Poisson brackets it is given by

$$\{f, g\}_\omega = -\omega(X_f, X_g), \quad f, g \in C^\infty(M),$$

where X_f, X_g denote the Hamiltonian vector fields of f and g with respect to ω. The corresponding bivector π_ω is determined by the fact that $\pi^\sharp_\omega : T^*M \to TM$ is inverse to $\omega^\flat : TM \to T^*M$. Of course one could define a non-degenerate bivector π_ω for a merely non-degenerate two-form ω on M. However, the Koszul formula implies that

$$d\omega(X_f, X_g, X_h) = -\mathrm{Jac}(f, g, h)$$

for all $f, g, h \in C^\infty(M)$, so that ω is closed precisely if π_ω is a Poisson bivector. This implies that the mapping $\omega \mapsto \pi_\omega$ defines a bijection between the symplectic structures and the non-degenerate Poisson structures on M. Note that the Hamiltonian vector fields with respect to ω coincide with those for π_ω. The anchor map $\pi^\sharp_\omega : T^*M \to TM$ is an isomorphism of Lie algebroids in this case. The isotropy Lie algebras are all trivial as vector spaces and there is just one symplectic leaf, namely (M, ω) itself.

Example 1.1.12. A Poisson structure $\{\cdot, \cdot\}$ on a real vector space V is called linear if the subalgebra V^* of $C^\infty(V)$ is closed under the Poisson bracket. Linear Poisson structures arise in the following way. Given a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$, we can define a linear Poisson structure on $\mathfrak{g}^*$, given by

$$\{f, g\}(\alpha) = \langle \alpha, [df_\alpha, dg_\alpha]_\mathfrak{g} \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*), \alpha \in \mathfrak{g}^*.$$

Here we consider df_α and dg_α as element of $\mathfrak{g}$ via the canonical isomorphisms $T^*_\alpha\mathfrak{g}^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$. One can show that this defines a bijection between Lie algebras and linear Poisson manifolds. Under these same isomorphisms we have that $T^*\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}$ and $\Gamma(T^*\mathfrak{g}^*) \cong C^\infty(\mathfrak{g}^*; \mathfrak{g})$. The anchor map at $\alpha \in \mathfrak{g}^*$ is:

$$\pi^\sharp_\alpha : \mathfrak{g} \to \mathfrak{g}^*, \quad v \mapsto \mathrm{ad}^*_v(\alpha) = \langle \alpha, [v, \cdot]_\mathfrak{g} \rangle$$

and the Lie bracket on $C^\infty(\mathfrak{g}^*; \mathfrak{g})$ is determined by the fact that

$$[f, g](\alpha) = [f(\alpha), g(\alpha)]_\mathfrak{g}$$

for all $\alpha \in \mathfrak{g}^*$ and all constant functions $f, g : \mathfrak{g}^* \to \mathfrak{g}$. The isotropy Lie algebra at $\alpha \in \mathfrak{g}^*$ is the Lie subalgebra:

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} |\ \mathrm{ad}^*_v(\alpha) = 0\}$$

of $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$. Given any connected Lie group G that integrates $\mathfrak{g}$, the symplectic leaves of $\mathfrak{g}^*$ are the orbits of the coadjoint action of G on $\mathfrak{g}^*$. These are endowed with the so-called KKS-symplectic form.

1.1.2 Moment maps

Infinitesimal symmetry

In this section we introduce the main objects of study in this thesis: moment maps. The omitted proofs in this section are rather straightforward.

We first need the notion of a Poisson map.

Definition 1.1.13. Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be two Poisson manifolds. A map $\varphi : M \to N$ is a **Poisson map** if:

$$\{f \circ \varphi, g \circ \varphi\}_M = \{f, g\}_N \circ \varphi$$

for all $f, g \in C^\infty(N)$.

A more practical characterization can be given in terms of the anchor maps.

Proposition 1.1.14. *Let (M, π_M) and (N, π_N) be two Poisson manifolds. A map $\varphi : M \to N$ is a Poisson map if and only if the diagram:*

$$\begin{tikzcd}
T^*_m M \arrow[r, "\pi_M^\sharp"] & T_m M \arrow[d, "d\varphi"] \\
T^*_{\varphi(m)} N \arrow[u, "d\varphi^*"] \arrow[r, "\pi_N^\sharp"] & T_{\varphi(m)} N
\end{tikzcd}$$

commutes for all $m \in M$.

Definition 1.1.15. Let (M, π) be a Poisson manifold. An **(M, π)-valued moment map** is a Poisson map $\mu : (S, \omega) \to (M, \pi)$ from a symplectic manifold into (M, π). Often we will just call μ a moment map and omit the adjective (M, π)-valued.

The reader familiar with classical mechanics will undoubtedly have heard of Noether's principle: symmetries correspond to conserved quantities. Moment maps are the mathematical incarnation of conserved quantities. From this viewpoint, Noether's principle dictates that there is a symmetry associated to a moment map. At least infinitesimally this symmetry is present: it is the canonical action of the Lie algebroid associated to (M, π). Let us make this precise.

Definition 1.1.16. Let $(\mathcal{A}, \rho, [\cdot,\cdot])$ be a Lie algebroid. A **Lie algebroid action** of $\mathcal{A}$ along a map $\mu : S \to M$ is a vector bundle map $a : \mu^*\mathcal{A} \to TS$ that induces a morphism of Lie algebras $\mathcal{A} \to \mathcal{X}(S)$ and makes the diagram

$$\begin{tikzcd}
\mathcal{A}_{\mu(p)} \arrow[rr, "a_p"] \arrow[dr, "\rho_{\mu(p)}"'] & & T_p S \arrow[dl, "d\mu_p"] \\
& T_{\mu(p)} M &
\end{tikzcd}$$

commute for every $p \in S$.

The following can be interpreted as Noether's principle for infinitesimal symmetries.

Proposition 1.1.17. *Let (M, π) be a Poisson manifold, (S, ω) a symplectic manifold and $\mu : S \to M$ a map. The vector bundle map $a : \mu^*(T^*M) \to TS$ determined by:*

$$\iota_{a_p(\alpha)}\omega = d\mu_p^*(\alpha), \quad \alpha \in T^*_{\mu(p)}M,$$

defines a Lie algebroid action of $T^*_\pi M$ along μ if and only if μ is a Poisson map.

Given a Lie algebroid $(\mathcal{A}, \rho, [\cdot,\cdot])$ and a Lie algebroid action of $\mathcal{A}$ along a map $\mu : S \to M$, we define the **isotropy Lie algebra of the action** at p

$$\mathfrak{g}_p := \mathrm{Ker}(a_p) \subset T^*_{\mu(p)}M$$

with Lie bracket given by:

$$[\alpha_{\mu(p)}, \beta_{\mu(p)}]_{\mathfrak{g}_p} = [\alpha, \beta](\mu(p)),$$

where $\alpha, \beta \in \Gamma(\mathcal{A})$ are arbitrary extensions of $\alpha_{\mu(p)}, \beta_{\mu(p)}$. One can show that this is well-defined directly. Alternatively, one can endow $\Gamma(\mu^*\mathcal{A})$ with a Lie bracket that turns $\mu^*\mathcal{A}$ into a Lie algebroid over S with anchor map a, and apply Proposition 1.1.6.

Given a moment map μ, there are two related singular distributions on S:

$$\mathcal{F}_\mu := \mathrm{Ker}(d\mu) \quad \text{and} \quad \mathcal{F}_\mu^\omega = \{(p, v) \in TS |\ \omega_p(v, w) = 0,\ \forall\ w \in (\mathcal{F}_\mu)_p\}.$$

The following basic observations are crucial in the study of moment maps.

Proposition 1.1.18. *Let $\mu : (S, \omega) \to (M, \pi)$ be a moment map and $p \in S$. Then:*

a) $\mathfrak{g}_p$ *is a Lie subalgebra of* $\mathfrak{g}_{\mu(p)}$.

b) $Im(a_p) = (\mathcal{F}_\mu^\omega)_p$.

c) $a_p(\mathfrak{g}_{\mu(p)}) = (\mathcal{F}_\mu)_p \cap (\mathcal{F}_\mu^\omega)_p$.

d) $Im(\pi^\sharp_{\mu(p)}) = d\mu_p((\mathcal{F}_\mu^\omega)_p)$.

e) $\mathfrak{g}_p = Im(d\mu_p)^0$.

These equalities can be captured by the following commutative diagram

$$\begin{tikzcd}
\mathfrak{g}_p \arrow[r] \arrow[d, equal] & \mathfrak{g}_{\mu(p)} \arrow[rr, "a_p"] \arrow[d] & & (\mathcal{F}_\mu)_p \cap (\mathcal{F}_\mu^\omega)_p \arrow[r] \arrow[d] & 0 \\
\mathfrak{g}_p \arrow[r] & T^*_{\mu(p)}M \arrow[rr, "a_p"] \arrow[dr, "\pi^\sharp_{\mu(p)}"'] & & (\mathcal{F}_\mu^\omega)_p \arrow[r] \arrow[dl, "d\mu_p"] & 0 \\
& & T_{\mu(p)}M & &
\end{tikzcd}$$

in which the rows and the sequences that strictly decrease in height are exact.

Corollary 1.1.19. *Let $\mu : (S, \omega) \to (M, \pi)$ be a moment map. If μ is a submersion at $p \in S$, then a_p is injective and*

$$dim(S) \geq 2dim(M) - rk(\pi_{\mu(p)}),$$

with equality precisely if $(\mathcal{F}_\mu)_p \subset (\mathcal{F}_\mu^\omega)_p$.

Proof. If μ is a submersion at p, then we have

$$\dim(S) = \dim((\mathcal{F}_\mu)_p) + \dim(M),$$

and by part e of the previous proposition it follows that a_p is injective. By part c this implies that

$$\begin{aligned}\dim(\mathfrak{g}_{\mu(p)}) &= \dim((\mathcal{F}_\mu)_p \cap (\mathcal{F}_\mu)_p^\omega) \\ &\leq \dim((\mathcal{F}_\mu)_p)\end{aligned}$$

and equality holds precisely if $(\mathcal{F}_\mu)_p \cap (\mathcal{F}_\mu^\omega)_p = (\mathcal{F}_\mu)_p$, or equivalently, if $(\mathcal{F}_\mu)_p \subset (\mathcal{F}^\omega)_p$. Hence the statement follows, because

$$\dim(\mathfrak{g}_{\mu(p)}) = \dim(M) - \mathrm{rk}(\pi_{\mu(p)}).$$

□

Some particular classes of moment maps

We end this section by defining a few classes of moment maps. Thus far, we have introduced moment maps in the way that they arise in symplectic geometry. However, one could as well study Poisson maps $(S,\omega) \to (M,\pi)$ with the aim of studying the Poisson structure on the base space. The non-weak versions of the following classes of moment maps arise as such in Poisson geometry, whereas the weak versions arise more naturally from the starting point of symplectic geometry.

Definition 1.1.20. Let (M,π) be a Poisson manifold. A (weak) **symplectic realization** of (M,π) is a Poisson map $\mu : (S,\omega) \to (M,\pi)$ which is a surjective submersion (on a dense subset). A (weak) **isotropic realization** of (M,π) is a (weak) symplectic realization $(S,\omega) \to (M,\pi)$ for which π is regular and

$$\dim(S) = 2\dim(M) - \mathrm{rk}(\pi).$$

The name isotropic realization stems from the following consequence of Corollary 1.1.19.

Proposition 1.1.21. *The fibers of an isotropic realization are isotropic.*

A particular subclass that we will study more extensively is that of the Lagrangian fibrations.

Definition 1.1.22. A **Lagrangian fibration** is a surjective submersion $\mu : (S,\omega) \to B$ the fibers of which are Lagrangian submanifolds.

Proposition 1.1.23. *Lagrangian fibrations are the same thing as isotropic realizations of the zero-Poisson structure.*

Proof. Let $\mu : (S,\omega) \to (B,0)$ be an isotropic realization. Then the fibers of μ have dimension

$$\dim(S) - \dim(B) = \frac{1}{2}\dim(S)$$

and they are isotropic due to the previous proposition. On the other hand, suppose that $\mu : (S,\omega) \to B$ is a Lagrangian fibration. Part b of Proposition 1.1.18 holds even if μ is not a Poisson map, and so we find that $\mathrm{Im}(a_p) = (\mathcal{F}_\mu)_p$ for all $p \in S$ since the fibers of μ

are Lagrangian. Therefore $d\mu_p \circ a_p = 0$, so that μ is a Poisson map into the zero-Poisson structure. Furthermore, we have

$$\dim(S) - \dim(B) = \mathrm{rk}(\mathcal{F}_\mu) = \frac{1}{2}\dim(S),$$

because the fibers of μ are Lagrangian. Hence $\dim(S) = 2\dim(B)$ and so μ is an isotropic realization of $(B, 0)$. $\square$

1.2 Lie groupoids

In the previous section we have defined Lie algebroids and their actions, which generalize the notion of Lie algebras and their actions. In this section we will introduce the objects that generalize Lie groups and integrate Lie algebroids. These are called Lie groupoids.

1.2.1 Definition and examples of Lie groupoids

Definition 1.2.1. A **groupoid** is a small category, all arrows of which are invertible.

Although this definition is nice and compact, in practice one should think about a groupoid as follows. A groupoid $\mathcal{G} \rightrightarrows M$ consists of a set of arrows $\mathcal{G}$, a set of objects M and a collection of structure maps $\{s, t, m, u, i\}$. More elaborately, one thinks of an element $g \in \mathcal{G}$ as an arrow $y \xleftarrow{g} x$ starting at the **source** $s(g) := x$ of g and ending at the **target** $t(g) := y$ of g. This defines the source and target maps $s, t : \mathcal{G} \to M$. In contrast with groups, not every pair of arrows can be composed. A pair of arrows (g, h) is **composable** precisely if g starts where h ends. That is, if

$$(g, h) \in \mathcal{G}_{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} |\ s(g) = t(h)\}.$$

In this case, we write gh for their composition, which we require to start at $s(h)$ and end at $t(g)$. We thus have a **multiplication map** $m : \mathcal{G}_{(2)} \to \mathcal{G}$ given by $m(g, h) = gh$. The multiplication is required to be associative:

$$f(gh) = (fg)h$$

whenever $s(f) = t(g)$ and $s(g) = t(h)$. Another difference between groups and groupoids are their **unit elements**: whereas a group has a single unit, a groupoid has a unit element for every element of M. That is, for every $x \in M$ there is a (necessarily unique) arrow 1_x starting and ending at x, with the property that:

$$g1_x = g \text{ and } 1_x h = h$$

for all $g \in s^{-1}(x)$ and $h \in t^{-1}(x)$. Thus there is an injection $u : M \to \mathcal{G}$ given by $u(x) = 1_x$, called the **unit map**. For this reason, M is also called the space of units. Finally, every element $g \in \mathcal{G}$ must have an **inverse** $g^{-1} \in \mathcal{G}$. That is, given an arrow $y \xleftarrow{g} x$ there is a (necessarily unique) arrow $x \xleftarrow{g^{-1}} y$ such that:

$$gg^{-1} = 1_y \text{ and } g^{-1}g = 1_x.$$

This gives rise to the **inversion map** $i : \mathcal{G} \to \mathcal{G}$ given by $i(g) = g^{-1}$.

Remark 1. The structure maps satisfy the following relations: $i \circ i = \mathrm{Id}_{\mathcal{G}}$, $s \circ i = t$ and $s \circ u = \mathrm{Id}_M$. Therefore i is a bijection, and s and t are surjections and (as remarked before) u is injective.

A groupoid $\mathcal{G} \rightrightarrows M$ comes with more intrinsic structure. The subset of arrows that start and end at $x \in M$ is a group:

$$\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x)$$

called the **isotropy group** at x. Furthermore, the groupoid comes with an equivalence relation on M defined by:

$$x \sim y \iff \text{there is a } g \in \mathcal{G} \text{ such that } s(g) = x \text{ and } t(g) = y.$$

The equivalence classes are called the **orbits** of $\mathcal{G}$. Explicitly, the orbit $\mathcal{O}_x$ through $x \in M$ is given by $t(s^{-1}(x))$. The **orbit space** $M/\mathcal{G}$ is the set of orbits.

Definition 1.2.2. A **Lie groupoid** is a groupoid for which both the space of arrows and the space of objects are smooth manifolds. Moreover, the source map is required to be a smooth submersion and the other structure maps are required to be smooth.

Note here that if the source map is a submersion, then $\mathcal{G}_{(2)}$ is a submanifold of $\mathcal{G} \times \mathcal{G}$ and so it makes sense to require the multiplication map to be smooth.

Convention 1. Throughout the literature, the space of arrows of a Lie groupoid is usually not assumed to be Hausdorff; only the source and target fibers are. For the purposes of this text it is however not necessary to consider such Lie groupoids and we will therefore take as convention that the space of arrows of a Lie groupoid is Hausdorff, unless explicitly stated otherwise.

Proposition 1.2.3. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then the following hold:*

a) The unit map is an embedding, the inversion map is a diffeomorphism and the target map is a submersion.

b) The isotropy groups are submanifolds of $\mathcal{G}$ and hence they are Lie groups.

c) For every $x \in M$, the orbit $\mathcal{O}_x$ admits a unique smooth structure such that

$$t : s^{-1}(x) \to \mathcal{O}_x$$

is a right principal $\mathcal{G}_x$-bundle. As such, each orbit is an initial submanifold of M.

The proof of part b hinges on the fact that $t : s^{-1}(x) \to M$ has constant rank. A full proof can be found in [MM03].

Definition 1.2.4. We say that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is:

- **proper** if $(s, t) : \mathcal{G} \to M \times M$ is a proper map (that is, pre-images of compacts are again compact),
- **s-connected** if it source-fibers are connected,

- **s-1-connected** if its source-fibers are connected and simply-connected.

The following is not difficult to prove.

Proposition 1.2.5. *Let $\mathcal{G} \rightrightarrows M$ is a proper Lie groupoid. Then the following hold:*

a) The isotropy groups of $\mathcal{G}$ are compact.

b) The orbits of $\mathcal{G}$ are closed submanifolds of M.

c) The orbit space of $\mathcal{G}$ is Hausdorff, locally compact and second countable.

This goes to show that proper Lie groupoids have considerably better properties than general Lie groupoids. Let us turn to some examples.

Example 1.2.6. A Lie group G is a Lie groupoid over a point. It is proper if and only if G is compact, and it is s-connected precisely if G is connected.

Example 1.2.7. Another example is the fundamental groupoid $\Pi(X) \rightrightarrows X$ of a topological space X. Its arrows are path-homotopy classes (with fixed end-points) in X. Given a path γ in X, the source of $[\gamma]$ is the starting point of the path γ, the target is the end point of γ and the multiplication is defined by concatenation of paths. The units are the path-homotopy classes of constant paths and inversion is the usual inversion of path-homotopy classes. The isotropy group at $x \in X$ is the fundamental group $\pi_1(X, x)$ and orbit through x is the path-component of X through x. If X is a manifold, then $\Pi(X)$ can be given the structure of a Lie groupoid.

Example 1.2.8. Let M be a G-space, that is, M is a manifold equipped with a smooth action of a Lie group G. Then $G \times M$ is the space of arrows for a Lie groupoid $G \ltimes M$ over M, called the action groupoid. The structure maps are given by:

$$\begin{aligned} s(g, m) &= m, \\ t(g, m) &= g \cdot m, \\ m((g, h \cdot m), (h, m)) &= (gh \cdot m, m), \\ u(m) &= (e, m), \\ i(g, m) &= (g^{-1}, g \cdot m). \end{aligned}$$

The isotropy group and the orbit at a point $m \in M$ are just the usual isotropy group G_m and orbit $\mathcal{O}_m$ of the action. The action groupoid is proper precisely if the action of G on M is proper and it is s-connected precisely if G is connected.

Example 1.2.9. A **bundle of Lie groups** is a Lie groupoid for which $s = t$. For instance, a vector bundle $E \to M$ is bundle of Lie groups with the source and target map the bundle projection, multiplication given by fiberwise addition, the unit map being the zero-section and the inversion being fiberwise scalar multiplication by -1. A **torus bundle** is a bundle of Lie groups for which the isotropy groups are all compact, abelian and connected. In general, the isotropy Lie groups of a bundle of Lie groups are the source-fibers and the orbits are the points in M. Obviously, vector bundles are not proper. On the other hand, torus bundles are proper since their source-map is proper, by a general result in differential geometry that asserts that a surjective submersion with compact and connected fibers is proper.

1.2.2 The Lie algebroid of a Lie groupoid

We will now show that Lie groupoids integrate Lie algebroids. That is, there is a functor from the category of Lie groupoids to that of Lie algebroids. For a more extensive account of the construction discussed in this section, see [CF11].

To construct this functor one mimicks the construction for Lie groups, as follows. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We define $\mathcal{A}_{\mathcal{G}} \to M$ to be the vector bundle $u^*(\mathrm{Ker} ds)$. That is, its fiber over $x \in M$ is $\mathrm{Ker}(ds_{1_x})$ or, in other words, it is the tangent space to $s^{-1}(x)$ at 1_x. The anchor map $\rho : \mathcal{A}_{\mathcal{G}} \to TM$ is given by $\rho_x(v) = dt_{1_x}(v)$. The Lie bracket on $\mathcal{A}_{\mathcal{G}}$ is induced by the Lie bracket of vector fields on $T\mathcal{G}$, by means of right-invariant vector fields. On a Lie groupoid, right translation by an arrow $y \xleftarrow{g} x$ is defined only as a map $R_g : s^{-1}(y) \to s^{-1}(x)$. A right invariant vector field is a vector field $X \in \mathcal{X}(\mathcal{G})$ that is tangent to the s-fibers and satisfies:

$$(dR_g)_h(X_h) = X_{hg}$$

for all $h \in s^{-1}(t(g))$ and $g \in \mathcal{G}$. The linear subspace $\mathcal{X}^R(\mathcal{G})$ of right-invariant vector fields on $\mathcal{G}$ is isomorphic to $\Gamma(\mathcal{A}_{\mathcal{G}})$. Explicitly, the isomorphism $\Gamma(\mathcal{A}_{\mathcal{G}}) \to \mathcal{X}^R(\mathcal{G})$ is given by $\alpha \mapsto \alpha^R$, where α^R is the right-invariant vector field defined by $\alpha^R_g = dR_g(\alpha(t(g)))$. We now transport the Lie bracket of vector fields on the Lie subalgebre $\mathcal{X}^R(\mathcal{G})$ of $\mathcal{X}(\mathcal{G})$ to $\Gamma(\mathcal{A}_{\mathcal{G}})$ via this isomorphism. Explicitly:

$$[\alpha, \beta](x) = [\alpha^R, \beta^R]_{\mathcal{X}(\mathcal{G})}(1_x).$$

We denote the triple $(\mathcal{A}_{\mathcal{G}}, \rho, [\cdot, \cdot])$ defined in this way by $\mathrm{Lie}(\mathcal{G})$.

To define the rest of the functor, we should first say what morphisms of Lie groupoids and Lie algebroids are. For Lie groupoids this is the obvious choice: given two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ a **morphism of Lie groupoids** is a pair (Φ, φ) of smooth maps $\Phi : \mathcal{G} \to \mathcal{H}$ and $\varphi : M \to N$ which is compatible with all structure maps. The compatibility can be rephrased as saying that (Φ, φ) defines a functor from the category $(\mathcal{G}, M)$ to $(\mathcal{H}, N)$. If $M = N$ and $\varphi = \mathrm{id}_M$ we say that Φ is a morphism of groupoids over M.

The notion of morphism of Lie algebroids is considerably less obvious if $M \neq N$. Since we will not really need this case in the rest of this thesis we will only give it in the case $M = N$ and $\varphi = \mathrm{id}_M$. A **morphism of Lie algebroids** $\mathcal{A} \to \mathcal{B}$ over M is a vector bundle map covering the identity, which intertwines the anchor maps and induces a morphism of Lie algebras on the spaces of sections.

Now, given a morphism of groupoids $(\Phi, \varphi) : \mathcal{G} \to \mathcal{H}$ we define $\mathrm{Lie}(\Phi)$ to be the vector bundle map $\mathcal{A}_{\mathcal{G}} \to \mathcal{A}_{\mathcal{H}}$ covering φ, given by $\mathrm{Lie}(\Phi)(x, v) = d\Phi_{1_x}(v)$. This is well-defined because $s_{\mathcal{H}} \circ \Phi = \varphi \circ s_{\mathcal{G}}$.

Proposition 1.2.10. *"Lie" defines a functor from the category of Lie groupoids to that of Lie algebroids.*

A proof of this can be found in [Mac05]. By comparing the constructions of Lie algebras[1] from Lie groups and Lie algebroids from Lie groupoids, one obtains:

[1] Here we define the Lie algebra of a Lie group by means of right-invariant vector fields, instead of left-invariant ones which are more commonly used for Lie groups. The resulting difference for the Lie bracket is a minus sign.

Proposition 1.2.11. *The isotropy Lie algebra of Lie($\mathcal{G}$) at $x \in M$ is the Lie algebra of the isotropy group $\mathcal{G}_x$.*

1.2.3 Lie groupoid actions

As for Lie groups, one can define the notion of an action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a manifold S. However, one bit of additional data is needed: a moment map $\mu : S \to M$.

Definition 1.2.12. A (left) **action of a Lie groupoid** $\mathcal{G} \rightrightarrows M$ along a smooth map $\mu : S \to M$ consists of a smooth map

$$m : \mathcal{G}\,{}_s\!\times_\mu S \to S, \quad (g,p) \mapsto g \cdot p,$$

with the property that:

a) $\mu(g \cdot p) = t(g)$ if $s(g) = \mu(p)$.

b) $g \cdot (h \cdot p) = (gh) \cdot p$ if $s(g) = t(h)$ and $s(h) = \mu(p)$.

c) $1_{\mu(p)} \cdot p = p$ for all $p \in S$.

The map μ is called the **moment map** of the action.

For the sake of overview, we will sometimes depict a Lie groupoid action as:

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & S \arrow[dl, "\mu"] \\ M & \end{tikzcd}$$

We realise that at this point the terminology conflicts with our earlier definition of moment map. However, as soon as we restrict our attention to Hamiltonian actions of symplectic groupoids (from Section 1.4 onwards), the two notions will coincide.

An action of a Lie groupoid can be encoded by a Lie groupoid over S (just as a Lie group action can be encoded by a Lie groupoid). The space of arrows is $\mathcal{G}\,{}_s\!\times_\mu S$, and the structure maps are defined by the same formulas that we used to define the action groupoid of a Lie group action in Example 1.2.8. This is called the **action groupoid** of a Lie groupoid action and denoted $\mathcal{G} \ltimes S$. We define the isotropy group at p of a groupoid action to be the isotropy group of the action groupoid. Strictly speaking, it is a Lie group of the form $\mathcal{G}_p \times \{p\}$, but we will always consider it to be just the closed Lie subgroup:

$$\mathcal{G}_p = \{g \in \mathcal{G}_{\mu(p)} |\ g \cdot p = p\} \subset \mathcal{G}_{\mu(p)}.$$

The orbits of the action are by definition the orbits of the action groupoid. Explicitly, they are:

$$\mathcal{O}_p = \{g \cdot p \in S |\ s(g) = \mu(p)\}.$$

An action of a Lie groupoid induces an action of its Lie algebroid $\mathcal{A} := \mathrm{Lie}(\mathcal{G})$, as follows. Its Lie algebroid acts along $\mu : S \to M$ via the vector bundle map

$$a : \mu^*(\mathcal{A}_\mathcal{G}) \to TS, \quad a_p = d(m_p)_{1_{\mu(p)}}, \tag{1.3}$$

where $m_p : s^{-1}(\mu(p)) \to S$ is given by $m_p(g) = g \cdot p$.

Proposition 1.2.13. *This defines a Lie algebroid action of $\mathcal{A}$ along $\mu : S \to M$. Moreover, the isotropy Lie algebra of this Lie algebroid action at $p \in S$ is the Lie algebra of $\mathcal{G}_p$.*

Finally, we note that:

Proposition 1.2.14. *Let $\mathcal{G} \rightrightarrows M$ be an s-connected Lie groupoid and suppose that $\mathcal{A} = Lie(\mathcal{G})$ acts along a map $\mu : S \to M$. Then there is at most one Lie groupoid action of $\mathcal{G}$ on $\mu : S \to M$ for which the given action of $\mathcal{A}$ coincides with* (1.3)*.*

The proofs of the previous two propositions are rather straightforward exercises.

Principal $\mathcal{G}$-bundles

As for Lie group actions, we have a notion of proper and free Lie groupoid actions.

Definition 1.2.15. The action of a Lie groupoid $\mathcal{G}$ along a map $\mu : S \to M$ is **free at** $p \in S$ if $\mathcal{G}_p$ is trivial. It is **free** if it is so at every $p \in S$. Secondly, the action is **proper** if the action groupoid $\mathcal{G} \ltimes S$ is proper.

Remark 2. If a Lie groupoid $\mathcal{G} \rightrightarrows M$ is proper, then any action of $\mathcal{G}$ is proper as well. This generalizes the fact that compact Lie groups act properly.

Definition 1.2.16. Given an equivalence relation R on a manifold M, we say that M/R is **smooth** if it is Hausdorff, second-countable and it admits a (necessarily unique) smooth structure for which the canonical projection $M \to M/R$ is a submersion.

The following generalizes a well-known result for proper and free Lie group actions.

Theorem 1.2.17. *The orbit space of a free and proper Lie groupoid action is smooth.*

To prove this, one can use a general result on smoothness of quotients of manifolds.

Theorem 1.2.18 (Godement, [Ser06, Thm 2, pg 92])**.** *Let M be a manifold and R an equivalence relation on M. Then M/R is smooth if and only if R is a closed submanifold of $M \times M$ and $pr_2 : R \to M$ is a submersion.*

Although the book we refer to treats analytic manifolds, the proof given there works for smooth manifolds as well. Using this, Theorem 1.2.17 follow from the fact that the map

$$(m, pr_2) : \mathcal{G} \,{}_s\!\times_\mu S \to S \times S$$

is a proper embedding if the action is proper and free, and its image is the orbit equivalence relation $R \subset S \times S$.

We end this section with the notion of a principal $\mathcal{G}$-bundle.

Definition 1.2.19. A (left) **principal $\mathcal{G}$-bundle** consists of a left action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ along a map $\mu : P \to M$, and a surjective submersion $\pi : P \to B$ such that the map:

$$(m, pr_2) : \mathcal{G} \,{}_s\!\times_\mu P \to P \,{}_\pi\!\times_\pi P$$

is a well-defined diffeomorphism.

Equivalently, one can require that the action is free and the fibers of π coincide with the orbits of the action. As for principal G-bundles, it holds that B is diffeomorphic to the orbit space of the $\mathcal{G}$-action. For the sake of overview, we will sometimes depict a principal $\mathcal{G}$-bundle as:

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & P \arrow[dl, "\mu"] \arrow[dr, "\pi"] & \\ M & & B \end{tikzcd}$$

1.2.4 Normal subgroupoids and short exact sequences

Normal subgroupoids are those subgroupoids by which one can take quotients to obtain another Lie groupoid. As we will show, these correspond to short exact sequences of Lie groupoids.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

Definition 1.2.20. By a **Lie subgroupoid** of $\mathcal{G}$ we mean a pair $(\mathcal{K}, j)$ consisting of a Lie groupoid $\mathcal{K} \rightrightarrows N$ and a morphism of Lie groupoids $j : \mathcal{K} \to \mathcal{G}$ which is an injective immersion. A Lie subgroupoid is called:

- **wide** if $N = M$,
- **embedded** if j is an embedding,
- **closed** if $j(\mathcal{K})$ is closed in $\mathcal{G}$.

Remark 3. If $\mathcal{K} \subset \mathcal{G}$ and we do not mention the map j, then we take j to be the inclusion of sets.

Definition 1.2.21. A **normal** Lie subgroupoid $(\mathcal{K}, j)$ of $\mathcal{G}$ is a Lie subgroupoid with the following properties:

- it is closed, embedded and wide,
- the source and target map of $\mathcal{K}$ coincide,
- for every $g \in \mathcal{G}$:
$$gj(\mathcal{K}_x)g^{-1} \subset j(\mathcal{K}_y),$$
where $x = s(g)$ and $y = t(g)$.

Normal Lie subgroupoids arise as follows.

Proposition 1.2.22. *Let $q : \mathcal{G} \to \mathcal{H}$ be a surjective and submersive morphism of Lie groupoids over M. Then*

$$Ker(q) := \{g \in \mathcal{G} |\ q(g) = 1_x \text{ for some } x \in M\}$$

is a normal Lie subgroupoid of $\mathcal{G}$.

This gives a relationship between normal Lie subgroupoids and short exact sequences of Lie groupoids.

Definition 1.2.23. A **short exact sequence of Lie groupoids** over a manifold M, depicted by

$$1 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 1,$$

is a sequence of morphisms of Lie groupoids over M, where $\mathcal{K} \to \mathcal{G}$ is required to be an embedding with image equal to $\mathrm{Ker}(\mathcal{G} \to \mathcal{H})$ and $\mathcal{G} \to \mathcal{H}$ is required to be a surjective submersion.

The following results, which are readily verified, complete the relationship between normal Lie subgroupoids and short exact sequences. On one hand, we have:

Proposition 1.2.24. *Let $(\mathcal{K}, j)$ be a normal Lie subgroupoid of $\mathcal{G}$. Then it acts on $\mathcal{G}$ in a free and proper way along $s : \mathcal{G} \to M$ by $h \cdot g = gj(h^{-1})$. Consequently, $\mathcal{G}/\mathcal{K}$ admits a unique Lie groupoid structure for which the sequence of Lie groupoids over M:*

$$1 \to \mathcal{K} \xrightarrow{j} \mathcal{G} \to \mathcal{G}/\mathcal{K} \to 1$$

is short exact.

Conversely, we have:

Proposition 1.2.25. *Suppose that a Lie groupoid $\mathcal{K}$ fits into a short exact sequence*

$$1 \to \mathcal{K} \xrightarrow{j} \mathcal{G} \xrightarrow{q} \mathcal{H} \to 1$$

of Lie groupoids over M. Then $(\mathcal{K}, j)$ is a normal Lie subgroupoid of $\mathcal{G}$, j is an isomorphism of Lie groupoids $\mathcal{K} \to Ker(q)$ and $q : \mathcal{G} \to \mathcal{H}$ descends to an isomorphism of Lie groupoids $\mathcal{G}/\mathcal{K} \to \mathcal{H}$.

1.2.5 Basic forms on Lie groupoids

We now give the notion of basic differential forms for Lie groupoids $\mathcal{G} \rightrightarrows M$, as in [PPT14]. A more detailed discussion than the one given here can be found in [Yud16] (although the discussion there is about action groupoids of Lie groupoid actions, instead of general Lie groupoids, the ideas are the same). A basic differential form on the base M should:

- generalize the classical notion in the case of the action groupoid of a Lie group action;
- coincide with the usual smooth differential forms on $M/\mathcal{G}$ under pull-back by the orbit projection $\pi : M \to M/\mathcal{G}$, in case the orbit space is smooth.

Although there is a very short and algebraic definition, let us first give the more geometric but longer one.

Definition 1.2.26. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We say that $\alpha \in \Omega^k(M)$ is horizontal if $\iota_{\rho(X)}\alpha = 0$ for each $X \in \Gamma(\mathcal{A}_\mathcal{G})$, where ρ is the anchor map of $\mathrm{Lie}(\mathcal{G})$.

For horizontal forms there is a notion of $\mathcal{G}$-invariance. This uses the normal bundle

$$\mathcal{N}_\mathcal{O} := \frac{TM|_\mathcal{O}}{T\mathcal{O}}$$

to an orbit $\mathcal{O}$ of $\mathcal{G}$. The normal bundle to $\mathcal{O}$ admits a canonical smooth $\mathcal{G}_{\mathcal{O}}$-action along $\mu : \mathcal{N}_{\mathcal{O}} \to \mathcal{O}$, $(x, [v]) \mapsto x$, where $\mathcal{G}_{\mathcal{O}}$ denotes the Lie subgroupoid $s^{-1}(\mathcal{O})$ of $\mathcal{G}$ and the action is given by:

$$g \cdot (s(g), [v]) = (t(g), [dt_g(v_0)]), \quad v_0 \in ds_g^{-1}(v).$$

That this is well-defined follows from the fact that

$$T_{s(g)}\mathcal{O} = ds_g(\mathrm{Ker}(dt_g)) \text{ and } T_{t(g)}\mathcal{O} = dt_g(\mathrm{Ker}(ds_g))$$

for each $g \in \mathcal{G}$. We denote by

$$m_g^{\mathcal{O}} : (\mathcal{N}_{\mathcal{O}})_{s(g)} \to (\mathcal{N}_{\mathcal{O}})_{t(g)}$$

the multiplication by a fixed $g \in \mathcal{G}_{\mathcal{O}}$. It is this $\mathcal{G}_{\mathcal{O}}$-action that allows us to generalize the notion of G-invariance for horizontal forms. Indeed, horizontality of $\alpha \in \Omega^k(M)$ is equivalent to asking that α vanishes if one of its input vectors is tangent to an orbit. Therefore every horizontal $\alpha \in \Omega^k(M)$ descends to a section $\overline{\alpha} \in \Gamma(\Lambda^k(\mathcal{N}\mathcal{O})^*)$ for each orbit $\mathcal{O}$ in M, and we can define:

Definition 1.2.27. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We call $\alpha \in \Omega^k(M)$ basic if it is horizontal and $\mathcal{G}$-invariant, where by the latter we mean that:

$$\overline{\alpha}_{t(g)} \circ m_g^{\mathcal{O}} = \overline{\alpha}_{s(g)}$$

for every $g \in \mathcal{G}_{\mathcal{O}}$ and every orbit $\mathcal{O}$. We denote the set of basic k-forms by $\Omega^k_{bas}(M)$.

This notion of basic forms indeed generalizes the old notion of basic forms for Lie group actions and by the same type of proof as for Lie group actions we have:

Proposition 1.2.28. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and suppose that $M/\mathcal{G}$ is smooth (in the sense of Definition 1.2.16). Then the pull-back by π induces an isomorphism of cochain-complexes:*

$$\pi^* : \Omega^\bullet(M/\mathcal{G}) \to \Omega^\bullet_{bas}(M)$$

The following equivalent algebraic condition used in [Wat13] and [Yud16] will turn out to be useful.

Proposition 1.2.29. *A differential form $\alpha \in \Omega^k(M)$ on the base of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is basic if and only if $s^*\alpha = t^*\alpha$.*

The proof is straightforward.

1.3 Symplectic groupoids

In this section we introduce the notion of symplectic groupoid. For proofs and more details, the reader can for instance see [MW88].

As we have seen, every Lie groupoid gives rise to a Lie algebroid via the Lie functor. If $\mathrm{Lie}(\mathcal{G})$ is isomorphic to a given Lie algebroid $\mathcal{A}$ over the space of units M, then we say that $\mathcal{G}$ **integrates** $\mathcal{A}$.

Remark 4 (Integrability of Lie algebroids)**.** Lie's third theorem asserts that every Lie algebra is the Lie algebra of a 1-connected Lie group, which is unique up to isomorphism. The analog for Lie algebroids is only partly true: even if we allow for Lie groupoids for which the space of arrows is non-Hausdorff, not every Lie algebroid is integrated by a Lie groupoid. However, if a Lie algebroid is integrable, then there is an s-1-connected Lie groupoid integrating it and this one is unique up to isomorphism. The problem of when a Lie algebroid is integrated by a (possibly non-Hausdorff) Lie groupoid was solved in [CF03].

From the viewpoint of Poisson geometry, the question arises: which s-1-connected Lie groupoids integrate the Lie algebroids coming from a Poisson structure? These turn out to be the ones that are symplectic groupoids.

Definition 1.3.1. A **symplectic groupoid** is pair $(\mathcal{G}, \Omega)$ consisting of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a symplectic form Ω on $\mathcal{G}$ which satisfies the multiplicativity condition:

$$m^*\Omega = pr_1^*\Omega + pr_2^*\Omega$$

where $m, pr_1, pr_2 : \mathcal{G}_2 \to \mathcal{G}$ denote the groupoid multiplication and the two projections. An **isomorphism of symplectic groupoids** is an isomorphism of Lie groupoids which is a symplectomorphism as well.

Theorem 1.3.2. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid. Then the following hold.*

a) The unit $u : M \to \mathcal{G}$ embeds M as a Lagrangian submanifold of $\mathcal{G}$. In particular,

$$dim(\mathcal{G}) = 2dim(M).$$

b) The inversion $\iota : \mathcal{G} \to \mathcal{G}$ is anti-symplectic: $\iota^*\Omega = -\Omega$.

c) The tangent spaces of the source and target fibers are symplectically orthogonal:

$$Ker(ds_g) = Ker(dt_g)^\Omega, \quad \forall g \in \mathcal{G}.$$

d) There is a unique Poisson structure π on M such that $t : (\mathcal{G}, \Omega) \to (M, \pi)$ is a Poisson map. Moreover, $s : (\mathcal{G}, \Omega) \to (M, \pi)$ is anti-Poisson.

The coming theorem says that a symplectic groupoid integrates the Lie algebroid of the Poisson structure that it induces on its space of objects.

Theorem 1.3.3. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid and π the induced Poisson structure on M. Then the map*

$$T^*_\pi M \to Lie(\mathcal{G}), \quad (x, \alpha) \mapsto \Omega^\sharp_{1_x}((dt_{1_x})^*\alpha)$$

is an isomorphism of Lie algebroids over M. Consequently, for each $x \in M$:

a) This restricts to an isomorphism between the isotropy Lie algebra of $T^*_\pi M$ at x and the Lie algebra $\mathfrak{g}_x$ of $\mathcal{G}_x$.

b) We have $Im(\pi^\sharp_x) = T_x\mathcal{O}$ and so, if the orbits of $\mathcal{G}$ are connected, then the symplectic leaves of (M, π) coincide with the orbits of $\mathcal{G}$.

We say that a Poisson manifold (M, π) is integrated by a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ if it induces the Poisson structure π on M. It turns out that (M, π) is integrated by some symplectic groupoid if and only if the Lie algebroid $T^*_\pi M$ is integrated by some Lie groupoid. In this case we call (M, π) **integrable** and there is a unique (up to isomorphism) s-1-connected symplectic groupoid that integrates both (M, π) and $T^*_\pi M$. This is called the **Weinstein groupoid** of (M, π).

In the rest of this thesis we will study moment maps into Poisson manifolds that admit a proper integration.

Definition 1.3.4. A **proper integration** of a Poisson manifold (M, π) is a proper, s-connected symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ that induces the Poisson structure π on M.

Example 1.3.5. The cotangent bundle $(T^*M, \Omega_{can} = d\lambda_{can})$, considered as a bundle of abelian Lie groups over M, is the Weinstein groupoid of $(M, 0)$. The canonical isomorphism of Lie algebroids $T^*M \to \text{Lie}(T^*M)$ is in this case just given by:

$$(b, \alpha) \mapsto \left.\frac{d}{dt}\right|_{t=0} (b, t\alpha).$$

The groupoid T^*M is not proper. In the coming chapter we will answer the question of when a Poisson manifold with the zero-Poisson structure admits a proper integration.

Example 1.3.6. Let $\mathfrak{g}$ be a Lie algebra and G a Lie group integrating $\mathfrak{g}$. The cotangent bundle $(T^*G, -\Omega_{can})$ is a symplectic groupoid integrating the linear Poisson manifold $\mathfrak{g}^*$. To see this, we observe the following. Being the cotangent bundle of a Lie group, it admits a global bundle trivialization:

$$\varphi : G \times \mathfrak{g}^* \to T^*G, \quad (g, \alpha) \mapsto \alpha \circ (dL_{g^{-1}})_g.$$

This provides $G \times \mathfrak{g}^*$ with a symplectic structure which we still denote by $-\Omega_{can}$. Recall that the left-invariant Maurer-Cartan form $\Theta \in \Omega^1(G; \mathfrak{g})$ of a Lie group G is defined as:

$$\Theta_g(v) = (dL_{g^{-1}})_g(v), \quad g \in G, \quad v \in T_gG.$$

The symplectic form $-\Omega_{can}$ on $G \times \mathfrak{g}^*$ can be expressed as:

$$-\Omega_{can(g,\alpha)}((v, \beta), (v', \beta')) = -\langle \alpha, [\Theta_g(v), \Theta_g(v')]_{\mathfrak{g}} \rangle + \langle \beta', \Theta_g(v) \rangle - \langle \beta, \Theta_g(v') \rangle. \tag{1.4}$$

where $(v, \beta), (v', \beta') \in T_{(g,\alpha)}(G \times \mathfrak{g}^*) = T_gG \times \mathfrak{g}^*$. If we consider $G \ltimes \mathfrak{g}^*$ as the action groupoid of the coadjoint action of G on $\mathfrak{g}^*$, then $(G \times \mathfrak{g}^*, -\Omega_{can})$ is a symplectic groupoid. It follows from the above formula for $-\Omega_{can}$ that the target map is a Poisson map into $\mathfrak{g}^*$ with the linear Poisson structure defined in Example 1.1.12. This is a proper integration precisely if the coadjoint action is proper and G is connected. In particular, it is so if G is compact and connected.

1.4 Hamiltonian $\mathcal{G}$-spaces

In this section we introduce the type of groupoid-actions that integrate the canonical Lie algebroid action of a moment map.

1.4.1 Definition, basic properties and examples

Definition 1.4.1. A **Hamiltonian $\mathcal{G}$-space** $((S,\omega),(\mathcal{G},\Omega) \rightrightarrows M,\mu,m)$ consists of:

- a symplectic manifold (S,ω),
- a symplectic groupoid $(\mathcal{G},\Omega) \rightrightarrows M$,
- a map $\mu : S \to M$,
- a $\mathcal{G}$-action m along μ.

This quadruple is required to satisfy the Hamiltonian multiplicativity condition:

$$m^*\omega = pr_1^*\Omega + pr_2^*\omega,$$

where $m, pr_2 : \mathcal{G}\ {}_s\!\times_\mu S \to S$ and $pr_1 : \mathcal{G}\ {}_s\!\times_\mu S \to \mathcal{G}$. We will often just say that $\mu : (S,\omega) \to M$ is a Hamiltonian $\mathcal{G}$-space and suppress Ω and m from the notation.

Let $\mu : (S,\omega) \to M$ be a Hamiltonian $\mathcal{G}$-space. As for any action of a Lie groupoid $\mathcal{G}$, there is an induced action of Lie($\mathcal{G}$). Since the groupoid that is acting is symplectic, its Lie algebroid is canonically isomorphic to that of the Poisson manifold (M,π) that it integrates. Therefore it induces an action of the Lie algebroid $T^*_\pi M$ along the moment map μ.

Proposition 1.4.2. *Under the canonical isomorphism of Lie($\mathcal{G}$) with $T^*_\pi M$, the Lie algebroid action induced by the $\mathcal{G}$-action becomes the canonical Lie algebroid action*

$$a : \mu^*(T^*M) \to TS$$

along μ.

Proof. Let $\alpha \in \Omega^1(M)$ and $e \in \Gamma(\mathcal{A}_\mathcal{G})$ defined by $e(x) = \Omega^\sharp_{1_x}(dt^*_{1_x}(\alpha_x))$. We have to show that

$$\iota_{d(m_p)(e)}\omega = \mu^*\alpha.$$

Let $p \in S$, $x = \mu(p)$ and let $\Phi^t_e(1_x)$ denote the time-t-flow through 1_x of the right-invariant vector field on $\mathcal{G}$ generated by e. Since right-invariant vector fields are tangent to the source-fibers, their flow preserves these fibers. Therefore $\gamma(t) = (\Phi^t_e(1_x), p)$ defines a path in $\mathcal{G}\ {}_s\!\times_\mu S$ through $(1_x, p)$. Secondly, let $w \in T_pS$, let $p(t)$ be a path through p with $\dot{p}(0) = w$ and let w be the path in $\mathcal{G}\ {}_s\!\times_\mu S$ defined by $w(t) = (1_{\mu(p(t))}, p(t))$. Then we have:

$$\begin{aligned} dm(e(x),0) &= \left.\frac{d}{dt}\right|_{t=0} m(\gamma(t)) = d(m_p)_{1_x}(e(x)), \\ dm(du_x(d\mu_p(w)),w) &= \left.\frac{d}{dt}\right|_{t=0} m(w(t)) = w. \end{aligned}$$

Using this and the Hamiltonian multiplicativity condition we derive:

$$\begin{aligned} (\iota_{d(m_p)(e(x))}\omega)(w) &= (m^*\omega)((e(x),0),(du_x(d\mu_p(w)),w)) \\ &= (pr_1^*\Omega + pr_2^*\omega)((e(x),0),(du_x(d\mu_p(w)),w)) \\ &= (\iota_{e(x)}\Omega)(du_x(d\mu_p(w)) \\ &= (dt^*_{1_x}\alpha_x)(du_x(d\mu_p(w))) \\ &= (\mu^*\alpha)_p(w) \end{aligned}$$

as desired. ∎

In combination with propositions 1.1.17 and 1.1.18 respectively, we get the following result.

Corollary 1.4.3. *Let $\mu : (S, \omega) \to M$ be a Hamiltonian $\mathcal{G}$-space and π the Poisson structure on M induced by the symplectic groupoid. Then $\mu : (S, \omega) \to (M, \pi)$ is a Poisson map and the following equalities hold for every $p \in S$.*

a) $T_p\mathcal{O} = Im(a_p) = (\mathcal{F}_\mu)_p^\omega$.

b) $a_p(\mathfrak{g}_{\mu(p)}) = T_p\mathcal{O} \cap T_p\mathcal{O}^\omega$.

c) $Im(\pi^\sharp_{\mu(p)}) = d\mu_p(T_p\mathcal{O})$.

d) $\mathfrak{g}_p = Im(d\mu_p)^0$.

In particular we see that, as anticipated, the moment map of a Hamiltonian $\mathcal{G}$-space is a moment map in the Poisson geometric sense as well. By combining propositions 1.4.2 and 1.2.14 we further deduce:

Corollary 1.4.4. *Let $\mu : (S, \omega) \to (M, \pi)$ be a moment map and $(\mathcal{G}, \Omega) \to M$ an s-connected symplectic groupoid integrating π. Then there is at most one Hamiltonian action of $(\mathcal{G}, \Omega)$ along μ.*

One can thus think of a Hamiltonian $\mathcal{G}$-action as being an intrinsic part of the moment map, the existence of which is a property of the moment map. We now turn to some important examples.

Example 1.4.5. Every symplectic groupoid $(\mathcal{G}, \Omega)$ acts on itself from the left in a Hamiltonian fashion along its target map $t : (\mathcal{G}, \Omega) \to M$.

Hamiltonian G-spaces form an important class of examples.

Definition 1.4.6. A **symplectic G-space** is a symplectic manifold (S, ω) equipped with an action of a Lie group G that acts by symplectomorphisms.

Definition 1.4.7. A **Hamiltonian G-space** is a symplectic G-space together with a map $\mu : S \to \mathfrak{g}^*$, which satisfies:

- The equivariance condition: $\mu(g \cdot p) = \mathrm{Ad}^*_g(\mu(p)), \quad \forall g \in G, p \in S$;
- The weak Hamiltonian condition: $d\mu_\xi = \iota_{\xi_S}\omega, \quad \forall \xi \in \mathfrak{g}$.

Here $\mu_\xi = \langle \mu, \xi \rangle \in C^\infty(S)$, where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ denotes the canonical pairing and ξ_S is the image of ξ under the infinitesimal action $\mathfrak{g} \to \mathcal{X}(S)$, given explicitly by

$$\xi_S(p) = d(m_p)_e(\xi) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot p.$$

The map μ is called the **moment map** of the Hamiltonian G-space.

Remark 5. Let us elaborate on the definition of a Hamiltonian G-space. On a symplectic G-space M, the vector field ξ_M generated by any $\xi \in \mathfrak{g}$ is a symplectic vector field, that is: $\iota_{\xi_M}\omega$ is a closed 1-form. The weak Hamiltonian axiom ensures that the vector fields ξ_M are in fact Hamiltonian vector fields, which is to say that $\iota_{\xi_M}\omega$ is an exact 1-form.

Hamiltonian G-spaces are in bijective correspondence with Hamiltonian $G \ltimes \mathfrak{g}^*$-spaces, for the following reason. Pairs (S, μ) consisting of a G-space S and an equivariant map $\mu : S \to \mathfrak{g}^*$ correspond to actions of the action groupoid $G \ltimes \mathfrak{g}^*$ along moment maps μ. Indeed, the two actions are related by the formula

$$(g, \mu(p)) \cdot p = g \cdot p, \quad g \in G, \ p \in S.$$

Note here that, besides the associativity and the unit axiom, the groupoid action is required to satisfy

$$\mu((g, \mu(p)) \cdot p) = t(g, \mu(p)), \quad g \in G, \ p \in S,$$

which by the above relation and the definition of t this is equivalent to:

$$\mu(g \cdot p) = \mathrm{Ad}^*_g(\mu(p)), \quad g \in G, \ p \in S.$$

This is precisely the equivariance axiom of μ. Under this correspondence, the Hamiltonian G-spaces correspond to Hamiltonian $G \ltimes \mathfrak{g}^*$-spaces, because it follows from equation (1.4) that the Lie group action is weakly Hamiltonian and symplectic if and only if the corresponding action of the symplectic groupoid $(G \ltimes \mathfrak{g}^*, -\Omega_{can})$ is Hamiltonian.

Let us give a couple of examples of Hamiltonian G-spaces.

Convention 2. Whenever we deal with a Hamiltonian $\mathbb{T}^n$-space, we identify the Lie algebra of $\mathbb{T}^n$ with $\mathbb{R}^n$ by choosing a basis of the Lie algebra consisting of the tangent vectors of the form

$$\left.\frac{d}{ds}\right|_{s=0} (1, ..., 1, e^{2\pi i s}, 1, ..., 1).$$

Accordingly, we identify the dual of the Lie algebra with $(\mathbb{R}^n)^*$, which in turn we identify with $\mathbb{R}^n$ via the standard inner product. In this way the moment map takes values in $\mathbb{R}^n$.

Example 1.4.8. The standard symplectic space $(\mathbb{C}^n, -\omega_0)$ is a Hamiltonian $\mathbb{T}^n$-space for the action

$$(t_1, ..., t_n) \cdot (z_1, ..., z_n) = (t_1 z_1, ..., t_n z_n)$$

and the moment map

$$\mu_0(z) = \pi(|z_1|^2, ..., |z_n|^2).$$

We call this the standard Hamiltonian $\mathbb{T}^n$-space structure on $\mathbb{C}^n$.

Example 1.4.9. Another standard symplectic space is $(\mathbb{T}^n \times \mathbb{R}^n, \omega_0)$ where

$$\omega_0 = \sum_{j=1}^{n} d\theta_j \wedge dx_j.$$

This is a Hamiltonian $\mathbb{T}^n$-space for the action

$$s \cdot (t, x) = (st, x), \quad s, t \in \mathbb{T}^n, x \in \mathbb{R}^n,$$

and the moment map

$$\mu_0(t, x) = x.$$

We call this the standard Hamiltonian $\mathbb{T}^n$-space structure on $\mathbb{T}^n \times \mathbb{R}^n$.

Example 1.4.10. Let (V,ω) be a symplectic vector space of dimension $2n$ and G a Lie group. By a **symplectic representation** of G we mean a morphism of Lie groups

$$r : G \to \mathrm{Sp}(V,\omega).$$

Let such a representation be given. We denote by $\rho = dr_e : \mathfrak{g} \to \mathfrak{sp}(V,\omega)$ the induced infinitesimal representation and we write $\xi \cdot v = \rho(\xi)(v)$ for $\xi \in \mathfrak{g}$ and $v \in V$. The map $\mu : V \to \mathfrak{g}^*$ defined by

$$\langle \mu(v), \xi \rangle = \frac{1}{2}\omega(\xi \cdot v, v), \quad \xi \in \mathfrak{g}, v \in V,$$

is a moment map for the symplectic G-action on (V,ω), called the moment map of the symplectic representation.

Example 1.4.11. Let $(S,\omega = d\lambda)$ be an exact symplectic manifold together with an action of a Lie group G that preserves λ. Then

$$\langle \mu, \xi \rangle = -\iota_{\xi_S}\lambda, \quad \xi \in \mathfrak{g},$$

defines a moment map for this symplectic G-space. An important case of this is the following. Let X be a G-space and $\pi : T^*X \to X$ the footpoint projection. Then $(T^*X, -\Omega_{can})$ satisfies the requirements of the above for the G-action

$$g \cdot \alpha = \alpha \circ (dm_{g^{-1}})_{g\cdot\pi(\alpha)}.$$

So it is a Hamiltonian G-space. It follows from G-equivariance of π that:

$$\langle \mu \circ \alpha, \xi \rangle = \iota_{\xi_X}\alpha, \quad \forall \alpha \in \Omega^1(X).$$

In particular, this applies to the cotangent bundle T^*G of a Lie group G. Indeed, both the actions

$$h \cdot g = hg, \quad h * g = gh^{-1}, \quad g, h \in G,$$

of G on itself lift to actions on the cotangent bundle T^*G. In this way, we obtain two moment maps: the target map and (up to a minus-sign) the source map of the symplectic groupoid $(T^*G, -\Omega_{can})$.

We end this section with a short discussion on another class of moment maps: the complete symplectic realizations.

Definition 1.4.12. A Poisson map $\varphi : (M,\pi_M) \to (N,\pi_N)$ is called **complete** if the vector field $X_{f\circ\varphi}$ is complete whenever X_f is complete.

One can show that the moment map of any Hamiltonian $\mathcal{G}$-space is a complete Poisson map. This brings up the question: which complete moment maps are the moment map of some Hamiltonian action? For complete symplectic realizations, this problem is solved: the Weinstein groupoid of (M,π) acts along any complete symplectic realization of (M,π) in a Hamiltonian fashion. We will only need this fact for the zero-Poisson structure. In that case, this takes the following form.

Example 1.4.13. Let $\mu : (S,\omega) \to (B,0)$ be a complete symplectic realization. The Weinstein groupoid of $(B,0)$ is (T^*B,Ω_{can}). We define an action of T^*B along μ by:

$$\alpha \cdot p = \Phi^1_{a(\alpha)}(p), \quad \alpha \in T^*_bB, \quad p \in \mu^{-1}(b),$$

where $a(\alpha) \in \mathcal{X}(\mu^{-1}(b))$ is the vector field determined by the equation:

$$\iota_{a(\alpha)_p}\omega = (d\mu_p)^*\alpha, \quad p \in \mu^{-1}(b).$$

This vector field is indeed tangent to the fiber of μ because μ is a Poisson map into the zero-Poisson structure. Using this once more, it is straightforward to verify that the above defines a smooth action of T^*B along μ, provided that the time-1-flow of $a(\alpha)$ through p exists. To see that it does, note that for every $\alpha \in T^*_bB$ there is a compactly supported $f \in C^\infty(B)$ such that $df_b = \alpha$, and by completeness of μ it holds that $X_{f\circ\mu}$ is complete. Since $X_{f\circ\mu}$ restricts to $a(\alpha)$ on $\mu^{-1}(b)$ it follows that $a(\alpha)$ is complete as well, as desired.

Proposition 1.4.14. *This action of (T^*B,Ω_{can}) along $\mu : (S,\omega) \to B$ is Hamiltonian.*

Proof. For $\tau \in \mathbb{R}$ we define

$$\varphi^\tau : T^*B \,{}_s\!\times_\mu S \to T^*B \,{}_s\!\times_\mu S, \quad (\alpha,p) \mapsto (\tau\alpha,p).$$

Here we denote the footpoint projection (and source-map) of T^*B by s. Observe that

$$m^*\omega - pr_2^*\omega = (\varphi^1)^*m^*\omega - (\varphi^0)^*m^*\omega = \int_0^1 \left.\frac{d}{dt}\right|_{t=\tau} (\varphi^t)^*m^*\omega \; d\tau.$$

Therefore it is enough to show that

$$\left.\frac{d}{dt}\right|_{t=\tau} (\varphi^t)^*m^*\omega = (pr_1)^*\Omega_{can} \tag{1.5}$$

for all $\tau > 0$. To this end, define the time-dependent vector field $\{X^\tau\}_{\{\tau>0\}}$ on $T^*B \,{}_s\!\times_\mu S$ by:

$$X^\tau_{(\alpha,p)} = \left.\frac{d}{dt}\right|_{t=0} \left(\frac{t}{\tau}\alpha + \alpha, p\right), \quad (\alpha,p) \in T^*B \,{}_s\!\times_\mu S.$$

The flow of $\{X^\tau\}_{\{\tau>0\}}$ starting at time 1 is given by $(\tau,p) \mapsto \varphi^\tau(p)$. We therefore find that:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=\tau} (\varphi^t)^*m^*\omega &= (\varphi^\tau)^*\mathcal{L}_{X_\tau}(m^*\omega)\\ &= d((\varphi^\tau)^*\iota_{X_\tau}(m^*\omega))\end{aligned}$$

where we have used a well-known formula in the first equality and Cartan's magic formula combined with closedness of ω in the second. Now note that

$$\begin{aligned}dm(X^\tau_{(\alpha,p)}) &= \left.\frac{d}{dt}\right|_{t=0} \left(\frac{t}{\tau}\alpha + \alpha\right)\cdot p\\ &= \left.\frac{d}{dt}\right|_{t=0} \left(\frac{t}{\tau}\alpha\right)\cdot(\alpha\cdot p)\\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi^t_{a(\alpha/\tau)}(\alpha\cdot p)\\ &= a(\alpha/\tau)_{(\alpha\cdot p)},\end{aligned}$$

from which we deduce that:

$$\begin{aligned}((\varphi^\tau)^*\iota_{X_\tau}(m^*\omega))_{(\alpha,p)} &= d(m\circ\varphi^\tau)^*_{(\alpha,p)}(\iota_{a(\alpha)}\omega)_{(\tau\alpha\cdot p)}\\ &= \alpha\circ d(\mu\circ m\circ\varphi^\tau)_{(\alpha,p)}\\ &= \alpha\circ ds_\alpha\circ d(pr_1)_{(\alpha,p)}\\ &= ((pr_1)^*\lambda_{can})_{(\alpha,p)}.\end{aligned}$$

for all $(\alpha,p)\in T^*B\,{}_s\times_\mu S$ and $\tau>0$. We thus see that (1.5) holds, as was left to show. $\square$

1.4.2 Quotients by Hamiltonian actions

The theory of quotients of symplectic manifolds by Hamiltonian $\mathcal{G}$-actions revolves around the Symplectic Reduction Theorem. From a physicist's perspective, this a tool to reduce the degrees of freedom of a mechanical system with symmetry. To a Poisson geometer, it is a tool to construct new Poisson manifolds. To us, it will be a tool to construct Hamiltonian G-spaces.

We start by considering symplectic reduction from the Poisson geometric perspective by the following observation, which is not very hard to prove.

Theorem 1.4.15. *Suppose that $\mu:(S,\omega)\to M$ is a Hamiltonian $\mathcal{G}$-space, the action of which is free and proper. Then $S/\mathcal{G}$ is smooth and there is a unique Poisson structure $\pi_{\mathcal{G}}$ on $S/\mathcal{G}$ for which the quotient map $(S,\omega)\to(S/\mathcal{G},\pi_{\mathcal{G}})$ is a Poisson map.*

The Symplectic Reduction Theorem provides a new perspective on the symplectic leaves of $S/\mathcal{G}$. It is originally due to Marsden-Weinstein [MW74] and Meyer [Mey73] in the case of Hamiltonian G-spaces and was later generalized to the realm of Hamiltonian $\mathcal{G}$-spaces by Mikami-Weinstein [MW88]. In order to state it, note that for each $x\in M$, the isotropy group $\mathcal{G}_x$ acts on $\mu^{-1}(x)$. Indeed, if $g\in\mathcal{G}_x$, then $\mu(g\cdot p)=t(g)=x$. Therefore we can consider the so-called reduced phase space

$$P_x=\frac{\mu^{-1}(x)}{\mathcal{G}_x}.$$

Theorem 1.4.16 (Symplectic reduction)**.** *Let $\mu:(S,\omega)\to M$ be a Hamiltonian $\mathcal{G}$-space. Suppose that $\mathcal{G}_x$ acts on $\mu^{-1}(x)$ in a free and proper fashion. Then the following hold:*

a) The point $x\in M$ is a regular value of μ and the reduced space P_x is smooth.

b) The reduced space P_x admits a unique symplectic form ω_x satisfying

$$\pi^*\omega_x=i^*\omega, \tag{1.6}$$

where $i:\mu^{-1}(x)\hookrightarrow M$ denotes inclusion.

We need a short lemma to prove this.

Lemma 1.4.17. *The isotropy groups of a symplectic groupoid are isotropic.*

Proof. Let $(\mathcal{G},\Omega)$ be a symplectic groupoid over M and $x\in M$. The tangent space to $\mathcal{G}_x$ at a point g is

$$T_g\mathcal{G}_x=\mathrm{Ker}(ds_g)\cap\mathrm{Ker}(dt_g).$$

Because $\mathrm{Ker}(ds_g)$ and $\mathrm{Ker}(dt_g)$ are Ω-orthogonal and the Ω-complement of an intersection is the sum of the Ω-complements, it follows that $T_g\mathcal{G}_x\subset(T_g\mathcal{G}_x)^\Omega$, as desired. $\square$

Proof of Theorem 1.4.16. Because the action of $\mathcal{G}_x$ on $\mu^{-1}(x)$ is free, it follows that $\mathcal{G}_p$ is trivial for every $p \in \mu^{-1}(x)$. From Corollary 1.4.3*d*, it therefore follows that x is a regular value of μ. The rest of statement a follows from the usual theory of free and proper Lie group actions. To see that a form as in (1.6) exists, we need to verify that the form $i^*\omega$ is basic with respect to the action groupoid $\mathcal{G}_x \ltimes \mu^{-1}(x)$. In other words, we need to check that $m_x^* i^*\omega = (pr_2)_x^* i^*\omega$, where

$$m_x, (pr_2)_x : \mathcal{G}_x \times \mu^{-1}(x) \to \mu^{-1}(x)$$

are the multiplication and projection. This follows by pulling back the equation

$$m^*\omega = pr_1^*\Omega + pr_2^*\omega$$

along the inclusion $\mathcal{G}_x \times \mu^{-1}(x) \hookrightarrow \mathcal{G} \,{}_s\!\times_\mu S$ and applying the previous lemma. The form ω_x is closed because ω is closed and π is a surjective submersion. So it remains to see that ω_x is non-degenerate, or equivalently, that $\mathrm{Ker}((i^*\omega)_p)$ is the tangent space at p to the orbit $\mathcal{O}_{\mathcal{G}_x}$ of the $\mathcal{G}_x$-action, for every $p \in \mu^{-1}(x)$. In view of parts a and b of Corollary 1.4.3 it holds that:

$$\begin{aligned} T_p\mathcal{O}_{\mathcal{G}_x} &= a_p(\mathfrak{g}_x) \\ &= (\mathcal{F}_\mu)_p \cap (\mathcal{F}_\mu^\omega)_p \\ &= \mathrm{Ker}(d\mu_p) \cap \mathrm{Ker}(d\mu_p)^\omega \\ &= \mathrm{Ker}((i^*\omega)_p) \end{aligned}$$

which ends the proof. □

If the $\mathcal{G}$-action is free and proper, then the assumptions of the Symplectic Reduction Theorem are satisfied for every $x \in M$. Proposition 1.1.9 can be used to show that the connected compontents of the reduced spaces (P_x, ω_x) are the symplectic leaves of $\mathcal{S}/\mathcal{G}$, in the following sense.

Theorem 1.4.18. *Suppose that we are in the setting of Theorem 1.4.15. Let $p \in S$ and $x = \mu(p)$. Then the inclusion $i : \mu^{-1}(x) \hookrightarrow S$ descends to a map*

$$\bar{i} : P_x \to S/\mathcal{G}.$$

that maps the connected component of (P_x, ω_x) through $[p]$ symplectomorphically onto the symplectic leaf of $S/\mathcal{G}$ through $\bar{i}([p])$.

The value of the Symplectic Reduction Theorem lies not in the complexity of its proof, but rather in its ability to construct interesting spaces in Poisson geometry. The final result of this section provides us with a way to equip the reduced phase spaces with a Hamiltonian action and so we can use it to construct new Hamiltonian spaces.

Proposition 1.4.19. *In addition to the hypotheses of the Symplectic Reduction Theorem, suppose that $(\mathcal{H}, \Omega_\mathcal{H}) \rightrightarrows N$ is another symplectic groupoid acting in a Hamiltonian fashion along a map $\varphi : S \to N$, in such a way that:*

- *φ is $\mathcal{G}_x$-invariant and $\mu^{-1}(x)$ is $\mathcal{H}$-invariant,*
- *the actions of $\mathcal{G}_x$ and $\mathcal{H}$ commute on $\mu^{-1}(x)$.*

Then the action of $\mathcal{H}$ descends to a Hamiltonian action on (P_x, ω_x) along the map

$$\varphi_x : P_x \to N$$

determined uniquely by the fact that:

$$\pi^*\varphi_x = i^*\varphi.$$

Proof. The invariance and commutativity assumptions imply that φ descends to the desired map φ_x and the $\mathcal{H}$-action descends to an $\mathcal{H}$-action along φ_x, given by:

$$h \cdot [p] = [h \cdot p]$$

for $h \in \mathcal{H}$ and $p \in \mu^{-1}(x)$ such that $s(h) = \varphi_x([p])$. This action is Hamiltonian because the action of $\mathcal{H}$ along φ is so, as is readily verified. □

1.4.3 The symplectic isotropy representations

We will see later that a large part of the local behaviour of the moment map of a Hamiltonian $\mathcal{G}$-space is determined by its symplectic isotropy representations. In this final section of the chapter we introduce these.

As we saw in Section 1.2.5, for any orbit $\mathcal{O}$ of a Lie groupoid $\mathcal{G}$, the Lie subgroupoid $\mathcal{G}_\mathcal{O}$ acts on the normal bundle $\mathcal{N}_\mathcal{O}$. In particular, for every $x \in M$ we have a representation of the isotropy group $\mathcal{G}_x$ on $\mathcal{N}_x$. This is called the **isotropy representation** of $\mathcal{G}$ at x. The Hamiltonian variant of this is the following. Applying the above to the action groupoid of a Hamiltonian $\mathcal{G}$-space we obtain a representation of $\mathcal{G}_p$ on $\mathcal{N}_p = T_pS/T_p\mathcal{O}$ for each $p \in S$. The linear symplectic form ω_p on T_pS does not descend to $\mathcal{N}_p$. It does, however, descend to a linear symplectic form on the **symplectic slice** at p:

$$\mathcal{S}_p := \frac{T_p\mathcal{O}^\omega}{T_p\mathcal{O} \cap T_p\mathcal{O}^\omega}.$$

We will denote this symplectic form by ω_p as well.

Proposition 1.4.20. *There is a unique symplectic representation of $\mathcal{G}_p$ on $(\mathcal{S}_p, \omega_p)$ for which the canonical injection $\mathcal{S}_p \to \mathcal{N}_p$ is $\mathcal{G}_p$-equivariant.*

Proof. The representation on $\mathcal{S}_p$ is given by:

$$g \cdot [v] = [dm_{(g,p)}(\hat{v})], \text{ where } \hat{v} \in (dpr_2)^{-1}_{(g,p)}(v) \cap dm^{-1}_{(g,p)}(T_p\mathcal{O}^\omega).$$

Such a $\hat{v}$ exists, for if $v \in T_p\mathcal{O}^\omega = \mathrm{Ker}(d\mu_p)$ and $g \in \mathcal{G}_p$, then $d\mu(v) = 0$, so that $(0, v)$ is tangent to $\mathcal{G}\ {}_s\!\times_\mu S$ at (g, p). Since

$$d\mu(dm(0, v)) = dt(dpr_1(0, v)) = 0$$

we see that $dm(0, v) \in T_p\mathcal{O}^\omega$ and so $\hat{v} = (0, v)$ satisfies the requirements. Moreover, that the above does not depend on any choices follows from the well-definedness of the isotropy representation of $\mathcal{G}_p$ on $\mathcal{N}_p$. Obviously, this is the unique representation on $\mathcal{S}_p$ for which the canonical injection $\mathcal{S}_p \to \mathcal{N}_p$ is $\mathcal{G}_p$-equivariant. So it remains to see that the action preserves ω_p. This holds because

$$\begin{aligned} \omega_p(g \cdot [v], g \cdot [w]) &= (m^*\omega)_{(g,p)}((0, v), (0, w)) \\ &= \omega_p([v], [w]), \end{aligned}$$

where in the last equality we used that the $\mathcal{G}$-action is Hamiltonian. □

Definition 1.4.21. We call this symplectic representation of $\mathcal{G}_p$ on $(\mathcal{S}_p, \omega_p)$ the **symplectic isotropy representation** of the Hamiltonian $\mathcal{G}$-space at p.

Chapter 2

Symplectic torus bundles and integral affine geometry

The first step in our study of Hamiltonian $\mathcal{G}$-spaces is to consider the simplest class of such spaces: the ones over the zero-Poisson structure. Before studying the actions of proper symplectic groupoids that integrate the zero-Poisson structure we explore the groupoids themselves. As we observed in the previous chapter, the Weinstein groupoid of $(M, 0)$ is just the cotangent bundle $(T^*M, \Omega_{can}) \rightrightarrows M$ considered as a symplectic groupoid over M. This fails to be proper, which leads one to wonder:

Which manifolds admit a proper integration of the zero-Poisson structure?

In this chapter, which is largely based on parts of [CFT16], we answer this question.

2.1 Proper integrations of the zero-Poisson structure

First, we provide a more detailed description of proper integrations of the zero-Poisson structure.

Proposition 2.1.1. *The proper integrations of* $(M, 0)$ *are precisely the symplectic torus bundles over* M.

Proof. By definition, a symplectic torus bundle is a symplectic groupoid, the source and target map of which coincide and the isotropy groups of which are tori (that is, they are compact, connected and abelian). Since its target map coincides with its source map, it is both Poisson and anti-Poisson. Therefore it must induce the zero-Poisson structure on its base. It is a fact that a surjective submersion with compact and connected fibers is proper, and so the source map of a symplectic torus bundle is proper[1], which in turn implies that a symplectic torus bundle is a proper groupoid. So a symplectic torus bundle is a proper integration of $(M, 0)$.

Conversely, suppose that $(\mathcal{G}, \Omega) \rightrightarrows M$ is a proper integration of the zero-Poisson structure. Then $t : (\mathcal{G}, \Omega) \to (M, 0)$ is an isotropic realization. So it is a Lagrangian fibration and Theorem 1.3.2c implies that $\mathrm{Ker}(ds) = \mathrm{Ker}(dt)$. Since the s-fibers are connected, they are

[1] Alternatively, in the coming section we will see that a torus bundle is a fiber bundle with compact fibers, hence its source-map (the bundle projection) is proper.

the leaves of the foliation that integrates the distribution $\mathrm{Ker}(ds)$. But the same holds for the t-fibers and therefore the s- and t-fibers coincide. Since every such leaf contains a unit, this means that $s = t$. In particular, the isotropy groups of $\mathcal{G}$ are the s-fibers and hence they are connected. Because $(\mathcal{G}, \Omega)$ integrates the zero-Poison structure, its isotropy Lie algebras are abelian. Therefore, by connectedness, so are its isotropy groups. Finally, since $\mathcal{G}$ is proper, its isotropy groups are compact as well. □

With the previous result in mind, we will henceforth restrict our attention to symplectic torus bundles.

2.2 Integral affine structures

The central theme of this section is the relationship between tori and integral affine geometry. As a highlight, we answer the main question of this chapter, by showing that the isomorphism classes of symplectic torus bundles over a given manifold B are in bijective correspondence with integral affine structures on B.

2.2.1 Integral affine vector spaces and tori

Lattices

Throughout, let V be a real, finite-dimensional vector space. A **lattice** Λ in V is a discrete additive subgroup of V. Every lattice Λ in V gives rise to a **dual lattice**:

$$\Lambda^* = \{\alpha \in V^* \mid \alpha(\Lambda) \subset \mathbb{Z}\}.$$

The **rank** of Λ is the dimension of the $\mathbb{R}$-linear subspace that it spans in V. A **full** lattice in V is a lattice with rank equal to the dimension of V.

Definition 2.2.1. An **integral affine vector space** is a pair (V, Λ) consisting of a vector space V and a full lattice Λ in V. A morphism of integral affine vector spaces

$$\varphi : (V, \Lambda) \to (V', \Lambda')$$

is a linear map that sends Λ into Λ'.

As we will now see, integral affine vector spaces are closely related to tori. Recall that:

Definition 2.2.2. A compact, connected and abelian Lie group T is called a torus.

The following is a standard result.

Proposition 2.2.3. *Let Λ be a lattice in V. Then V/Λ is a torus if and only if Λ is full as a lattice in V.*

By this proposition, we see that every integral affine vector space (V, Λ) gives rise to a torus V/Λ. Conversely, every torus T gives rise to a lattice Λ_T in its Lie algebra $\mathfrak{t}$. Indeed, since T is abelian its exponential map is a morphism of Lie groups and its kernel is a subgroup of $\mathfrak{t}$. We will always write $\Lambda_T := \mathrm{Ker}(\exp_T)$. Since $\exp_T$ is locally injective, its kernel is a lattice in $\mathfrak{t}$. Because T is connected, its exponential map is surjective and hence it induces an isomorphism of Lie groups:

$$\exp_T : \mathfrak{t}/\Lambda_T \xrightarrow{\sim} T.$$

Because T is compact, it follows from Proposition 2.2.3 that Λ_T is a full lattice in $\mathfrak{t}$. Summing this up, we have:

Theorem 2.2.4. *There is a bijective correspondence:*

$$\left\{\begin{array}{l}\text{Isomorphism classes of integral}\\ \text{affine vector spaces } (V,\Lambda)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of tori } T\end{array}\right\}$$

which associates to a class $[(V,\Lambda)]$ *the class* $[V/\Lambda]$ *and conversely associates to a class* $[T]$ *the class* $[(\mathfrak{t},\Lambda_T)]$.

We say that $B = \{b_1, ..., b_n\} \subset V$ is a **basis** of the lattice Λ if B is linearly independent over $\mathbb{R}$ and $\mathrm{Span}_{\mathbb{Z}}(B) = \Lambda$. The cardinality of such a basis is equal to the rank of Λ. The proof of Proposition 2.2.3 hinges on the fact that every lattice Λ in V admits a basis. Using such a basis one can construct:

- An isomorphism $V \cong \mathbb{R}^n$ that identifies Λ with $\mathbb{Z}^k \times \{0\}$, where $k = \mathrm{rk}(\Lambda)$.
- An isomorphism of Lie groups of $V/\Lambda \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$.

Here $\mathbb{T}^k$ denotes the standard torus: the product of k copies of S^1. Consequently, every integral affine vector space is isomorphic to $(\mathbb{R}^n, \mathbb{Z}^n)$ and every n-dimensional torus is isomorphic to $\mathbb{T}^n$. This justifies the terminology of Definition 2.2.2. Moreover, it shows that the previous theorem is rather trivial, as both sets consist of a single element. However, the bijection in the theorem is canonical and therefore generalizes to integral affine vector bundles as we shall see in the next section.

Primitive sublattices

Let (V,Λ) be an integral affine vector space. A natural question that arises from the notion of a basis is: when does a basis of a sublattice $\widetilde{\Lambda}$ in Λ extend to one of Λ? A sublattice $\widetilde{\Lambda}$ for which some (or equivalently any) basis extends to one of Λ is called **primitive**. The following characterization of primitive sublattices is well-known.

Proposition 2.2.5. *Let* (V,Λ) *be an integral affine vector space,* $\widetilde{\Lambda}$ *a sublattice of* Λ *and* $\widetilde{V} = \mathrm{Span}_{\mathbb{R}}(\widetilde{\Lambda})$. *Then the following are equivalent:*

a) The sublattice $\widetilde{\Lambda}$ *is primitive.*

b) The equality $\widetilde{V} \cap \Lambda = \widetilde{\Lambda}$ *holds.*

In terms of tori, we can characterize primitive sublattices of Λ_T as follows.

Corollary 2.2.6. *Let* T *be a torus. There is a bijective correspondence:*

$$\{\text{Subtori of } T\} \longleftrightarrow \{\text{Primitive sublattices of } \Lambda_T\}$$

which associates to a subtorus H *the primitive sublattice* Λ_H.

Proof. Given a subtorus H of T, the exponential map of H is the restriction of that of T to $\mathfrak{h}$ and Λ_H is a full lattice in $\mathfrak{h}$. Therefore we have

$$\Lambda_H = \mathfrak{h} \cap \Lambda_T = \mathrm{Span}_{\mathbb{R}}(\Lambda_H) \cap \Lambda,$$

so that $\Lambda_H \subset \Lambda_T$ is indeed primitive. We further have

$$H = \exp_T(\mathfrak{h}) = \exp_T(\mathrm{Span}_{\mathbb{R}}(\Lambda_H)),$$

so that $H \mapsto \Lambda_H$ is injective. For surjectivity, let a primitive sublattice $\widetilde{\Lambda}$ of Λ_T be given. The connected subgroup

$$H := \exp_T(\mathrm{Span}_{\mathbb{R}}(\widetilde{\Lambda}))$$

is compact (because it is the image of a compact parallelipiped) and hence defines a subtorus of T with Lie algebra $\mathfrak{h} = \mathrm{Span}_{\mathbb{R}}(\widetilde{\Lambda})$. As before, we have $\Lambda_H = \mathfrak{h} \cap \Lambda_T$ which equals $\widetilde{\Lambda}$ by primitivity. This proves surjectivity, as was left to show. □

2.2.2 Integral affine vector bundles and torus bundles

In this section we generalize the notion of integral affine vector space to vector bundles. Throughout, let $E \to M$ be a real vector bundle of rank k.

Definition 2.2.7. A smooth lattice $\Lambda \subset E$ is a normal Lie subgroupoid of E with the property that each isotropy group Λ_x is a full lattice in E_x. A pair (E, Λ) is called an integral affine vector bundle.

Proposition 2.2.8. *Let $\{\Lambda_x\}_{x \in M}$ be a collection of full lattices $\Lambda_x \subset E_x$ and set*

$$\Lambda = \bigsqcup_{x \in M} \Lambda_x.$$

The following are equivalent.

a) The subspace Λ is a smooth lattice in E.

b) Every $x \in M$ admits an open neighbourhood U and local frame e for E over U for which

$$\Lambda_y = \mathbb{Z}e_1(y) \oplus \dots \oplus \mathbb{Z}e_k(y), \quad \forall y \in U. \tag{2.1}$$

c) The subset $\Lambda \subset E$ is closed and the bundle projection restricts to a topological covering map $\Lambda \to M$.

We need the following lemma to prove this.

Lemma 2.2.9. *Let (V, Λ) be an integral affine vector space and $e = \{e_1, \dots, e_n\}$ an $\mathbb{R}$-basis of V such that $e_i \in \Lambda$ for each i. Then*

$$\Lambda \subset \mathit{Span}_{\mathbb{Q}}\{e_1, \dots, e_n\}.$$

Proof. Pick a basis γ of the lattice Λ. It is enough to show that the matrix A defined by $\gamma_i = \sum_{j=1}^n A_i^j e_j$ has rational coefficients. Its inverse satisfies $e_i = \sum_{j=1}^n (A^{-1})_i^j \gamma_j$. Since $e \subset \Lambda$, it follows that A^{-1} has integer coefficients and so A has rational coefficients by Cramer's rule. □

Proof of Proposition 2.2.8. We prove the implications from up to down. First assume a. The argument that we give here comes from [Luk08, Lemma 1.2.8]. Let $x \in M$ and choose a basis $\gamma_1, \dots, \gamma_k$ of the lattice Λ_x in E_x. Since Λ is a Lie subgroupoid, the restriction of π to Λ is still a surjective submersion. So there are smooth local sections $e_1, \dots, e_k$ of π, defined on some connected open neighbourhood U of x, that map into Λ and satisfy $e_i(x) = \gamma_i$ for each i. Since $e := \{e_1, \dots, e_k\}$ is linearly independent at x, we can shrink U such that e is frame of E over U. By the previous lemma, every element of $\Lambda|_U$ has rational coefficients with respect to e. So because the connected components of $\mathbb{Q}$ are single points, every continuous local section of $\pi : \Lambda|_U \to U$ must have locally constant coefficient functions with respect to e. This implies that for every $q \in \mathbb{Q}^n$ the set

$$U_q = \left\{ y \in U | \sum_{i=1}^{k} q^i e_i(y) \in \Lambda_y \right\}$$

is open in U. By continuity and the assumption that Λ is closed in E, each U_q is closed in U as well. Connectedness of U and the fact that (2.1) holds at x thus imply that (2.1) holds at all $y \in U$. So b follows from a.

Suppose now that b holds. Every local frame as in (2.1) induces a trivialization of $E|_U$ that identifies $\Lambda|_U$ with $U \times \mathbb{Z}^n$. This implies that $\Lambda \to M$ is a covering map since the projection $U \times \mathbb{Z}^n \to U$ is a covering map for every open U in M, and it implies that Λ is closed in E because $\mathbb{Z}^n$ is closed in $\mathbb{R}^n$.

Finally, suppose that c holds. Every topological covering of a smooth manifold M is a topological manifold and admits a unique smooth structure for which the covering projection is a local diffeomorphism. This turns Λ into a smooth lattice in E, which shows that c implies a. $\square$

As in the vector space case, there is a bijective correspondence between integral affine vector bundles and torus bundles over a given manifold. However, our true interest lies in the analogue of this for symplectic torus bundles, which is the content of the coming section.

2.2.3 Integral affine structures and symplectic torus bundles

Recall that $\mathbb{R}^n \ltimes \mathrm{GL}_n(\mathbb{Z})$ is the group of integral affine transformations of $\mathbb{R}^n$. These are maps $\mathbb{R}^n \to \mathbb{R}^n$ of the form:

$$x \mapsto p + Ax, \quad p \in \mathbb{R}^n,\ A \in \mathrm{GL}_n(\mathbb{Z}).$$

Further recall that an integral affine map $\mathbb{R}^n \to \mathbb{R}^k$ is a map of the form:

$$x \mapsto p + Ax, \quad p \in \mathbb{R}^n,\ A \in \mathrm{M}(k \times n, \mathbb{Z}).$$

Definition 2.2.10. An **integral affine atlas** on a manifold B is an atlas $\mathcal{I} = \{(U, \chi_i)\}_{i \in I}$ for which each transition map $\chi_i \circ \chi_j^{-1}$ is the restriction of an element of $\mathbb{R}^n \ltimes \mathrm{GL}_n(\mathbb{Z})$. An **integral affine structure** is a maximal such atlas. Given an integral affine manifold $(B, \mathcal{I})$ we call a chart $(U, \chi) \in \mathcal{I}$ an **integral affine chart**. A **morphism** of integral affine manifolds

$$\varphi : (B, \mathcal{I}) \to (B', \mathcal{I}')$$

is a smooth map which in integral affine coordinates has the following property: on each connected component of the coordinate domain it is the restriction of an integral affine map $\mathbb{R}^n \to \mathbb{R}^k$.

An integral affine structure can viewed as a type of lattice in the cotangent bundle.

Proposition 2.2.11. *There is a bijective correspondence:*

$$\left\{\begin{array}{c}\textit{Integral affine structures}\\ \mathcal{I} \textit{ on } B\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Smooth Lagrangian lattices}\\ \Lambda \subset (T^*B, \Omega_{can})\end{array}\right\}$$

that associates to an integral affine structure $\mathcal{I}$ the smooth lattice Λ given by:

$$\Lambda_b = \mathbb{Z}\ d\chi^1_b \oplus ... \oplus \mathbb{Z}\ d\chi^n_b, \quad b \in B, \tag{2.2}$$

where χ is any integral affine chart for $(B, \mathcal{I})$ around b.

Proof. First note that the lattice Λ is well-defined, because for any two integral affine charts $(U, \chi), (V, \varphi) \in \mathcal{I}$ it holds that:

$$\mathrm{Jac}(\chi \circ \varphi^{-1})(x) \in \mathrm{GL}_n(\mathbb{Z}), \quad \forall x \in \varphi(U \cap V).$$

By construction, Λ is a smooth lattice in T^*B. The fact that it is a Lagrangian submanifold of (T^*B, Ω_{can}) is equivalent to saying that all local sections of Λ are Lagrangian sections of (T^*B, Ω_{can}) (that is, their image is a Lagrangian submanifold), because each such section is a diffeomorphism onto an open in Λ. A local section σ of T^*B is Lagrangian if and only if $\sigma^*\Omega_{can} = 0$ and this is in turn is equivalent to σ being a closed 1-form, because $\sigma^*\Omega_{can} = d\sigma$ by the defining property of the Liouville 1-form. In conclusion: Λ is a Lagrangian submanifold if and only if all of its local sections are closed as 1-forms. Any local section σ of Λ can locally be written as $\sigma = \sum_{i=1}^n k_i d\chi^i$ for some constants $k_i \in \mathbb{Z}$ and an integral affine chart χ, so it is locally exact and hence closed. This proves that Λ is indeed a smooth Lagrangian lattice in (T^*B, Ω_{can}).

Now suppose that we are given a smooth Lagrangian lattice Λ in (T^*B, Ω_{can}). Let $b \in B$ and choose a local frame e of T^*B, defined over an open neighbourhood U of x, such that:

$$\Lambda_b = \mathbb{Z}e_1(b) \oplus ... \oplus \mathbb{Z}e_n(b), \quad \forall b \in U.$$

By our previous discussion, each e_i is closed as a 1-form on U. Hence by the Poincare Lemma we can shrink U so that there are $\chi_1, ..., \chi_n \in C^\infty(U)$ such that $e_i = d\chi_i$ for each i. By the inverse function theorem, we can further shrink U such that $(U, \chi = (\chi_1, ..., \chi_n))$ is a chart for B around b. Doing this for all $b \in B$, we obtain an atlas $\mathcal{A}$ of B with the property that (2.2) holds for every $(U, \chi) \in \mathcal{A}$ and $b \in U$. Moreover, as any open cover of a manifold can be refined by a good open cover, we can assume that each two chart domains in $\mathcal{A}$ have connected intersection. Let $(U, \chi), (V, \varphi) \in \mathcal{A}$. By (2.2) it follows that:

$$\mathrm{Jac}(\chi \circ \varphi^{-1})(x) \in \mathrm{GL}_n(\mathbb{Z}), \quad \forall x \in \varphi(U \cap V).$$

Since $\mathrm{GL}_n(\mathbb{Z})$ is discrete and $U \cap V$ is connected, this implies by continuity that there is an $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $\mathrm{Jac}(\chi \circ \varphi^{-1})(x) = A$ for all $x \in \varphi(U \cap V)$. Using the connectedness of $U \cap V$ once more, we find by integration that there is a $p \in \mathbb{R}^n$ such that:

$$\chi \circ \varphi^{-1}(x) = p + Ax, \quad \forall x \in \varphi(U \cap V).$$

Thus the transition maps in the atlas $\mathcal{A}$ are restrictions of elements of $\mathbb{R}^n \ltimes \mathrm{GL}_n(\mathbb{Z})$. Letting $\mathcal{I}$ be the unique maximal such atlas that contains $\mathcal{A}$, we have constructed a map from the smooth Lagrangian lattices over B to the integral affine structures on B. The reader can directly verify that this is inverse to the previously defined map. $\square$

From now on we shall denote an integral affine manifold by (B, Λ), where $\Lambda \subset T^*B$ is the lattice that encodes the integral affine structure. Using similar arguments as in the above proof one can derive the following.

Proposition 2.2.12. *A map $\varphi : (B, \Lambda) \to (B', \Lambda')$ is a morphism of integral affine manifolds if and only if*

$$(d\varphi_b)^* : (T^*_{\varphi(b)}B', \Lambda'_{\varphi(b)}) \to (T^*_b B, \Lambda_b)$$

is a morphism of integral affine vector spaces for every $b \in B$.

Before we turn to the main result of this section, we give a few examples.

Example 2.2.13. Any integral affine vector space (V, Λ) has the trivial integral affine structure:

$$V \times \Lambda^* \subset V \times V^* = T^*V.$$

Example 2.2.14. Let G be a discrete Lie group, acting on an integral affine manifold (B, Λ) in a free and proper way and by integral affine maps. Then B/G is smooth and there is a unique integral affine structure $\overline{\Lambda}$ on B/G with the property that:

$$(dq_b)^*(\overline{\Lambda}_{q(b)}) = \Lambda_b$$

for every $b \in B$. Here $q : B \to B/G$ denotes the orbit projection. Using the standard action of the lattice Λ_T of a torus on its Lie algebra $\mathfrak{t}$, we can thus endow any torus T with an integral affine structure, which we call the standard one. By using different actions one can obtain integral affine structures on a torus that are not isomorphic to the standard one. See [CFT16] for such an example. Another space that can be equipped with an integral affine structure in this way is the Klein bottle.

We will now work towards to the main result of this section. Given a smooth lattice Λ, the quotient

$$\mathcal{T}_\Lambda := T^*B/\Lambda$$

is a Lie groupoid, because Λ is a normal Lie subgroupoid of T^*B.

Lemma 2.2.15. *A smooth lattice $\Lambda \subset (T^*B, \Omega_{can})$ is Lagrangian if and only if Ω_{can} descends to a form Ω_Λ on $\mathcal{T}_\Lambda$. In this case, $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ is a symplectic torus bundle over B.*

Proof. By definition, Ω_{can} descends to $\mathcal{T}_\Lambda$ precisely if there is a 2-form Ω_Λ on $\mathcal{T}_\Lambda$ such that

$$q^*\Omega_\Lambda = \Omega_{can} \tag{2.3}$$

where $q : T^*B \to \mathcal{T}_\Lambda$ is the quotient map. This is equivalent to Ω_{can} being basic with respect to the action groupoid $\Lambda \ltimes T^*B$, which means that:

$$m_\Lambda^*\Omega_{can} = pr_2^*\Omega_{can}$$

where $m_\Lambda, pr_2 : \Lambda \; {}_\pi\!\times_\pi T^*B \to T^*B$ denote the source and target map of the action groupoid $\Lambda \ltimes T^*B$. By the multiplicativity of Ω_{can} this is equivalent to:

$$pr_1^* i^*\Omega_{can} = 0$$

where $pr_1 : \Lambda\ {}_\pi\!\times_\pi T^*B \to \Lambda$ is the projection and $i : \Lambda \to T^*B$ the inclusion. Since pr_1 is a surjective submersion and Λ has half the dimension of T^*B, this is equivalent to Λ being Lagrangian in (T^*B, Ω_{can}). This proves the desired equivalence. In this case, Ω_Λ inherits multiplicativity and closedness from Ω_{can} via equation (2.3) and the fact that q is a surjective submersion. Moreover, for dimensional reasons, q is in fact a local diffeomorphism and therefore (2.3) implies as well that Ω_Λ inherits non-degeneracy from Ω_{can}. Thus $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ is a symplectic torus bundle. ☐

We are now ready to state and prove the main result of this section.

Theorem 2.2.16. *There is a bijective correspondence:*

$$\left\{\begin{matrix}\textit{Integral affine structures}\\ \textit{on } B\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Isomorphism classes of symplectic}\\ \textit{torus bundles over } B\end{matrix}\right\}$$

which associates to an integral affine structure the symplectic torus bundle $(\mathcal{T}_\Lambda, \Omega_\Lambda)$, where $\Lambda \subset T^*B$ is the lattice encoding the integral affine structure.

Proof. We will first show that the map defined above is injective. Let Λ_0 and Λ_1 be two smooth Lagrangian lattices in (T^*B, Ω_{can}) and let $\varphi : (\mathcal{T}_{\Lambda_0}, \Omega_{\Lambda_0}) \to (\mathcal{T}_{\Lambda_1}, \Omega_{\Lambda_1})$ be an isomorphism of symplectic torus bundles covering the identity. By applying the Lie functor we obtain an isomorphism of vector bundles $\hat{\varphi}$ that makes

$$\begin{tikzcd} T^*B \arrow[r, "\hat{\varphi}"] \arrow[d, "q_0"] & T^*B \arrow[d, "q_1"] \\ \mathcal{T}_{\Lambda_0} \arrow[r, "\varphi"] & \mathcal{T}_{\Lambda_1} \end{tikzcd}$$

commute. By commutativity we find that $\hat{\varphi}(\Lambda_0) = \Lambda_1$ and $\hat{\varphi}^*\Omega_{can} = \Omega_{can}$. Now, as can be verified in canonical coordinates, the only vector bundle automorphism of T^*B that covers the identity and preserves Ω_{can} is the identity map. Therefore $\hat{\varphi} = \mathrm{Id}_{T^*B}$ and $\Lambda_0 = \Lambda_1$.

For surjectivity, suppose that we are given a symplectic torus bundle $(\mathcal{T}, \Omega)$ over B. We are looking for a smooth Lagrangian lattice Λ in (T^*B, Ω_{can}) such that $(\mathcal{T}, \Omega)$ is isomorphic to $(\mathcal{T}_\Lambda, \Omega_\Lambda)$. The map $t : (\mathcal{T}, \Omega) \to B$ is a Lagrangian fibration with compact and connected fibers. As will be shown in Section 3.1, this comes with a Hamiltonian action of the symplectic groupoid (T^*B, Ω_{can}), the orbits of which coincide with the t-fibers. Consider the map:

$$p : T^*B \to \mathcal{T}, \quad (b, \alpha) \mapsto (b, \alpha) \cdot 1_b.$$

It follows from right-invariance of the vector fields of the form $a(\alpha)$, for $\alpha \in \Omega^1(B)$, that this map is a morphism of groupoids. Using that $p = m \circ (\mathrm{Id}_{T^*B}, u \circ \pi)$, the fact that the T^*B-action is Hamiltonian and that u is a Lagrangian section of t, it follows that $p^*\Omega = \Omega_{can}$. In particular p is an immersion and for dimensional reasons it must be a local diffeomorphism. Therefore $\Lambda := \mathrm{Ker}(p)$ is a normal Lie subgroupoid of T^*B with discrete isotropy groups. Furthermore, for each $b \in B$ we have:

$$p(T_b^*B) = \mathcal{O}_{1_b} = \mathcal{T}_b.$$

So p is surjective and by compactness of $\mathcal{T}_b$ the lattice Λ_b in T_b^*B is full. Hence Λ is a smooth lattice in T^*B and p factors through an isomorphism of Lie groupoids $\mathcal{T}_\Lambda \to \mathcal{T}$. In other words, we have a commutative diagram:

$$\begin{tikzcd} T^*B \arrow[r, "p"] \arrow[d, "q"] & \mathcal{T} \\ \mathcal{T}_\Lambda \arrow[ur, "\simeq"'] & \end{tikzcd}$$

By commutativity and the fact that $p^*\Omega = \Omega_{can}$, we find that Ω_{can} descends to $\mathcal{T}_\Lambda$. Therefore Λ is Lagrangian, $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ is a symplectic torus bundle and p factors through is an isomorphism of symplectic torus bundles. This proves the theorem. □

2.2.4 Monodromy of integral affine structures

In Section 2.2.2 we have seen that a smooth lattice Λ in a vector bundle E over B can be interpreted as a covering space. As for any covering space, we have a notion of monodromy. We will show that the triviality of the torus bundle E/Λ is equivalent to the triviality of the monodromy representations of Λ.

The **monodromy representation of** Λ is a morphism of groupoids

$$\rho : \Pi(B) \to \mathrm{Aut}(\Lambda),$$

defined as follows. First of all, $\mathrm{Aut}(\Lambda)$ is a groupoid with arrows $y \xleftarrow{\varphi} x$ the group isomorphisms $\varphi : \Lambda_x \to \Lambda_y$, with multiplication and inversion the composition and inversion of maps and units the identity maps. Now, given a class $[\gamma] \in \Pi(B)$ starting at b_0 and ending at b_1, we define $\rho([\gamma])(g) = \hat{\gamma}_g(1)$, where $\hat{\gamma}_g : [0,1] \to \Lambda$ is the unique lift of γ with starting point $\hat{\gamma}_g(0) = g \in \Lambda_{b_0}$. Note that $\rho([\gamma]) : \Lambda_{b_0} \to \Lambda_{b_1}$ is a bijection, with inverse $\rho([\gamma]^{-1})$. In fact, it is an isomorphism of groups. Indeed, given $g, h \in \Lambda_{b_0}$, the path $\hat{\gamma}_g + \hat{\gamma}_h$ defines a lifts of γ, with starting point $g + h$ and so

$$\rho([\gamma])(g+h) = \hat{\gamma}_g(1) + \hat{\gamma}_h(1) = \rho([\gamma])(g) + \rho([\gamma])(h).$$

This defines ρ. For each $b \in B$ we call

$$\rho_b : \pi_1(B, b) \to \mathrm{Aut}(\Lambda_b)$$

the **monodromy reprentation of Λ at** b.

Proposition 2.2.17. *Let (E, Λ) be an integral affine vector bundle over B. Then the following are equivalent.*

a) There is a vector bundle isomorphism $E \cong B \times \mathbb{R}^n$ that identifies Λ with $B \times \mathbb{Z}^n$.

b) The torus bundle E/Λ is isomorphic to $B \times \mathbb{T}^n$.

c) The monodromy representation of Λ at b is trivial for each $b \in B$.

Proof. We prove the implication from c to a and leave the rest to the reader. The argument for this is standard. We can assume that B is path-connected, since the argument below works component-wise. It is enough to construct a global frame e of E for which

$$\Lambda_b = \mathbb{Z}e_1(b) \oplus ... \oplus \mathbb{Z}e_n(b), \quad \forall b \in B. \tag{2.4}$$

Fix a $b_0 \in B$ and a basis $\lambda_1, ..., \lambda_n$ of the lattice Λ_{b_0} in E_{b_0}. We define $e_j : B \to \Lambda$ as follows: given a $b \in B$ we choose a continuous path $\gamma : [0,1] \to B$ from b_0 to b and let

$$e_j(b) = \rho([\gamma])(\lambda_j) \in \Lambda_b.$$

Since the monodromy representation at b_0 is trivial, $e_j(b)$ does not depend on the choice of γ. As $\rho([\gamma])$ is a group isomorphism $\Lambda_{b_0} \to \Lambda_b$, it follows that (2.4) holds for all $b \in B$. It remains to show that each e_j is smooth. For every $b_1 \in B$ there is an open neighbourhood U and a smooth homotopy $F : U \times [0,1] \to U$ such that $F(\cdot, 0) = b_1$ and $F(\cdot, 1) = \mathrm{Id}_U$. Since Λ is a covering space over B, there is a unique smooth lift $\hat{F} : U \times [0,1] \to \Lambda$ that satisfies $\hat{F}(b, 0) = e_j(b_1)$ for all $b \in U$. Now observe that

$$e_j(b) = \hat{F}(b, 1)$$

for all $b \in U$. So e_j is smooth at b_1, as desired. □

Of course, we can apply this to integral affine manifolds as well, by declaring the monodromy of an integral affine manifold (B, Λ) to be that of the lattice Λ in T^*B.

Chapter 3

Hamiltonian $\mathcal{T}_\Lambda$-spaces

Having studied the geometry of symplectic torus bundles in the previous chapter, we now continue with the study of their Hamiltonian actions. From this point onward we will shift our perspective slightly: instead of viewing symplectic torus bundles as Poisson geometric objects, we will consider them as part of an integral affine manifold. After discussing some classical classes of Hamiltonian $\mathcal{T}_\Lambda$-space, we prove a local normal form for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-space, by using the integral affine structure to reduce to a normal form for Hamiltonian T-spaces. We will use this normal form to study the local convexity properties of the moment map.

3.1 Some classical classes of Hamiltonian $\mathcal{T}_\Lambda$-spaces

Hamiltonian T-spaces

As follows from the general correspondence between Hamiltonian G-spaces and Hamiltonian $G \ltimes \mathfrak{g}^*$-spaces, the Hamiltonian T-spaces correspond to Hamiltonian $T \ltimes \mathfrak{t}^*$-spaces. Here we view $T \times \mathfrak{t}^*$ as the trivial symplectic torus bundle over $\mathfrak{t}^*$ that corresponds to the integral affine structure Λ_T^* on $\mathfrak{t}^*$ (or to be more precise, the trivial smooth Lagrangian lattice bundle $\Lambda_T \times \mathfrak{t}^*$ in the cotangent bundle $\mathfrak{t} \times \mathfrak{t}^*$ of $\mathfrak{t}^*$). So, a Hamiltonian T-space is an example of a Hamiltonian $\mathcal{T}_\Lambda$-space over the integral affine vector space $(\mathfrak{t}^*, \Lambda_T^*)$.

Conversely, are all Hamiltonian $\mathcal{T}_\Lambda$-spaces over a vector space are of this form? To answer this, let V be a real, finite-dimensional vector space and suppose that we are given an integral affine structure Λ on V. Since V is contractible, Λ has trivial monodromy. Therefore Λ is isomorphic to a trivial smooth Lagrangian lattice $\Lambda_{V^*} \times V$ in the cotangent bundle $V^* \times V$ of V, where Λ_{V^*} is a full lattice in the vector space V^*, and the symplectic torus bundle $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ is canonically isomorphic to the symplectic groupoid $(T \ltimes \mathfrak{t}^*, -\Omega_{can})$, where $T = V^*/\Lambda_{V^*}$ and $\mathfrak{t}^* = V$, which provides a positive answer to our question. Note here that a minus sign appears before the canonical symplectic form, because the fiber of the bundle $\mathcal{T}_\Lambda$ corresponds to the base of the cotangent bundle $T \times \mathfrak{t}^*$ and vice versa.

Quasi-Hamiltonian T-spaces

Quasi Hamiltonian T-spaces are very similar to Hamiltonian T-spaces, the only difference being that their moment map takes values in the torus $T^* := \mathfrak{t}^*/\Lambda_T^*$ instead of $\mathfrak{t}^*$ itself. In order to define them, let $\Theta \in \Omega^1(T^*; \mathfrak{t}^*)$ denote the Maurer-Cartan form on T^*. This is given by:

$$\Theta_g(\alpha) = (dL_{g^{-1}})_g(\alpha), \quad \alpha \in T_g(T^*).$$

The choice of a basis of Λ_T induces isomorphisms $T \cong \mathbb{T}^n$ and $T^* \cong \mathbb{T}^n$, under which the Maurer-Cartan form becomes the 1-form $(d\theta^1, ..., d\theta^n) \in \Omega^1(\mathbb{T}^n; \mathbb{R}^n)$.

Definition 3.1.1. A **quasi-Hamiltonian T-space** is a triple (S, ω, μ) consisting of a symplectic T-space (S, ω), together with a map $\mu : S \to T^*$ that satisfies:

- The invariance condition: $\mu(g \cdot p) = \mu(p), \quad \forall g \in T, \quad p \in S.$
- The weak Hamiltonian condition: $\mu^*(\langle \Theta, \xi \rangle) = \iota_{\xi_S}\omega, \quad \forall \xi \in \mathfrak{t}.$

Proposition 3.1.2. *Every Hamiltonian T-space gives rise to a quasi-Hamiltonian T-space, by composing the moment map $\mu : S \to \mathfrak{t}^*$ with the exponential map $\mathfrak{t}^* \to T^*$.*

An important difference between the two is the following: a Hamiltonian T-action on a compact symplectic manifold cannot be free, for otherwise the moment map would be a submersion from the compact S into the connected space $\mathfrak{t}^*$, hence it would have to map S onto $\mathfrak{t}^*$ and $\mathfrak{t}^*$ would have to be compact, which is absurd. On the other hand, there are examples of free quasi-Hamiltonian T-actions on compact manifolds and their orbit spaces give interesting examples of compact Poisson manifolds.

The action groupoid $T \ltimes T^*$ of the trivial action of T on T^* inherits a symplectic structure from $T \times \mathfrak{t}^*$ and in this way becomes a symplectic groupoid integrating $(T^*, 0)$. In a way completely analogous to the story for Hamiltonian T-spaces, the quasi-Hamiltonian T-spaces correspond to Hamiltonian $T \ltimes T^*$-spaces over the torus T^*, where we view $T \times T^*$ simultaneously as the action groupoid, and as the trivial symplectic torus bundle corresponding to the trivial smooth Lagrangian lattice bundle $\Lambda_T \times T^*$ in the cotangent bundle $\mathfrak{t} \times T^*$ of T^*. So, quasi-Hamiltonian T-spaces provide examples of Hamiltonian $\mathcal{T}_\Lambda$-spaces over the torus T^* endowed with its standard integral affine structure Λ. Conversely, all Hamiltonian $\mathcal{T}_\Lambda$-spaces over a torus with its standard integral affine structure Λ are of this form, by an argument similar to that in the previous example. It may be interesting to see what kind of $\mathcal{T}_\Lambda$-spaces there are for non-standard integral affine structures on the torus. We will however not address this question.

Lagrangian fibrations with compact and connected fibers

We have already seen that Lagrangian fibrations $\mu : (S, \omega) \to B$ are the same thing as isotropic realizations of the zero-Poisson structure on B and that the complete ones come with a unique Hamiltonian T^*B-action. We will now show that a Lagrangian fibration with compact and connected fibers induces an integral affine structure Λ on B and that it comes with a free Hamiltonian $\mathcal{T}_\Lambda$-action. In fact, we will show that such Lagrangian fibrations correspond to principal Hamiltonian $\mathcal{T}_\Lambda$-bundles for which the principal bundle projection is the moment map.

To this end, let $\mu : (S, \omega) \to B$ be a Lagrangian fibration with compact and connected fibers. Then μ is complete, by compactness of its fibers. In view of Example 1.4.13, there is a unique Hamiltonian T^*B-action along μ, given by:

$$\alpha \cdot p = \Phi^1_{a(\alpha)}(p), \quad \alpha \in T^*_bB, \quad p \in \mu^{-1}(b), \tag{3.1}$$

where $a(\alpha)$ is the vector field on $\mu^{-1}(b)$ determined by the equation:

$$\iota_{a(\alpha)_p}\omega = (d\mu_p)^*\alpha, \quad p \in \mu^{-1}(b).$$

Since μ is a submersion, the infinitesimal action $a : \mu^*(T^*B) \to TS$ is injective. Because the bundle of isotropy Lie algebras of the action is the kernel of a, this means that the isotropy Lie groups of the action are all discrete. Therefore, the isotropy Λ_p of the action at $p \in S$ is a lattice in $T^*_{\mu(p)}B$. Moreover, this implies that:

Proposition 3.1.3. *The orbits of the T^*B-action coincide with the fibers of μ.*

Proof. Let $b \in B$ and $p \in \mu^{-1}(b)$. Then the map

$$\varphi_p : T^*_bB \to \mathcal{O}_p \subset \mu^{-1}(b), \quad \alpha \mapsto \alpha \cdot p$$

factors through a diffeomorphism $T^*_bB/\Lambda_p \to \mathcal{O}_p$. Therefore $\mathcal{O}_p$ is an immersed submanifold of codimension 0 in $\mu^{-1}(b)$ and hence an open subset. This holds for each orbit in $\mu^{-1}(b)$. Since the source and target map of T^*B coincide, the fiber $\mu^{-1}(b)$ is partitioned by such orbits and hence it must coincide with a single orbit, because it is connected. $\square$

For any two points $p, q \in S$ in a single orbit $\mathcal{O}$ of the action, there is an $\alpha \in T^*_bB$ such that $\alpha \cdot p = q$. The isotropy groups Λ_p and Λ_q are related by conjugation by α and must therefore be equal as subgroup of T^*_bB, for T^*_bB is abelian. The previous proposition therefore shows that Λ_p is independent of $p \in \mu^{-1}(b)$ and hence we can define:

$$\Lambda_b := \Lambda_p = \{\alpha \in T^*_bB | \ \Phi^1_{a(\alpha)}(p) = p\}$$

where p is any element of $\mu^{-1}(b)$. This defines a wide subgroupoid Λ of T^*B, the isotropy groups of which are discrete. Now, because T^*_bB/Λ_b is diffeomorphic to $\mathcal{O}_p = \mu^{-1}(b)$, compactness of $\mu^{-1}(b)$ is equivalent to the lattice Λ_b being full in T^*_bB. Therefore, one would hope that under this compactness assumption, Λ becomes a smooth Lagrangian lattice in T^*B. By Proposition 2.2.11 this would mean that a Lagrangian fibration with compact and connected fibers induces an integral affine structure on its base B. The following confirms this.

Proposition 3.1.4. *A Lagrangian fibration $\mu : (S, \omega) \to B$ with compact and connected fibers gives rise to an integral affine structure Λ on B. Moreover, the symplectic torus bundle $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ acts along μ in a free and Hamiltonian fashion, and the orbits of this action coincide with the fibers of μ.*

Proof. By the previous discussion, it is enough for the first statement to show that every $b \in B$ admits an open neighbourhood U such that $\Lambda|_U$ is a closed Lagrangian submanifold of $T^*B|_U$. To this end, let $b \in B$ and choose a local section $\sigma : U \to S$ of the surjective submersion μ defined on an open neighbourhood of b. Then $\sigma(U)$ is a closed submanifold of $\mu^{-1}(U)$. Moreover, the map

$$\varphi : T^*B|_U \to \mu^{-1}(U), \quad \alpha_b \mapsto \alpha_b \cdot \sigma(b)$$

is a local diffeomorphism with the property that $\varphi^{-1}(\sigma(U)) = \Lambda|_U$. Therefore $\Lambda|_U$ is a closed submanifold of $T^*B|_U$. For dimensional reasons it remains to verify that $\Lambda|_U$ is isotropic. Let $i : \Lambda|_U \to T^*B$ be the inclusion. Since the action of T^*B is Hamiltonian, it follows that

$$\varphi^*\omega = \Omega_{can} + \pi^*\sigma^*\omega,$$

where $\pi : T^*B \to B$ denotes the bundle projection. Pulling this back by i and using that $\varphi \circ i = \sigma \circ \pi \circ i$, we derive that $i^*\Omega_{can} = 0$, as was left to be shown.

For the second statement, note that by construction of Λ, the T^*B-action along μ descends to a free action of $\mathcal{T}_\Lambda$ along μ. This action is Hamiltonian and its orbits coincide with the fibers of μ, because the same holds for the T^*B-action. □

In other words, a Lagrangian fibration with compact and connected fibers gives rise to a principal Hamiltonian $\mathcal{T}_\Lambda$-bundle:

$$\begin{tikzcd}
(\mathcal{T}_\Lambda,\Omega_\Lambda) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & (S,\omega) \arrow[dl, "\mu"'] \arrow[dr, "\mu"] & \\
B & & B
\end{tikzcd}$$

Proposition 3.1.5. *Conversely, any principal Hamiltonian $\mathcal{T}_\Lambda$-bundle with moment map $\mu : (S,\omega) \to B$ equal to the principal bundle projection is of this form.*

Proof. Since μ is a principal bundle projection, it is a surjective submersion and the $\mathcal{T}_\Lambda$-orbits equal the fibers of μ. In particular, this implies that $\dim(S) = 2\dim(B)$. Being the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-action, $\mu : (S,\omega) \to (B,0)$ is a Poisson map. Thus μ is a Lagrangian fibration with compact and connected fibers. It remains to show that the integral affine structure induced by μ coincides with Λ. Via the quotient map $T^*B \to \mathcal{T}_\Lambda$, the action of $\mathcal{T}_\Lambda$ induces a Hamiltonian action of T^*B along μ with isotropy group at p given by $\Lambda_{\mu(p)}$. By uniqueness this action must be given by (3.1) and hence the lattice Λ coincides with the one induced by μ, as desired. □

3.2 A normal form for Hamiltonian $\mathcal{T}_\Lambda$-spaces

The main tool in the rest of this chapter and Section 5.1 is a normal form for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-space. The normal form is a rather straightforward consequence of the following normal form for Hamiltonian T-spaces. This is a particular case of the Marle-Guillemin-Sternberg normal form for proper Hamiltonian G-spaces, which is explained and proved in detail in [OR04].

Theorem 3.2.1. *Let (S,ω,μ) be a Hamiltonian T-space, $x \in S$ and $\mathcal{O}$ the orbit through x. Let $p : \mathfrak{t} \to \mathfrak{t}_x$ be a linear projection. Then there exists an embedding of Hamiltonian T-spaces*

$$\psi : (T \times_{T_x} (\mathfrak{t}_x^0 \oplus \mathcal{S}_x), \Omega_p, \mathcal{M}_p + \mu(x)) \supset\!\!\!\to (S,\omega,\mu)$$

from a T-invariant open neighbourhood of $\mathcal{O}$ onto a T-invariant open neighbourhood of $\mathcal{O}$ in S, that restricts to the identity on $\mathcal{O}$. Here $\mathcal{M}_p$ is given by:

$$\mathcal{M}_p([t,\alpha,v]) = \alpha + p^*(\mu_{\mathcal{S}_x}(v)).$$

Remark 6. Let us elaborate briefly on the normal form above. A more detailed explanation and proof of Theorem 3.2.1 is given in the appendix. Given a proper G-space S and an orbit $\mathcal{O}$ through a point $x \in S$, the G-space S can be modeled in an invariant neighbourhood of $\mathcal{O}$ by the normal bundle to $\mathcal{O}$. The normal bundle can be realized as the G-vector bundle:

$$G \times_{G_x} \mathcal{N}_x$$

over $\mathcal{O}$, associated to the principal G_x-bundle $G \to \mathcal{O}$ and the isotropy representation $\mathcal{N}_x$ at x. When using this model we view $\mathcal{O}$ as the zero-section of this vector bundle and x as the point $[e, 0]$. In our case, the Lie group G is the torus T and the isotropy representation $\mathcal{N}_x$ decomposes into the subrepresentations $\mathfrak{t}_x^0$ (on which T_x acts trivially) and the symplectic isotropy representation $\mathcal{S}_x$. Recall here from Definition 1.4.21 that

$$\mathcal{S}_x = \frac{T_x\mathcal{O}^\omega}{T_x\mathcal{O} \cap T_x\mathcal{O}^\omega}.$$

Since we are considering the action of a Lie group, multiplication $m_t : S \to S$ by $t \in T$ is defined on all of S and one easily verifies that the action of T_x on $\mathcal{S}_x$ is given by:

$$t \cdot [v] = [d(m_t)_x(v)], \quad t \in T_x, \quad [v] \in \mathcal{S}_x.$$

An auxiliary linear projection $p : \mathfrak{t} \to \mathfrak{t}_x$ can be used to define a symplectic structure Ω_p and moment map $\mathcal{M}_p$ on the model. Equipped with this extra structure, it then models a neighbourhood of $\mathcal{O}$ in S as a Hamiltonian T-space.

The local normal form for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-space is essentially the same. Using the next lemma, which states that Hamiltonian $\mathcal{T}_\Lambda$-spaces are locally isomorphic to Hamiltonian T-spaces, we can reduce the proof of the local normal form to the previous theorem.

For this lemma we need to introduce a new notion, inherent to an affine structure on a manifold. The definition of an affine structure on a manifold is analogous to that of an integral affine manifold: one just replaces the group $\mathrm{GL}_n(\mathbb{Z})$ by $\mathrm{GL}_n(\mathbb{R})$. So, any integral affine manifold is in particular an affine manifold. Now, the notion that we need is the following. Let $(B, \mathcal{A})$ be an affine manifold. For each $b \in B$ there is an open neighbourhood U of b and an affine embedding[1]

$$\log_b : U \to T_bB$$

with the property that $\log_b(b) = 0$ and $d(\log_b)_b = \mathrm{Id}_{T_bB}$. For a given open neighbourhood U of b, there is at most one such map. To see the existence of an open U and such an affine embedding, choose any affine chart (U, χ) around b that sends b to 0 and let $\log_b = d\chi_b^{-1} \circ \chi$. If the affine structure is in fact an integral affine structure, then

$$\log_b : (U, \Lambda|_U) \to (T_bB, \Lambda_b^*)$$

is in fact an integral affine embedding, so that

$$d(\log_b)_x^*(\Lambda_b) = \Lambda_x$$

for all $x \in U$.

[1] Morphisms and embeddings of affine manifolds are defined completely analogously to those for integral affine manifolds.

Remark 7. Alternatively (and perhaps more naturally) one can show the existence of $\log_b$ as follows. One can associate to an affine structure $\mathcal{A}$ a canonical flat, torsion free connection $\Delta_{\mathcal{A}}$ on B, uniquely determined by the condition that it is the standard flat connection with respect to any affine chart. For every $b \in B$, the exponential map associated to $\Delta_{\mathcal{A}}$ induces a map $\exp_b : T_bB \to B$ which sends 0 to b and satisfies $d(\exp_b)_b = \mathrm{Id}_{T_bB}$. This is a local diffeomorphism at 0 and hence, locally at b, has an inverse $\log_b : U \to T_bB$ defined on some open neighbourhood U of b. This is explains our notation as well.

Returning to the normal form for the moment map, we can now formulate the aforementioned lemma.

Lemma 3.2.2. *Let (B, Λ) be an integral affine manifold, $b \in B$, $T = T_b^*B/\Lambda_b$ and U an open neighbourhood of b on which $\log_b$ is defined. We canonically identify T_bB with $\mathfrak{t}^*$. Then $\log_b$ induces an isomorphism of symplectic groupoids:*

$$\begin{aligned}
(T \ltimes \log_b(U), -\Omega_{can}) &\to (\mathcal{T}_\Lambda|_U, \Omega_{can}) \\
(\alpha \mod \Lambda_b, \log_b(x)) &\mapsto d(\log_b)_x^*(\alpha) \mod \Lambda_x
\end{aligned}$$

where the groupoid $T \ltimes \log_b(U) \rightrightarrows \log_b(U)$ is the action groupoid for the (trivial) co-adjoint action of T on $\log_b(U) \subset \mathfrak{t}^*$. Consequently, given a symplectic manifold (S, ω), an action of $\mathcal{T}_\Lambda|_U$ on a map $\mu : S \to U$ is Hamiltonian if and only if the induced action of T on (S, ω) is Hamiltonian with moment map

$$\mu_b = \log_b \circ \mu : S \to \mathfrak{t}^*.$$

Proof. The first statement is straightforward to verify by means of formula (1.4). The second statement follows from the first and the correspondence between Hamiltonian G-spaces and Hamiltonian $G \times \mathfrak{g}^*$-spaces. $\square$

Remark 8. In principle the previous lemma holds for any integral affine embedding $(U, \Lambda|_U) \to (T_bB, \Lambda_b^*)$ instead of $\log_b$. However, the property that its differential at b is Id_{T_bB} ensures that, for every $p \in \mu^{-1}(b)$, the isotropy representation of the induced action of $T \ltimes \log_b(U)$ at p coincides with the isotropy representation of the $\mathcal{T}_\Lambda$-action at p.

In combination with the normal form for Hamiltonian T-spaces this immediately gives rise to the following normal form for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-space. Let:

- $\mu : (S, \omega) \to B$ be a Hamiltonian $\mathcal{T}_\Lambda$-space,
- $x \in S$, $b = \mu(x)$ and $T = T_b^*B/\Lambda_b$,
- U be an open neighbourhood of b on which $\log_b$ is defined,
- $p : \mathfrak{t} \to \mathfrak{t}_x$ be a linear projection.

We canonically identify T_bB with $\mathfrak{t}^*$ and consider the induced Hamiltonian T-action on $(\mu^{-1}(U), \omega)$ as in the previous lemma.

Theorem 3.2.3 (Normal form)**.** *There exists a T-equivariant symplectic embedding*

$$\varphi : (\mu^{-1}(U), \omega) \supset\!\!\to (T \times_{T_x} (\mathfrak{t}_x^0 \oplus S_x), \Omega_p)$$

from a T-invariant open neighbourhood of $\mathcal{O}$ in $\mu^{-1}(U)$ onto a T-invariant open neighbourhood of $\mathcal{O}$ that restricts to the identity on $\mathcal{O}$ and makes the diagram

$$\begin{array}{ccc} \mu^{-1}(U) & \xrightarrow{\varphi} & T \times_{T_x} (\mathfrak{t}_x^0 \oplus \mathcal{S}_x) \\ \downarrow \mu & & \downarrow \mathcal{M}_p \\ U & \xrightarrow{\log_b} & \mathfrak{t}^* \end{array}$$

commute wherever φ is defined.

3.3 Local convexity properties of the moment map

In this section we derive local convexity properties for the moment map of a Hamiltonian $\mathcal{T}_\Lambda$-spaces from its local normal form. The local normal form shows that the local behaviour of the moment map is governed by the symplectic isotropy representations of the $\mathcal{T}_\Lambda$-action. This leads us to study symplectic torus representations in the coming section. It will become clear from this that the moment map is what we call locally polyhedral. We will define what this means and we will give sufficient conditions for the image of a locally polyhedral map to be locally polyhedral. This allows us to derive global convexity for the image of a certain class of Hamiltonian T-spaces and in particular we derive the classical Atiyah-Guillemin-Sternberg convexity theorem.

3.3.1 Symplectic torus representations

Real and complex representations

Before going into symplectic representations, we remind the reader on the weight-classification of real and complex torus-representations. Throughout, by a representation (V, r) of a Lie group G we mean pair consisting of a finite-dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$, and a morphism of Lie groups $r : G \to \mathrm{GL}(V)$. We will often suppress r from the notation an just write V for the representation.

A representation V is called:

- **Irreducible** if $V \neq \{0\}$ and the only invariant subspaces of V are $\{0\}$ and V.
- **Completely decomposable** if every invariant linear subspace $W \subset V$ admits an invariant linear complement in V.

A straightforward induction argument shows that every completely decomposable representation can be written as a direct sum of irreducible subrepresentations. If a representation admits a G-invariant inner product (real if V is real and Hermitian if V is complex), then it is completely decomposable. Indeed, the orthogonal complement to an invariant linear subspace with respect to an invariant inner product is an invariant linear complement. By averaging over the group one shows that every representation of a compact Lie group admits a G-invariant inner product. Therefore:

Proposition 3.3.1. *Every representation of a compact Lie group is completely decomposable.*

To understand representations of tori it is therefore enough to understand their irreducible representations and the ways in which they can decompose into irreducibles. Let us begin with the latter. Decompositions into irreducible subrepresentations need not be unique. However, by grouping the irreducible subrepresentations that are isomorphic one does obtain a canonical decomposition, called the **decomposition into isotypical components**.

Definition 3.3.2. Let W be an irreducible representation of a Lie group G. Let V be a completely decomposable representation of G and let

$$V = V_1 \oplus ... \oplus V_n$$

be a decomposition into irreducible subrepresentations. We define the **W-isotypical summand** of V to be:

$$V_W = \bigoplus_{i:V_i \cong W} V_i.$$

The following proposition implies that the isotypical summands do not depend on the choice of decomposition into irreducibles.

Proposition 3.3.3. *Let W be an irreducible representation of G and let V, $\widetilde{V}$ be isomorphic representations of G. Suppose that $V = V_1 \oplus ... \oplus V_n$ and $\widetilde{V} = \widetilde{V}_1 \oplus ... \oplus \widetilde{V}_m$ are decompositions into irreducible subrepresentations and $\varphi : V \to \widetilde{V}$ is an isomorphism of representations. Then*

$$\varphi\left(\bigoplus_{i:V_i \cong W} V_i\right) = \bigoplus_{i:\widetilde{V}_i \cong W} \widetilde{V}_i.$$

This result follows from Schur's Lemma:

Lemma 3.3.4 (Schur)**.** *Let $V, \widetilde{V}$ be two irreducible representations of a Lie group G and $\varphi : V \to \widetilde{V}$ an equivariant linear map. Then $\varphi = 0$ or φ is an isomorphism. Moreover, if V is a complex representation, $V = \widetilde{V}$ and $\varphi \neq 0$, then $\varphi \in \mathbb{C} \cdot Id_V$.*

As a consequence of Proposition 3.3.3 we have:

Corollary 3.3.5. *Every completely decomposable representation V admits a canonical decomposition into its isotypical summands:*

$$V = \bigoplus_{W \in Irr(G,V)} V_W$$

where the sum runs over the isomorphism classes of irreducible subrepresentations of V.

With this result in mind, we will now focus on irreducible representations. The following is another consequence of Schur's Lemma:

Proposition 3.3.6. *Every irreducible complex representation of an abelian Lie group is one-dimensional.*

By viewing a real representation as a real subrepresentation of its complexification, one can deduce from the previous proposition that:

Proposition 3.3.7. *The irreducible real representations of a torus are either* 1*-dimensional or* 2*-dimensional.*

Whereas isomorphism classes of finite-dimensional vector spaces are classified by their dimension, finite-dimensional representations are not classified by just their dimension. What we need in addition are their weights. Let T be a torus. By a weight of T we mean an element of Λ_T^* (the dual of the lattice $\Lambda_T = \mathrm{Ker}(\exp_T)$ in $\mathfrak{t}$). To each non-zero $\alpha \in \Lambda_T^*$ we can associate a representation:

$$\hat{\alpha} : T \to \mathrm{GL}(\mathbb{C}), \quad \hat{\alpha}(\exp(\xi))(z) = e^{2\pi i \alpha(\xi)} z. \tag{3.2}$$

which we denote by $\mathbb{C}_\alpha$. We can view this both as a complex or a real representation and, as we assumed α to be non-zero, $\mathbb{C}_\alpha$ is irreducible in both cases. Using the same definition for $\alpha = 0$, $\mathbb{C}_\alpha$ would be irreducible as complex representation, but not as real representation. In light of this, we define $\mathbb{R}_0$ to be the trivial representation on $\mathbb{R}$, and denote $V_\alpha = \mathbb{C}_\alpha$ if $\alpha \neq 0$ and $V_0 = \mathbb{R}_0$ when considering real representations.

We can go the other way as well. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathrm{Rep}_\mathbb{K}(T)$ denote the set of isomorphism classes of $\mathbb{K}$-representations of a torus T.

Proposition 3.3.8. *Let T be a torus. Then:*

a) For every irreducible $[V] \in Rep_\mathbb{C}(T)$ there is a unique weight α such that $[V] = [\mathbb{C}_\alpha]$.

b) For every irreducible $[V] \in Rep_\mathbb{R}(T)$ there is a weight α such that $[V] = [V_\alpha]$. Such a weight is unique up to sign.

The uniqueness parts are the content of the following lemma, which can by verified by a direct approach.

Lemma 3.3.9. *Let $\alpha, \beta \in \Lambda_T^*$.*

a) In the case of complex representations, $\mathbb{C}_\alpha \cong \mathbb{C}_\beta$ if and only if $\alpha = \beta$.

b) In the case of real representations, $V_\alpha \cong V_\beta$ if and only if $\alpha = \pm\beta$.

We now prove the remainder of the proposition.

Proof of Proposition 3.3.8. We prove part a, the proof of b is similar. Let (r, V) be an irreducible complex representation of T. Then V is 1-dimensional. By averaging over T we can choose an invariant hermitian metric h on V. Then $r : T \to \mathrm{U}(V, h)$ and so

$$\rho = dr_e : \mathfrak{t} \to \mathfrak{u}(V, h).$$

Notice that $\mathrm{End}(V) = \mathbb{C} \cdot I_V$, as V is 1-dimensional. So because

$$\mathfrak{u}(V, h) = \{A \in \mathrm{End}(V) |\ h(Au, v) + h(u, Av) = 0\ \forall u, v \in V\},$$

it follows that $\mathfrak{u}(V, h) = (i\mathbb{R})I_V$. Define α to be the composition of ρ with the isomorphism

$$\mathfrak{u}(V, h) \to \mathbb{R}, \quad ixI_V \mapsto \frac{1}{2\pi} x.$$

By construction

$$\rho(\xi)v = 2\pi i \alpha(\xi)v, \quad \forall \xi \in \mathfrak{t}, v \in V.$$

Using the power series for the exponential, this integrates to:

$$r(\exp(\xi))(v) = e^{2\pi i \alpha(\xi)} v, \quad \forall \xi \in \mathfrak{t}, v \in V.$$

Therefore $\alpha \in \Lambda_T^*$ and any complex linear isomorphism $V \cong \mathbb{C}$ is an isomorphism of representations $V \cong \mathbb{C}_\alpha$. By the previous lemma, the weight α is uniquely determined by the fact that $V \cong \mathbb{C}_\alpha$. $\square$

By means of the isotypical decomposition, one can extend this story to general torus representations and classify representations of tori in terms of their weight-tuples. This is the content of the following theorem.

Theorem 3.3.10. *Let T be a torus. The following classification results hold.*

a) There is a bijection[2]

$$Rep_{\mathbb{C}}(T) \to \{\textit{unordered tuples of weights of } T\}$$

which associates to every class $[V]$ the unique unordered tuple $(\alpha_1, ..., \alpha_n)$ of weights such that

$$[V] = [\mathbb{C}_{\alpha_1} \oplus ... \oplus \mathbb{C}_{\alpha_n}].$$

b) There is a bijection

$$Rep_{\mathbb{R}}(T) \to \{\textit{unordered tuples of weights-modulo-sign of } T\}$$

which associates to every class $[V]$ the unique unordered tuple $([\alpha_1], ..., [\alpha_n])$ of weights-modulo-sign such that

$$[V] = [V_{\alpha_1} \oplus ... \oplus V_{\alpha_n}].$$

Symplectic representations

Our true interest is the classification of *symplectic* torus-representations. Symplectic representations are in particular real representations. However, in analogy with the case of complex representations, the symplectic structure allows for a classification in terms of weights, and not just weights up to sign. This is the content of this paragraph. The proofs here are based on the appendix of [LT97].

The notions of irreducibility and complete decomposability are analogous to those for real or complex representations.

Proposition 3.3.11. *Every symplectic representation is completely decomposable.*

Proof. By G-invariance of ω, the ω-complement to an invariant, symplectic linear subspace of V is an invariant symplectic linear complemen to it in V. □

It follows that every symplectic representation decomposes into a direct sum of irreducible symplectic subrepresentations, even if G is not compact. Contrary to the case of real or complex representations, it is not clear anymore whether there is a well-defined notion of isotopical summands. Symplectic representations do however come with an extra piece of structure: their moment map. Due to the following result, many invariant of the moment map are in fact invariants of the isomorphism class of the symplectic representation.

Lemma 3.3.12. *Let (V, ω_V) and (W, ω_W) be symplectic representations. If $\varphi : V \to W$ is an isomorphism of symplectic representations, then it intertwines the standard moment maps:*

$$\mu_W \circ \varphi = \mu_V.$$

[2] We allow for repetitions in an unordered tuple.

Proof. Both $\mu_W \circ \varphi$ and μ_V are moment maps for the symplectic G-space (V, ω_V), because φ is an equivariant symplectomorphism. Since φ is linear, we have

$$\mu_W(\varphi(0)) = \mu_W(0) = 0 = \mu_V(0).$$

So $\mu_W \circ \varphi = \mu_V$ by Lemma A.8. □

As will become apparent later, the invariants of the moment map will allow for a weight-classification. As for real and complex representations, the weights of a torus T provide the building blocks for isomorphism classes of symplectic T-representations. For each $\alpha \in \Lambda_T^*$ we define $\mathbb{C}_\alpha$ by (3.2), which we view as a symplectic representation on $(\mathbb{C}, -\omega_0)$. Each $\mathbb{C}_\alpha$ is an irreducible symplectic representation, for dimensional reasons.

Lemma 3.3.13. *Let $\alpha, \beta \in \Lambda_T^*$.*

a) The standard moment map $\mu_\alpha : \mathbb{C}_\alpha \to \mathfrak{t}^*$ is given by $\mu_\alpha(z) = \pi|z|^2\alpha$.

b) As symplectic representations it holds that: $\mathbb{C}_\alpha \cong \mathbb{C}_\beta$ if and only if $\alpha = \beta$.

Proof. Let h_0 denote the standard Hermitian inner product on $\mathbb{C}$. We compute:

$$\begin{aligned}
\langle \mu_\alpha(z), \xi \rangle &= -\frac{1}{2}\omega_0(\xi \cdot z, z) \\
&= -\frac{1}{2}\omega_0(2\pi i \alpha(\xi) z, z) \\
&= \frac{1}{2}\mathrm{Im}(h_0(2\pi i \alpha(\xi) z, z)) \\
&= \pi |z|^2 \alpha(\xi),
\end{aligned}$$

for all $\xi \in \mathfrak{t}$ and $z \in \mathbb{C}$. Hence a holds.

For b, the implication from right to left is immediate. Let us prove the other implication. Suppose there is a linear symplectomorphism $\varphi : \mathbb{C}_\alpha \to \mathbb{C}_\beta$. Then it holds that $\mu_\beta \circ \varphi = \mu_\alpha$ by the previous lemma and so

$$\mathbb{R}_{\geq 0} \cdot \alpha = \mathrm{Im}(\mu_\alpha) = \mathrm{Im}(\mu_\beta) = \mathbb{R}_{\geq 0} \cdot \beta$$

by part a of this lemma. But $\alpha = \pm\beta$ by Lemma 3.3.9, so it must be that $\alpha = \beta$, as was to be shown. □

Had we used the symplectic form ω_0 on $\mathbb{C}$, then we would have obtained a minus-sign in the expression for μ_α. This explains our choice of $-\omega_0$. As for complex representations, we have the following result. Let us denote by $\mathrm{SpRep}(T)$ the set of isomorphism classes of symplectic representations on a torus T.

Proposition 3.3.14. *Let T be a torus. For every irreducible $[V] \in SpRep(T)$ there is a unique weight α such that $[V] \cong [\mathbb{C}_\alpha]$.*

Proof. Let $r : G \to \mathrm{Sp}(V, \omega)$ be an irreducible symplectic representation of T. By compactness of T and T-invariance of ω, we can choose a T-equivariant, ω-compatible complex structure J on V. The equivariance means that r is a complex representation on (V, J). By ω-compatibility, every complex linear subspace of (V, J) is a symplectic linear subspace of (V, ω). So it follows from this that r is irreducible as complex representation on (V, J), hence Proposition 3.3.6 implies that V has real dimension 2. Now, observe that

$$h(\cdot, \cdot) = \omega(\cdot, J\cdot) + i\omega(\cdot, \cdot)$$

defines an invariant Hermitian metric on $\mathbb{C}$ and a choice of orthonormal basis with respect to h induces a complex linear symplectomorphism $(V, \omega) \to (\mathbb{C}, -\omega_0)$. Since h is invariant, r is a unitary representation on (V, J, h) and we can define α as in the proof of Proposition 3.3.8. A complex linear symplectomorphism as above will then be an isomorphism of symplectic representations $V \to \mathbb{C}_\alpha$. □

We are now ready to prove the weight-classification for symplectic representations.

Theorem 3.3.15. *Let T be a torus. There is a bijection*

$$SpRep(T) \to \{\text{unordered tuples of weights of } T\},$$

which associates to every class $[V]$ the unique unordered tuple $(\alpha_1, ..., \alpha_n)$ of weights such that

$$[V] = [V_{\alpha_1} \oplus ... \oplus V_{\alpha_n}].$$

We need one lemma.

Lemma 3.3.16. *Let T be a torus, (V, ω) a symplectic representation of T and*

$$\varphi : (V, \omega) \to (\mathbb{C}_{\alpha_1}, -\omega_0) \oplus ... \oplus (\mathbb{C}_{\alpha_n}, -\omega_0)$$

an isomorphism of symplectic representations. Then φ intertwines the standard moment map of the symplectic representation (V, ω) with the moment map:

$$\mu : \mathbb{C}^n \to \mathfrak{t}^*, \quad \mu(z) = \pi \sum_{j=1}^{n} |z_j|^2 \alpha_j.$$

Proof. By Lemma 3.3.13a and Proposition A.2c the map $\mu : \mathbb{C}^n \to \mathfrak{t}^*$ is indeed a moment map for the given T-action. In view of Proposition A.2b, the map $\mu \circ \varphi$ is a moment map for the T-action on V. So both $\mu \circ \varphi$ and μ_V are moment maps on the connected Hamiltonian T-space V, and

$$\mu(\varphi(0)) = 0 = \mu_V(0).$$

Hence $\mu_V = \mu \circ \varphi$ as claimed. □

Proof. In view of Proposition 3.3.14 and the fact that every symplectic representation splits as a direct sum of irreducible symplectic subrepresentations, the only part of the theorem that still needs proof is the fact that the above map is well-defined. So let V a symplectic representations of T, let $V = V_1 \oplus ... \oplus V_n$ be a decomposition into irreducible symplectic subrepresentations and let α_i be the unique weights satisfying $V_i \cong \mathbb{C}_{\alpha_i}$. We need to show that the unordered tuple

$$(\alpha_1, ..., \alpha_n)$$

depends only on the isomorphism class of V and not on the choice of representative or of the decomposition. To this end, given $\alpha \in \Lambda_T^*$, we denote by W_α the isotypical summand of the real representation underlying V for the irreducible real representation V_α. First note that

$$\#\{j|\ \alpha_j = 0\} = \frac{1}{2}\dim_{\mathbb{R}}(W_0)$$

which is an invariant of the isomorphism class $[V]$ by Proposition 3.3.3. Secondly, suppose that $\alpha \in \Lambda_T^*$ is non-zero. Then there is a $\xi \in \mathfrak{t}$ such that $\alpha(\xi) > 0$. Note that

$$W_\alpha = \bigoplus_{j:\alpha_j=\pm\alpha} V_j,$$

so by the previous lemma the map $\langle \mu_V, \xi\rangle : V \to \mathbb{R}$ restricts to a non-degenerate quadratic form of positive index

$$2\#\{j|\alpha_j = \alpha\} = \dim_{\mathbb{R}}\left(\bigoplus_{j:\alpha_j=\alpha} V_j\right)$$

on the real vector space W_α. The positive index is an invariant of the quadratic form $\langle \mu_V, \xi\rangle|_{W_\alpha}$ (as is the positive index of any quadratic form) and by combining Proposition 3.3.3 with Lemma 3.3.12 we see that it is in fact an invariant of the isomorphism class $[V]$. All in all, we conclude that the unordered tuple $(\alpha_1, ..., \alpha_n)$ is an invariant of $[V]$, as desired. □

We will refer to the image of a symplectic T-representation under the above bijection as the **weight-tuple** of the symplectic representation. The symplectic representations that we will come across may not be those of tori, but the connected component G^0 containing the identity element of the Lie group G will always be a torus. By the weight-tuple of a symplectic representation of such a Lie group G we will always mean the weight-tuple of the induced symplectic G^0-representation. Although in this case the weight-tuple need not completely determine the G-representation, it still determines its associated Lie algebra representation and hence its moment map. We therefore have:

Corollary 3.3.17. *Let G be a Lie group with the property that G^0 is a torus, (V, ω) a symplectic representation of G and $(\alpha_1, ..., \alpha_n)$ its weight-tuple. Then there is an isomorphism of symplectic G^0-representations:*

$$(V, \omega) \to (\mathbb{C}_{\alpha_1}, -\omega_0) \oplus ... \oplus (\mathbb{C}_{\alpha_n}, -\omega_0).$$

that intertwines the standard moment map of the symplectic G-representation (V, ω) with the moment map:

$$\mu : \mathbb{C}^n \to \mathfrak{t}^*, \quad \mu(z) = \pi \sum_{j=1}^n |z_j|^2 \alpha_j.$$

Corollary 3.3.18. *Let G be a Lie group with the property that G^0 is a torus, (V, ω) a symplectic representation of G, μ_V its moment map and $(\alpha_1, ..., \alpha_n)$ its weight-tuple. Then the moment image is:*

$$\mu_V(V) = Cone(\alpha_1, ..., \alpha_n) := \left\{ \sum_{j=1}^n t_j\alpha_j |\ t_j \geq 0 \right\}.$$

This is the first instance of polyhedrality of the moment image that we have come across. In the coming sections we will use this to derive local polyhedrality of more general moment maps.

3.3.2 Locally polyhedral maps into vector spaces

The definitions and results in this section are adaptations of parts of the results in [BOR09]. Throughout, let X be a topological space and V a finite-dimensional, real vector space. We first introduce some notions regarding polyhedrality.

Definition 3.3.19. A **polyhedral cone** $C \subset V$ is a subset of the form:

$$C = \mathrm{Cone}(v_1, ..., v_n) := \left\{ \sum_{j=1}^{n} t_j v_j | \ t_j \geq 0 \right\}$$

where $v_1, ..., v_n \in V$. We say that the tuple $(v_1, ..., v_n)$ **generates** the polyhedral cone C.

Definition 3.3.20. We call a continuous map $f : X \to V$ **locally polyhedral** at $x \in X$, if there is an open neighbourhood U of x in X and there is a polyhedral cone C in V such that

$$f(U) \subset f(x) + C$$

and $f|_U$ is open as a map into $f(x) + C$. We call f locally polyhedral if it is so at every $x \in X$.

Definition 3.3.21. A subset $A \subset V$ is called **locally polyhedral** at $a \in A$ if the germ of A at a can be represented by a set of the form $a + C$, where C is a polyhedral cone in V. A subset is called locally polyhedral if it is so at each of its points.

Remark 9. Note that for each $x \in X$, a cone C for which there exists an open neighbourhood U of x in X such that $f(U) \subset f(x) + C$ and $f|_U$ is open as map into $f(x) + C$, is automatically unique. Indeed, this follows from the fact that two cones in V with the same germ at 0 must be equal, as they are invariant under multiplication by positive scalars. For a given $x \in X$, we shall refer to this cone as **the cone of f at x**. Similarly, a cone representing the germ of a locally polyhedral set A at a point a is unique and we shall refer to it as **the cone of A at a**.

Our aim is to derive sufficient conditions for the image of a locally polyhedral map to be locally polyhedral.

Proposition 3.3.22. *Let $f : X \to V$ be a locally polyhedral map. Suppose that f is closed as a map onto its image and that its fiber over a point $v \in f(X)$ is connected. Then the image of f is locally polyhedral at v and the cone of $f(X)$ at v is the cone of f at x for any $x \in f^{-1}(v)$.*

The following preliminary lemma states that, under connectedness conditions, the cone C in the definition of a locally polyhedral map depends only on $f(x)$.

Lemma 3.3.23. *Let $f : X \to V$ be locally polyhedral, $v \in f(X)$ and suppose that the fiber $f^{-1}(v)$ is connected. Then there is a unique polyhedral cone C in V, such that for every $x \in f^{-1}(v)$ there is an open neighbourhood U of x in X for which $f(U) \subset C$ and $f|_U$ is open as map into C. This cone is the cone of f at x for each $x \in f^{-1}(v)$.*

Proof. For each $x \in f^{-1}(v)$, let C_x be the unique cone in V for which there exists an open neighbourhood U_x of x in X, such that $f(U_x) \subset C_x$ and $f|_{U_x}$ is open as map into C_x. Notice that

$$x \sim y \iff C_x = C_y$$

defines an equivalence relation on $f^{-1}(v)$. Since each equivalence class is an open subset of $f^{-1}(v)$, the connectedness of $f^{-1}(v)$ implies that $C_x = C_y$ for all $x, y \in f^{-1}(v)$. So we can take $C = C_x$ for any $x \in f^{-1}(v)$. $\square$

Secondly, we will use an elementary topological lemma, which we state without proof.

Lemma 3.3.24. *A continuous map $f : X \to Y$ is closed onto its image if and only if for every $y \in f(X)$ and every open neighbourhood U of $f^{-1}(y)$ in X there is an open neighbourhood W of y in Y such that $f^{-1}(W) \subset U$.*

Proof of Proposition 3.3.22. Let $C \subset V$ be a polyhedral cone as in the first of the previous two lemmata, and for each $x \in f^{-1}(v)$ let U_x and W_x be open neighbourhoods of x and v in X and V respectively, such that $f(U_x) = (v + C) \cap W_x$. Let U be the union of the U_x and W that of the W_x. Then $f^{-1}(v) \subset U$ and $f(U) = (v + C) \cap W$. According to the last lemma, there is an open neighbourhood $\widetilde{W} \subset W$ of v in V such that $f^{-1}(\widetilde{W}) \subset U$. By construction we have:

$$f(X) \cap \widetilde{W} = (v + C) \cap \widetilde{W},$$

and the proposition follows. $\square$

A natural question is: when is a locally convex subset of V convex? A sufficient (but not necessary) condition is provided by the Tietze-Nakajima theorem.

Theorem 3.3.25 (Tietze-Nakajima)**.** *Let $A \subset V$ be a closed, connected, locally convex set. Then A is convex.*

Proof. See for instance [BK10] for an elementary proof. $\square$

Corollary 3.3.26. *Let $f : X \to V$ be a closed, locally polyhedral map with connected fibers and image. Then $f(X)$ is convex.*

3.3.3 Locally polyhedral maps into affine manifolds

We will now generalize the definitions and results of the previous paragraph to maps into affine manifolds. Throughout, let X be a topological space and $(B, \mathcal{A})$ an affine manifold.

Definition 3.3.27. We call a continuous map $f : X \to B$ locally polyhedral at $x \in X$, if for every open neighbourhood U of $f(x) \in B$ on which $\log_{f(x)}$ is defined, the map

$$\log_{f(x)} \circ f : f^{-1}(U) \to T_{f(x)}B$$

is locally polyhedral at x as map into the vector space $T_{f(x)}B$. We call f locally polyhedral if it is so at every $x \in X$.

Definition 3.3.28. A subset $A \subset B$ is called locally polyhedral (respectively locally convex) at $b \in A$ if for every open neighbourhood U of b on which $\log_b$ is defined, the set $\log_b(U \cap A)$ is locally polyhedral (respectively locally convex) at 0 as subset of the vector space T_bB. It is called locally polyhedral if it is so at all $b \in A$.

Remark 10. The conditions for a map or a subset to be locally polyhedral (or convex) at a point can be verified on a single open neighbourhood U as above (instead of all such open neighbourhoods). One could as well phrase these conditions in terms of affine charts, instead of using log. Furthermore, the definitions given in the previous section for a vector space V coincide with the ones given here when interpreting V as affine manifold with its standard affine structure.

Let $f : X \to B$ be locally polyhedral at x. Then the cone $C_{f,x} \subset T_{f(x)}B$ of

$$\log_{f(x)} \circ f : f^{-1}(U) \to T_{f(x)}B$$

at x (in the sense of Remark 9) does not depend on the choice of open neighbourhood U of $f(x)$ (on which $\log_{f(x)}$ is defined). Similarly, if $A \subset B$ is locally polyhedral at b, then the cone $C_{A,b} \subset T_bB$ of $\log_b(U \cap A)$ at 0 does not depend on the choice of open neighbourhood U of b. Therefore it makes sense to define the following.

Definition 3.3.29. Let $f : X \to B$ be locally polyhedral at x. We call the polyhedral cone $C_{f,x} \subset T_{f(x)}B$ above **the cone of f at x**.

Definition 3.3.30. Let $A \subset B$ be locally polyhedral at b. We call the polyhedral cone $C_{A,b} \subset T_bB$ above **the cone of A at b**.

Proposition 3.3.22 implies its own generalization to maps into integral affine manifolds. Explicitly:

Proposition 3.3.31. *Let $f : X \to B$ be a locally polyhedral map. Suppose that f is closed as a map onto its image and that its fiber over a point $v \in f(X)$ is connected. Then the image of f is locally polyhedral at v and the cone of $f(X)$ at v is the cone of f at x for any $x \in f^{-1}(v)$. In particular, $f(X)$ is locally convex at v.*

3.3.4 Local polyhedrality of the moment map

Our interest in locally polyhedral maps originates from the next result. Let $\mu : (S, \omega) \to B$ be a Hamiltonian $\mathcal{T}_\Lambda$-space. Throughout, given a point $x \in S$, we let $(\alpha_1, ..., \alpha_k)$ denote the weight-tuple of the symplectic isotropy representation at x. Moreover, we let $b = \mu(x)$, $T = T_b^*B/\Lambda_b$ and we canonically identify $\mathfrak{t}^*$ with T_bB.

Theorem 3.3.32. *The moment map μ is locally polyhedral and the cone of μ at $x \in S$ is*

$$\pi^{-1}(Cone(\alpha_1, ..., \alpha_k)) \subset \mathfrak{t}^* = T_bB$$

where $\pi : \mathfrak{t}^* \to \mathfrak{t}_x^*$ is the dual to the inclusion map $\mathfrak{t}_x \to \mathfrak{t}$.

Proof. Let $x \in S$. We will prove that μ is locally polyhedral at x, with the desired cone at x. By the local normal form for μ we can assume without loss of generality that $M = T \times_{T_x} (\mathfrak{t}_x^0 \times \mathcal{S}_x)$ and $\mu : T \times_{T_x} (\mathfrak{t}_x^0 \times \mathcal{S}_x)$ is given by $[t, \alpha, v] \mapsto \alpha + p^*\mu_{\mathcal{S}_x}(v)$, where $x = [e, 0, 0]$ and $p : \mathfrak{t} \to \mathfrak{t}_x$ is a linear projection. Observe that p^* is a linear embedding of $\mathfrak{t}_x^*$ onto a linear complement to $\mathfrak{t}_x^0$ in $\mathfrak{t}^*$ and that

$$\pi^{-1}(\mathrm{Cone}(\alpha_1, ..., \alpha_k)) = \mathfrak{t}_x^0 + p^*\mathrm{Cone}(\alpha_1, ..., \alpha_k),$$

because $\pi \circ p^* = \mathrm{Id}_{\mathfrak{t}_x}$ and $\mathfrak{t}_x^0 = \mathrm{Ker}(\pi)$. Therefore it suffices to show that $\mu_{\mathcal{S}_x}$ is an open map into $\mathrm{Cone}(\alpha_1, ..., \alpha_k)$. In view of Corollary 3.3.17 we can assume that $\mathcal{S}_x = \mathbb{C}^k$ and $\mu_{\mathcal{S}_x}$ is given by

$$\mathbb{C}^k \to \mathfrak{t}_x^*, \quad z \mapsto \sum_{j=1}^k |z_j|^2 \alpha_j.$$

This map is the composition of the map $f : \mathbb{C}^k \to \mathbb{R}^k$, $z \mapsto (|z_1|^2, ..., |z_k|^2)$ and the linear map $g : \mathbb{R}^k \to \mathfrak{t}_x^*$, $x \mapsto \sum_{i=1}^k x_i\alpha_i$. It is clear that the map f is open onto its image $\mathbb{R}_+^k$, so it remains to verify that $g : \mathbb{R}_+^k \to \mathrm{Cone}(\alpha_1, ..., \alpha_k)$ is open. This follows from linearity of g by means of elementary analysis, the details of which we leave to the reader. $\square$

This and Proposition 3.3.22 leads us to conclude the following.

Theorem 3.3.33. *Let $\mu : (S, \omega) \to B$ be a Hamiltonian $\mathcal{T}_\Lambda$-space and $x \in S$. Suppose that μ is closed as a map into its image and its fiber over $b = \mu(x)$ is connected. Then $\mu(S)$ is locally polyhedral at b and the cone of $\mu(S)$ at b is:*

$$\pi^{-1}(Cone(\alpha_1, ..., \alpha_k)).$$

In combination with Corollary 3.3.26, this leads to the following global convexity theorem for the class of Hamiltonian T-spaces.

Corollary 3.3.34. *Let (S, ω, μ) be a Hamiltonian T-space. Suppose that μ is a closed map with connected fibers and image. Then $\mu(S)$ is convex.*

3.3.5 A note on connectedness of the fibers

Atiyah [Ati82] proved that the moment map of a compact and connected Hamiltonian T-space always has connected fibers. He simultaneously proved this and the convexity of the moment image, by a clever induction argument on the dimension of T. To prove the base case $\dim(T) = 1$, he used Morse-Bott theory (which uses the compactness of S).

Simultaneously and independently, Guillemin and Sternberg [GS82] proved the same theorem, although they did not prove or use the connectedness of the fibers. They used the same local normal form as we did (but for Hamiltonian T-spaces and just around fixed points of the action) to prove Theorem 3.3.32 for Hamiltonian T-spaces. They then used Morse-Bott theory as well, to derive from this that the image of the moment map of a compact and connected Hamiltonian T-space is a convex polytope.

The case where the Hamiltonian T-space is non-compact was only studied later. The convexity theorem no longer holds in this case: one can always start with a compact and connected Hamiltonian T-space the moment image of which has non-empty interior and remove a fiber over a point in the interior to destroy the convexity of the moment image (and the compactness of S). The so-called Local-to-Global principle gives local conditions on a map from a topological space into a vector space to have locally or even globally convex image. The results that we have presented here on locally polyhedral maps into vector spaces are a particular case of a more general version of the Local-to-Global principle in [BOR09] (compare [BOR09, Cor 2.18; Thm 2.13]). As is shown there, the assumption in Proposition 3.3.22 that f is closed onto its image and has connected fibers can be replaced by the weaker condition that f is open as a map into its image (and consequently the same goes for Theorem 3.3.33). The following example illustrates this.

Example 3.3.35. Consider the map $\mu : (\mathbb{T}^2, \omega_0) \to S^1$, $(\lambda_1, \lambda_2) \mapsto \lambda_1^2$. This is a Lagrangian fibration and hence it is a free Hamiltonian $\mathcal{T}_\Lambda$-space for the induced integral affines structure Λ on S^1. Since μ is a submersion, it is open. Therefore it satisfies the conclusions of Theorem 3.3.33. Furthermore, the map μ is closed because $\mathbb{T}^2$ is compact. On the other hand, each of its fibers has two connected components, so the connectedness of the fibers is not a necessary condition for the conclusions of this theorem.

Contrary to the previous example, the openness is not always easy to verify and in Chapter 5 we will have other reasons to assume that the moment map is closed onto its image and has connected fibers. Therefore these assumptions are sufficient for our purposes.

It was proved in [BOR09, Cor 3.4], by purely topological means, that the fibers of a closed moment map of a connected Hamiltonian T-space are necessarily connected. Therefore, for closed moment maps of Hamiltonian T-spaces, Theorem 3.3.33 and the corollary following it hold under much milder assumptions. This however does not generalize to moment maps into general affine manifolds, as the coming example will show.

Example 3.3.36. What follows is an example of a compact and connected quasi-Hamiltonian $\mathbb{T}^2$-space, the moment image of which is not locally convex (let alone locally polyhedral). Consider the Hamiltonian $\mathbb{T}^2$-space[3] $(\mathbb{C}\mathrm{P}^2, -\frac{1}{\pi}\omega_0, \mu_0)$ where the moment map is given by

$$\mu_0([z_0 : z_1 : z_2]) = \frac{1}{||z||^2}(|z_1|^2, |z_2|^2)$$

and the action by

$$(\lambda_1, \lambda_2) \cdot [z_0 : z_1 : z_2] = [z_0 : \lambda_1 z_1 : \lambda_2 z_2].$$

Its moment image is the triangle in $\mathbb{R}^2$ with vertices $(1,0)$, $(0,0)$ and $(0,1)$. By composing μ_0 with the exponential map of $\mathbb{T}^2$ (that is, we compose each component with $t \mapsto e^{2\pi i t}$) we obtain a moment map $\mu : (S, \omega) \to \mathbb{T}^2$ that turns the above symplectic $\mathbb{T}^2$-space into a quasi-Hamiltonian $\mathbb{T}^2$-space. The moment image is the image of the above triangle under the exponential map. Observe that the only points in the triangle that are identified with other points are the three vertices, which are mapped onto the single point $(1,1) \in \mathbb{T}^2$. Clearly the moment image is not locally convex at $(1,1)$ with respect to the trivial integral affine structure on $\mathbb{T}^2$. According to Theorem 3.3.32 the fiber over $(1,1)$ is therefore disconnected. Indeed, it consists of the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. This shows that, contrary to the case of Hamiltonian T-spaces, the moment image of a compact and connected Hamiltonian $\mathcal{T}_\Lambda$-space need not be locally convex.

3.3.6 The Atiyah-Guillemin-Sternberg Convexity Theorem

In the previous section we briefly mentioned the Convexity Theorem due to Atiyah and Guillemin-Sternberg. We end this chapter by giving a more detailed account of this theorem. In full detail, the Atiyah-Guillemin-Sternberg Convexity Theorem is as follows.

Theorem 3.3.37 (Classical Convexity Theorem)**.** *Let (M, ω, μ) be a compact and connected Hamiltonian T-space. Then the set $\mu(M^T)$ is finite and the image $\mu(M)$ is the convex hull of $\mu(M^T)$.*

With the convexity theorem in mind, people commonly refer to the image of the moment map of a compact and connected Hamiltonian T-space as its **moment polytope**. This is because, by definition, a polytope is a compact and convex subset of a vector space. We will take it as a fact that the moment map of a compact and connected Hamiltonian T-space has connected fibers. The interested reader can consult either [Ati82] or [BOR09] for a proof. To prove the classical convexity theorem, we need one more lemma.

Lemma 3.3.38. *Let G be a compact Lie group and (M, ω) a symplectic G-space. Then M^G is a symplectic submanifold of (M, ω) (the connected components of which may have*

[3] To verify that this is indeed a Hamiltonian $\mathbb{T}^2$-space, it is easiest to consider $\mathbb{C}P^2$ as S^5/S^1 via symplectic reduction of $(\mathbb{C}^3\backslash\{0\}, -\frac{1}{\pi}\omega_0)$.

varying dimension) with tangent space given by $T_x(M^G) = (T_xM)^G$ for all $x \in M^G$. Moreover, if μ is a moment map for this G-action, then it is constant on each connected component of M^G.

Proof. Notice that M^G is a submanifold with the required tangent space as a consequence of Theorem A.1. Because G is compact and acts symplectically, we can choose a G-equivariant ω-compatible almost complex structure J on (M, ω). Let $x \in M^G$. Since

$$g \cdot Jv = J(g \cdot v) = Jv, \quad \forall g \in G, \quad v \in (T_xM)^G,$$

it follows that $T_x(M^G)$ is J-invariant and hence is a symplectic subspace of (T_xM, ω_x). Thus M^G is a symplectic submanifold. At last, let μ be a moment map for the given G-action. Let $x \in M^G$, $(M^G)_x$ the connected component of M^G containing x and let $\xi \in \mathfrak{g}$. If $y \in M^G$, then $G_y = G$, so $\mathfrak{g}_y = \mathfrak{g}$ and $\xi_M(y) = 0$. Therefore, $d\langle \mu, \xi \rangle = \iota_{\xi_M}\omega = 0$ on M^G, so that $\langle \mu, \xi \rangle$ is constant on $(M^G)_x$. This being valid for all $\xi \in \mathfrak{g}$ implies that μ is constant on $(M^G)_x$, as desired. □

We can now prove the classical convexity theorem.

Proof of Theorem 3.3.37. From Corollary 3.3.34 it follows that $\mu(M)$ is convex and since M is compact it is a polytope. In particular $\mu(M)$ contains the convex hull of $\mu(M^T)$. We will now prove that $\mu(M^T)$ is finite and that its convex hull is equal to $\mu(M)$. Since M^T is a submanifold of M, it is locally connected and so its components are open subsets. Since M^T is closed in the compact M, it is compact as well and so it has finitely many connected components. But μ is constant on each connected component of M^T, hence $\mu(M^T)$ is finite. By Theorem 3.3.33, the image of the moment map contains an open neighbourhood of $\mu(x)$ in $\mu(x) + \mathfrak{t}_x^0$, for each $x \in M$. Therefore $\mathfrak{t}_x^0 = \{0\}$, if $\mu(x)$ is an extreme point of $\mu(M)$. This means that $\mathfrak{t}_x = \mathfrak{t}$ and hence

$$T = \exp_T(\mathfrak{t}) = \exp_T(\mathfrak{t}_x) \subset T_x \subset T,$$

so that $T = T_x$. So every extreme point of $\mu(M)$ lies in $\mu(M^T)$. Because a convex polytope is the convex hull of its extreme points (see [Brø83, Theorem 5.10]), this implies that $\mu(M)$ is contained in the convex hull of $\mu(M^T)$. □

It is natural to wonder what properties of a Hamiltonian T-space can be read off of its moment polytope. For instance, as we have shown in the proof of the convexity theorem, the pre-image of the set of vertices of the moment polytope is contained in the fixed point set M^T. As a particular consequence we have:

Corollary 3.3.39. *The number of fixed points of a Hamiltonian torus action on a compact and connected manifold is strictly greater than the dimension of its moment polytope.*

As can be seen from the example below, it is not true that every fixed point has to be mapped to a vertex of the moment polytope. Similarly, if the isotropy group T_x of a point $x \in M$ is trivial (or even discrete), then μ is a submersion at x and so $\mu(x)$ must lie in the interior of the moment polytope. It may be tempting to believe that every point in the interior of the moment polytope is a regular value of μ. This however is false as well, as the following simple example shows.

Example 3.3.40. Consider the Hamiltonian S^1-action on $(\mathbb{C}P^2, -\frac{1}{\pi}\omega_0)$, given by

$$\lambda \cdot [z_0 : z_1 : z_2] = [z_0 : \lambda z_1 : \lambda^{-1} z_2],$$

with moment map defined by:

$$\mu([z_0 : z_1 : z_2]) = \frac{|z_1|^2 - |z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

This has moment image $\mu(\mathbb{C}P^2) = [-1, 1]$. However, $x = [1 : 0 : 0]$ is a fixed point (and hence μ is singular at x) while $\mu(x) = 0$. Therefore not every fixed point is mapped to a vertex of $[-1, 1]$ and not every interior point of $[-1, 1]$ is a regular value of μ.

In Chapter 5 we will come across a class of compact and connected Hamiltonian T-spaces, called toric manifolds, for which fixed points are always mapped to vertices of the moment polytope and its the interior does consist entirely of regular values of the moment map. In fact, for this class the moment polytope completely encodes the Hamiltonian T-space.

Chapter 4

Intermezzo: stratified spaces by examples

In this chapter we recall the notion of a stratification and its regular part, and we provide some examples of stratified spaces that we will come across in the coming chapter. Roughly speaking, a stratification of a topological space is a partition of the space by manifolds that fit together in a good way. Many singular spaces admit natural stratifications, although they might not be a smooth manifold themselves.

Sections 4.1, 4.3 and 4.4 are based on [CM17], while Section 4.2 is based on [Joy10].

4.1 The definition of a stratification and its regular part

Stratifications are defined as follows.

Definition 4.1.1. Let X be a Hausdorff, second-countable and paracompact topological space. A **stratification** of X is a locally finite partition $\mathcal{S} = \{S_i\}_{i\in I}$ of X satisfying:

- Each S_i is a connected manifold endowed with the subspace topology, which is locally closed in X.
- For each $i \in I$, the closure of S_i in X is a union of S_i and some other $S_j{}'s$ of dimension strictly smaller than that of S_i.

The second condition is known as the frontier condition. An element of $\mathcal{S}$ is called a **stratum** of the stratification and $(X, \mathcal{S})$ is called a stratified space.

At a first glance, the frontier condition does not seem to be of a local nature. The following lemma sheds a different light on this.

Lemma 4.1.2. *Let X be a Hausdorff, second-countable and paracompact topological space and $\mathcal{S}$ a partition of X that satisfies all the axioms of a stratification, except for the frontier condition. Then $\mathcal{S}$ is a stratification if and only if every $x \in X$ admits an open neighbourhood U with the property that $U \cap S_x \subset \overline{S}$ and $dim(S_x) < dim(S)$ for each $S \in \mathcal{S}$ that intersects U and is different from the member S_x through x.*

Proof. First suppose that $\mathcal{S}$ is a stratification and let $x \in X$. Then x admits an open neighbourhood U that intersects only finitely many strata. By removing the finite union of those sets $\overline{S}$ (for $S \in \mathcal{S}$) that intersect U but do not contain x, we can assume that x is contained in the closure of each stratum that intersects U. The desired condition on U is then immediate from the frontier condition.

Conversely, suppose that every $x \in X$ admits an open neighbourhood with the above property. To verify the frontier condition, let $S, S' \in \mathcal{S}$ be distinct strata such that $S \cap \overline{S'} \neq \emptyset$. We need to show that $\dim(S) < \dim(S')$ and $S \subset S'$. For each $x \in S \cap \overline{S'}$ there is an open neighbourhood U of x with the above property. Since S' intersects U, it then follows that $\dim(S) < \dim(S')$ and $U \cap S \subset \overline{S'}$ for each such x and U. The latter shows that $S \cap \overline{S'}$ is open in S. Since it is closed in the connected space S as well, it follows that $S \cap \overline{S'} = S$. Thus $S \subset \overline{S'}$, as desired. □

Each stratification comes with a partial ordering: for $S, S' \in \mathcal{S}$, we say that $S \leq S'$ if and only if $S \subset \overline{S'}$. This allows one to speak of maximal strata. The following characterizes these.

Proposition 4.1.3. *Let $(X, \mathcal{S})$ be a stratified space. Then $S \in \mathcal{S}$ is maximal if and only if it is open in X. Moreover, the union of all maximal strata is open and dense in X.*

Proof. Let $S \in \mathcal{S}$ be open and contained in $\overline{S'}$ for some $S' \in \mathcal{S}$. Fix an $x \in S$. Then S is an open neighbourhood of x in X and $x \in \overline{S'}$, so S' must intersect S and hence $S = S'$. Conversely, suppose that $S \in \mathcal{S}$ is maximal. Let $x \in S$. As in the proof of the previous lemma we choose an open neighbourhood U of x in X such that the closure of any stratum that intersects U must contain x. By the frontier condition this implies that $S \subset \overline{S'}$ for any stratum S' that intersects U, so that $S = S'$ by maximality. We conclude that S is the only stratum that intersects U, or in other words, $U \subset S$. Therefore S is open in X.

It remains to show that the union of all maximal strata is dense in X. Let $x \in X$ and S_x the stratum through x. By locally finiteness of $\mathcal{S}$, there are only finitely many strata S such that $S_x \leq S$. Therefore we can inductively construct a strict chain

$$S_x := S_0 < S_1 < ... < S_k$$

where S_k is a maximal stratum. Then $x \in \overline{S_k}$ and in particular x is contained in the closure of the union of all maximal strata, which completes the proof. □

Definition 4.1.4. Let $(X, \mathcal{S})$ be a stratified space. The union of all maximal strata is called the **$\mathcal{S}$-regular part of** X.

A priori, we know that the $\mathcal{S}$-regular part of X is a union of open strata which is dense in X. In each of the coming sections we will introduce a different example of a stratified space that will return in the coming chapter and in each example we will describe the regular part more explicitly.

4.2 Manifolds with corners

The connected components of a smooth manifold form a stratification. More generally, every manifold with boundary is stratified by the connected components of its interior and of its boundary. One can generalize this even further to manifolds with corners. This is the aim of this section.

Let X be a topological space. We use the notation:

$$\mathbb{R}^n_k := [0,\infty[^k \times \mathbb{R}^{n-k}.$$

By an n-chart with corners on X we mean a pair (U,χ) consisting of an open U in X and an open topological embedding $\chi : U \to \mathbb{R}^n_k$, for some $k \in \{0,...,n\}$. Given two subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we say that a map $f : A \to B$ is smooth if it extends to a smooth map from an open neighbourhood of A into $\mathbb{R}^m$. Two n-charts with corners (U,χ) and (V,φ) on X are called smoothly compatible if both transition maps between them are smooth maps in this sense. Just as for smooth manifolds without corners, this leads to a notion of (maximal) smooth atlas for manifolds with corners.

Definition 4.2.1. A **smooth n-manifold with corners** is a second countable Hausdorff space X together with a maximal smooth atlas $\mathcal{A}$ consisting of n-charts with corners.

Since a manifold with corners is locally compact (as well as Hausdorff and second-countable), it is paracompact. A manifold with corners comes with a natural stratification, defined as follows. Let X be an n-manifold with corners. Given a point $x \in X$ and a chart (U,χ) around x that maps onto an open subset of $\mathbb{R}^n_k$, we define $\mathrm{depth}_X(x)$ to be the number of $i \in \{1,...,k\}$ for which $\chi^i(x) = 0$.

Lemma 4.2.2. *The number $\mathrm{depth}_X(x)$ is independent of the choice of chart.*

Proof. Let (U,χ) and (V,φ) be two charts around x that are open embeddings into $\mathbb{R}^n_k$ and $\mathbb{R}^n_l$ respectively. We can assume that $\chi^i(x) = 0$ for all $i \leq k$ and $\varphi^i(x) = 0$ for all $i \leq l$. If $i \in \{1,...,k\}$ and $j \in \{l+1,...,n\}$, then the path

$$t \mapsto \chi^i(\varphi^{-1}(\varphi(x) + te_j))$$

in $\mathbb{R}$ is defined on an open neighbourhood of 0 in $\mathbb{R}$ and attains a minimum value of 0 at $t = 0$. It follows that $\mathrm{Jac}(\chi\circ\varphi^{-1})^i_j(\varphi(x)) = 0$ for all such i and j. But $\mathrm{Jac}(\chi\circ\varphi^{-1})(\varphi(x))$ is invertible by the chain rule, which still holds since it does so on an open in $\mathbb{R}^n$ that is dense in $\varphi(U)$. So the rows of its first $k \times l$-block are linearly independent, which implies that $k \leq l$. Reversing the roles of χ and φ, we obtain that $l \leq k$ as well. □

We thus obtain a well-defined number $\mathrm{depth}_X(x) \in \{0,...,n\}$ for each $x \in X$. Points of depth 0 are called interior points of X, whereas points of depth $n = \dim(X)$ are called vertices.

Definition 4.2.3. For each $i \in \{0,...,n\}$, let

$$\mathcal{P}^i(X) = \{x \in X |\ \mathrm{depth}_X(x) = i\}.$$

The sets $\mathcal{P}^i(X)$ form the **partition by depths** of X. The **open faces** of X are the connected components of each of the subspaces $\mathcal{P}^i(X)$ of X and the **canonical stratification** of X is the collection of all of its open faces, denoted by $\mathcal{S}(X)$.

Proposition 4.2.4. *The collection $\mathcal{S}(X)$ is indeed a stratification of X.*

Proof. Let $x \in \mathcal{P}^k(X)$. Then there is a chart (U, χ) for X around x for which $\chi(U)$ is open in $\mathbb{R}^n_k$. From the definition of depth it follows that

$$\chi(U \cap \mathcal{P}^k(X)) = \chi(U) \cap (\{0\} \times \mathbb{R}^{n-k}).$$

As for manifolds without corners, this property implies that $\mathcal{P}^k(X)$ inherits the structure of an $(n-k)$-manifold without corners from X. It implies as well that the open faces of X are locally closed in X. To see that $\mathcal{S}(X)$ is locally finite and satisfies the frontier condition, note that, in fact, the definition of depth implies more. Namely, we have that:

$$\chi(U \cap \mathcal{P}^j(X)) = \left(\bigsqcup_{I \subset \{1,...,k\}:\ |I|=j} \chi(U) \cap F_I \right),$$

where

$$F_I = \{x \in \mathbb{R}^n_k |\ x_i = 0 \text{ if } i \in I \text{ and } x_i \neq 0 \text{ if } i \in \{1, ..., k\} \backslash I\}.$$

By shrinking U, we can assume that $\chi(U)$ is convex. By convexity of the $F_I's$, it follows that $\chi(U) \cap F_I$ is convex, and hence connected, for each $I \subset \{0, ..., k\}$. Therefore $\chi^{-1}(\chi(U) \cap F_I)$ is contained in a single open face of X for each I, which implies that U intersects finitely many open faces. So $\mathcal{S}(X)$ is locally finite. We further see that $U \cap \mathcal{P}^k(X) = U \cap F$ where F is the open face of X through x. So if F' is a different face that intersects U, then $\text{depth}(F') < k$. Therefore $\dim(F') > \dim(F)$. The fact that $\{0\} \times \mathbb{R}^{n-k} \subset \overline{F_I}$ for each $I \subset \{1, ..., k\}$ implies that $U \cap F \subset \overline{F'}$. So by lemma 4.1.2 the desired follows. $\square$

Remark 11. There is a unique open and dense member of the partition by depths of X: the set of interior points $\mathcal{P}^0(X)$. From the fact that the maximal strata are the open ones, it is clear that the set of interior points is the $\mathcal{S}(X)$-regular part of X.

4.3 Proper G-spaces and their orbit spaces

Proper G-spaces and their orbit spaces admit stratifications. From now on, let M denote a proper G-space and X its orbit space. As for manifold with corners, we first define a partition of M into possibly disconnected manifolds.

Definition 4.3.1. The **isotropy type** of a point $x \in M$ is the conjugacy class (G_x) of the subgroup G_x in G. The relation $x \sim y$ if and only if x and y have the same isotropy type is an equivalence relation. Its equivalence classes are called the **orbit types** of the G-space M.

Remark 12. Let us make a few short remarks on the orbit types. First of all, each orbit type is a set of the form

$$M_{(H)} = \{x \in M |\ (G_x) = (H)\},$$

for some subgroup H of G. Secondly, because $G_{g \cdot x} = gG_xg^{-1}$, each orbit is contained in a single orbit type. Finally, the orbit types need not be connected. We illustrate this in the next example.

Example 4.3.2. Consider the S^1-action on S^2, by rotation around the axis through the north and south pole. There are two orbit types: the disconnected set consisting of the north and south pole is the orbit type $M_{(S^1)}$, while its complement $M_{(\{e\})}$ is the other.

Using the Slice Theorem for proper G-spaces (Theorem A.1 in the appendix), one can show that the orbit types are submanifolds of M, the connected components of which may have varying dimension. By passing to the connected components of the partition by orbit types, we obtain a partition of M by connected submanifolds, which we call the **canonical stratification** of M and denote by $\mathcal{S}_G(M)$. Moreover, since each orbit is contained in a single orbit type, the orbit projection $\pi : M \to X$ sends the partition into orbit types to a partition of the orbit space X. By passing to connected component we obtain a partition of X into connected subspaces, which we call the canonical stratification of X and denote by $\mathcal{S}_G(X)$.

Theorem 4.3.3. *The collection $\mathcal{S}_G(M)$ is a stratification of M. Furthermore, for each orbit type $M_{(H)}$ on M there is a unique smooth structure on $\pi(M_{(H)})$ for which $\pi : M_{(H)} \to \pi(M_{(H)})$ is a submersion. This provides the members of $\mathcal{S}_G(X)$ with a smooth structure for which the collection is a stratification of X.*

The proof of this uses the Slice Theorem for proper G-spaces. This provides a local normal form for the action around each orbit, which allows one to prove the above theorem by induction on the dimension of M. A full proof can be found in [DK00].

Next, we will express the $\mathcal{S}_G(M)$-regular part of M and the $\mathcal{S}_G(X)$-regular part of X in terms of the orbit types of M. The orbit types of M come with a natural partial ordering defined by:

$$M_{(H_0)} \leq M_{(H_1)} \iff H_1 \text{ is } G\text{-conjugate to a subgroup of } H_0.$$

The anti-symmetry of this relation can be shown using the fact that the isotropy groups of a proper action are compact Lie groups. Having this partial ordering, one can talk about maximal orbit types. The following theorem describes the maximal orbit types of M and its orbit space X.

Theorem 4.3.4 (Principal orbit type theorem)**.** *Let M be a proper G-space and suppose that its orbit space X is connected. Then there is a unique orbit type M^{prin}, called the **principal orbit type**, that is open and dense in M. The principal orbit type is a greatest element with respect to the partial ordering that we just defined. Furthermore, M^{prin}/G is connected.*

The proof of this again hinges on the Slice Theorem. See for instance [tD87, Thm 5.14] for details.

Remark 13. The fact that M^{prin} is the greatest element can also be expressed by saying that amongst the isotropy types occurring in M, there is a unique one $(H)^{\text{prin}}$, called the **principal isotropy type**, such that H is G-conjugate to a subgroup of G_x for every $x \in M$.

The principal orbit type theorem has the following consequence for the regular parts.

Corollary 4.3.5. *Let M be a proper G-space and suppose that its orbit space X is connected. Then the $\mathcal{S}_G(M)$-regular part of M is the principal orbit type of M and the $\mathcal{S}_G(X)$-regular part of X consists of the single stratum M^{prin}/G.*

Suppose that the Lie group G is abelian. In this case, two subgroups of G are conjugate precisely if they are equal, so that each isotropy type is represented by a unique subgroup of G. Therefore it makes sense to speak of the principal isotropy group of M.

Corollary 4.3.6. *Let G be an abelian Lie group, acting properly on a manifold M. Suppose M/G is connected. Then the principal isotropy group is trivial if and only if the action is effective.*

Proof. The implication from left to right is immediate. Now suppose that action is effective. Because G is abelian the principal isotropy group is not just conjugate, but equal to a subgroup of G_x, for every $x \in M$. Therefore the principal isotropy group fixes all of M, which by effectiveness of the action means that it is the trivial group. $\square$

4.4 The base and orbit space of a proper Lie groupoid

We will now generalize the canonical stratifications in the previous section to stratifications on the base and orbit space of a proper Lie groupoid. Throughout, let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and X its orbit space. Once again, we start by defining a partition of M. Two representations (G, V) and (H, W) are called equivalent if there is an isomorphism of Lie groups $\varphi : G \to H$ and a linear isomorphism $L : V \to W$ such that:

$$L(g \cdot v) = \varphi(g) \cdot L(v)$$

for all $g \in G$ and $v \in V$. We write $(G, V) \sim (H, W)$ if (G, V) and (H, W) are equivalent representations.

Definition 4.4.1. The relation

$$x \sim y \iff (\mathcal{G}_x, \mathcal{N}_x) \sim (\mathcal{G}_y, \mathcal{N}_y)$$

is an equivalence relation on M. Its equivalence classes are called the **Morita types** of the Lie groupoid $\mathcal{G} \rightrightarrows M$. We denote the Morita type through $x \in M$ by $M_{(x)}$.

The Morita types turn out to be submanifolds of M, the connected components of which may have varying dimensions. By passing to the connected components we obtain a partition of M by connected submanifolds, which we call the **canonical stratification** and denote by $\mathcal{S}_\mathcal{G}(M)$. As before, the partition by Morita types is mapped to a partition of the orbit space X by the orbit projection and by passing to connected components we obtain a partition of X by connected subspaces that we call the **canonical stratification** of X and denote by $\mathcal{S}_\mathcal{G}(X)$. To see that the partition by Morita types indeed descends to a partition on X, one needs to verify that all points on a single orbit $\mathcal{O}$ belong to the same Morita type. This holds because, for any two points $x, y \in \mathcal{O}$ there is a $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$, and conjugation by g is a Lie group isomorphism $\mathcal{G}_x \to \mathcal{G}_y$ which is compatible with the isomorphism of vector spaces $m_g : \mathcal{N}_x \to \mathcal{N}_y$ coming from the Lie groupoid action of $\mathcal{G}_\mathcal{O}$ on the normal bundle $\mathcal{N}_\mathcal{O} \to \mathcal{O}$.

Theorem 4.4.2. *The collection $\mathcal{S}_\mathcal{G}(M)$ is a stratification of M. Furthermore, for each Morita type $M_{(x)}$ of M there is a unique smooth structure on the Morita type $\pi(M_{(x)})$ of X for which $\pi : M_{(x)} \to \pi(M_{(x)})$ is a smooth submersion. This provides members of $\mathcal{S}_\mathcal{G}(X)$ with a smooth structure for which the collection is stratification of X.*

A full proof can be found in [CM17]. It hinges on the Linearization Theorem for proper Lie groupoids. This is a generalization of the Slice Theorem for proper G-spaces that provides a local normal form for the Lie groupoid around each orbit.

Lastly, we discuss what the regular parts of these stratifications are. Contrary to the partition of orbit types, there is no obvious partial order on the Morita types. We can however still relate the regular parts to the Morita types by a theorem analogous to the principal orbit type theorem for proper G-spaces. The characterization of maximal strata as being the open strata in X and an analysis of a local model for the proper Lie groupoid around a point $x \in M$ leads to:

Proposition 4.4.3 (Lemma 4.31, [CM17])**.** *A point $x \in M$ belongs to the $\mathcal{S}_{\mathcal{G}}(M)$-regular part of M if and only if the isotropy action of $\mathcal{G}_x$ on $\mathcal{N}_x$ is trivial.*

As an immediate corollary we have:

Corollary 4.4.4. *The $\mathcal{S}_{\mathcal{G}}(M)$-regular part is a union of Morita types.*

The principal Morita type theorem gives an even stronger specification of the regular parts.

Theorem 4.4.5 (Principal Morita type theorem)**.** *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and suppose that the orbit space X is connected. Then the $S_{\mathcal{G}}(X)$-regular part of X is connected and hence it is a single stratum of X.*

This is proved in [CM17]. Using this and the previous result, one can derive:

Corollary 4.4.6. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and suppose that the orbit space X is connected. Then on both M and X there is a unique open and dense Morita type (which coincides with the regular part).*

In analogy with the case of proper group actions, we will call the unique open and dense Morita types above the **principal Morita types** of M and X.

Remark 14. We end this chapter with some closing remarks. See [CM17] for proofs.

1. Using the Principal Morita type theorem it is not hard to show that the principal Morita type of M is connected if both the orbits and the orbit space of $\mathcal{G}$ are connected.

2. If M is a proper G-space, then the action groupoid $G \ltimes M \rightrightarrows M$ is a proper Lie groupoid and its orbit space is that of the G-action. The canonical stratifications on M and M/G that we have defined in the last two sections coincide, although the partitions into orbit types and Morita types of M do not necessarily coincide.

Chapter 5

A classification of toric $\mathcal{T}_\Lambda$-spaces

In this chapter we focus on the class of toric $\mathcal{T}_\Lambda$-spaces. We classify the toric $\mathcal{T}_\Lambda$-spaces over a given integral affine manifold (B, Λ) in terms of the moment image and a cohomology class. Both the classical notions of toric manifolds and Lagrangian fibrations belong to this class and when applied to these special cases our classification result reduces to Delzant's classification of toric manifolds and Duistermaat's classification of Lagrangian fibrations, respectively.

5.1 Main properties of toric $\mathcal{T}_\Lambda$-spaces

Throughout the rest of this chapter, let (B, Λ) denote an integral affine manifold.

Definition 5.1.1. A toric $\mathcal{T}_\Lambda$-space is a Hamiltonian $\mathcal{T}_\Lambda$-space $\mu : (S, \omega) \to B$, for which:

- The action is free on an open and dense subset of S.
- $\dim(S) = 2\dim(B)$.
- The moment map μ is closed as a map onto its image and its fibers are connected.

Let us shed some light on the above conditions. If $S/\mathcal{T}_\Lambda$ is connected, then in view of the Principal Morita type Theorem, the first condition is equivalent to the condition that the action is free at some point of S. It ensures in particular that the moment map is a weak isotropic realization of $(B, 0)$. Therefore the dimension of S is at least twice the dimension of B. The case in which $\dim(S) = 2\dim(B)$ is the most ideal situation. This will become clearer in the coming section, where we will study toric representations. Finally, the conditions for μ to be closed as a map onto its image and have connected fibers ensures that the moment map descends to a homeomorphism from the orbit space of the action onto its moment image. Moreover, it implies that the moment image is locally polyhedral and has the structure of a Delzant submanifold, as will be shown in Section 5.1.3.

Throughout this chapter there are two main examples to keep in mind, which we will return to at the end of the chapter.

Example 5.1.2. A Hamiltonian T-space (S, ω, μ) is called a **toric manifold** if S is compact and connected, $\dim(S) = 2\dim(T)$ and the T-action is effective. Due to Corollary 4.3.6 this class of Hamiltonian T-spaces corresponds to the compact and connected toric $\mathcal{T}_{\Lambda_T}$-spaces over $(\mathfrak{t}^*, \Lambda_T^*)$.

Example 5.1.3. As we saw in Section 3.1, a Lagrangian fibration $\mu : (S, \omega) \to B$ with compact and connected fibers induces an integral affine structure on B. Those that induce the given integral affine structure Λ on the base B correspond to principal $\mathcal{T}_\Lambda$-bundles over (B, Λ). Therefore they are examples of toric $\mathcal{T}_\Lambda$-spaces.

5.1.1 Toric representations

In Section 3.2 we found that the local behaviour of a Hamiltonian $\mathcal{T}_\Lambda$-space is governed by its symplectic isotropy representations. For toric $\mathcal{T}_\Lambda$-spaces the symplectic isotropy representations belong to a special class: the toric T-representations. In this section we classify the toric T-representations in terms of their moment image. This should be viewed as a preliminary version of the classification of toric $\mathcal{T}_\Lambda$-spaces. The results and proofs in this section are mostly reformulations of those in [LT97].

Definition 5.1.4. Let G be a compact and abelian Lie group and $G \to \mathrm{Sp}(V, \omega)$ a symplectic representation. We say that the representation is **dependency-free** if its standard moment map is a submersion on an open and dense subset of V and

$$\dim(G) = \frac{1}{2}\dim(V).$$

Secondly, we say that it is **toric** if the action is free on an open and dense subset of V and the above condition on dimensions holds.

Remark 15. Observe that a toric G-representation is in particular dependency-free. Furthermore, the action is free on an open and dense subset of V if and only if it is effective. This follow from Corollary 4.3.6.

The term toric is justified by the next lemma.

Lemma 5.1.5. *Let (V, ω) be a toric representation of a compact, abelian Lie group G. Then G is a torus.*

Proof. Since the action of G is effective, the morphism of Lie groups $G \to \mathrm{Sp}(V, \omega)$ is injective. Any injective morphism of Lie groups is immersive, so the above on is an embedding by compactness of G. Therefore we can assume that G is a compact Lie subgroup of $\mathrm{Sp}(V, \omega)$. Since G is compact we can choose an equivariant ω-compatible complex structure J on V. Then the representation of G is unitary with respect to this complex structure and the Hermitian inner-product h defined by

$$h(v, w) = \omega(v, Jw) - i\omega(v, w), \quad v, w \in V.$$

Thus $G \subset \mathrm{U}(V, J, h)$. Choose a unitary decomposition of V into irreducible complex subrepresentations and choose unitary basis of V which is compatible with this decomposition. This basis induces an isomorphism of Lie groups $\mathrm{U}(V, J, h) \to \mathrm{U}(n)$ that sends G into the closed matrix subgroup $\mathrm{U}(1)^n \subset \mathrm{U}(n)$ consisting of diagonal matrices in $\mathrm{U}(n)$. This matrix subgroup is a torus of dimension n, because it consists of the diagonal matrices with diagonal entries in $\mathrm{U}(1)$. Since G has dimension n, it must be a codimension-0 (and hence open) submanifold of $\mathrm{U}(1)^n$. On the other hand, it is compact and hence closed in $\mathrm{U}(1)^n$. So by connectedness $\mathrm{U}(1)^n = G$ and hence G is a torus. $\square$

The following proposition shows that the toric T-representations are classified by those polyhedral cones in $\mathfrak{t}^*$ that are generated by a basis of Λ_T^*. Such polyhedral cones are called **smooth**.

Proposition 5.1.6. *Let T be an n-dimensional torus. The following holds.*

a) A symplectic T-representation is dependency-free if and only if its weight-tuple is a basis of $\mathfrak{t}^*$.

b) A symplectic T-representation is toric if and only if its weight-tuple forms a basis of the lattice Λ_T^*.

c) The map that associates to each unordered basis $\{\alpha_1, ..., \alpha_n\}$ of Λ_T^* the polyhedral cone generated by $(\alpha_1, ..., \alpha_n)$ is a bijection from the set of such n-tuples to the set of smooth polyhedral cones.

Consequently, we have a bijection:

$$\{\textit{Isomorphism classes of toric } T\textit{-representations}\} \to \{\textit{Smooth polyhedral cones in } \mathfrak{t}^*\}$$
$$[(V, \omega)] \mapsto \mu_V(V)$$

Proof. Suppose that we are given a symplectic T-representation (V, ω). Let $(\alpha_1, ..., \alpha_n)$ be its weights and let μ be its standard moment map. For part a, assume first that the representation is dependency-free. Then μ must be a submersion at some point of $v \in V$. As a consequence of the submersion theorem, $\mu(v)$ lies in the interior of $\mu(V)$. By corollary 3.3.17 and the assumption that $n = \dim(T) = \frac{1}{2}\dim(V)$ we have:

$$\mu(V) = \text{Cone}(\alpha_1, ..., \alpha_n).$$

Hence the weight-tuple must be linearly independent, for otherwise $\mu(V)$ would be contained in a hyperplane in $\mathfrak{t}^*$ and its interior would be empty. Conversely, assume that the weights form a basis of $\mathfrak{t}^*$. Then since the weight-tuple has length $\frac{1}{2}\dim(V)$, it follows that $\dim(T) = \frac{1}{2}\dim(V)$. Moreover, one easily computes that the map

$$\mathbb{C}^n \to \mathfrak{t}^*, \quad z \mapsto \sum_{j=1}^{n} |z_j|^2 \alpha_j$$

is submersion at those points where $z_j \neq 0$ for all j, because the weights form a basis of $\mathfrak{t}^*$. So it follows from Corollary 3.3.17 that μ is a submersion on an open and dense subset of V.

Next, we adress part b. By a and the fact that Λ_T^* is a full lattice in $\mathfrak{t}^*$ we know that, under either of the two assertions in b, the weights form a basis of $\mathfrak{t}^*$ and the representation is dependency-free. Suppose this is the case. Let $\{\alpha^1, ..., \alpha^n\}$ be the dual basis of $\mathfrak{t}$ to $\{\alpha_1, ..., \alpha_n\}$ and let Λ be their $\mathbb{Z}$-span. Then Λ^* is $\mathbb{Z}$-spanned by $\{\alpha_1, ..., \alpha_n\}$, hence $\Lambda^* \subset \Lambda_T^*$ which implies $\Lambda_T \subset \Lambda$. By the weight-classification theorem we may assume that $(V, \omega) = (\mathbb{C}^n, \omega_0)$, where T acts on $\mathbb{C}^n$ as

$$\exp(\xi) \cdot (z_1, ..., z_n) = (e^{2\pi i \alpha_1(\xi)} z_1, ..., e^{2\pi i \alpha_n(\xi)} z_n), \quad \xi \in \mathfrak{t}.$$

Consider the open subset $U = \{z \in \mathbb{C}^n \mid z_j \neq 0 \text{ for all } j\}$ of $\mathbb{C}^n$. From the above expression for the action one directly verifies that

$$\exp(\Lambda) = T_z$$

for each $z \in U$. Hence T_z is trivial for all z in the open and dense subset U of $\mathbb{C}^n$ if and only if $\Lambda \subset \Lambda_T$. So part b follows.

Finally we prove assertion c. The surjectivity of the map in question holds by our definition of a smooth polyhedral cone. Hence it remains to verify injectivity. To this end, let C be a smooth polyhedral cone in $\mathfrak{t}^*$, generated by both the bases $(\alpha_1, ..., \alpha_n)$ and $(\beta_1, ..., \beta_n)$ of the lattice Λ_T^*. Using that both tuples form a basis of $\mathfrak{t}^*$, it is straightforward to deduce from this that (after a possible reordering) there are $t_1, ..., t_n \in \mathbb{R}_{>0}$ such that $\alpha_j = t_j\beta_j$ for each j. Geometrically speaking, this can be phrased as saying that the polyhedral cone C is pointed and it has n extreme rays given by both $\{\mathbb{R}_{\geq 0}\cdot\alpha_1, ..., \mathbb{R}_{\geq 0}\cdot\alpha_n\}$ and $\{\mathbb{R}_{\geq 0}\cdot\beta_1, ..., \mathbb{R}_{\geq 0}\cdot\beta_n\}$. Since $(\beta_1, ..., \beta_n)$ forms a basis of Λ_T^* and $\alpha_j \in \Lambda_T^*$ for each j it follows that $t_j \in \mathbb{Z}$ for each j. Reversing the roles of the α_j and β_j, the same argument shows that $\frac{1}{t_j} \in \mathbb{Z}$ for each j. Hence $t_j = 1$ for each j and the proposition is proven. $\square$

The following lemma allows us to apply the preceding proposition to general Hamiltonian T-spaces.

Proposition 5.1.7. *The symplectic isotropy representations of a toric $\mathcal{T}_\Lambda$-space are toric representations.*

Proof. Let $\mu : (S, \omega) \to B$ be the toric $\mathcal{T}_\Lambda$-space, $x \in S$, $b = \mu(x)$ and $T = T_b^*B/\Lambda_b$. First we observe that

$$\begin{aligned}\dim(\mathcal{S}_x) &= \dim(S) - (\dim(\mathcal{O}_x) + \dim(\mathfrak{t}_x^0))\\ &= \dim(S) - 2(\dim(T) - \dim(T_x))\\ &= 2\dim(T_x).\end{aligned}$$

since $\dim(T) = \dim(B)$. Secondly, by the local normal form for Hamiltonian $\mathcal{T}_\Lambda$-spaces there is an open neighbourhood of the orbit through $[e, 0, 0]$ in $T \times_{T_x} (\mathfrak{t}_x^0 \times \mathcal{S}_x)$ inside which the points $[t, \alpha, v]$ with trivial isotropy group lie open and dense. Observe that

$$T_{[t,\alpha,v]} = T_{[e,\alpha,v]} = (T_x)_v$$

since T is abelian and the action of T_x on $\mathfrak{t}_x^0$ is trivial. One could now conclude the existence of a point in $\mathcal{S}_x$ with trivial isotropy group and appeal to the Principal Orbit Type Theorem to conclude that the action is free on an open and dense subset of $\mathcal{S}_x$. Alternatively, one can use that $(T_x)_{sv} = (T_x)_v$ for all $s > 0$ (by linearity of the action on $\mathcal{S}_x$) to derive directly from the above that the action on $\mathcal{S}_x$ is free on an open and dense subset. Anyhow, the desired representation is toric. $\square$

Together with Lemma 5.1.5 this yields:

Corollary 5.1.8. *The isotropy groups of a toric $\mathcal{T}_\Lambda$-space are tori.*

5.1.2 Delzant submanifolds

As we will see throughout this chapter, toric $\mathcal{T}_\Lambda$-spaces are closely related to Delzant submanifolds. In this section we introduce these and discuss their properties.

Definition 5.1.9. A Delzant submanifold of an integral affine manifold (B, Λ) is a subset $\Delta \subset B$ such that for every point $p \in \Delta$ there is an integral affine chart (U, χ) of (B, Λ) around p for which $\chi(p) = 0$ and $\chi(\Delta \cap U)$ is an open neighbourhood of 0 in

$$\mathbb{R}_k^n = [0, \infty[^k \times \mathbb{R}^{n-k}$$

for some $k \in \{0, ..., n\}$.

A Delzant submanifold is a manifold with corners and therefore comes with a stratification by open faces. These faces are integral affine submanifolds of (B, Λ).

Definition 5.1.10. We say that a subspace $F \subset B$ is an integral affine submanifold if there is a $k \in \mathbb{N}$ such that for every $x \in F$ there is an integral affine chart (U, χ) for (B, Λ) around x and an open $V \subset \mathbb{R}^k$ for which:

$$\chi(U \cap F) = V \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

Proposition 5.1.11. *Let (B, Λ) be an n-dimensional integral affine manifold. A Delzant submanifold $\Delta \subset (B, \Lambda)$ is an n-manifold with corners. Furthermore, each $\mathcal{P}^i(\Delta)$ is an $(n-i)$-dimensional integral affine submanifold of (B, Λ).*

Proof. By definition of a Delzant submanifold, for each $p \in \Delta$, there is an integral affine chart (U_p, χ_p) around p such that $\chi_p(\Delta \cap U_p)$ is an open neighbourhood of $\chi(p) = 0$ in $\mathbb{R}^n_k$ for some k. Restricting each of these charts (U, χ) to $U \cap \Delta$ yields an atlas of n-charts with corners for Δ. This atlas is smooth due to the fact that all charts (U, χ) are smoothly compatible. Thus Δ is an n-manifold with corners. Using an atlas of Δ consisting of charts that are restrictions of integral affine charts for (B, Λ), the proof of Proposition 4.2.4 shows that each $\mathcal{P}^i(\Delta)$ is an $(n-i)$-dimensional integral affine submanifold of (B, Λ). $\square$

The next result gives part of a relationship between integral affine submanifolds, sublattices of Λ and sub-torus bundles of $\mathcal{T}_\Lambda$.

Definition 5.1.12. Let F be an integral affine submanifold of (B, Λ). We denote by Λ_F the lattice $j^*(\Lambda|_F)$ in T^*F, where $j : F \to B$ is the inclusion map.

Proposition 5.1.13. *An integral affine submanifold F of (B, Λ) admits an integral affine structure, encoded by the lattice Λ_F. Moreover, $\Lambda \cap TF^0$ is a smooth lattice in the conormal bundle TF^0 and the obvious sequence:*

$$0 \to \frac{TF^0}{\Lambda \cap TF^0} \to \mathcal{T}_\Lambda|_F \xrightarrow{j^*} \mathcal{T}_{\Lambda_F} \to 0 \tag{5.1}$$

is a short exact sequence of torus bundles over F.

Proof. As for ordinary submanifolds, the integral affine charts in the definition of an integral affine submanifold give rise to an atlas $\mathcal{A}$ that turns F into a manifold. For each such chart (U, χ) for F that comes from an integral affine chart $(\hat{U}, \hat{\chi})$ for (B, Λ), we have:

$$\Lambda_b = \mathbb{Z}\, d\hat{\chi}^1_b \oplus ... \oplus \mathbb{Z}\, d\hat{\chi}^n_b \tag{5.2}$$

and hence:

$$(\Lambda_F)_b = \mathbb{Z}\, d\chi^1_b \oplus ... \oplus \mathbb{Z}\, d\chi^k_b \tag{5.3}$$

for all $b \in U$, because $\hat{\chi}^j|_U = \chi^j$ if $j \leq k$ and $\hat{\chi}^j|_U$ is constant if $j > k$. This implies that Λ_F is a smooth Lagrangian lattice in (T^*F, Ω_{can}), as becomes clear from a closer inspection of the first part of the proof of Proposition 2.2.11. Therefore F has an integral affine structure encoded by Λ_F. By (5.2) it further follows that:

$$\Lambda_b \cap T_bF^0 = \mathbb{Z}\, d\hat{\chi}^{k+1}_b \oplus ... \oplus \mathbb{Z}\, d\hat{\chi}^n_b$$

for all $b \in U$. Since $d\hat{\chi}^{k+1}, ..., d\hat{\chi}^n$ forms a local frame of TF^0, this shows that $\Lambda \cap TF^0$ is a smooth lattice in TF^0. The fact that the given sequence is a short exact sequence of Lie groupoids is now straightforward and so this completes the proof. $\square$

Let us just mention that one can rephrase the condition that a submanifold F is an integral affine submanifold, purely in terms of the lattices Λ_F and $\Lambda \cap TF^0$, by generalizing the notion of primitive sublattices to the realm of vector bundles.

5.1.3 Morita types and the open faces of the moment image

In this section we study the relationship between the orbit space of a toric $\mathcal{T}_\Lambda$-space and its moment image.

Notation 1. Throughout this section, let $\mu : (S, \omega) \to B$ be a toric $\mathcal{T}_\Lambda$-space and $\Delta = \mu(S)$. Moreover, given a fixed $x \in S$ we will always denote $b = \mu(x)$, $T = T_b^*B/\Lambda_b$ and we canonically identify $\mathfrak{t}^*$ with T_bB. Finally, we let $(\alpha_1, ..., \alpha_k)$ denote the weight-tuple of the symplectic isotropy representation at x and $\pi : \mathfrak{t}^* \to \mathfrak{t}_x^*$ the dual map to the inclusion $\mathfrak{t}_x \to \mathfrak{t}$.

The following theorem is a first sign of the relationship between toric $\mathcal{T}_\Lambda$-spaces and Delzant submanifolds.

Theorem 5.1.14. *Let $x \in S$ be given. The following statements hold true.*

a) The subset Δ is locally polyhedral (in the sense of Definition 3.3.28) and the polyhedral cone of Δ at b is

$$\pi^{-1}(Cone(\alpha_1, ..., \alpha_n)) \subset \mathfrak{t}^* = T_bB.$$

b) The moment image Δ is a Delzant submanifold of (B, Λ).

c) The Lie algebra of $(\mathcal{T}_\Lambda)_x$ is T_bF^0, where F is the open face of Δ through b.

Proof. Assertion a is immediate from Theorem 3.3.33. From this assertion it follows that there is an open neighbourhood U of b on which $\log_b$ is defined and maps $U \cap \Delta$ onto an open neighbourhood of 0 in $\pi^{-1}(\text{Cone}(\alpha_1, ..., \alpha_n))$. Since the symplectic isotropy representation at x is toric, its weights form a basis of the lattice $\Lambda_{T_x}^*$. Hence the dual basis $\{\alpha^1, ..., \alpha^k\}$ of $\mathfrak{t}_x$ is a basis of Λ_{T_x}. By Corollary 2.2.6, Λ_{T_x} is a primitive sublattice of Λ_b, so this basis extends to one of Λ_b. This in turn dualizes to a basis $\{\gamma_1, ..., \gamma_n\}$ of Λ_b^* with the property that

$$\pi(\gamma_j) = \begin{cases} \alpha_j & \text{if } j \leq k \\ 0 & \text{if } j > k \end{cases}$$

Let $A : (T_bB, \Lambda_b^*) \to (\mathbb{R}^n, \mathbb{Z}^n)$ be the corresponding isomorphism of integral affine vector spaces. Then A maps the cone $\pi^{-1}(\text{Cone}(\alpha_1, ..., \alpha_k))$ onto $\mathbb{R}_k^n$ and therefore $(U, A \circ \log_b)$ is an integral affine chart that maps b to 0 and $U \cap \Delta$ onto an open neighbourhood of 0 in $\mathbb{R}_k^n$. This proves that Δ is a Delzant submanifold. From the construction it is clear as well that:

$$\begin{aligned} \mathfrak{t}_x^0 &= \text{Ker}(\pi) \\ &= \text{span}_{\mathbb{R}}\{\gamma_{k+1}, ..., \gamma_n\} \\ &= T_bF \end{aligned}$$

where in the last step we used that $d(A \circ \log_b)_b = A$. So part c follows as well. $\square$

Corollary 5.1.15. *Let $x \in S$, $b = \mu(x)$ and let F be the open face of Δ through b. Then the exponential map of $(\mathcal{T}_\Lambda)_b$ descends to an isomorphism from $T_bF^0/(\Lambda_b \cap T_bF^0)$ to the isotropy group $(\mathcal{T}_\Lambda)_x$.*

Proof. Since the isotropy group $(\mathcal{T}_\Lambda)_x$ is a subtorus of $(\mathcal{T}_\Lambda)_b$, the exponential map of $(\mathcal{T}_\Lambda)_b$ maps its Lie algebra T_bF^0 onto $(\mathcal{T}_\Lambda)_x$. The result now follows because

$$\Lambda_b \cap T_bF^0 = \mathrm{Ker}(\exp|_{T_bF^0}).$$

$\square$

As it turns out, the canonical stratification on the orbit space of a toric $\mathcal{T}_\Lambda$-space and the stratification by open faces of its moment image Δ are closely related via the moment map. This the content of the next theorem.

Theorem 5.1.16. *The moment map factors through an isomorphism of stratified spaces:*

$$\bar{\mu} : S/\mathcal{T}_\Lambda \to \Delta.$$

That is, it is a homeomorphism that maps each stratum of $S/\mathcal{T}_\Lambda$ diffeomorphically onto a stratum of Δ.

In the remainder of this section we aim the prove this. We need two intermediate results, starting with a lemma.

Lemma 5.1.17. *Two points $x, y \in S$ have the same Morita type if and only if $\mathrm{depth}_\Delta(\mu(x)) = \mathrm{depth}_\Delta(\mu(y))$. In other words, the partition of S by Morita types coincides with the pre-image under μ of the partition of Δ by depth.*

Proof. If two points $x, y \in S$ have the same Morita type, then their isotropy groups have the same dimension and so $\mathrm{depth}_\Delta(\mu(x)) = \mathrm{depth}_\Delta(\mu(y))$. For the converse it will be enough to show that, for each $x \in S$, the isotropy representation of $\mathcal{T}_\Lambda$ at x is equivalent to the direct sum of the trivial representation of $\mathbb{T}^k$ on $\mathbb{R}^{n-k}$ and the standard representation of $\mathbb{T}^k$ on $\mathbb{C}^k$, where $k = \mathrm{depth}_\Delta(\mu(x))$ and $2n = \dim(S)$. Let $x \in S$, $b = \mu(x)$ and $T = T_b^*B/\Lambda_b$. As we have seen, the isotropy representation on $\mathcal{N}_x$ decomposes as $\mathfrak{t}_x^0 \oplus \mathcal{S}_x$, where the action is trivial on the first summand. By the weight-classification theorem, $\mathcal{S}_x$ is isomorphic to $\mathbb{C}_{\alpha_1} \oplus ... \oplus \mathbb{C}_{\alpha_k}$ as symplectic T_x-representation. The dual basis to the weight-tuple $(\alpha_1, ..., \alpha_k)$ of the symplectic isotropy representation at x forms a basis of the lattice Λ_{T_x}. This basis induces an isomorphism of T_x with $\mathbb{T}^k$, which interwines the T_x-representation $\mathbb{C}_{\alpha_1} \oplus ... \oplus \mathbb{C}_{\alpha_k}$ with the standard representation on $\mathbb{C}^k$, as can be easily verified. This concludes the proof. $\square$

Proposition 5.1.18. *Let F be an open face of Δ, $k = \dim(F)$ and $i : \mu^{-1}(F) \to S$ the inclusion. Then the pre-image $\mu^{-1}(F)$ is a symplectic submanifold of dimension $2k$. Moreover, the map $\mu : (\mu^{-1}(F), i^*\omega) \to F$ is a principal Hamiltonian $\mathcal{T}_{\Lambda_F}$-bundle:*

$$\begin{tikzcd}
(\mathcal{T}_{\Lambda_F}, \Omega_{\Lambda_F}) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowleft"] & (\mu^{-1}(F), i^*\omega) \arrow[dl, "\mu"'] \arrow[dr, "\mu"] & \\
F & & F
\end{tikzcd}$$

with Λ_F as in Definition 5.1.12. Consequently, the fibers of μ coincide with the $\mathcal{T}_\Lambda$-orbits.

Proof. We first show that $\mu^{-1}(F)$ is a symplectic submanifold. Let $x \in \mu^{-1}(F)$, $b = \mu(x)$ and $T = T_b^*B/\Lambda_b$. Fix a linear projection $p : \mathfrak{t} \to \mathfrak{t}_x$. In view of theorems 3.2.3 and 5.1.14*c* it is enough to prove that $\mathcal{M}^{-1}(\mathfrak{t}_x^0)$ is a symplectic submanifold of $(T \times_{T_x} (\mathfrak{t}_x^0 \oplus \mathcal{S}_x), \Omega)$ at $[e, 0, 0]$, where $\mathcal{M}$ is given by

$$T \times_{T_x} (\mathfrak{t}_x^0 \times \mathcal{S}_x) \to \mathfrak{t}^*, \quad [t, \alpha, s] \mapsto p^*(\mu_{\mathcal{S}_x}(s)) + \alpha.$$

Now we notice that

$$\mathcal{M}^{-1}(\mathfrak{t}_x^0) = T \times_{T_x} \mathfrak{t}_x^0. \tag{5.4}$$

Indeed, $[t, \alpha, s] \in \mathcal{M}^{-1}(\mathfrak{t}_x^0)$ if and only if $\mu_{\mathcal{S}_x}(s) = 0$, because p^* is injective and its image is a linear complement to $\mathfrak{t}_x^0$ in $\mathfrak{t}^*$. Moreover, because the weights of the symplectic isotropy representation at x are linearly independent, it follows from Corollary 3.3.17 that $\mu_{\mathcal{S}_x}(s) = 0$ if and only if $s = 0$. Hence (5.4) holds. So we see that $\mathcal{M}^{-1}(\mathfrak{t}_x^0)$ is indeed a submanifold and by Proposition A.4 it follows that the tangent space to $\mathcal{M}^{-1}(\mathfrak{t}_x^0)$ at $[e, 0, 0]$ is a symplectic linear subspace of the tangent space to the model at $[e, 0, 0]$, as was left to be shown.

Now let us turn to the second statement. Let $\iota : F \to B$ denote the inclusion. By Proposition 5.1.13 we have a short exact sequence of torus bundles over F:

$$0 \to \frac{TF^0}{(\Lambda \cap TF^0)} \to \mathcal{T}_\Lambda|_F \xrightarrow{\iota^*} \mathcal{T}_{\Lambda_F} \to 0.$$

By exactness and the fact that $TF^0/(\Lambda \cap TF^0)$ is the isotropy bundle of the $\mathcal{T}_\Lambda|_F$-action along $\mu : \mu^{-1}(F) \to F$, this action descends to a free action of $\mathcal{T}_{\Lambda_F}$ along $\mu : \mu^{-1}(F) \to F$. In particular, the $\mathcal{T}_{\Lambda_F}$-orbits are k-dimensional. The $\mathcal{T}_{\Lambda_F}$-action is Hamiltonian, because $\mu^{-1}(F)$ is a symplectic submanifold of (S, ω) and ι^* pulls back the canonical symplectic form on $\mathcal{T}_{\Lambda_F}$ back to the restriction of the canonical symplectic form on $\mathcal{T}_\Lambda$ to $\mathcal{T}_\Lambda|_F$. Consequently, $\mu : \mu^{-1}(F) \to F$ is a submersion, being the moment map of a free Hamiltonian groupoid space. It follows that the fibers of μ are k-dimensional. The moment map μ is $\mathcal{T}_\Lambda$ invariant since the source and target map of $\mathcal{T}_\Lambda$ coincide. Therefore the fibers of μ are unions of $\mathcal{T}_{\Lambda_F}$ orbits. Since each such orbit is a codimension 0 submanifold of such a fiber, it must be an open subset thereof. So since we assumed the fibers to be connected, each fiber must coincide with a single $\mathcal{T}_{\Lambda_F}$-orbit. All in all, this proves that the desired $\mathcal{T}_{\Lambda_F}$-bundle is principal. The final claim follows from the remark that the $\mathcal{T}_\Lambda$-orbit through a point in $\mu^{-1}(F)$ coincides with that of $\mathcal{T}_{\Lambda_F}$, as the action of $\mathcal{T}_{\Lambda_F}$ descends from that of $\mathcal{T}_\Lambda|_F$ via the above short exact sequence. This proves the proposition. □

We can now derive the desired theorem.

Proof of Theorem 5.1.16. An equivalent way of saying that the fibers of μ are the $\mathcal{T}_\Lambda$-orbits, is that μ factors through a bijection

$$\bar{\mu} : S/\mathcal{T}_\Lambda \to \Delta.$$

Since the map μ is continuous and closed onto its image, so is $\bar{\mu}$ and hence it is a homeomorphism. By the previous lemma, $\bar{\mu}$ maps each Morita type of $S/\mathcal{T}_\Lambda$ homeomorphically onto a member of the partition of Δ by depth, and so the same holds for the strata. It thus remains to verify that it does so diffeomorphically. The restriction $\mu : \mu^{-1}(F) \to F$ being a principal $\mathcal{T}_{\Lambda_F}$-bundle, it factors through a diffeomorphism:

$$\mu^{-1}(F)/(\mathcal{T}_\Lambda|_F) = \mu^{-1}(F)/\mathcal{T}_{\Lambda_F} \to F.$$

Now recall that for each stratum R the projection $\pi : \pi^{-1}(R) \to R$ is a surjective submersion. Here $\pi : S \to S/\mathcal{T}_\Lambda$ is the orbit projection. Together with the fact that $\mu^{-1}(F)$ is $\mathcal{T}_\Lambda$-invariant, this implies that the canonical inclusion $\mu^{-1}(F)/(\mathcal{T}_\Lambda|_F)$ into $S/\mathcal{T}_\Lambda$ is a diffeomorphism onto the stratum $\mu^{-1}(F)/\mathcal{T}_\Lambda$. Therefore $\bar{\mu} : \mu^{-1}(F)/\mathcal{T}_\Lambda \to F$ is the composition of two diffeomorphisms and hence a diffeomorphism itself. $\square$

5.1.4 Local isomorphism types

With the understanding of toric $\mathcal{T}_\Lambda$-spaces that we have gained thus far, we can show that the local structure of a toric $\mathcal{T}_\Lambda$-space is fully encoded in the local properties of its moment image. This is the content of the theorem below.

Theorem 5.1.19. *Let $\mu_i : (S_i, \omega_i) \to B$ be two toric $\mathcal{T}_\Lambda$-spaces and let $\Delta_i = \mu(S_i)$ for $i \in \{1, 2\}$. If the germs of Δ_1 and Δ_2 at a point $b \in \Delta_1 \cap \Delta_2$ agree, then there is an open neighbourhood U of b in B and an isomorphism of Hamiltonian $\mathcal{T}_\Lambda|_U$-spaces:*

$$\begin{tikzcd}
(\mu_1^{-1}(U), \omega_1) \arrow[rr, "\simeq"] \arrow[dr, "\mu_1"'] & & (\mu_2^{-1}(U), \omega_2) \arrow[dl, "\mu_2"] \\
& U &
\end{tikzcd}$$

Proof. Fix some $x_i \in \mu_i^{-1}(b)$. Because the germs of Δ_1 and Δ_2 at b agree, so do their polyhedral cones and the tangent spaces to their open faces at b. From the latter and Corollary 5.1.15 it follows that

$$(\mathcal{T}_\Lambda)_{x_1} = (\mathcal{T}_\Lambda)_{x_2}. \tag{5.5}$$

We will denote this group by H. From this and Theorem 5.1.14*c* we conclude that the weights of the symplectic isotropy representations at x_1 and x_2 span the same cone in $\mathfrak{h}^*$. Therefore, the classification theorem for toric representations implies that there is an isomorphism of symplectic H-representations $\varphi : (\mathcal{S}_{x_1}, \omega_{x_1}) \to (\mathcal{S}_{x_2}, \omega_{x_2})$. This in turn provides an isomorphism of Hamiltonian T-spaces:

$$\psi : T \times_H (\mathfrak{h}^0 \oplus \mathcal{S}_{x_1}) \to T \times_H (\mathfrak{h}^0 \oplus \mathcal{S}_{x_2}), \quad [t, \alpha, s] \mapsto [t, \alpha, \varphi(s)].$$

Here we use the same choice of linear projection $p : \mathfrak{t} \to \mathfrak{h}$ to equip both models with a symplectic structure and moment map. One can directly verify that the above map is symplectic by chasing through the construction of the symplectic form on the model in the Appendix.

As in the local normal form theorem, choose an open neighbourhood U_b of b on which $\log_b$ is defined and consider the induced Hamiltonian T-action on $\mu_i^{-1}(U_b)$, for both $i \in \{1, 2\}$. By that theorem each orbit $\mathcal{O}_{x_i}$ admits a T-invariant open neighbourhood $V_i \subset \mu_i^{-1}(U_b)$ which is isomorphic to a T-invariant open neighbourhood W_i of $\mathcal{O}_{x_i}$ in $T \times_H (\mathfrak{h}^0 \oplus \mathcal{S}_{x_i})$, by an isomorphism that restricts to the identity on $\mathcal{O}_{x_i}$. Replacing W_1 by $W_1 \cap \psi^{-1}(W_2)$ and shrinking W_2, V_1 and V_2 accordingly, we can assume that ψ maps W_1 onto W_2. By the fact that μ_i is closed as map onto Δ_i and that $\mu_i^{-1}(b) = \mathcal{O}_{x_i}$, we can find open neighbourhoods U_i of b in U_0 such that $\mu_i^{-1}(U_i) \subset V_i$. Since isomorphisms of Hamiltonian T-spaces intertwine moment maps, the composition of the three isomorphisms above yields an isomorphism from $\mu_1^{-1}(U)$ onto $\mu_2^{-1}(U)$, for $U = U_1 \cap U_2$. This is the desired isomorphism. □

5.2 Constructing a toric $\mathcal{T}_\Lambda$-space out of a Delzant submanifold

In the next section we will classify the toric $\mathcal{T}_\Lambda$-spaces with moment image a given Delzant submanifold $\Delta \subset (B, \Lambda)$. The method for doing so closely resembles the classification of principal $\mathbb{T}^n$-bundles by means of their cocycles. In fact, our method is a generalization of this. The idea of this method is to fix a principal $\mathbb{T}^n$-bundle, called **the reference bundle**, and to cook up a Čech cohomology class that measures whether or not a given principal $\mathbb{T}^n$-bundle is isomorphic to the reference bundle. In the case of principal $\mathbb{T}^n$-bundles, the choice of reference bundle comes for free: we can just take the trivial principal $\mathbb{T}^n$-bundle. Duistermaat generalized this to a classification of Lagrangian fibrations, or in other words, he classified the principal Hamiltonian $\mathcal{T}_\Lambda$-bundles over a given integral affine manifold (B, Λ). In that case, the choice of reference bundle comes for free as well: it is $\mathcal{T}_\Lambda$, considered as a principal Hamiltonian $\mathcal{T}_\Lambda$-bundle by the left action on itself. We will generalize this classification by very similar means. In contrast with the previous two cases, in our generality the choice of a reference space does not present itself immediately, as was already the case in Delzant's classification of toric manifolds. The goal of this section is to construct it. In other words, we aim to prove:

Theorem 5.2.1. *Let $\Delta \subset (B, \Lambda)$ be a Delzant submanifold. Then there is a toric $\mathcal{T}_\Lambda$-space*

$$\mu_\Delta : (S_\Delta, \omega_\Delta) \to B$$

with the property that $\mu_\Delta(S_\Delta) = \Delta$.

When he classified toric manifolds, Delzant already proved this theorem for Delzant polytopes, which coincide with the compact and connected Delzant submanifolds of $(\mathbb{R}^n, \mathbb{Z}^n)$. The construction that we give is however different from Delzant's construction.

5.2.1 The topology of S_Δ

We have seen that a toric $\mathcal{T}_\Lambda$-space with moment image Δ is partitioned into principal Hamiltonian $\mathcal{T}_{\Lambda_F}$-bundles over each of the open faces F of Δ. At the level of sets, we define $\mu_\Delta : S_\Delta \to B$ as the simplest such candidate. Namely, in light of Proposition 5.1.13, we can define:

$$\mu_\Delta : S_\Delta := \bigsqcup_{F \in \mathcal{S}(\Delta)} \mathcal{T}_{\Lambda_F} \to B$$

where μ_Δ restricts to the bundle projection $\mathcal{T}_{\Lambda_F} \to F$ for each open face F of Δ. This has image Δ, as desired. To obtain a suitable topology on S_Δ, we realise it as a quotient of $\mathcal{T}_\Lambda|_\Delta$, as follows. For each open face F of Δ, we let

$$\mathcal{K}_F = \frac{TF^0}{\Lambda \cap TF^0},$$

and we let

$$\mathcal{K}_\Delta = \bigsqcup_{F \in \mathcal{S}(\Delta)} \mathcal{K}_F.$$

This is a (discrete) subgroupoid of $\mathcal{T}_\Lambda|_\Delta$. From Proposition 5.1.13 it follows that S_Δ can be realised canonically as the quotient:

$$\frac{\mathcal{T}_\Lambda|_\Delta}{\mathcal{K}_\Delta}$$

We endow S_Δ with the quotient topology. Under this identification, the bundle projection $\mathcal{T}_\Lambda|_\Delta \to \Delta$ descends to the map μ_Δ, which is therefore continuous. Moreover, the canonical left action of $\mathcal{T}_\Lambda$ on $\mathcal{T}_\Lambda|_\Delta \to \Delta$ commutes with the action of $\mathcal{K}_\Delta$ along $\mathcal{T}_\Lambda|_\Delta \to \Delta$, and hence descends to a left action of $\mathcal{T}_\Lambda$ along $\mu_\Delta : S_\Delta \to B$.

Proposition 5.2.2. *The map μ_Δ is continuous, closed onto its image and has connected fibers. Moreover, the $\mathcal{T}_\Lambda$-action is free on the open and dense subset $S_{\mathring{\Delta}}$ of S_Δ.*

Notation 2. Given an open subset U of B or of Δ, we denote by $(S_\Delta)_U$ the open subset $\mu_\Delta^{-1}(U)$. We do however make one exception: if $U = \mathring{\Delta}$ is the interior of the manifold with corners Δ, then we write $S_{\mathring{\Delta}}$ for $\mu_\Delta^{-1}(U)$.

Proof. We have already shown that μ_Δ is continuous. By construction its fibers are tori and so they are connected. It is clear as well that the action is free on $S_{\mathring{\Delta}}$, so it remains to verify that this is an open and dense subset, and that μ is closed onto its image. To this end, note that $\mathcal{T}_\Lambda \to B$ is a fiber bundle and hence $\mathcal{T}_\Lambda|_\Delta \to \Delta$ is a continuous fiber bundle. The bundle projection of a continuous fiber bundle with compact fibers is closed onto its image (by local triviality and the Tube Lemma). So because closedness is preserved when factoring through a quotient map, μ_Δ is closed onto its image. Moreover, since the interior $\mathring{\Delta} = \mathcal{P}^0(\Delta)$ is open and dense in Δ and $\mathcal{T}_\Lambda|_\Delta \to \Delta$ is a continuous fiber bundle, so is $\mathcal{T}_\Lambda|_{\mathring{\Delta}}$ in $\mathcal{T}_\Lambda|_\Delta$. Since it is saturated as well, this in turn implies that its image $S_{\mathring{\Delta}}$ under the quotient map is open and dense in S_Δ. $\square$

In the remaining sections we equip S_Δ with a suitable smooth and symplectic structure.

5.2.2 The local structure and symplectic cuts

In this section we show how to equip S with the desired structure in the special case in which $(B, \Lambda) = (\mathbb{R}^n, \mathbb{Z}^n)$ and $\Delta = \mathbb{R}^n_k$. We will do this by means of a procedure that is called the symplectic cut. What we will use is only a special case of the more general symplectic cutting procedure introduced in [Ler95]. This will later serve as a local model for the general case.

Suppose that $\mathcal{T}_{\mathbb{Z}^n}$ acts along a map $\mu : (M^{2n}, \omega) \to \mathbb{R}^n$ in a Hamiltonian way. The standard coordinates on $\mathbb{R}^n$ induce an isomorphism of symplectic groupoids:

$$(\mathcal{T}_{\mathbb{Z}^n}, \Omega_{\mathbb{Z}^n}) \cong (\mathbb{T}^n \times \mathbb{R}^n, \omega_0)$$

where the symplectic groupoid on the right-hand side is the action groupoid of the trivial action of $\mathbb{T}^n$ on $\mathbb{R}^n$. So we can interpret this action as a Hamiltonian $\mathbb{T}^n$-action on (M, ω)

with moment map μ. Let us fix some notation. Let $k \in \{0, ..., n\}$. For $I \subset \{1, ..., k\}$ (which by convention is empty if $k = 0$) we denote

$$\mathbb{T}_I = \{\lambda \in \mathbb{T}^n | \ \lambda_i = 1 \text{ if } i \notin I\}.$$

Furthermore, we write:

- $\mathbb{T}_j := \mathbb{T}_{\{j\}}$,
- $\mathbb{T}^k := \mathbb{T}_{\{1,...,k\}} = \{\lambda \in \mathbb{T}^n | \ \lambda_j = 1 \text{ if } j > k\}$.

The diagonal action of $\mathbb{T}^k$ on $M \times \mathbb{C}^k$, by the action of $\mathbb{T}^k$ on (M, ω) and the anti-standard action of $\mathbb{T}^k$ on $(\mathbb{C}^k, -\omega_0)$, is Hamiltonian with moment map

$$\hat{\mu} : M \times \mathbb{C}^k \to \mathbb{R}^k, \quad \hat{\mu}_j(x, z) = \mu_j(x) - \pi |z_j|^2.$$

We denote the reduced space $\hat{\mu}^{-1}(0)/\mathbb{T}^k$ by C^k_μ.

Proposition 5.2.3. *Suppose that the induced $\mathbb{T}_j$-action on $\mu_j^{-1}(0)$ is free for each $j \in \{1, ..., k\}$. Then the action of $\mathbb{T}^k$ on $\hat{\mu}^{-1}(0)$ is free as well. In particular, this holds if $\mathbb{T}^n$ acts freely on M.*

Proof. Let $(x, z) \in \hat{\mu}^{-1}(0)$ and $\lambda \in \mathbb{T}^k$ such that $\lambda \cdot (x, z) = (x, z)$. If $z_j \neq 0$ then it follows directly that $\lambda_j = 1$, while if $z_j = 0$, then $\mu_j(x) = \pi |z_j|^2 = 0$ and our assumption implies that $\lambda_j = 1$. □

From now on, we assume that the hypotheses of the previous proposition hold. Then the symplectic reduction theorem applies, so that the reduced space C^k_μ is smooth and the symplectic structure on $M \times \mathbb{C}^k$ descends to one on C^k_μ. Moreover, the moment map μ descends to a moment map for the induced action of $\mathbb{T}^n$ on C^k_μ, given by:

$$C^k_\mu \to \mathbb{R}^n, \quad [x, z] \mapsto \mu(x).$$

The image of this moment map is $\mu(M) \cap \mathbb{R}^n_k$.

It is enlightening to interpret this reduced space in the framework that we have developed so far. The cone $\Delta := \mathbb{R}^n_k$ is a Delzant submanifold of $(\mathbb{R}^n, \mathbb{Z}^n)$, with open faces given by

$$F_I = \{x \in \mathbb{R}^n_k | \ x_i = 0 \text{ if } i \in I \text{ and } x_i \neq 0 \text{ if } i \in \{0, ..., k\} \backslash I\},$$

where $I \subset \{1, ..., k\}$. The groupoid $\mathcal{K}_{\mathbb{R}^n_k}$ defined in the previous section acts on $\mu^{-1}(\mathbb{R}^n_k)$. One can interpret this action in terms of group-actions of subtori of $\mathbb{T}^k$ on each subset $\mu^{-1}(F_I)$. As was observed before, the standard coordinates on $\mathbb{R}^n$ induce a trivialization

$$\mathcal{T}_{\mathbb{Z}^n} \cong \mathbb{T}^n \times \mathbb{R}^n.$$

For each $I \subset \{1, ..., k\}$, the subgroupoid $\mathcal{K}_{F_I}$ of $\mathcal{T}_{\mathbb{Z}^n}$ is identified with $\mathbb{T}_I \times F_I$ and the action of $\mathcal{K}_{F_I}$ along $\mu^{-1}(F_I) \to F_I$ is just the action of $\mathbb{T}_I$. The orbit space of the $\mathcal{K}_{\mathbb{R}^n_k}$-action is related to C^k_μ, as follows.

Proposition 5.2.4. *The map*

$$h : \mu^{-1}(\mathbb{R}^n_k)/\mathcal{K}_{\mathbb{R}^n_k} \to C^k_\mu, \quad [x] \mapsto \left[\left(x, \sqrt{\frac{\mu_1(x)}{\pi}}, ..., \sqrt{\frac{\mu_k(x)}{\pi}}\right)\right]. \tag{5.6}$$

is a homeomorphism.

Proof. By interpreting the action of $\mathcal{K}_{\mathbb{R}^n_k}$ in terms of group-actions of subtori of $\mathbb{T}^k$ on the pre-image of each face, as was just discussed, one can verify that the given map is a continuous bijection. It is a homeomorphism, since it is closed as well. To see this, let $\pi : \hat{\mu}^{-1}(0) \to C^k_\mu$ denote the canonical projection and note that for $C \subset \mu^{-1}(\mathbb{R}^n_k)$ we have:

$$\pi^{-1}(h(C/\mathcal{K}_{\mathbb{R}^n_k})) = (\mathbb{T}^k \cdot (C \times \mathbb{R}^k_{\geq 0})) \cap \hat{\mu}^{-1}(0)$$

which is closed in $\hat{\mu}^{-1}(0)$ if C is closed in $\mu^{-1}(\mathbb{R}^n_k)$, by compactness of $\mathbb{T}^k$. □

We leave it to the reader to check that:

Proposition 5.2.5. *The composition*

$$\mu^{-1}(\overset{\circ}{\mathbb{R}}{}^n_k) \to \mu^{-1}(\mathbb{R}^n_k)/\mathcal{K}_{\mathbb{R}^n_k} \xrightarrow{h} C^k_\mu$$

is a symplectic embedding.

Example 5.2.6. We apply this procedure to the free Hamiltonian $\mathbb{T}^n$-space $(\mathbb{T}^n \times \mathbb{R}^n, \omega_0, \mu_0)$. The image of the moment map of the Hamiltonian $\mathbb{T}^n$-space $C^k_{\mu_0}$ will be $\mathbb{R}^n_k$, and so in the specific case $\Delta = \mathbb{R}^n_k \subset (\mathbb{R}^n, \mathbb{Z}^n)$ this provides us with the desired structure on our candidate topological space S_Δ. In the general case, this will be the local model. In order to gain a better understanding, we now give a more explicit description of it. Observe that we have an isomorphism of Hamiltonian $\mathbb{T}^n$-spaces:

$$\mathbb{T}^{n-k} \times \mathbb{R}^{n-k} \times \mathbb{C}^k \to C^k_{\mu_0}, \quad (\lambda, x, z) \mapsto [((1, \lambda), \pi|z_1|^2, ..., \pi|z_k|^2, x, z)],$$

which has inverse given by:

$$[\lambda, x, z] \mapsto (\lambda_{k+1}, ..., \lambda_n, x_{k+1}, ..., x_n, \lambda_1 z_1, ..., \lambda_k z_k)\,.$$

Here the Hamiltonian action of $\mathbb{T}^n = \mathbb{T}^k \times \mathbb{T}^{n-k}$ on $\mathbb{T}^{n-k} \times \mathbb{R}^k \times \mathbb{C}^k$ is the cross-diagonal action of the standard action of $\mathbb{T}^{n-k}$ on $(\mathbb{T}^{n-k} \times \mathbb{R}^{n-k}, \omega_0)$ and that of $\mathbb{T}^k$ on $(\mathbb{C}^k, -\omega_0)$. We get the cross-diagonal action instead of the diagonal action, because we chose to require the first k coordinates in $\mathbb{R}^n_k$ to be positive, instead of the last k coordinates. The moment map is therefore given by:

$$\mu_0 : \mathbb{T}^{n-k} \times \mathbb{R}^{n-k} \times \mathbb{C}^k \to \mathbb{R}^n, \quad (\lambda, x, z) \mapsto (\pi|z_1|^2, ..., \pi|z_k|^2, x).$$

For this to truly serve as a local model, one can for instance replace $\mathbb{R}^{n-k}$ and $\mathbb{C}^k$ by the opens balls B^{n-k}_ε and B^{2k}_ε of radius $\varepsilon > 0$, centred at the origin. Accordingly, one can replace $\mathbb{R}^n$ by $P^k_\varepsilon \times B^{n-k}_\varepsilon$, where:

$$P^k_\varepsilon = \left\{ x \in \mathbb{R}^k | \ \sum_{i=1}^k |x_i| < \pi\varepsilon^2 \right\},$$

because

$$\mu_0^{-1}(P^k_\varepsilon \times B^{n-k}_\varepsilon) = \mathbb{T}^{n-k} \times B^{n-k}_\varepsilon \times B^{2k}_\varepsilon.$$

The choice of these specific opens is of course merely made to please the eye; what is important is that the collection $\{P^k_\varepsilon \times B^{n-k}_\varepsilon\}_{\varepsilon>0}$ is a basis of open neighbourhoods of $0 \in \mathbb{R}^n$.

5.2.3 The smooth and symplectic structure

For general $\Delta \subset (B,\Lambda)$, the procedure of symplectic cutting can be applied locally, by means of the following lemma.

Lemma 5.2.7. *Let (U,χ) be an integral affine chart for (B,Λ). Then χ induces an isomorphism of symplectic groupoids:*

$$(\mathbb{T}^n \times \chi(U), \omega_0) \to (\mathcal{T}_\Lambda|_U, \Omega_{can}), \quad (e^{2\pi i\theta}, x) \mapsto \sum_{j=1}^{n} \theta_j d\chi^j_{\chi^{-1}(x)} \mod \Lambda,$$

where the groupoid $\mathbb{T}^n \times \chi(U) \rightrightarrows \chi(U)$ is the action groupoid for the trivial action of $\mathbb{T}^n$ on $\chi(U)$. Consequently, given a symplectic manifold (S,ω), an action of $\mathcal{T}_\Lambda|_U$ on a map $\mu : S \to U$ is Hamiltonian if and only if the induced action of $\mathbb{T}^n$ on (S,ω) is Hamiltonian with moment map

$$\mu_\chi = \chi \circ \mu : S \to \mathbb{R}^n.$$

Proof. The proof is essentially that of Lemma 3.2.2. The first statement is straightforward to verify; the second statement follows from the first, the correspondence between Hamiltonian G-spaces and Hamiltonian $G \times \mathfrak{g}^*$-spaces and the observation that ω_0 coincides with the canonical symplectic form on the action groupoid $\mathbb{T}^n \times \mathfrak{t}^* = \mathbb{T}^n \times \mathbb{R}^n$, because of formula (1.4) and triviality of the Lie bracket on $\mathfrak{t}^*$. □

We call a triple (p,U,χ) consisting of an integral affine chart (U,χ) for (B,Λ) and a point $p \in U \cap \Delta$ **admissible**, if $\chi(p) = 0$ and

$$\chi(U \cap \Delta) = \chi(U) \cap \mathbb{R}^n_k,$$

where $k = \text{depth}(p)$. Observe that for any such triple $\chi(U \cap \Delta)$ is an open neighbourhood of 0 in $\mathbb{R}^n_k$ and p has maximal depth amongst all points in $U \cap \Delta$. Further note that every $p \in \Delta$ belongs to some admissible triple (p,U,χ).

Since the action of $\mathcal{T}_\Lambda$ on itself is free, so is the action of $\mathbb{T}^n$ on $\mathcal{T}_\Lambda|_U$, induced by an integral chart (U,χ) as in the above lemma. Therefore, for each admissible triple (p,U,χ) we can apply the symplectic cutting procedure to the free Hamiltonian $\mathbb{T}^n$-space $(\mathcal{T}_\Lambda|_U, \Omega_{can}, \mu_\chi)$. We will henceforth denote the reduced space $C^k_{\mu_\chi}$ that we obtain in this way by $C_{(p,U,\chi)}$.

Lemma 5.2.8. *Every admissible triple (p,U,χ) gives rise to a homeomorphism*

$$h_{(p,U,\chi)} : (S_\Delta)_U \to C_{(p,U,\chi)}.$$

This induces a structure of a symplectic manifold on $(S_\Delta)_U$ for which the inclusion

$$\mathcal{T}_\Lambda|_{\mathring{\Delta}\cap U} \hookrightarrow (S_\Delta)_U$$

is a symplectic embedding and the action of $\mathcal{T}_\Lambda|_U$ on $\mu : (S_\Delta)_U \to U$ is Hamiltonian.

Proof. Because the triple (p,U,χ) is admissible we have that

$$\mu_\chi^{-1}(\mathbb{R}^n_k) = \mu^{-1}(U \cap \Delta) = \mathcal{T}_\Lambda|_{U\cap\Delta},$$

and if $\chi(b) \in F_I \subset \mathbb{R}^n_k$ for some $I \subset \{1,...,k\}$, then

$$T_bF^0 = \text{span}_\mathbb{R}\{d\chi^i_b |\ i \in I\}.$$

This implies that

$$\mu_\chi^{-1}(\mathbb{R}^n_k)/\mathcal{K}_{\mathbb{R}^n_k} = (\mathcal{T}_\Lambda|_{U\cap\Delta})/(\mathcal{K}_\Delta|_U).$$

Therefore, (5.6) provides a homeomorphism $(\mathcal{T}_\Lambda|_{U\cap\Delta})/(\mathcal{K}|_{U\cap\Delta}) \to C_{(p,U,\chi)}$. We can canonically identify $(S_\Delta)_U$ with $(\mathcal{T}_\Lambda|_{U\cap\Delta})/(\mathcal{K}_\Delta|_U)$ as sets. This identification is in fact a homeomorphism, because $\mathcal{T}_\Lambda|_{U\cap\Delta} \subset \mathcal{T}_\Lambda|_\Delta$ is a saturated open subset with respect to the quotient by $\mathcal{K}_\Delta$. By composing these homeomorphisms we obtain the homeomorphism $h_{(p,U,\chi)}$. Therefore $(S_\Delta)_U$ admits a unique structure of symplectic manifold for which $h_{(p,U,\chi)}$ is a symplectomorphism. It follows from Proposition 5.2.5 that the inclusion $\mathcal{T}_\Lambda|_{\mathring{\Delta}\cap U} \hookrightarrow (S_\Delta)_U$ is a symplectic embedding. The $\mathcal{T}_\Lambda$-action on S_Δ restricts to an action:

$$\begin{tikzcd}
\mathcal{T}_\Lambda|_U \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & (S_\Delta)_U \arrow[dl, "\mu_\Delta"] \\
U &
\end{tikzcd}$$

This action is Hamiltonian if and only if the $\mathbb{T}^n$-action:

$$\begin{tikzcd}
\mathbb{T}^n \times \mathbb{R}^n \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & (S_\Delta)_U \arrow[dl, "\mu_\chi"] \\
\mathbb{R}^n &
\end{tikzcd}$$

induced by χ as in Lemma 5.2.7 is so. The symplectomorphism $h_{(p,U,\chi)}$ interwines this $\mathbb{T}^n$-action with the Hamiltonian $\mathbb{T}^n$-action:

$$\begin{tikzcd}
(\mathbb{T}^n \times \mathbb{R}^n, \omega_0) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & (C_{(p,U,\chi)}, \omega_{red}) \arrow[dl] \\
\mathbb{R}^n &
\end{tikzcd}$$

coming from the symplectic cut. We conclude from this that the action of $\mathcal{T}_\Lambda|_U$ on $\mu_\Delta : (S_\Delta)_U \to U$ is Hamiltonian (which includes the fact that μ_Δ is smooth). $\square$

The previous proposition shows that, locally, S_Δ has the desired structure. The following lemma shows that the local pieces form a global structure.

Lemma 5.2.9. *Let (p, U, χ) and (q, V, φ) be two admissible triples. The induced smooth and symplectic structures on $(S_\Delta)_U$ and $(S_\Delta)_V$ restrict to one and the same smooth structure on their overlap $(S_\Delta)_{U\cap V}$.*

Proof. Throughout, let $k = \text{depth}_\Delta(p)$ and $l = \text{depth}_\Delta(q)$. Let us first address the case in which $p = q$. In this case, $\chi \circ \varphi^{-1}$ maps an open neighbourhood of 0 in $\mathbb{R}^n_k$ onto an open neighbourhood of 0 in $\mathbb{R}^n_k$ and is given by $x \mapsto \text{Jac}(\chi\circ\varphi^{-1})(p)\cdot x$, where $\text{Jac}(\chi\circ\varphi^{-1})(p) \in \text{GL}_n(\mathbb{Z})$. As in the proof of Lemma 4.2.2 we find that the Jacobian at p is of the form:

$$\text{Jac}(\chi\circ\varphi^{-1})(p) = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

for $A \in \text{M}_{k\times k}(\mathbb{R})$, $B \in \text{M}_{(n-k)\times k}(\mathbb{R})$ and $C \in \text{M}_{(n-k)\times(n-k)}(\mathbb{R})$. Consequently, A maps $[0,\infty[^k$ onto $[0,\infty[^k$ and belongs to $\text{GL}_k(\mathbb{Z})$. In other words,

$$\text{Cone}(e_1, ..., e_k) = \text{Cone}(Ae_1, ..., Ae_k),$$

and both generating sets form a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^k$. By Proposition 5.1.6*c* we deduce that the tuples $(e_1, ..., e_k)$ and $(Ae_1, ..., Ae_k)$ coincide up to a reordering. Therefore, the same holds for the k-tuples $(\chi^1, ..., \chi^k)$ and $(\varphi^1, ..., \varphi^k)$. Let $\sigma \in S_k$ be the permutation such that $\chi^{\sigma(i)} = \varphi^i$ for each $i \in \{1, ..., k\}$. Then the map $h_\varphi \circ h_\chi^{-1}$ is given by

$$C_{(p,U,\chi)} \to C_{(p,U,\varphi)}, \quad [(m, z)] \mapsto [(m, z_\sigma)],$$

where $(z_\sigma)_i = z_{\sigma(i)}$ for each $i \in \{1, ..., k\}$. The inverse of this map is obtained by reversing the roles of χ and φ, hence it is a diffeomorphism. We conclude that χ and φ indeed induce the same smooth structure on $(S_\Delta)_U$.

We shall now show how to reduce the general case to the previous one. Let $r \in U \cap V \cap \Delta$ and $m = \text{depth}_\Delta(r)$. Then $m \leq \min\{k, l\}$, so by rearranging the first k and l components of χ and φ respectively, we may assume that $\chi^i(r) = 0$ if $i \leq m$, while $\chi^i(r) > 0$ if $m < i \leq k$ and similarly for φ. Indeed, as we showed just now, the smooth structure induced by an admissible triple is invariant under a permutation of the coordinates. Now, choose an open neighbourhood W of r in $U \cap V$ such that for every $x \in W$ we have:

$$\chi^i(x) > 0 \text{ if } m < i \leq k, \tag{5.7}$$

$$\varphi^i(x) > 0 \text{ if } m < i \leq l. \tag{5.8}$$

Write $\tilde{\chi} = \chi|_W - \chi(r)$ and similarly for φ. By construction, the triples $(r, W, \tilde{\chi})$ and $(r, W, \tilde{\varphi})$ are admissible and by the previous case they induce the same smooth structure on $(S_\Delta)_W$. We will show that the smooth structure on $(S_\Delta)_U$ induced by (p, U, χ) restricts to this one and that the same holds for $(S_\Delta)_V$ and (q, V, φ). Since $r \in U \cap V$ was arbitrary, this would prove smoothness part of the lemma. To this end, observe that the map $h_{(p,U,\chi)} \circ h_{(r,W,\tilde{\chi})}^{-1}$ is given by

$$C_{(r,W,\tilde{\chi})} \to C_{(p,U,\chi)}, \quad [(x, z)] \mapsto \left[\left(x, z, \sqrt{\frac{\chi^{m+1}(\mu(x))}{\pi}}, ..., \sqrt{\frac{\chi^k(\mu(x))}{\pi}}\right)\right].$$

This is smooth by (5.7). Moreover, one can directly verify that this is an embedding. Therefore the smooth structure on $(S_\Delta)_U$ restricts to that on $(S_\Delta)_W$. The same argument goes for $(S_\Delta)_V$ and (q, V, φ).

Next, we address the symplectic structure. The inclusion

$$\mathcal{T}_\Lambda|_{U \cap V \cap \mathring{\Delta}} \hookrightarrow (S_\Delta)_{U \cap V}$$

is a symplectic embedding with respect to both symplectic structures. Therefore the symplectic forms on $(S_\Delta)_U$ and $(S_\Delta)_V$ coincide on an open and dense subset of their overlap, and hence they agree on the entire overlap. This finishes the proof. □

Corollary 5.2.10. *The topological space S_Δ is Hausdorff, second-countable and it admits a unique smooth structure that restricts to the one on $(S_\Delta)_U$ for each open U belonging to an admissible triple (p, U, χ). Moreover, there is a unique symplectic form ω_Δ on S_Δ that restricts to the one on $(S_\Delta)_U$ for each such U. Furthermore, the action of $\mathcal{T}_\Lambda$ on $\mu_\Delta : (S_\Delta, \omega) \to B$ is Hamiltonian.*

Proof. To see that S_Δ is Hausdorff, let $p, q \in S_\Delta$ be distinct. If $\mu_\Delta(p) = \mu_\Delta(q)$, then $p, q \in (S_\Delta)_U$ for some open U belonging to an admissible triple and p and q can be separated by opens contained in the Hausdorff subspace $(S_\Delta)_U$. Otherwise, $\mu_\Delta(p) \neq \mu_\Delta(q)$ and by Hausdorffness of Δ and continuity of μ we can separate p and q by opens in S_Δ. Therefore S_Δ is Hausdorff. Since Δ is second-countable it admits a countable cover $\mathcal{U}$ by opens that belong to admissible triples, hence S_Δ admits a countable open cover $\{(S_\Delta)_U | \ U \in \mathcal{U}\}$ by second-countable subspaces. This implies that S_Δ is itself second-countable. The statements about the smooth and symplectic structure are an immediate consequence of the previous lemma. Finally, because the action of $\mathcal{T}_\Lambda$ on $\mu : S \to B$ restricts to a Hamiltonian action of $\mathcal{T}_\Lambda|_U$ on $\mu : (S_\Delta)_U \to U$ for each $U \in \mathcal{U}$, the $\mathcal{T}_\Lambda$-action is Hamiltonian itself. This completes the proof. $\square$

By combining the previous Corollary with Proposition 5.2.2 we conclude that the Hamiltonian $\mathcal{T}_\Lambda$-space $\mu_\Delta : (S_\Delta, \omega_\Delta) \to B$ satisfies the requirements of Theorem 5.2.2.

Remark 16. Let us point out that our construction would have worked as well if we would have started with any principal $\mathcal{T}_\Lambda$-bundle over (B, Λ), instead of $\mathcal{T}_\Lambda$. As we will soon see, there may or may not exist any others. In any case, we believe that there should be a condition that characterizes the one that comes out when starting with $\mathcal{T}_\Lambda$ as the simplest one. In the classifications of principal Hamiltonian $\mathcal{T}_\Lambda$-bundles in terms of cocycles there is an obvious choice of such a simplest one: the bundle $\mathcal{T}_\Lambda$. Its isomorphism class is characterized as the simplest one by the fact that it admits a global Lagrangian section. It is a characterization of this flavour that we would be looking for.

5.3 The classification

Throughout, let $\Delta \subset (B, \Lambda)$ be a fixed Delzant submanifold. In this section we will classify the toric $\mathcal{T}_\Lambda$-spaces with moment image Δ.

5.3.1 The Lagrangian Chern class

Mimicking the work of Duistermaat [Dui80] (who classified Lagrangian fibrations over a fixed integral affine manifold), to a general toric $\mathcal{T}_\Lambda$-space we will associate a cohomology class that measures the failure of being isomorphic to $\mu_\Delta : (S_\Delta, \omega_\Delta) \to B$.

First we will relate the smooth local sections of $\mathcal{T}_\Lambda|_\Delta \to \Delta$ to local automorphisms of S_Δ. Let $U \subset \Delta$ open. We call a map $\sigma : U \to M$ into a manifold (without corners) M **smooth** if it extends to a smooth map into M on an open neighbourhood of U in B. To each smooth local section $\sigma : U \to \mathcal{T}_\Lambda$ we can associate a $\mathcal{T}_\Lambda$-equivariant diffeomorphism:

$$\psi_\sigma : (S_\Delta)_U \to (S_\Delta)_U, \quad x \mapsto \sigma(\mu_\Delta(x)) \cdot x.$$

Conversely, we have:

Proposition 5.3.1. *Let $U \subset \Delta$ open. For every $\mathcal{T}_\Lambda$-equivariant diffeomorphism*

$$\psi : (S_\Delta)_U \to (S_\Delta)_U$$

there is a unique smooth local section $\sigma : U \to \mathcal{T}_\Lambda$ for which $\psi = \psi_\sigma$.

Here by $\mathcal{T}_\Lambda$-equivariance we mean that $\mu_\Delta \circ \psi = \mu_\Delta$ and $\psi(t \cdot p) = t \cdot \psi(p)$ for all $t \in \mathcal{T}_\Lambda$ in the fiber over $\mu_\Delta(p)$. In other words, ψ is a morphism of $\mathcal{T}_\Lambda$-spaces over B.

Proof. Uniqueness follows from the fact that the action is free on the dense subset $S_{\mathring{\Delta}}$ of S_Δ. Now let an open U and an equivariant diffeomorphism ψ be given. Having established the uniqueness for arbitrary opens, to prove the existence of the desired smooth local section, it is now enough to show that every point $x_0 \in U$ admits an open neighbourhood U_0 of x in U and a smooth local section $\sigma_0 : U_0 \to \mathcal{T}_\Lambda$ for which $\psi|_{(S_\Delta)_{U_0}} = \psi_{\sigma_0}$. To this end, let $x_0 \in U$ and let (x_0, V, χ) be an admissible triple with the property that

$$\chi(V) = P^k_\varepsilon \times B^{n-k}_\varepsilon$$

for some $\varepsilon > 0$. By construction of S_Δ, the Hamiltonian $\mathcal{T}_\Lambda|_V$-space $(S_\Delta)_V$ is isomorphic to the Hamiltonian $\mathbb{T}^n$-space of Example 5.2.6. Therefore, Lemma 5.3.2 below provides the desired section σ_0, defined on $U_0 = V \cap \Delta$. □

Lemma 5.3.2. *Let* $\varepsilon > 0$ *and*

$$\varphi : \mathbb{T}^{n-k} \times B^{n-k}_\varepsilon \times B^{2k}_\varepsilon \to \mathbb{T}^{n-k} \times B^{n-k}_\varepsilon \times B^{2k}_\varepsilon$$

be an automorphism of the $\mathbb{T}^k \times \mathbb{T}^{n-k}$*-space of Example 5.2.6 that preserves the moment map* μ_0*. Then there is a map* $\sigma : (P^k_\varepsilon \times B^{n-k}_\varepsilon) \cap \mathbb{R}^n_k \to \mathbb{T}^n$ *such that* φ *is given by:*

$$\varphi(x) = \sigma(\mu_0(x)) \cdot x$$

and σ *extends to a smooth map into* $\mathbb{T}^n$ *on an open neighbourhood of its domain in* $\mathbb{R}^n$*.*

Proof. This proof is inspired on that of [Del88, Lemma 2.6]. Let φ_j denote the j^{th} component of φ in $\mathbb{C}^k$ and denote by q the projection from $\mathbb{T}^{n-k} \times \mathbb{R}^{n-k} \times \mathbb{C}^k$ to $\mathbb{R}^{n-k}$. Then $q \circ \varphi = q$, because φ preserves μ_0. Hence we can write:

$$\varphi = (\varphi_1, ..., \varphi_{n-k}, q, \psi_1, ..., \psi_k).$$

In combination with the equivariance of φ this gives:

$$\varphi(t, \alpha, z) = (t_1 \cdot \varphi_1(1, \alpha, z), ..., t_k \cdot \varphi_{n-k}(1, \alpha, z), \alpha, \psi_1(1, \alpha, z), ..., \psi_k(1, \alpha, z)). \qquad (5.9)$$

Let us now study the components φ_j and ψ_j. It follows from equivariance of φ that

$$\varphi_j(1, \alpha, \lambda \cdot z) = \varphi_j(1, \alpha, z),$$
$$\psi_j(1, \alpha, \lambda \cdot z) = \lambda_j \psi_j(1, \alpha, z),$$

for all $(\alpha, z) \in B^{n-k}_\varepsilon \times B^{2k}_\varepsilon$ and $\lambda \in \mathbb{T}^k$. In particular, $(\alpha, x) \mapsto \psi_j(1, \alpha, x)$ restricts to a smooth function on $B^{n-k}_\varepsilon \times B^k_\varepsilon \subset \mathbb{R}^{n-k} \times \mathrm{Re}(\mathbb{C}^k)$ which is odd in the x_j variable and even in the other x-variables, while $(\alpha, x) \mapsto \varphi_j(1, \alpha, x)$ restricts to a smooth function on this domain as well, but is even in all x-variables. Therefore, a theorem by Hassler Whitney [Whi43, Thm 1; Thm 2] implies that there are continuous functions

$$f_j, g_j : U := (P^k_\varepsilon \times B^{n-k}_\varepsilon) \cap \mathbb{R}^n_k \to \mathbb{C}$$

that satisfy

$$\varphi_j(1, \alpha, x) = f_j(\pi x_1^2, ..., \pi x_k^2, \alpha),$$
$$\psi_j(1, \alpha, x) = x_j g_j(\pi x_1^2, ..., \pi x_k^2, \alpha),$$

for all $(\alpha, x) \in B_\varepsilon^{n-k} \times B_\varepsilon^k$, and extend to smooth functions on some open neighbourhood of U in $\mathbb{R}^n$. The fact that φ preserves μ_0 implies that:

$$x_j^2|g_j(\pi x_1^2, ..., \pi x_k^2, \alpha)|^2 = |\psi_j(1, \alpha, x)|^2 = x_j^2$$

for all $(\alpha, x) \in B_\varepsilon^{n-k} \times B_\varepsilon^k$. Therefore g_j takes values in S^1 on the dense subset $U \cap \mathring{\mathbb{R}}_k^n$ of its domain, hence it must do so on its entire domain. Note that f_j does so as well, since φ_j does. Finally, observe that for $(\alpha, z) \in B_\varepsilon^{n-k} \times B_\varepsilon^{2k}$, writing $z_m = e^{i\theta_m}|z_m|$ one finds:

$$\begin{aligned}\psi_j(1, \alpha, z) &= e^{i\theta_j}\psi_j(1, \alpha, |z_1|, ..., |z_k|)\\ &= e^{i\theta_j}|z_j|g_j(\pi|z_1|^2, ..., \pi|z_k|^2, \alpha)\\ &= z_j g_j(\mu_0(1, \alpha, z))\end{aligned} \tag{5.10}$$

by equivariance of φ. Similarly,

$$\varphi_j(1, \alpha, z) = f_j(\mu_0(1, \alpha, z)). \tag{5.11}$$

Now define

$$\begin{aligned}f &: U \to \mathbb{T}^{n-k},\\ g &: U \to \mathbb{T}^k,\end{aligned}$$

to have j^{th} component f_j and g_j respectively. As maps into $\mathbb{C}^{n-k}$ and $\mathbb{C}^k$ respectively, f and g extend to smooth maps on an open neighbourhood of U in $\mathbb{R}^n$. Since their components do not vanish on a small enough such open neighbourhood, by normalizing them we can find such extensions that map smoothly into $\mathbb{T}^{n-k}$ and $\mathbb{T}^k$, respectively. By combining (5.9) and (5.11) with (5.10) and $\mathbb{T}^n$-invariance of μ_0 one deduces that:

$$\begin{aligned}\varphi(t, \alpha, z) &= (t \cdot f(\mu_0(t, \alpha, z)), \alpha, g(\mu_0(t, \alpha, z)) \cdot z)\\ &= (g(\mu_0(t, \alpha, z)), f(\mu_0(t, \alpha, z))) \cdot (t, \alpha, z).\end{aligned}$$

We conclude that

$$\sigma = g \times f : U \to \mathbb{T}^k \times \mathbb{T}^{n-k}$$

satisfies the requirements of the lemma. □

Proposition 5.3.1 shows that equivariant self-diffeomorphisms of $(S_\Delta)_U$ correspond to smooth local sections of the fiber bundle $\mathcal{T}_\Lambda|_\Delta$. We are however interested in the symplectic structure ω_Δ as well. The next result below shows that equivariant *symplectomorphisms* correspond to *Lagrangian* sections.

Definition 5.3.3. A **Lagrangian section** σ of $\mathcal{T}_\Lambda$ over Δ is a smooth local section of $\mathcal{T}_\Lambda|_\Delta$ for which $\sigma^*\Omega_\Lambda = 0$.

Remark 17. Having defined the notion of smooth maps defined on Δ, we have as well defined the differential forms on Δ: they are the smooth section $\Delta \to \Lambda^k T^*B$. One can define pull-backs by smooth maps $f : \Delta \to M$ into a manifold (without corners) M as follows: choose a smooth extension $\hat{f}$ defined on an open in B and for $\alpha \in \Omega^k(M)$ define

$$f^*\alpha : \Delta \to \Lambda^k T^*B, \quad (f^*\alpha)_x = ((\hat{f})^*\alpha)_x$$

for $x \in \Delta$. This does not depend on the choice of extension, since it does not do so on the open subset $\mathring{\Delta}$ of B, which is dense in Δ. Similarly, one can define the exterior derivative and wedge product of smooth forms on Δ, and the usual identities such as "d commutes with pull-back" still hold, because they do so on the open subset $\mathring{\Delta}$ of B, which is dense in Δ. Furthermore, one can define pull-backs of differential forms on Δ by smooth maps $g : M \to B$ with $g(M) \subset \Delta$ by the usual formula, and the chain rule still holds. In particular, given smooth maps $g : N \to B$ with $g(N) \subset \Delta$ and $f : \Delta \to M$, it holds that $f \circ g : N \to M$ is smooth and:

$$(f \circ g)^* \alpha = g^* f^* \alpha$$

for $\alpha \in \Omega^k(M)$. Of course, in the above, we could replace Δ by any of its open subsets. An alternative (perhaps more natural, but more time-consuming) approach would be to generalize the theory of differential forms to manifold with corners.

Returning to Lagrangian sections, we have:

Proposition 5.3.4. *Let $U \subset \Delta$ open and $\sigma : U \to \mathcal{T}_\Lambda$ a smooth local section. Then ψ_σ is a symplectomorphism if and only if σ is a Lagrangian section.*

Proof. Because the action of $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ is on μ_Δ is Hamiltonian, we have

$$m^* \omega_\Delta = pr_1^* \Omega_\Lambda + pr_2^* \omega_\Delta.$$

In combination with the fact that:

$$i \circ \psi_\sigma = m \circ ((\sigma \circ \mu_\Delta \circ i), i)$$

where i is the inclusion of $(S_\Delta)_U$ into S_Δ, this yields

$$\psi_\sigma^* i^* \omega_\Delta = (\mu_\Delta \circ i)^* \sigma^* \Omega_\Lambda + i^* \omega_\Delta,$$

from which the result follows, since the interior of Δ is dense and consists of regular values of μ_Δ. $\square$

The Lagrangian sections of $\mathcal{T}_\Lambda$ over Δ give rise to a sheaf (of abelian groups) $\mathcal{L}(\mathcal{T}_\Lambda)$ over Δ. Analogously, we can define a sheaf of Lagrangian sections of T^*B over Δ, denoted: $\mathcal{L}(T^*B)$. With this at hand, we are in position to generalize Duistermaat's definition of the Lagrangian Chern class of a Lagrangian fibration to that of a toric $\mathcal{T}_\Lambda$-space. Let $\mu : (S, \omega) \to B$ be a toric $\mathcal{T}_\Lambda$-space with moment image Δ. Due to Theorem 5.1.19 we can choose an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Δ that admits a collection $\{\psi_i\}_{i \in I}$ of $\mathcal{T}_\Lambda$-equivariant symplectomorphisms $\psi_i : \mu^{-1}(U_i) \to (S_\Delta)_{U_i}$. For each $i, j \in I$ we let $s_{ij} \in \mathcal{L}(U_{ij}; \mathcal{T}_\Lambda)$ be the unique smooth local section satisfying:

$$\psi_{s_{ij}} = \psi_i \circ \psi_j^{-1} : (S_\Delta)_{U_{ij}} \to (S_\Delta)_{U_{ij}}.$$

This defines a 1-cocycle $s \in \check{C}^1(\mathcal{U}, \mathcal{L}(\mathcal{T}_\Lambda))$ the cohomology class of which is independent of the choice of $\{\psi_i\}_{i \in I}$. The corresponding class $c^{lag} \in \check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$ depends only on the isomorphism class of the Hamiltonian $\mathcal{T}_\Lambda$-space.

Definition 5.3.5. The class $c^{lag} \in \check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$ is called the **Lagrangian Chern class** of the toric $\mathcal{T}_\Lambda$-space $\mu : (S, \omega) \to B$.

The main result of this chapter is:

Theorem 5.3.6 (Classification of toric $\mathcal{T}_\Lambda$-spaces)**.** *The toric $\mathcal{T}_\Lambda$-spaces $\mu : (S, \omega) \to B$ with moment image Δ are classified (up to isomorphism) by the set $\check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$. More precisely, the map that associates to every isomorphism class of such spaces its Lagrangian Chern class in $\check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$ is a bijection.*

Proof. The map is well-defined since, as noted before, c^{lag} depends only on the isomorphism class of a toric $\mathcal{T}_\Lambda$-space. Now we prove injectivity. Suppose that $\mu_j : (S_j, \omega_j) \to B$, $j \in \{1, 2\}$, are two toric $\mathcal{T}_\Lambda$-spaces, the Lagrangian Chern classes of which coincide. Then by construction of these classes, there is an open cover $\{U_i\}_{i\in I}$ of Δ and there are two collections $\{\psi_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ consisting of isomorphisms $\psi_i : \mu_1^{-1}(U_i) \to (S_\Delta)_{U_i}$ and $\varphi_i : \mu_2^{-1}(U_i) \to (S_\Delta)_{U_i}$ such that:

$$[s^\psi] = [s^\varphi] \in \check{H}^1(\mathcal{U}, \mathcal{L}(\Delta; \mathcal{T}_\Lambda)),$$

where s^ψ is the unique element of $\check{C}^1(\mathcal{U}, \mathcal{L}(\mathcal{T}_\Lambda))$ satisying $\psi_i \circ \psi_j^{-1} = \psi_{s^\psi_{ij}}$ for all $i, j \in I$, and analogously for s^φ. Therefore there is a collection $\{\sigma_i\}_{i\in I}$ of Lagrangian sections $\sigma_i : U_i \to \mathcal{T}_\Lambda$ such that

$$s^\psi_{ij} - s^\varphi_{ij} = \sigma_j - \sigma_i$$

for all $i, j \in I$. Rewriting this we get $\sigma_i = s^\varphi_{ij} + \sigma_j - s^\psi_{ij}$ which implies:

$$\varphi_i^{-1} \circ \psi_{\sigma_i} \circ \psi_i = \varphi_j^{-1} \circ \psi_{\sigma_j} \circ \psi_j \quad \text{ on } (S_\Delta)_{U_{ij}}.$$

Therefore

$$\Psi : S_1 \to S_2, \quad \Psi|_{\mu_1^{-1}(U_i)} = \varphi_i^{-1} \circ \psi_{\sigma_i} \circ \psi_i$$

is well-defined. By the analogous properties of φ_i, ψ_{σ_i} and ψ_i, this is a morphism of Hamiltonian $\mathcal{T}_\Lambda$-spaces, a local diffeomorphism and it is a fiberwise bijection. By the latter it is a global bijection and hence it is in fact a diffeomorphism, as desired. This proves injectivity.

Finally, we address surjectivity. Suppose that $c \in \check{H}^1(\Delta; \mathcal{L}(\mathcal{T}_\Lambda))$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of Δ, $s \in \check{C}^1(\mathcal{U}, \mathcal{L}(\mathcal{T}_\Lambda))$ a 1-cocycle representing the class c and

$$\psi_{ij} := \psi_{s_{ij}} : (S_\Delta)_{U_{ij}} \to (S_\Delta)_{U_{ij}}.$$

We define the topological space

$$S = \frac{\bigsqcup_{i\in I} (S_\Delta)_{U_i}}{\sim}$$

where we quotient by the equivalence relation

$$x \sim y \iff x \in (S_\Delta)_{U_i},\ y \in (S_\Delta)_{U_j} \text{ and } x = \psi_{ij}(y).$$

This is indeed an equivalence relation, because s is a cocycle. For each $i \in I$, there is a canonical injection

$$\psi_i : (S_\Delta)_{U_i} \to S, \quad x \mapsto [x].$$

Each ψ_i is a topological embedding onto an open subset of S. Therefore each ψ_i endows its image with the structure of a toric $\mathcal{T}_\Lambda|_{U_i}$-space. Remark that $\psi_{ij} = \psi_i^{-1} \circ \psi_j$ for each $i, j \in I$. This and the fact that each ψ_{ij} is an isomorphism of Hamiltonian $\mathcal{T}_\Lambda$-spaces guarantees that the structures induced by ψ_i and by ψ_j coincide on the overlap of their images. Hence S admits a smooth structure, a symplectic structure and a toric $\mathcal{T}_\Lambda$-action, which is uniquely determined by the fact that each ψ_i is an embedding of Hamiltonian $\mathcal{T}_\Lambda$-spaces. The topology on S is Hausdorff and second-countable, by the same arguments as in the proof of Corollary 5.2.10. Denoting by μ the moment map for the $\mathcal{T}_\Lambda$-action on S we have:

$$\mu(S) = \bigcup_{i\in I} \mu_\Delta((S_\Delta)_{U_i}) = \bigcup_{i\in I} U_i = \Delta.$$

Moreover, using the cover $\mathcal{U}$ and the collection $\{\psi_i\}_{i\in I}$ to compute the Lagrangian Chern class of the toric $\mathcal{T}_\Lambda$-space $\mu : (S,\omega) \to B$, it is immediate that this must be c. This completes the proof of the theorem. $\square$

5.3.2 Computing $\check{H}^1(\Delta;\mathcal{L}(\mathcal{T}_\Lambda))$

So far we have classified the toric $\mathcal{T}_\Lambda$-spaces in terms of their moment image Δ and their Lagrangian Chern class. We will now derive sufficient conditions on the topology of Δ under which $\check{H}^1(\Delta;\Lambda(\mathcal{T}_\Lambda))$ vanishes, so that the toric $\mathcal{T}_\Lambda$-spaces with such moment image are unique (up to isomorphism).

Once more, let Δ be a given Delzant submanifold of (B,Λ). We begin by introducing a new sheaf over Δ: the sheaf C^∞_Λ is the subsheaf of C^∞ consisting of integral affine smooth functions. That is, $f \in C^\infty_\Lambda(U)$ if and only if $f \in C^\infty(U)$ and $df_b \in \Lambda_b$ for every $b \in U$. We have a short exact sequence of sheaves:

$$0 \to C^\infty_\Lambda \to C^\infty \to \mathcal{L}(\mathcal{T}_\Lambda) \to 0,$$

where the first map is the inclusion, whereas the second map is the composition of exterior derivative $d : C^\infty \to \mathcal{L}(T^*B)$ and the map $\mathcal{L}(T^*B) \to \mathcal{L}(\mathcal{T}_\Lambda)$ induced by the projection $T^*B \to \mathcal{T}_\Lambda$. It follows from the defining property of $\lambda_{can} \in \Omega^1(T^*B)$ that a smooth local section of T^*B is Lagrangian, precisely if it is closed as a 1-form. This implies that the second map in the short exact sequence maps into $\mathcal{L}(\mathcal{T}_\Lambda)$ and, together with the Poincar Lemma and the fact that $q : T^*B \to \mathcal{T}_\Lambda$ is a surjective submersion, it implies its surjectivity at the level of stalks. The rest of the exactness is obvious, so we obtain a long exact sequence in cohomology. Because C^∞ is a fine sheaf over Δ (a property which it inherits from the smooth functions on B), the cohomology $\check{H}^k(\Delta;C^\infty)$ vanishes if $k \geq 1$. So the connecting homomorphism in this long exact sequence is an isomorphism:

$$\delta : \check{H}^1(\Delta;\mathcal{L}(\mathcal{T}_\Lambda)) \xrightarrow{\sim} \check{H}^2(\Delta;C^\infty_\Lambda).$$

Secondly, observe that we have a short exact sequence of sheaves over Δ:

$$0 \to \underline{\mathbb{R}} \to C^\infty_\Lambda \xrightarrow{d} \Gamma(\Lambda) \to 0,$$

where $\underline{\mathbb{R}}$ denotes the sheaf of locally constant functions with values in $\mathbb{R}$. The second map is surjective at the level of stalks, because of the Poincar Lemma and the fact that all smooth local sections of Λ are Lagrangian. Again, the rest of the exactness is rather obvious, so in particular we obtain part of the long exact sequence:

$$\check{H}^2(\Delta;\underline{\mathbb{R}}) \to \check{H}^2(\Delta;C^\infty_\Lambda) \to \check{H}^2(\Delta;\Gamma(\Lambda)) \to \check{H}^3(\Delta;\underline{\mathbb{R}}).$$

Since Δ is paracompact, Hausdorff and locally contractible, the cohomology $\check{H}^\bullet(\Delta;\underline{A})$ is isomorphic to $H^\bullet_{sing}(\Delta,A)$ for any abelian group A. In particular, this holds for $A=\mathbb{R}$. If $\Lambda|_\Delta$ has trivial monodromy as covering space over Δ, then we can recognize the third cohomology group as singular cohomology as well.

Proposition 5.3.7. *If $\Lambda|_\Delta$ has trivial monodromy, then:*

$$\check{H}^\bullet(\Delta;\Gamma(\Lambda)) \cong H^\bullet_{sing}(\Delta,\mathbb{Z})^n$$

where $n = dim(B)$.

Proof. By the same proof as for Proposition 2.2.17 the triviality of the monodromy representations implies that $\Lambda|_\Delta$ is trivializable by a frame of smooth sections $\Delta \to \Lambda$. Such a trivialization induces an isomorphism of sheafs over Δ between $\Gamma(\Lambda)$ and $(\underline{\mathbb{Z}})^n$ (the n-fold direct sum of the sheaf of locally constant functions with values in $\mathbb{Z}$). This implies that:

$$\check{H}^\bullet(\Delta;\Gamma(\Lambda)) \cong \check{H}^\bullet(\Delta;(\underline{\mathbb{Z}})^n) \cong \check{H}^\bullet(\Delta;\underline{\mathbb{Z}})^n \cong H^\bullet_{sing}(\Delta,\mathbb{Z})^n$$

as claimed. □

From this discussion we conclude:

Proposition 5.3.8. *Suppose $\Lambda|_\Delta$ has trivial monodromy and that $H^2_{sing}(\Delta,\mathbb{Z})$, $H^2_{sing}(\Delta,\mathbb{R})$ and $H^3_{sing}(\Delta,\mathbb{R})$ vanish. Then $\check{H}^1(\Delta;\mathcal{L}(\mathcal{T}_\Lambda))$ vanishes. Consequently, up to isomorphism, there is a unique toric $\mathcal{T}_\Lambda$-space over B with moment image Δ.*

As an immediate consequence, we obtain:

Corollary 5.3.9. *If Δ is contractible, then, up to isomorphism, there is a unique toric $\mathcal{T}_\Lambda$-space over B with moment image Δ.*

5.3.3 Derivation of classical classification theorems

In this section we will show how Duistermaat's classification of Lagrangian fibrations and Delzant's classification of toric manifolds are particular examples of our classification of toric $\mathcal{T}_\Lambda$-spaces.

Duistermaat's classification of Lagrangian fibrations with compact and connected fibers

Let (B,Λ) be an integral affine manifold. We have shown in Section 3.1 that the Lagrangian fibrations with compact and connected fibers that induce Λ on B are in bijective correspondence with principal Hamiltonian $\mathcal{T}_\Lambda$-bundles. Because the moment image of a Lagrangian fibration is all of B, Theorem 5.3.6 leads to:

Theorem 5.3.10 (Duistermaat, [Dui80])**.** *The isomorphism classes of Lagrangian fibrations with compact and connected fibers that induce Λ on B are classified by*

$$\check{H}^1(B;\mathcal{L}(\mathcal{T}_\Lambda))$$

via their Lagrangian Chern class.

Delzant's classification of toric manifolds

Delzant showed that a class of Hamiltonian T-spaces, called toric manifolds, is classified by Delzant polytopes in $(\mathfrak{t}^*, \Lambda_T^*)$. Here by a Delzant polytope we mean a compact and connected Delzant submanifold of $(\mathfrak{t}^*, \Lambda_T^*)$. In the literature one will probably find the following equivalent definition: it is a polytope in $\mathfrak{t}^*$ for which the germ of each its vertices is a smooth polyhedral cone. We have seen that there is a bijective correspondence between toric T-spaces and compact, connected toric $\mathcal{T}_{\Lambda_T}$-spaces over $(\mathfrak{t}^*, \Lambda_T^*)$. Since, moreover, convex spaces are contractible, Corollary 5.3.9 and Theorem 5.3.6 reduce to Delzant's Theorem for toric manifolds:

Theorem 5.3.11 (Delzant,[Del88])**.** *Let T be a torus. Then the following hold.*

a) If (M, ω, μ) is a toric T-space, then $\mu(M)$ is a Delzant polytope in $(\mathfrak{t}^*, \Lambda_T^*)$.

b) Two toric Hamiltonian T-spaces are isomorphic if and only if their moment polytopes are equal.

c) For every Delzant polytope Δ in $(\mathfrak{t}^*, \Lambda_T^*)$, there is a toric T-space the moment polytope of which is Δ.

In other words, we have a bijection:

$$\{\textit{Isomorphism classes of toric } T\textit{-spaces}\} \to \{\textit{Delzant polytopes in } (\mathfrak{t}^*, \Lambda_T^*)\}$$
$$[(M, \omega, \mu)] \mapsto \mu(M)$$

Chapter 6

Hamiltonian $\mathcal{G}$-spaces over simple Poisson manifolds

In this chapter we consider Hamiltonian $\mathcal{G}$-spaces over simple Poisson manifolds. These are those Poisson manifolds for which the leaf space is smooth. We will extend our results for moment maps of Hamiltonian $\mathcal{T}_\Lambda$-spaces to those for Hamiltonian $\mathcal{G}$-spaces over simple Poisson manifolds, by considering their leaf space as a Poisson manifold with the zero-Poisson structure.

6.1 Proper integrations of simple Poisson manifolds and integral affine structures on the leaf space

In [CFT16] it is shown that the leaf space of a regular Poisson manifold that admits a proper integration is an integral affine orbifold. If the leaf space is smooth, then this implies that the leaf space admits an integral affine structure. The aim of this section is to prove this.

Definition 6.1.1. A Poisson manifold (M, π) is called **simple** if its leaf space $B := M/\mathcal{F}_\pi$ is smooth. Moreover, a groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ is called **simple** if its orbit space is smooth.

Remark 18. A simple Poisson manifold is in particular regular. Moreover, an s-connected symplectic groupoid is simple if and only if the Poisson structure that it induces is simple. This holds because its orbits are the symplectic leaves of this Poisson structure.

An important role will be played by the isotropy subgroupoid $\mathcal{G}_M$ of $\mathcal{G}$ and the source-connected part of $\mathcal{G}_M$.

Definition 6.1.2. The **isotropy subgroupoid** of a groupoid $\mathcal{G} \rightrightarrows M$ is the bundle of groups $\mathcal{G}_M$ over M defined by:

$$(\mathcal{G}_M)_x = \mathcal{G}_x, \quad x \in M.$$

Definition 6.1.3. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ we define the **source-connected part of** $\mathcal{G}$ to be the wide subgroupoid of $\mathcal{G}$ given by:

$$\mathcal{G}^0 = \bigsqcup_{x \in M} (s^{-1}(x))_{1_x},$$

where by $(\cdot)_{1_x}$ we mean the connected component through 1_x.

Proposition 6.1.4. *The source-connected part of a Lie groupoid is an open subset and hence it is a Lie subgroupoid.*

Proof. See [Mac05, Proposition 1.5.1]. □

In [Moe03] it is shown that if $\mathcal{G}$ is a regular Lie groupoid over M (which is to say that its orbit foliation on M is regular), then the source-connected part of $\mathcal{G}_M$ is a Lie subgroupoid of $\mathcal{G}$. We will now show this for the particular case of simple Lie groupoids.

Proposition 6.1.5. *Let $\mathcal{G} \rightrightarrows M$ be a simple Lie groupoid. Then $\mathcal{G}_M$ is a Lie subgroupoid. Consequently, $\mathcal{G}^0_M$ is so too.*

Proof. Since $q : M \to M/\mathcal{G}$ is a submersion, the subspace

$$M \; {}_q\!\times_q M = \{(m,n) \in M \times M \mid q(m) = q(n)\}$$

is a submanifold of $M \times M$. In fact, this is a Lie groupoid called the submersion groupoid, with source and target maps the first and second projection onto M and multiplication defined by:

$$(x,y) \cdot (z,x) = (z,y), \quad x,y,z \in M.$$

Consider the morphism of Lie groupoids:

$$(s,t) : \mathcal{G} \to M \; {}_q\!\times_q M. \tag{6.1}$$

We claim that this is a surjective submersion. This would prove the proposition, for then $\mathcal{G}_M = \mathrm{Ker}(s,t)$ is a Lie subgroupoid of $\mathcal{G}$, as desired. Let $(x,y) \in M \; {}_q\!\times_q M$. Then x,y belong the same orbit, so there is a $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$. Further suppose that (v,w) is a tangent vector to $M \; {}_q\!\times_q M$ at (x,y), then $dq_x(v) = dq_y(w)$. Since s is a surjective submersion, there is a $\hat{v} \in T_g\mathcal{G}$ such that $ds_g(\hat{v}) = v$. Because $q \circ s = q \circ t$, we have that $dq_y(dt_g(\hat{v})) = dq_y(w)$ and hence

$$w - dt_g(\hat{v}) \in T_y\mathcal{O} = dt_g(\mathrm{Ker}(ds_g)).$$

This implies that $w = dt_g(\hat{v} + \hat{w})$ for some $\hat{w} \in \mathrm{Ker}(ds_g)$. Now observe that

$$d(s,t)_g(\hat{v} + \hat{w}) = (v,w).$$

We conclude that (6.1) is indeed a surjective submersion. □

The main result of this section is:

Theorem 6.1.6 ([CFT16]). *The orbit space B of a simple and proper symplectic groupoid $(\mathcal{G},\Omega) \rightrightarrows M$ admits an integral affine structure Λ.*

Lemma 6.1.7. *Let π be the Poisson structure induced by $(\mathcal{G},\Omega)$ on M. The map $q : (M,\pi) \to (B,0)$ is a complete Poisson map. Consequently,*

$$\bar{t} = q \circ t : (\mathcal{G},\Omega) \to (B,0)$$

is a complete symplectic realization and (T^*B, Ω_{can}) acts along $\bar{t}$ in a Hamiltonian way.

Proof. That q is Poisson follows from the fact that

$$\mathrm{Im}(\pi_x^\sharp) = T_xL = \mathrm{Ker}(dq_x), \tag{6.2}$$

where L is the symplectic leaf through $x \in M$. Therefore $\bar{t}$ is Poisson as well, being the composition of two Poisson maps. Equality (6.2) implies as well that $q^*\alpha$ is a section of $\mathrm{Im}(\pi^\sharp)^0 = \mathrm{Ker}(\pi^\sharp)$ for any $\alpha \in \Omega^1(B)$, so it follows that $X_{f\circ q} = 0$ for any $f \in C^\infty(B)$ and (trivially) q is a complete Poisson maps. Appealing to the fact that the target map of a symplectic groupoid is a complete Poisson map, it follows that $\bar{t}$ is a complete symplectic realization. Alternatively, one can use that for any $\alpha \in \Omega^1(B)$ the vector field $a(\alpha)$ is tangent to the source- and target-fibers (as will be shown in the next lemma), which by properness of $\mathcal{G}$ implies that $a(\alpha)$ is complete (due to compactness of the submanifolds $s^{-1}(x) \cap t^{-1}(y)$ for $x, y \in M$). $\square$

Lemma 6.1.8. *The T^*B-action preserves $\mathcal{G}_x^0$ for each $x \in M$ and induces a surjective, local diffeomorphism of Lie groupoids:*

$$\varphi : q^*(T^*B) \to \mathcal{G}_M^0, \quad (x, \alpha) \mapsto \alpha \cdot 1_x.$$

Consequently, the kernel of φ is a smooth isotropic lattice in $q^*(T^*B)$ with respect to the pull-back of Ω_{can} along the canonical map $q^*(T^*B) \to T^*B$.

Proof. Note that for every $\alpha \in \Omega^1(B)$, the vector field $a(\alpha)$ on $\mathcal{G}$ is tangent to both the source- and target-fibers of $\mathcal{G}$, because of the fact that

$$\iota_{a(\alpha)}\Omega = t^*(q^*\alpha) = s^*(q^*\alpha),$$

the fact that $s, t : (\mathcal{G}, \Omega) \to (M, \pi)$ are anti-Poisson and Poisson and the fact that $q^*\alpha$ is a section of $\mathrm{Ker}(\pi^\sharp)$. This implies that the flow of $a(\alpha)$ preserves $\mathcal{G}_x^0$ for each $x \in M$. Hence the map φ is well-defined. The only non-obvious part of verifying that φ is a morphism of groupoids is its compatibility with the multiplication on $\mathcal{G}_M^0$; this follows from right-invariance of $a(\alpha)$, which in turn follows from multiplicativity of Ω. So φ is a morphism of Lie groupoids. We will now show that φ is a surjective local diffeomorphism. To this end, note first that the Hamiltonian T^*B-action along $\bar{t}$ is infinitesimally free, since $\bar{t}$ is a submersion. Therefore its orbits are immersed submanifolds of dimension that of B and, for each $x \in M$, the map

$$\varphi_x : T^*_{q(x)}B \to \mathcal{G}_x^0, \quad (x, \alpha) \mapsto \alpha \cdot 1_x \tag{6.3}$$

is an immersion. Since $\mathcal{G}_x^0$ is preserved by the action, it is partitioned by such orbits. As

$$\dim(\mathcal{G}_x^0) = \dim(\mathrm{Ker}(\pi_x^\sharp)) = \dim(B),$$

each such orbit is open in $\mathcal{G}_x^0$, so by connectedness $\mathcal{G}_x^0$ consists of a single orbit for each $x \in M$. We conclude that (6.3) is in fact a surjective local diffeomorphism. Therefore $\mathrm{Ker}(\varphi_x)$ is a discrete lattice in $T^*_{q(x)}B$, which is full by compactness of $\mathcal{G}_x^0$. We conclude from this that φ is surjective and, because φ is a morphism of Lie groupoids over M, it is a local diffeomorphism itself, the kernel of which is a smooth lattice in $q^*(T^*B)$.

Finally, let $\hat{q}$ be the canonical map $q^*(T^*B) \to T^*B$, $i : \mathcal{G}_M^0 \to \mathcal{G}$ the inclusion and $pr : q^*(T^*B) \to M$ the bundle projection. Because the T^*B-action is Hamiltonian and

$$i \circ \varphi = m \circ (\hat{q}, u \circ pr)$$

we find that:

$$\varphi^*(i^*\Omega) = (\hat{q})^*\Omega_{can} + pr^*(u^*\Omega) = (\hat{q})^*\Omega_{can}. \tag{6.4}$$

Because the space of units is isotropic in $(\mathcal{G}, \Omega)$, this implies that $\mathrm{Ker}(\varphi)$ is isotropic with respect to $(\hat{q})^*\Omega_{can}$. □

Proof of Theorem 6.1.6. By the previous lemma it is enough to show $\mathrm{Ker}(\varphi_x) = \mathrm{Ker}(\varphi_y)$ if $x, y \in M$ belong to the same orbit of $\mathcal{G}$. Indeed, it would then follow that there is a subgroupoid Λ of T^*B with the property that $q^*\Lambda = \mathrm{Ker}(\varphi)$. By the properties of $\mathrm{Ker}(\varphi)$ that were proved in the previous lemma and the fact that $\Lambda|_U = \sigma^*(\mathrm{Ker}(\varphi))$ for every local section $\sigma : U \to M$ of q, it would then follow that Λ is a smooth Lagrangian lattice in (T^*B, Ω_{can}), as desired.

To this end, let x and y belong to the same orbit. Then there is a $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$. Observe that $a(\alpha)$ is both left- and right-invariant for every $\alpha \in \Omega^1(B)$, which follows from multiplicativity of Ω and the fact that $q \circ s = q \circ t$. This in turn implies that:

$$\begin{aligned} g(\alpha_{q(x)} \cdot 1_x)g^{-1} &= g(\Phi^1_{a(\alpha)}(1_x))g^{-1} \\ &= \Phi^1_{a(\alpha)}(g1_xg^{-1}) \\ &= \alpha_{q(x)} \cdot 1_y. \end{aligned}$$

It follows from this that $\mathrm{Ker}(\varphi_x) = \mathrm{Ker}(\varphi_y)$, as desired. □

From the proof the Theorem 6.1.6 we can conclude more. Namely:

Corollary 6.1.9. *The integral affine structure in Theorem 6.1.6 is uniquely characterized by the fact that*

$$\varphi : q^*(\mathcal{T}_\Lambda) \to \mathcal{G}^0_M, \quad (x, [\alpha]) \mapsto \alpha \cdot 1_x$$

is a well-defined isomorphism of Lie groupoids that satisfies:

$$\varphi^*(i^*\Omega) = (\hat{q})^*\Omega_\Lambda.$$

In particular, $\mathcal{G}^0_M$ is a torus bundle over M.

6.2 Local polyhedrality properties of the moment map

In this section we will derive locally polyhedrality results for the moment map of Hamiltonian actions by the symplectic groupoids that we studied in the previous section. Throughout this section, let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a simple, proper symplectic groupoid acting along a map $\mu : (S, \omega) \to M$ in a Hamiltonian fashion. Let Λ be the induced integral affine structure on the orbit space B. We write:

$$\bar{\mu} = q \circ \mu : (S, \omega) \to B$$

Our first proposition allows us to reduce to the results that we have for Hamiltonian $\mathcal{T}_\Lambda$-spaces.

Proposition 6.2.1. *The symplectic torus bundle $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ acts along $\bar{\mu}$ in a Hamiltonian fashion as well.*

Proof. First observe that $q^*(\mathcal{T}_\Lambda)$ acts along μ, since we can identify it with the Lie subgroupoid $\mathcal{G}^0_M$ of $\mathcal{G}$ via the isomorphism φ of Corollary 6.1.9. The action of $\mathcal{T}_\Lambda$ along $\bar{\mu}$ is given by:

$$(b, \alpha) \cdot p = (\mu(p), \alpha) \cdot p,$$

where the action on the right hand side denotes that of $q^*(\mathcal{T}_\Lambda)$. This action is Hamiltonian. Indeed, this follows from the fact that the $\mathcal{G}$-action is Hamiltonian, $\hat{q} : q^*(\mathcal{T}_\Lambda) \to \mathcal{T}_\Lambda$ is a surjective submersion, $\varphi^* i^* \Omega = (\hat{q})^* \Omega_\Lambda$ and the relationship:

$$m_{\mathcal{T}_\Lambda} \circ (\hat{q}, \mathrm{Id}_S) = m_{\mathcal{G}} \circ (i \circ \varphi, \mathrm{Id}_S),$$

in the notation of Corollary 6.1.9. □

We thus obtain a normal form and local polyhedrality for the moment map $\bar{\mu} : (S, \omega) \to (B, 0)$ directly from the results of Chapter 3. In the rest of this section we will express the assumptions and consequences of these theorems in terms of the $\mathcal{G}$-action, instead of the $\mathcal{T}_\Lambda$-action. Our first objective will be to describe the weights of the symplectic isotropy representations of the $\mathcal{T}_\Lambda$-action in terms of those of the $\mathcal{G}$-action. To this end, let $p \in S$. We write $x = \mu(p)$ and $b = q(x)$. We have two symplectic isotropy representations: one of $\mathcal{G}_p$ on

$$\mathcal{S}^{\mathcal{G}}_p := \frac{T_p \mathcal{O}^\omega_{\mathcal{G}}}{T_p \mathcal{O}_{\mathcal{G}} \cap T_p \mathcal{O}^\omega_{\mathcal{G}}},$$

and one of $(\mathcal{T}_\Lambda)_p$ on

$$\mathcal{S}^{\mathcal{T}_\Lambda}_p := \frac{T_p \mathcal{O}^\omega_{\mathcal{T}_\Lambda}}{T_p \mathcal{O}_{\mathcal{T}_\Lambda} \cap T_p \mathcal{O}^\omega_{\mathcal{T}_\Lambda}} = \frac{T_p \mathcal{O}^\omega_{\mathcal{T}_\Lambda}}{T_p \mathcal{O}_{\mathcal{T}_\Lambda}}.$$

Here we have distinguished the notation for the orbits of the $\mathcal{G}$-action and those of the $\mathcal{T}_\Lambda$-action and we have used that those of the $\mathcal{T}_\Lambda$-action are isotropic by Corollary 1.4.3 and the fact that $\mathfrak{t}_b = T^*_b B$. The isomorphism of Lie groupoids $\varphi : q^*(\mathcal{T}_\Lambda) \to \mathcal{G}^0_M$ restricts to an isomorphism of Lie groups:

$$\varphi_x : (\mathcal{T}_\Lambda)_b \to \mathcal{G}^0_x.$$

This maps $(\mathcal{T}_\Lambda)_p$ onto the open Lie subgroup $\mathcal{G}^0_x \cap \mathcal{G}_p$ of $\mathcal{G}_p$ and hence it induces a linear symplectic action of $(\mathcal{T}_\Lambda)_p$ on $\mathcal{S}^{\mathcal{G}}_p$ and a Lie algebra isomorphism $(\varphi_x)_* : T^*_b B \to \mathfrak{g}_x$ which maps $\mathfrak{t}_p$ onto $\mathfrak{g}_p$.

Proposition 6.2.2. *The map*

$$j : (\mathcal{S}^{\mathcal{G}}_p, \omega_p) \to (\mathcal{S}^{\mathcal{T}_\Lambda}_p, \omega_p), \quad [v] \mapsto [v],$$

is a well-defined embedding of symplectic $(\mathcal{T}_\Lambda)_p$-representations. Moreover, the symplectic representation $(\mathcal{S}^{\mathcal{T}_\Lambda}_p, \omega_p)$ splits as

$$\mathcal{S}^{\mathcal{T}_\Lambda}_p = j(\mathcal{S}^{\mathcal{G}}_p) \oplus j(\mathcal{S}^{\mathcal{G}}_p)^{\omega_p}$$

and the action on the second component is trivial.

Proof. Recall that the actions of $\mathcal{G}$ and $\mathcal{T}_\Lambda$ are related via:

$$m_{\mathcal{T}_\Lambda} \circ (\hat{q}, \mathrm{Id}_S) = m_{\mathcal{G}} \circ (i \circ \varphi, \mathrm{Id}_S). \tag{6.5}$$

This implies that their infinitesimal actions are related as:

$$a_p^{\mathcal{T}_\Lambda} = a_p^{\mathcal{G}} \circ (\varphi_x)_* : T_b^* B \to T_p S,$$

In combination with Corollary 1.4.3 this implies that:

$$\begin{aligned} T_p \mathcal{O}_{\mathcal{T}_\Lambda} &= a_p^{\mathcal{T}_\Lambda}(T_b^* B) \\ &= a_p^{\mathcal{G}}(\mathfrak{g}_x) \\ &= T_p\mathcal{O}_{\mathcal{G}} \cap T_p\mathcal{O}_{\mathcal{G}}^{\omega}. \end{aligned}$$

Therefore i is a well-defined, injective map. Clearly it is a linear symplectic map as well. It further follows from (6.5) that

$$\begin{aligned} [\alpha] \cdot j([v]) &= [(dm_{\mathcal{T}_\Lambda})_{([\alpha],p)}(0,v)] \\ &= [(dm_{\mathcal{G}})_{(\varphi_x([\alpha]),p)}(0,v)] \\ &= j(\varphi_x([\alpha]) \cdot [v]) \end{aligned}$$

for all $[\alpha] \in (\mathcal{T}_\Lambda)_b$ and $v \in \mathcal{S}_p^{\mathcal{G}}$, hence j is $(\mathcal{T}_\Lambda)_p$-equivariant. It remains to prove that the action on $j(\mathcal{S}_p^{\mathcal{G}})^{\omega_p}$ is trivial. To this end, let $[v] \in j(\mathcal{S}_p^{\mathcal{G}})^{\omega_p}$. Then $\omega_p(v,w) = 0$ for all $w \in T_p\mathcal{O}_{\mathcal{G}}^{\omega}$ and therefore $v \in T_p\mathcal{O}_{\mathcal{G}}$. Now observe that $m_{\mathcal{T}_\Lambda}$ restricts to the projection onto the second component on the submanifold

$$(\mathcal{T}_\Lambda)_p \times \mathcal{O}_{\mathcal{G}} \subset \mathcal{T}_\Lambda\ {}_{pr}\!\times_{\bar{\mu}} S.$$

Here the first is indeed a subset of the latter due to Proposition 6.2.4 below. It follows that

$$[\alpha] \cdot [v] = [(dm_{\mathcal{T}_\Lambda})_{([\alpha],p)}(0,v)] = [v]$$

for all $[\alpha] \in (\mathcal{T}_\Lambda)_p$, as desired. □

As an immediate consequence we have:

Corollary 6.2.3. *The weight-tuple $(\alpha_1, ..., \alpha_k)$ of the symplectic isotropy action of $\mathcal{G}$ at p is mapped onto that of $\mathcal{T}_\Lambda$ at p by the linear isomorphism $\mathfrak{g}_p^* \to \mathfrak{t}_p^*$ induced by φ_x. Consequently, the moment map $\bar{\mu}$ is locally polyhedral and its cone at a point $p \in S$ is*

$$\pi_p^{-1}(Cone(\alpha_1, ..., \alpha_k))$$

where $\pi_p : T_b B \to \mathfrak{g}_p^*$ is the dual of the composition $\mathfrak{g}_p \xrightarrow{(\varphi_x)_*^{-1}} \mathfrak{t}_p \to \mathfrak{t}_b = T_b^* B$.

Next, we will relate the images of the moment maps μ and $\bar{\mu}$.

Proposition 6.2.4. *Let $\mu : (S, \omega) \to M$ be a Hamiltonian $\mathcal{G}$-space, $p \in S$, $\mathcal{O}_p$ the orbit of the $\mathcal{G}$-action through p and $\mathcal{O}_x$ the $\mathcal{G}$-orbit through $x = \mu(p)$. Then*

$$\mu(\mathcal{O}_p) = \mathcal{O}_x.$$

Proof. This holds because $y \in \mathcal{O}_x$ if and only if there is a $g \in \mathcal{G}$ such that $s(g) = x$ and

$$\mu(g \cdot p) = t(g) = y.$$

□

Corollary 6.2.5. *Let $\mu : (S, \omega) \to M$ be a Hamiltonian $\mathcal{G}$-space. Then*

$$\mu(S) = q^{-1}(\bar{\mu}(S)).$$

We thus see that the image of μ is determined by that of $\bar{\mu}$. Moreover, we can consider the map:

$$\bar{\bar{\mu}} : S/\mathcal{G} \to B.$$

Because the canonical quotient map $S \to S/\mathcal{G}$ is open (as is the orbit projection of any Lie groupoid), it follows that $\bar{\mu}$ is locally polyhedral if and only if $\bar{\bar{\mu}}$ is so and in this case the cone of $\bar{\mu}$ at $p \in S$ is the cone of $\bar{\bar{\mu}}$ at $[p] \in S/\mathcal{G}$. We can therefore conclude the following.

Theorem 6.2.6. *Suppose that $\bar{\bar{\mu}}$ is closed as a map onto its image and its fiber over a point $\bar{\bar{\mu}}([p])$ is connected. Then the image of $\bar{\bar{\mu}}$ is locally polyhedral at $\bar{\bar{\mu}}([p])$ and its cone at this point is*

$$\pi_p^{-1}(Cone(\alpha_1, ..., \alpha_k)).$$

6.3 Toric $\mathcal{G}$-spaces

We now suggest a definition of toric $\mathcal{G}$-spaces over simple Poisson manifolds and we show that the moment image of such a space is a Delzant submanifold of the leaf space.

Definition 6.3.1. A Hamiltonian $\mathcal{G}$-space $\mu : (S, \omega) \to (M, \pi)$ over a simple Poisson manifold (M, π) is called **toric** if the following conditions hold:

- $\mathcal{G}$ is proper and s-connected.
- The $\mathcal{G}$-action is free on an open and dense subset of S.
- $\dim(S) = 2\dim(M) - \mathrm{rk}(\pi)$.
- $\bar{\bar{\mu}}$ is closed as a map onto its image and has connected fibers.

The motivation for these conditions is similar to that for toric $\mathcal{T}_\Lambda$-spaces. The first condition ensures that the leaf space of (M, π) admits an integral affine structure induced by $(\mathcal{G}, \Omega)$. If $S/\mathcal{G}$ is connected, the second condition is equivalent to the triviality of the Principal Morita type of the action groupoid. It ensures that the moment map is a weak isotropic realization which forces the inequality $\dim(S) \geq 2\dim(M) - \mathrm{rk}(\pi)$ to hold. The third condition assumes that this is an equality, which is the most ideal situation. Together with the second and last condition it ensures that the image of $\bar{\bar{\mu}}$ is a Delzant submanifold of (B, Λ), as we will now show. To this end, let $\mu : (S, \omega) \to (M, \pi)$ be a toric $\mathcal{G}$-space and let Δ denote the image of $\bar{\bar{\mu}}$.

Proposition 6.3.2. *The symplectic isotropy representation of $(\mathcal{T}_\Lambda)_p$ on $(\mathcal{S}_p^{\mathcal{G}}, \omega_p)$ is toric. Consequently, the isotropy groups of the $\mathcal{T}_\Lambda$-action are tori.*

Proof. Using the decomposition $\mathcal{S}_p^{\mathcal{T}_\Lambda}$ from Proposition 6.2.2 and the normal form for the $\mathcal{T}_\Lambda$-action in the same way as in the proof of Proposition 5.1.7 one derives that the action of $(\mathcal{T}_\Lambda)_p$ on $\mathcal{S}_p^{\mathcal{G}}$ is free on an open and dense subset. Moreover, we have:

$$\begin{aligned}\dim(\mathcal{S}_p^{\mathcal{G}}) &= \dim(T_p\mathcal{O}_{\mathcal{G}}^{\omega}) - \dim(T_p\mathcal{O}_{\mathcal{G}} \cap T_p\mathcal{O}_{\mathcal{G}}^{\omega})\\ &= (\dim(S) - \dim(a_p(T_x^*M))) - \dim(a_p(\mathfrak{g}_x))\\ &= \dim(S) - (\dim(M) - \dim(\mathfrak{g}_p)) - (\dim(\mathfrak{g}_x) - \dim(\mathfrak{g}_p))\\ &= \dim(M) - \mathrm{rk}(\pi) - \dim(\mathrm{Ker}(\pi_x^\sharp)) + 2\dim(\mathfrak{g}_p)\\ &= 2\dim(\mathfrak{g}_p)\\ &= 2\dim((\mathcal{T}_\Lambda)_p).\end{aligned}$$

Here we used Corollary 1.4.3 throughout and we used that $\dim(S) = 2\dim(M) - \mathrm{rk}(\pi)$ in the fourth step. We conclude that the $(\mathcal{T}_\Lambda)_p$-representation on $(\mathcal{S}_p^{\mathcal{G}}, \omega_p)$ is indeed toric. □

Together with Theorem 6.2.6 and the same arguments as in the $\pi = 0$ case, this leads to a generalization of Theorem 5.1.14. In particular, we find:

Theorem 6.3.3. *The moment image Δ of a toric $\mathcal{G}$-space is a Delzant submanifold of its orbit space (B, Λ).*

A brief outlook

We conclude this chapter with an outlook on what we expect or hope to be true for toric $\mathcal{G}$-spaces. Although we do expect the map $\bar{\bar{\mu}} : S/\mathcal{G} \to \Delta$ to be a homeomorphism, it probably need not be an isomorphism of stratified spaces, unless the isotropy groups of the $\mathcal{G}$-action (and not just those of the $\mathcal{T}_\Lambda$-action) are connected.

Apart from this, the question remains whether or not the classification of toric $\mathcal{T}_\Lambda$-spaces could be generalized to toric $\mathcal{G}$-spaces. A similar classification has already been given for proper isotropic realizations with connected fibers, in [DD87]. The conclusion in that case is that every such isotropic realization induces an integral affine structure on the leaf space and, given a Poisson manifold (M, π) with an integral affines structure Λ on its leaf space, there is an obstruction class associated to π and Λ which vanishes precisely if (M, π) admits such an isotropic realization inducing the given integral affine structure. It is however not yet known whether or not this class vanishes if (M, π) admits a proper integration that induces the integral affines structure Λ on its leaf space, which is the situation that would be of interest to us. Anyhow, such an obstruction class will certainly play a role when classifying toric $\mathcal{G}$-spaces.

Appendix A

Proof of the normal form for Hamiltonian T-spaces

The Moser-Weinstein theorem is a classical tool in symplectic geometry that helps to prove many (if not all) symplectic normal form theorems. Guillemin and Sternberg noticed in [GS82] that this theorem and its proof generalize to give a normal form for a moment map around the fixed points of a proper Hamiltonian G-space. Later this was generalized to a normal form around arbitrary orbits. In this appendix we will prove a version of this around isotropic orbits, following the exposition in [Kar93]. Theorem 3.2.1 is a direct consequence of this.

The normal form for proper Lie group actions

Let M be a proper G-space, $x \in M$ and $\mathcal{O}$ the orbit through x. If x is a fixed point of the action, then $G_x = G$ and the G-space M is modeled by the isotropy representation T_xM in a G-invariant neighbourhood of x. To obtain a local model of the G-space M in a neighbourhood of a non-fixed point one is forced to consider G-invariant neighbourhoods of x. In particular, such a neighbourhood contains the entire orbit $\mathcal{O}$ through x. The idea is now to use the normal bundle to this orbit as the local model. Since the zero-section of such a normal bundle is an orbit $\mathcal{O}$, which is the base of the principal G_x-bundle $G \to \mathcal{O}$, we can use the G-vector bundle

$$G \times_{G_x} \mathcal{N}_x$$

associated to the isotropy representation at x to realize the normal bundle as a G-space. This is a vector bundle over $\mathcal{O}$, so that we can (and always will) view $\mathcal{O}$ as the zero-section in $G \times_{G_x} \mathcal{N}_x$ and x as $[e, 0, 0]$.

Theorem A.1. *There exists a G-equivariant embedding*

$$\psi : G \times_{G_x} \mathcal{N}_x \supset\!\!\to M$$

from a G-invariant open neighbourhood of $\mathcal{O}$ onto a G-invariant open neighbourhood of $\mathcal{O}$ in M, such that ψ restricts to the identity on $\mathcal{O}$. Given a G_x-equivariant linear injection

$$i : \mathcal{N}_x \to T_xM$$

onto a linear complement to $T_x\mathcal{O}$ in T_xM, we can choose ψ such that $d\psi_{[e,0]}$ is given by:

$$(\bar{a}_x, i) : \mathfrak{g}/\mathfrak{g}_x \oplus \mathcal{N}_x \to T_xM, \quad ([\alpha], v) \mapsto a_x(\alpha) + i(v).$$

Proof. See for instance the proof of [OR04, Thm 2.3.28]. $\square$

Remark 19. In the above theorem we can (by the same proof) replace $\mathcal{N}_x$ by any representation V of G_x that admits an equivariant linear injection $i : V \to T_xM$ such that $i(V) \oplus T_x\mathcal{O} = T_xM$. We will take V to be $\mathfrak{g}_x^0 \oplus \mathcal{S}_x$ in the coming sections.

Remark 20. Let G be a Lie group, H a closed subgroup and V a representation of H. Every G-invariant open subset U of $G \times_H V$ is of the form $G \times_H W$ for some H-invariant open $W \subset V$. Indeed, let $\pi : G \times V \to G \times_H V$ denote the canonical projection. Then for every $v \in V$ such that $(e, v) \in \pi^{-1}(U)$, there is an open neighbourhood W_v of v in V such that $\{e\} \times W_v \subset \pi^{-1}(U)$. Taking the union of the opens W_v over all such v we obtain an open $W \subset V$, which by G-invariance of U is such that $G \times W = \pi^{-1}(U)$. It follows that W is H-invariant and $U = G \times_H W$, as claimed.

The local model of a Hamiltonian G-space

As for any local normal form, one needs a model that describes the local behaviour of the structure under consideration. Suppose that we are given a proper Hamiltonian G-space (M, ω, μ) and a point $x \in M$ for which the orbit $\mathcal{O}$ through x is isotropic. We will build a new Hamiltonian G-space out of the following data:

- The Lie group G,
- the isotropy group G_x of the action,
- the symplectic isotropy representation: $G_x \to \mathrm{Sp}(\mathcal{S}_x, \omega_x)$.

Recall here from Definition 1.4.21 that

$$\mathcal{S}_x = \frac{T_x\mathcal{O}^\omega}{T_x\mathcal{O} \cap T_x\mathcal{O}^\omega}.$$

By means of the symplectic reduction theorem we will now construct the model. First of all, the symplectic isotropy representation at x comes with its standard moment map $\mu_{\mathcal{S}_x}$. Secondly, we consider $(G \times \mathfrak{g}^*, \omega_G)$ as a G_x-space with the action of G_x given by

$$h \cdot (g, \alpha) = (gh^{-1}, \mathrm{Ad}_h^*\alpha), \quad g \in G, h \in G_x.$$

In view of Example 1.4.11 and part a of the next lemma this action is Hamiltonian with moment map given by:

$$(g, \alpha) \mapsto -\alpha|_{\mathfrak{g}_x}.$$

Lemma A.2. *Let (M, ω, μ) be a Hamiltonian G-space. The following properties hold.*

a) If H is another Lie group and $\varphi : H \to G$ an morphism of Lie groups, then H acts on M via φ and $(M, \omega, (d\varphi_e)^* \circ \mu)$ is a Hamiltonian H-space with this action.

b) If (M', ω', μ') is another Hamiltonian G-space, then $(M \times M', \omega \oplus \omega', \mu + \mu')$ is a Hamiltonian G-space for the diagonal action of G.

The proof of this is straightforward. By part b, the cartesian product of $G \times \mathfrak{g}^*$ and $\mathcal{S}_x$ is a Hamiltonian G_x-space for the diagonal action, with moment map given by

$$\mu_{G_x}(g, \alpha, v) = \mu_{\mathcal{S}_x}(v) - \alpha|_{\mathfrak{g}_x}. \tag{A.1}$$

As the action of G_x on $G \times \mathfrak{g}^* \times \mathcal{S}_x$ is proper and free, it is so as well on the closed invariant subspace $\mu_{G_x}^{-1}(0)$. Therefore, by the symplectic reduction theorem, we obtain the reduced symplectic manifold

$$(P_0, \omega_0).$$

Using Proposition 1.4.19, we will now equip this with a Hamiltonian G-action. We endow $(\mathcal{S}_x, \omega_x)$ with the trivial Hamiltonian G-action (with moment map constantly 0) and equip $(G \times \mathfrak{g}^*, -\Omega_{can})$ with the Hamiltonian G-action induced by left translation on G, as in Example 1.4.11. Then the diagonal action turns the product $G \times \mathfrak{g}^* \times \mathcal{S}_x$ into a Hamiltonian G-space with moment map given by:

$$\mu_G(g, \alpha, v) = \mathrm{Ad}_g^* \alpha.$$

Because μ_G is G_x-invariant, μ_{G_x} is G-invariant and the actions of G and G_x commute, it follows that the Hamiltonian G-space structure on $G \times \mathfrak{g}^* \times \mathcal{S}_x$ descends to one on P_0 with moment map μ_0 uniquely determined by the fact that:

$$\mu_0 \circ \pi = \mu_G|_{\mu^{-1}(0)}.$$

Here $\pi : \mu_{G_x}^{-1}(0) \to P_0$ is the canonical quotient map.

In principal, this model would do. However in its current form the quotient space P_0 is still rather mysterious, while one would hope to end up with a model in the familiar form of a tubular neighbourhood. As we will now show, the G-space P_0 is equivariantly diffeomorphic to such a tube. For this we need an auxiliary G_x-equivariant linear projection $p : \mathfrak{g} \to \mathfrak{g}_x$, which can for instance be obtained by selecting a G_x-invariant inner product on $\mathfrak{g}$ and letting p be the orthogonal projection onto $\mathfrak{g}_x$. It follows from expression (A.1) that

$$\varphi_p : G \times \mathfrak{g}_x^0 \times \mathcal{S}_x \to G \times \mathfrak{g}^* \times \mathcal{S}_x, \quad (g, \alpha, v) \mapsto (g, p^*(\mu_{\mathcal{S}_x}(v)) + \alpha, v)$$

is a G_x-equivariant embedding with image $\mu_{G_x}^{-1}(0)$, where the G_x-action on the left-hand space is given by

$$h \cdot (g, \alpha, v) = (gh^{-1}, \mathrm{Ad}_h^* \alpha, hv).$$

Therefore φ_p descends to a diffeomorphism

$$\overline{\varphi}_p : G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x) \to P_0.$$

Because φ_p is G-equivariant as well, it follows that $\overline{\varphi}_p$ is an isomorphism of G-spaces. Pulling-back the symplectic form ω_0 and the moment map μ_0 turns $G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x)$ into a Hamiltonian G-space, with symplectic form $\Omega_p = \overline{\varphi}_p^* \omega_0$ uniquely determined by the fact that

$$q^* \Omega_p = \varphi_p^*(-\Omega_{can} \oplus \omega_x) \tag{A.2}$$

where $q : G \times \mathfrak{g}_x^0 \times \mathcal{S}_x \to G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x)$ is the canonical quotient map. The moment map $\mathcal{M}_p = \overline{\varphi}_p^* \mu_0$ is given by:

$$\mathcal{M}_p([g, \alpha, v]) = \mathrm{Ad}_g^*(p^*(\mu_{\mathcal{S}_x}(v)) + \alpha).$$

This ends the construction of the model. Note that the isomorphism class of the model does not depend on the choice of p. The local normal form that we aim to prove is as follows.

Theorem A.3. *Let (M, ω, μ) be a proper Hamiltonian G-space and $x \in M$ such that the orbit $\mathcal{O}$ through x is isotropic. Let $p : \mathfrak{g} \to \mathfrak{g}_x$ be a G_x-equivariant linear projection. Then there exists an embedding of Hamiltonian G-spaces*

$$\psi : (G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x), \Omega_p, \mathcal{M}_p + Ad^*(\mu(x))) \supset\!\to (M, \omega, \mu)$$

from a G-invariant open neighbourhood of $\mathcal{O}$ onto a G-invariant open neighbourhood of $\mathcal{O}$ in M that restricts to the identity on $\mathcal{O}$.

For the proof of the normal form theorem, it will be essential to obtain a more explicit expression for Ω_p at points of $\mathcal{O}$. Since Ω_p is G-invariant, we can restrict ourselves to the point $[e, 0, 0]$. The tangent space to the slice $G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x)$ at $[e, 0, 0]$ is canonically isomorphic to

$$\mathfrak{g}/\mathfrak{g}_x \oplus \mathfrak{g}_x^0 \oplus \mathcal{S}_x.$$

Because $\alpha \in \mathfrak{g}^*$ factors through an element of $(\mathfrak{g}/\mathfrak{g}_x)^*$ precisely if $\alpha \in \mathfrak{g}_x^0$, we have a canonical G_x-equivariant linear isomorphism

$$\mathfrak{g}_x^0 \to (\mathfrak{g}/\mathfrak{g}_x)^*, \quad \alpha \mapsto \overline{\alpha}. \tag{A.3}$$

Hence we can pull the canonical linear symplectic form on $\mathfrak{g}/\mathfrak{g}_x \times (\mathfrak{g}/\mathfrak{g}_x)^*$ back to a G_x-invariant linear symplectic form η on $\mathfrak{g}/\mathfrak{g}_x \times \mathfrak{g}_x^0$, given by:

$$\eta(([\xi], \alpha), ([\xi'], \alpha')) = \alpha'(\xi) - \alpha(\xi').$$

Proposition A.4. *Let Ω_p be the symplectic form as in (A.2). Then:*

$$(\mathfrak{g}/\mathfrak{g}_x \oplus \mathfrak{g}_x^0 \oplus \mathcal{S}_x, (\Omega_p)_{[e,0,0]}) = (\mathfrak{g}/\mathfrak{g}_x \oplus \mathfrak{g}_x^0, \eta) \oplus (\mathcal{S}_x, \omega_x).$$

Proof. The proof is a straightforward computation, the details of which are left to the reader. Let us at least give directions. First one uses the fact that Ω_p is determined by equation (A.2). To work out the right-hand side of (A.2) one computes the differential of φ by using the fact that $(d\mu_{\mathcal{S}_x})_0 = 0$, which holds because $\mu_{\mathcal{S}_x}$ is quadratic. Finally one applies formula (1.4). □

Proof of the local normal form

To prove that the model that we have constructed indeed describes the Hamiltonian G-space structure of a neighbourhood of an isotropic orbit, we combine the tube theorem for proper Lie group actions with an equivariant version of the Moser-Weinstein theorem.

Our first aim is to prove:

Theorem A.5. *Let (M, ω) be a symplectic G-space and $x \in M$ such that the orbit $\mathcal{O}$ through x is isotropic. Let $p : \mathfrak{g} \to \mathfrak{g}_x$ be a G_x-equivariant linear projection. Then there exists an embedding of symplectic G-spaces*

$$\varphi : (G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x), \Omega_p) \supset\!\to (M, \omega)$$

from a G-invariant open neighbourhood of $\mathcal{O}$ onto a G-invariant open neighbourhood of $\mathcal{O}$ in M that restricts to the identity on $\mathcal{O}$.

Lemma A.6. *Let (V, ω) be a symplectic vector representation of a compact Lie group H.*

a) *Let $L \subset V$ be an H-invariant Lagrangian linear subspace. Then there is an isomorphism of symplectic representations*

$$(V, \omega) \cong (L \oplus L^*, \omega_L)$$

that restricts to the identity map on L.

b) *Suppose that $W \subset V$ is an H-invariant linear subspace. Set $K = W \cap W^\omega$. Then there is an isomorphism of symplectic representations*

$$(V, \omega) \cong (K \oplus K^*, \omega_K) \oplus (W/K, \overline{\omega}) \oplus (W^\omega/K, \overline{\omega})$$

that restricts to the identity map on K.

Proof. We first address part a. Choose an H-equivariant, ω-compatible complex structure J on V. Then JL is an H-invariant Lagrangian linear complement to L. The map

$$\varphi : V = L \oplus JL \to L \oplus L^*, \quad (v, w) \mapsto (v, \omega(\cdot, w))$$

satisfies the requirements.
We turn to part b. Choose an H-invariant linear complement C to K in V. Then $W \cap C$ and $W^\omega \cap C$ are H-invariant symplectic subspaces of V. Indeed, if $w \in (W \cap C) \cap (W \cap C)^\omega$, then for all $w' = u + v \in W = K \oplus (W \cap C)$ we have

$$\omega(w, w') = \omega(w, u) + \omega(w, v) = 0.$$

Hence $w \in K \cap C$, which implies that $w = 0$. This shows that $W \cap C$ is symplectic and the argument for $W^\omega \cap C$ is analogous. Therefore V decomposes into H-invariant symplectic subspaces:

$$V = ((W \cap C) + (W^\omega \cap C))^\omega \oplus (W \cap C) \oplus (W^\omega \cap C)$$

Note further that the map $W \cap C \to W/K$, $w \mapsto [w]$ is an equivariant linear symplectomorphism and the same holds for $W^\omega \cap C \to W\omega/K$, $w \mapsto [w]$. Moreover, it follows directly from the definition of K that it is isotropic and contained in $((W \cap C) + (W^\omega \cap C))^\omega$. A straightforward dimension count shows that

$$\dim((W \cap C) + (W^\omega \cap C))^\omega = 2\dim K,$$

so that K is Lagrangian in $((W \cap C) + (W^\omega \cap C))^\omega$. So application of part a to K, in combination with the above, completes the proof of the lemma. □

Lemma A.7 (Equivariant version of the Moser-Weinstein theorem)**.** *Let M be a proper G-space and $\mathcal{O}$ an orbit. Suppose that ω_0 and ω_1 are G-invariant symplectic forms on M that coincide at every $x \in \mathcal{O}$. Then there are G-invariant open neighbourhoods U_0 and U_1 of $\mathcal{O}$ and an isomorphism of symplectic G-spaces $\varphi : (U_0, \omega_0) \to (U_1, \omega_1)$ such that φ restricts to the identity map on $\mathcal{O}$.*

Proof. Using an equivariant normal bundle to the orbit, the usual proof of the original non-equivariant version by means of the Moser-path method works in this situation as well. See for example [OR04, Thm 7.3.1] for details. □

Proof of Theorem A.5. Lemma A.6 yields a G_x-equivariant linear symplectomorphism

$$(T_x\mathcal{O} \oplus T_x\mathcal{O}^*, \omega_{T_x\mathcal{O}}) \oplus (\mathcal{S}_x, \omega_x) \to (T_xM, \omega_x)$$

that restricts to the identity on $T_x\mathcal{O}$. The infinitesimal action induces a G_x-equivariant linear isomorphism $\overline{a}_x : \mathfrak{g}/\mathfrak{g}_x \to T_x\mathcal{O}$. Combining this with the above we obtain a G-equivariant linear symplectomorphism

$$(\mathfrak{g}/\mathfrak{g}_x \oplus (\mathfrak{g}/\mathfrak{g}_x)^*, \omega_{\mathfrak{g}/\mathfrak{g}_x}) \oplus (\mathcal{S}_x, \omega_x) \to (T_xM, \omega_x)$$

that restricts to $\overline{a}_x$ on $\mathfrak{g}/\mathfrak{g}_x$. Finally, by using the isomorphism (A.3) we obtain a G_x-equivariant linear symplectomorphism

$$\Psi : (\mathfrak{g}/\mathfrak{g}_x \oplus \mathfrak{g}_x^0, \eta) \oplus (\mathcal{S}_x, \omega_x) \to (T_xM, \omega_x)$$

that restricts to $\overline{a}_x$ on $\mathfrak{g}/\mathfrak{g}_x$. By restricting this to $\mathfrak{g}_x^0 \oplus \mathcal{S}_x$ we obtain a G_x-equivariant linear injection

$$i : \mathfrak{g}_x^0 \oplus \mathcal{S}_x \to T_xM,$$

the image of which is complementary to $T_x\mathcal{O}$. So by Theorem A.1 there is a G-equivariant embedding

$$\psi : G \times_{G_x} (\mathfrak{g}_x^0 \oplus \mathcal{S}_x) \supset\!\rightarrow M$$

from a G-invariant open neighbourhood of U of $\mathcal{O}$ onto a G-invariant open neighbourhood of $\mathcal{O}$ in M, that restricts to the identity on $\mathcal{O}$ and satisfies:

$$d\psi_{[e,0,0]} = (\overline{a}_x, i) = \Psi.$$

It follows from Proposition A.4 that

$$(\psi^*\omega)_{[e,0,0]} = (\Omega_p)[e, 0, 0]$$

and hence, by G-invariance, both agree at all points of $\mathcal{O}$. Therefore, Lemma A.7 implies that there is an embedding of symplectic G-spaces

$$\varphi : (U, \Omega_p) \supset\!\rightarrow (U, \psi^*\omega)$$

from a G-invariant open neighbourhood of $\mathcal{O}$ in U to another such open neighbourhood, such that φ is the identity on $\mathcal{O}$. The required embedding of symplectic G-spaces is now given by $\psi \circ \varphi$. □

Finally, we derive Theorem A.3 from this. We need one short lemma.

Lemma A.8. *Let (M, ω) be a symplectic G-space and suppose that μ and ν are both moment maps for it. Then their difference $\mu - \nu$ is a locally constant function $M \to \mathfrak{g}^*$.*

Proof. For all $x \in M$, $v \in T_xM$ and $\xi \in \mathfrak{g}$ we have:

$$\langle d\mu_x(v) - d\nu_x(v), \xi\rangle = d\langle \mu, \xi\rangle_x(v) - d\langle \nu, \xi\rangle_x(v) = (\iota_{\xi_M}\omega_x)(v) - (\iota_{\xi_M}\omega_x)(v) = 0.$$

So $d(\mu - \nu) = 0$, which implies the lemma. □

Proof of Theorem A.3. Let φ be an embedding as in Theorem A.5, defined on a G-invariant open neighbourhood $U = G \times_{G_x} B$ of $\mathcal{O}$. By means of a G_x-invariant inner product (which exists because G_x is compact) we find a G_x-invariant open ball around $(0,0)$ inside B, so we can assume B to be such an open ball. Since φ is an equivariant symplectic embedding, both $\mathcal{M}_p$ and $\mu \circ \varphi$ define a moment map on the symplectic G-space U. By equivariance of the moment maps and the fact that $\varphi([e,0,0]) = x$ we find:

$$\mu(\varphi([g,0,0])) - \mathcal{M}_p([g,0,0]) = \mathrm{Ad}_g^*(\mu(x)),$$

for all $g \in G$. By convexity of B, $[g,\alpha,v]$ lies in the same connected component of U as $[g,0,0]$, for each $[g,\alpha,v] \in U$. Since by the previous lemma $\mu \circ \varphi - \mathcal{M}$ is constant on each such connected component, the map φ is the desired embedding of Hamiltonian G-spaces. □

Remark 21. The local normal form that we have given here can, in an entirely similar spirit, be generalized to a local normal form around arbitrary orbits of proper Hamiltonian G-spaces. This is often named the Marle-Guillemin-Sternberg normal form, after its discoverers. For the statement and proof of this generalization see for instance [Kar93].

For Hamiltonian T-spaces Theorem A.3 takes the following simpler form, as desired.

Corollary A.9. *Let T be a torus, (M,ω,μ) a Hamiltonian T-space, $x \in M$ and $\mathcal{O}$ the orbit through x. Let $p : \mathfrak{t} \to \mathfrak{t}_x$ be a linear projection. Then there exists an embedding of Hamiltonian T-spaces*

$$\psi : (T \times_{T_x} (\mathfrak{t}_x^0 \oplus \mathcal{S}_x), \Omega_p, \mathcal{M}_p + \mu(x)) \supset\!\!\to (M,\omega,\mu)$$

from a T-invariant open neighbourhood of $\mathcal{O}$ onto a T-invariant open neighbourhood of $\mathcal{O}$ in M, that restricts to the identity on $\mathcal{O}$. Here $\mathcal{M}_p$ is given by:

$$\mathcal{M}_p([t,\alpha,v]) = \alpha + p^*(\mu_{\mathcal{S}_x}(v)).$$

Proof. This follows from Theorem A.3 by two observations. First, notice that every orbit $\mathcal{O}$ is isotropic because μ is then T-invariant, and hence constant on $\mathcal{O}$, so that

$$T_y\mathcal{O} \subset \mathrm{Ker}(d\mu_y) = T_y\mathcal{O}^\omega$$

for all $y \in \mathcal{O}$. Secondly, the (co-)adjoint action of an abelian group is trivial, so any linear projection $p : \mathfrak{t} \to \mathfrak{t}_x$ is T_x-equivariant and the expression for $\mathcal{M}_p$ simplifies. □

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