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The Nash-Moser theorem and applications

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Chapter 1

Introduction

This thesis reviews the theory of global analysis, using Fréchet spaces and manifolds, as first described in Richard Hamilton's paper [Ham82b] on the Nash-Moser inverse function theorem, with a particular focus on real geometry. In addition, a chapter is devoted to applications, some of which are original work.

The inverse function theorem, smooth maps are locally invertible at regular points, is an important and well-known tool in finite dimensional analysis. It readily generalizes to smooth maps between Banach spaces and, accordingly, to maps between Banach manifolds. Typical examples of Banach spaces are the $C^k(M)$ of k-times differentiable functions on a compact manifold, while typical examples of smooth maps between Banach spaces are partial differential operators, seen as maps $C^{k+r}(M) \to C^k(M)$. The inverse function theorem, produces solutions under some conditions, but is typically rather restrictive: Even if the function in the codomain is smooth, solutions are at most $k+r$-times differentiable.

For Fréchet spaces, most typically the spaces of smooth functions $C^\infty(M)$, the inverse function theorem generally fails. Moser [Mos66] suggested an algorithm, or method of proof, similar to Newton-Raphson iteration, to find solutions to PDEs. Hamilton used this method to prove a general theorem, and a useful implicit function variant, and gave several applications, see e.g. [Ham82b, Ham77, Ham82a].

The theorem has some disadvantages over the Banach inverse function theorem. Most clamorously, in the Banach case it is sufficient for the map under scrutiny to just be regular in a point. It then follows that it is invertible in a neighborhood – in particular it is regular. While for Fréchet spaces the map is required to be regular in a neighborhood of the point. There are examples of smooth maps whose set of singular points has regular limit points, there is a sequence of singular points converging to a regular point, hence this condition *is* necessary.

As an example consider the Fréchet space $C^\infty[-1,1]$ of smooth functions $f : [-1,1] \to \mathbb{R}$, whose semi-norms are given by the C^k-norms,

$$\|f\|_n = \sum_{k \le n} \sup_{x \in [-1,1]} \left| \frac{d^k}{dx^k} f(x) \right| .$$

The partial differential operator

$$P : C^\infty[-1,1] \to C^\infty[-1,1], \quad Pf = f - xf\frac{df}{dx}$$

is a smooth Fréchet map and its derivative is given by

$$DP(f)g = g - xg\frac{df}{dx} - xf\frac{dg}{dx}.$$

It is clearly invertible at $f = 0$, since $DP(0) = \mathrm{id}$ is the identity. Yet the sequence of smooth maps $g_n = \frac{1}{n} + \frac{x^n}{n!}$ converges to 0 in $C^\infty[-1,1]$ and none of the g_n lie in the image of P. The latter is seen by a simple computation with formal power series. This and more examples can be found in [Ham82b].

The second glaring difference is that only a particular (non-full) subcategory of Fréchet manifolds will do. The Fréchet spaces require extra structure in the form of a *'grading'*, a choice of incremental order of seminorms generating the topology. Fittingly maps are required to suitably interact with this grading. In addition of requiring a linear map $L : E \to F$ to be continuous, which means one can estimate $\|Le\|_m \leq C\|e\|_n$ if n is sufficiently larger then m (recall that the grading is incremental), one requires an upper bound on the difference $n - m$ independent of m. Such maps are called *'tame'*. A similar notion of tameness exists for non-linear maps, and manifolds are required to be tame in the sense that the local model has a canonical grading and the transition maps are tame. This gives rise to an abundance of technical conditions that must be checked in applications.

The tameness condition on Fréchet manifolds is restrictive. Although in general the space of smooth maps $C^\infty(M, \mathbb{R}^n)$ is a Fréchet space, there is no canonical choice of grading if M is not compact. When M is compact however, this problem is easily solved and the transition maps are tame. Accordingly, many applications require manifolds to be compact.

An additional requirement on the graded Fréchet spaces is the existence of *'smoothing operators'*. They are a family of linear automorphisms that allows one to 'truncate' the elements of a graded Fréchet space in a controlled manner. In the Newton-Raphson iteration, a linear automorphism $L : C^k(M) \to C^k(M)$ is repeatedly applied to a starting element, one shows that L is a contraction and hence the resulting sequence $L^k f$ converges. In the setting of graded Fréchet spaces, say $C^\infty(M)$, one can at most prove an estimate of the form $\|Lf\|_n \leq \theta\|f\|_{n+r}$. As the index tends to infinity, this doesn't show that $L^k f$ is Cauchy. This phenomenom is named *'loss of derivatives'* by Moser. The smoothing operators allow for a different iteration that counters the effects of loss of derivative. It is the main ingenuity in the Nash-Moser inverse function theorem. In contrary to the choice of grading, it is sufficient for the smoothing operators to merely exist – the local model of a tame Fréchet manifold must merely allow the existence of smoothing operators.

The Nash-Moser theorem is most notably applicable in geometry. It provides an analytical tool to answer question revolving around deformations and stability. The primary example, of the method, not the theorem, is found in Nash' paper [Nas56] on the embbedding problem of Riemannian manifolds. Hamilton gave several example applications, in particular to embedding manifolds with positive curvature into $\mathbb{R}^n$, the shallow water equations, and the stability of symplectic and contact strutctures in [Ham82b, Ham77]. In an unpublished paper, he additionally applies it to stability of foliations. The resulting conditions for a foliation to be stable are sadly too strict to be useful. In this thesis we apply Nash-Moser to the classical example of stability of smooth maps between manifolds, as first conceived and proven by Mather [Mat69], and group actions for a fixed manifold and compact Lie group, see e.g. [Pal61]. The latter is also proven to work for compact Lie groupoids actions with a fixed moment map.

Chapter 2

Graded Fréchet spaces and examples

We begin with a short introduction to only the most relevant parts of the theory of Fréchet spaces and Fréchet calculus. Most statements in this section are given without proof. For a more detailed discussion of Fréchet spaces we refer to one of the many textbooks on functional analysis. As the main focus of this thesis is understanding, and applying to geometry, the theory presented in [Ham82b, Mos66, Nas56], we will quickly continue towards the notions of gradings and tame maps. All vector spaces are assumed to be over $\mathbb{R}$.

2.1 Fréchet spaces

Recall that a semi-norm on a vector space F is a function $\|-\| : F \to \mathbb{R}_+$, where, from now on, $\mathbb{R}_+$ indicates the set of $r \in \mathbb{R}$ with $r \geq 0$, satisfying the usual norm properties

- *subadditivity*: $\|f + g\| \leq \|f\| + \|g\|$,
- *positive homogeneity*: $\|\lambda f\| = |\lambda| \, \|f\|$,

except possibly failing to separate points, that is, $\|f\| = 0$ doesn't necessarily imply $f = 0$. A family of semi-norms $\{\|-\|_k\}_{k \in I}$ induces a topology on F with a basis of topology given by the intersection of finitely many open balls,

$$B_{k_1}^{r_1}(f) \cap \ldots \cap B_{k_n}^{r_n}(f), \quad B_k^r(f) = \{g \in F | \, \|f - g\|_k < r\}.$$

This topology is usually called the initial topology of the seminorms. It makes F a topological vector space of the *locally convex* kind. A sequence in F converges if and only if it converges with repect to each of the semi-norms, that is, $f_n \to f$ exactly when $\|f_n - f\|_k \to 0$ for all indices $k \in I$.

Definition 2.1.1 (Fréchet space). *A* Fréchet space *is a locally convex space with the following additional properties:*

1. *F is Hausdorff;*
2. *The topology can be induced by countably many semi-norms;*
3. *F is complete.* △

Note that the first property is equivalent to: if $\|f\|_k = 0$ for all k, then $f = 0$. The second property holds if and only if the topology is first countable. Note that every locally convex vector space F satisfying the first property is metrizable. For let the topology of F be given by a countable family $\{\|-\|_k\}_{k\in\mathbb{N}}$ of seminorms. Then

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|f-g\|_k}{1+\|f-g\|_k},$$

defines a metric on F that induces the same topology. A sequence $\{f_n\} \subseteq F$ is Cauchy with respect to this metric if and only if for every $k \in \mathbb{N}$ the seminorm

$$\|f_m - f_n\|_k \to 0$$

as $m, n \to \infty$.

Next we point out some facts about differentiable caluculus on Fréchet spaces.

Definition 2.1.2. *Let E and F be Fréchet spaces, $U \subseteq E$ an open, and $P : U \to F$ a* continuous *possibly non-linear map. Then P is called* differentiable at $e \in U$ *if the limit*

$$DP(e)h := \lim_{t\to 0} \frac{P(e+th) - P(e)}{t},$$

exists for every $h \in E$. In this case $D_e Ph = DP(e)h$ is the directional derivative of P at e in the direction of h. P is differentiable *if the limit exists for every $e \in U$ and $h \in E$.* △

Note that this is the usual Gâteaux differential. One should emphasize that P is assumed to be continuous in the definition of differentiability. This implies that the notions of Fréchet differential and Gâteaux differential coincide, see for example [Tay37].

Definition 2.1.3. *Let E and F be Fréchet spaces, $U \subseteq E$ an open, and $P : U \to F$ a continuous map. Then P is called* continuously *differentiable, or $P \in C^1(U, F)$, if the derivative is continuous as a map*

$$DP : U \times E \to F.$$

By recursion, it is of class C^k if $DP \in C^{k-1}(U \times E, F)$, and P is called smooth, *or of class C^∞, if it is C^k for every $k \in \mathbb{N}$.* △

Remark 2.1.4. *The directional derivative of $P : U \to F$ is linear in the sense that if P is C^1, and $e \in U$, then the map $DP(e) : E \to F$ is a linear map. For a proof of this statement we refer to lemma 3.2.3 and theorem 3.2.5 of [Ham77]. The requirement that P is* continuously *differential seems to be essential to the proof.* △

Some caution is needed with the above definitions. A map $Q : B \supseteq V \to C$ between Banach spaces is called continuously differentiable in the *Banach sense* if its derivative

$$DQ : V \to \mathcal{B}(B, B')$$

is well-defined and continuous. If Q is C^2 in the Fréchet sense defined above, then

$$\|DQ(g)h - DQ(f)h\| = \left\| \int_0^1 D^2Q(f + t(g-f))[h, g-f]dt \right\| \leq C \|g-f\| \|h\|$$

for all $g \in V$ sufficiently close to f. Hence it is C^1 in the Banach sense. By induction on k, Q is of class C^k in the Banach sense if it is of class C^{k+1} in the Fréchet sense. On the other hand, it is of class C^k in the Fréchet sense if it is of class C^k in the Banach sense. Both notions are, however, not equivalent.

Since the space $\mathcal{F}(E, F)$ of continuous maps between Fréchet spaces is not a Fréchet space in a natural way, this ambiguity isn't as relevant in the category of Fréchet spaces.

Remark 2.1.5. *In this thesis we use, both for maps between Fréchet spaces as between finite dimensional spaces, the notation $D_f Ph = DP(f)h$ to indicate the derivative of P at f in the direction of h. Hence DP is seen as a map*

$$DP : U \times E \to F.$$

The tangent map *of P is then the map*

$$TP : U \times E \to F \times F$$

with $TP = (P, DP)$. In the more intrinsic setting of Fréchet manifolds this translates, for $P : \mathcal{M} \to \mathcal{N}$, to

$$DP : T\mathcal{M} \longrightarrow P^*T\mathcal{N}$$

and

$$TP : T\mathcal{M} \longrightarrow T\mathcal{N}.$$

For a map $P : E \supseteq U \to \mathbb{R}$ we will sometimes write $dP = DP$; although viewing the derivative as a differential $dP : U \to E^*$ is less useful in the Fréchet setting, since the cotangent bundle of a Fréchet manifold doesn't have a natural structure of a Fréchet space.

Eventhough a Fréchet space isn't necessarily a normed space, the Hahn-Banach theorem still holds for Fréchet spaces. In particular, points of F are separated by the continuous linear functionals on F in the sense that if $f \in F$ is non-zero then then there is a continuous linear functional $l : F \to \mathbb{R}$ such that $l(f) = 1$. This can be used to extend some basic results from real finite dimensional analysis to Fréchet spaces. First we introduce the space $C^0([a, b], F)$ of continuous paths in F.

Proposition 2.1.6. *Let $[a, b] \subseteq \mathbb{R}$ be a compact interval, and F a Fréchet space with seminorms $\{\|-\|_i\}_{i \in I}$. Then the space $C^0([a, b], F)$ of continuous paths in F is a Fréchet space with seminorms defined by*

$$\|f\|_i = \sup_{t \in [a,b]} \|f(t)\|_i$$

for all $f \in C^0([a, b], F)$.

Proposition 2.1.7 (Integration). *Let $[a,b] \subseteq \mathbb{R}$ be a compact interval, and F a Fréchet space. Let $C^0([a,b],F)$ denote the space of continuous paths in F. Then there is a unique continuous linear map $\int_a^b dt : C^0([a,b],F) \to F$ such that*

$$l\left(\int_a^b f(t)dt\right) = \int_a^b l(f(t))dt$$

for every continuous linear functional $l : F \to \mathbb{R}$, and every $f \in C^0([a,b],F)$. It satisfies

$$\left\| \int_a^b f(t)dt \right\|_i \leq |b-a| \cdot \|f\|_i$$

for every $f \in C^0([a,b],F)$.

Moreover, if $f \in C^1([a,b],F)$ is continuous differentiable, then

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Proof. It follows directly from Hahn-Banach that there is at most one such map. To prove existence, first note that it is obvious how to define the integral for piecewise linear paths. Suppose that $a = a_0 < a_1 < \ldots < a_n = b$ is a partition of $[a,b]$, and $f \in C^0([a,b],F)$ is a continuous, piecewise linear path in F given by

$$f(t) = \alpha_k t + \beta_k,$$

for $t \in [a_{k-1}, a_k]$, $\alpha_k, \beta_k \in F$, and $1 \leq k \leq n$. Then the integral should be given by

$$\int_a^b f(t)dt := \sum_{k=1}^{n} (a_k - a_{k-1}) \left(\beta_k + \frac{1}{2}(a_{k-1} + a_k)\alpha_k \right).$$

Note that $\int_a^b dt$ is linear on the space of continuous, piecewise linear paths, and

$$\left\| \int_a^b f(t)dt \right\|_i \leq \sum_{k=1}^{n} |a_k - a_{k-1}| \left\| f\left(\frac{1}{2}(a_{k-1} + a_k) \right) \right\|_i \leq |b-a| \cdot \|f\|_i.$$

Since the continuous, piecewise linear paths lie dense in $C^0([a,b],F)$, conclude that $\int_a^b dt$ extends to a continuous linear map on $C^0([a,b],F)$. □

Integration along continuous paths allows one to prove many basic results about the directional derivative of a smooth map $P : E \to F$. For us the most essential ones are listed below.

Lemma 2.1.8. *Let E and F be Fréchet spaces, $U \subseteq E$ an open, and $P : E \supseteq U \longrightarrow F$ a continuous map. Then P is of class C^1 if there is a continuous map*

$$l : U \times U \times E \to F,$$

linear in the last component, such that

$$P(g) - P(f) = l(f,g)(g-f).$$

Proof. The proof is the same as in Banach calculus. If P is C^1, l is given by

$$l(f,g)h = \int_0^1 DP(f + t(g - f))h\, dt,$$

and, conversely, $DP(f)h = l(f, f)h$. $\square$

Corollary 2.1.9. *A continuous map $P : E \oplus F \supseteq U \longrightarrow G$ from the direct sum of two Fréchet spaces is smooth if and only if $P(e, -)$ and $P(-, f)$ are smooth for every $e \in E$ and $f \in F$.*

Finally, Fréchet calculus shares many properties with ordinary calculus in $\mathbb{R}^n$, such as the chain rule, linearity of the directional derivative $D_f P$, that higher order derivatives $D_f^k P$ are symmetric multi-linear, and Taylor's theorem with integral remainder:

$$P(f + h) = P(f) + \sum_{k=1}^{n-1} \frac{1}{k!} D^k P(f)h^k + R_n(f, h),$$

where

$$R_n(f, h) = \frac{1}{(n-1)!} \int_0^1 (1 - t)^{n-1} D^n P(f + th)h^n\, dt.$$

One proves these statements by using that the linear functionals separate the points of a Fréchet space F.

2.2 Graded Fréchet spaces

Let $\|-\|$ and $\|-\|'$ be two seminorms on a vector space F. We say that $\|-\| \leq \|-\|'$ if $\|f\| \leq \|f\|'$ for every $f \in F$. This defines a partial order on the family of seminorms on the vector space F.

Definition 2.2.1 (Graded Fréchet space)**.** *Let F be a Fréchet space. A* grading *on F is a particular choice of an increasing sequence*

$$\|-\|_0 \leq \|-\|_1 \leq \|-\|_2 \leq \cdots$$

of semi-norms which induce the topology on F. A graded Fréchet space *is a Fréchet space together with a fixed choice of grading.* △

In foresight, the archetypal example of a graded Fréchet space will be the space $C^\infty(V)$ of smooth functions $f : V \to \mathbb{R}$ on the closure V of a relatively compact, open subset of $\mathbb{R}^n$. The grading is given by the so-called C^k-norms, that is, the norms

$$\|f\|_k = \sum\nolimits_{|\alpha| \leq k} \sup_{x \in V} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right|.$$

Each of these semi-norms is actually a norm, since it takes the supremum of f on V, and $C^\infty(V)$ is easily seen to be complete and Hausdorff. It is, however, not complete to any of the norms individually; the completions $\overline{(C^\infty(V), \|-\|_k)}$ are just the regular Banach spaces $C^k(V)$.

Every Fréchet space can be seen as a graded Fréchet space, for if $\{\|-\|_n : n \in \mathbb{N}\}$ gives the topology of F, then the semi-norms

$$\|-\|'_n = \sum_{k=0}^{n} \|-\|_k$$

induce the same topology and are ordered increasingly. It is, however, obvious that such a choice of grading is highly non-canonical. The grading will turn out to play an essential role in the statement and proof of the Nash-Moser inverse function theorem; it allows one to define tame maps, which are, in some sense, a nice generalisation of continuous maps between Banach spaces.

Remark 2.2.2. *Any Banach space can be seen as a graded Fréchet space with the constant grading $\|-\| \leq \|-\| \leq \ldots$.*

A graded Fréchet space comes with a sequence of topologies $W_1 \supseteq W_2 \supseteq \ldots$ of increasing refinement.

Definition 2.2.3 (W_k-topologies)**.** *Let F be a graded Fréchet space. Then for every $k \in \mathbb{N}$ the W_k-topology on F is defined as the topology induced by the family of open balls*

$$B^r_k(f) := \{g \in F : \|g - f\|_k < r\}$$

for all $f \in F$ and $r \in \mathbb{R}_{>0}$. For every $l \geq k$ we have that W_l is a refinement of W_k. △

These topologies give additional means to prove the continuity of maps between graded Fréchet spaces. In later chapters we give a more geometrical description of the W_k topologies in the case of the graded Fréchet spaces $\Gamma_M E$ of sections of a vector bundle equipped with the C^k-norms.

2.2.1 Tame maps

Hamilton [Ham82b] introduces the notion of a *tame map* between graded Fréchet spaces. It is a continuous map with a clear upper bound on the increase of index of the semi-norms.

Definition 2.2.4 (Tame linear maps)**.** *Let E and F be graded Fréchet spaces. A linear map $L : E \to F$ is called* tame linear *of base b and degree r, in short r-tame of base b, if it satisfies the following estimates:*
For every $n \geq b$ there is a constant $C_n > 0$ such that

$$\|Le\|_n \leq C_n \|e\|_{n+r}$$

*for all $e \in E$. Let **TameL** denote the category of graded Fréchet spaces with tame linear maps.* $\triangle$

In words, a tame linear map has an upper bound to the increase of index and this bound may fail for only a finite number of semi-norms. Since this has no influence on the convergence of sequences, not even on the topology of E and F, we allow some leeway in the tameness condition by means of the base. Note that, with this definition, the degree refers only to an *upper bound* of the growth of index. We do not assume it to be the smallest of these upper bounds by definition, but this can be done in practice. Note that it also makes sense to speak of a negative degree when the index actually lowers.

We will always assume that finite dimensional manifolds are in the smooth category, and that the spaces of sections consists of the smooth sections. Let $E \to M$ and $F \to M$ be two vector bundles over a finite dimensional manifold. A map $P : \Gamma_M E \to \Gamma_M F$ between the respective spaces of sections is said to be local if $\operatorname{supp}(P\sigma) \subseteq \operatorname{supp}(\sigma)$ for all $\sigma \in \Gamma_M E$. A local operator is by definition a local, linear morphism of sheafs $\Gamma_M E \to \Gamma_M F$. The Peetre theorem [J.P59] states the following.

Theorem 2.2.5 (Peetre)**.** *Every local operator $P : \Gamma_M E \to \Gamma_M F$ is locally a differential operator in the sense that around every point $x \in M$ there is an open subset $U \subseteq M$ such that the restriction of P to U is the composition*

$$i \circ j^k : \Gamma_M E \to J^k(E) \to \Gamma_M F$$

of a linear map i and the k-th jet for some k depending on U.

Remarkably enough, no continuity conditions on P are necessary. In section 2.3.1 on page 23 we describe a graded Fréchet space structure on the spaces $\Gamma_M E$ of sections given that the base manifold M is compact. In this case it follows that any local operator $P : \Gamma_M E \to \Gamma_M F$ is a differential operator of order k for a fixed $k \in \mathbb{N}$. It is in particular a k-tame linear map.

Definition 2.2.6 (Smooth tame maps)**.** *Let E and F be graded Fréchet spaces, $U \subseteq E$ an open subset, and $P : U \to F$ a continuous possibly non-linear map. Then P is tame if there is a covering $\{U_i\}$ of U such that:*

For every U_i there is a base b_i and degree r_i so that for every $n \geq b_i$ there is a $C^i_n > 0$ such that the following estimates are fulfilled:

$$\|Pf\|_n \leq C^i_n(1 + \|f\|_{n+r_i}), \quad \forall f \in U_i.$$

A smooth map $P : U \to G$ is called smooth tame *if it and all its derivatives are tame. Let **TameS** denote the category of graded Fréchet spaces with smooth tame maps.* △

Remark 2.2.7. *For the tameness of the derivatives of P one needs to view $U \times E$ as an open subset of a graded Fréchet space $E \times E$. The grading on the latter space is given by taking the sum $\|e_1\|_n + \|e_2\|_n$ for every index $n \in \mathbb{N}$ and $e_1, e_2 \in E$. In section 2.2.3 on page 19 we approach this more systematically.*

Note that we do not require that P satisfies the tameness estimates globally on U but only on the open sets in an open cover of U. In later chapters we work with the notion of tame maps between tame Fréchet manifolds, manifolds which locally look like graded Fréchet spaces and whose transition maps are also tame. Then the notion of tameness for continuous non-linear maps directly globalizes to these tame Fréchet manifolds.

Remark 2.2.8. *We assume that P is continuous as part of the definition of tameness. For non-linear maps continuity doesn't necessarily follow from the tameness estimates.*

Note that we allow the degree, base and the constants C^i_n to vary from open to open. In most examples we are able to obtain a bound on the degrees r_i and bases b_i. In such a case it makes sense to speak of an r-tame (non-linear) map of base b. Where possible, we will keep track of the degree and base of a map. These computations are often not very challenging once one has established that the map in question *is* tame. One can make, for example, the following observation that helps tremendously in computing degrees and bases.

Lemma 2.2.9. *Both **TameL** and **TameS** are categories. More specifically, the composition $P \circ Q$ of an r-tame map of base b and an $\tilde{r}$-tame map of base $\tilde{b}$ respectively, with $r, \tilde{r} \geq 0$, is $(r + \tilde{r})$-tame of base $\max\left\{b, \tilde{b} - r\right\}$.*

Proof. We will only look at the smooth tame maps. For $k \geq \max\left\{b, \tilde{b} - r\right\}$ one can make estimates of the form

$$\begin{aligned}\|P(Q(f))\|_k &\leq C_1(1 + \|Q(f)\|_{k+r}) \\ &\leq C_1(1 + C_2(1 + \|f\|_{k+r+\tilde{r}})) \\ &\leq C_3(1 + \|f\|_{k+r+\tilde{r}}).\end{aligned}$$ □

This leads to the usual definitions of isomorphisms in **TameL** and **TameS**.

Definition 2.2.10. *A* tame linear isomorphism, *or* tame isomorphism *is a tame linear map with a tame inverse.*

Likewise, a smooth tame isomorphism, *or* tame diffeomorphism *is a smooth tame map with a smooth tame inverse.* △

Clearly, by the above corollary, such tame isomorphisms do not always preserve the notion of degree and base. If one wishes to preserve these notions nonetheless, one should work with the notion of a 0-tame isomorphism as defined below.

Definition 2.2.11 (0-tame maps)**.** *A* linear 0-tame isomorphism *is a tame linear isomorphism of degree* 0 *with an inverse of degree* 0*. The graded Fréchet spaces with the* 0*-tame linear maps form a category denoted by* $\boldsymbol{TameL}_0$*.*

A smooth 0-tame map *is a smooth map which and whose derivatives are all* 0*-tame. Consequently, a* 0-tame diffeomorphism *is a smooth* 0*-tame map with a smooth* 0*-tame inverse. The graded Fréchet spaces with smooth* 0*-tame maps form a category denoted by* $\boldsymbol{TameS}_0$*.* △

Analogously, one can give the definitions of r-tame linear isomorphisms and smooth r-tame maps. However, they do not form a category, as per lemma 2.2.9 on the facing page. Moreover note that a smooth r-tame map is not the same as smooth tame map of degree r. The former requires that all derivatives are at most of degree r, while the latter only restricts the degree of the map itself.

Remark 2.2.12. *It often seems interesting to restrict our attention to the categories* $\boldsymbol{TameL}_0$ *and* $\boldsymbol{TameS}_0$*. In the next chapter we encounter several examples of tame Fréchet manifolds. These are manifolds whose charts take values in graded Fréchet spaces, and whose transition maps are smooth tame maps. In nearly all of these examples the transition maps turn out to be* 0*-tame diffeomorphisms. This allows us to speak of the degree of a tame map between these manifolds.*

Since there are now two different definitions of tameness for linear maps, namely in **TameL** and **TameS**, we should better argue that this gives no ambiguity. Moreover, if F is a graded Fréchet space, recall that W_k denotes the topology on F induced by the k-th seminorm. We will also give a description of the tame linear maps in terms of continuity conditions with relation to the family of topologies $\{W_k\}_{k\in\mathbb{N}}$.

Proposition 2.2.13. *A map* $L : E \to F$ *between graded Fréchet spaces is tame linear if and only if it is linear and tame. Moreover, a linear map* $L : E \to F$ *is tame linear of degree* r *and base* b *if and only if it is continuous as a map*

$$L : (E, W_{k+r}) \longrightarrow (F, W_k)$$

for every $k \geq b$*.*

Proof. Suppose L is linear and tame. Then it satisfies tame estimates, $\forall n \geq b$,

$$\|Lf\|_n \leq C(1 + \|f\|_{n+r}), \quad \forall f \in U$$

for some neighborhood $U \subseteq E$ of the origin. We may assume $U = \left\{f \in E : \|f\|_{b+r} \leq \varepsilon\right\}$ for some $\varepsilon > 0$ and a sufficiently large $b \in \mathbb{N}$. Let $g \in E$ and $\delta > 0$ be arbitrary, and define $f = \varepsilon g/(\|g\|_{b+r} + \delta)$. Then $\|f\|_{b+r} \leq \varepsilon$, so the tame estimate holds for such an $f \in E$. Use the linearity of L and multiply on both sides with $(\|g\|_{b+r} + \delta)/\varepsilon$ to get

$$\|Lg\|_n \leq C\left(\frac{\|g\|_{b+r} + \delta}{\varepsilon} + \|g\|_{n+r}\right) \leq C\,\|g\|_{n+r}\,.$$

Now use that $\|g\|_{b+r} \leq \|g\|_{n+r}$ and take the limit $\delta \to 0$. The converse of the first statement is trivial.

For the second statement, consider the set $U = \{f \in E : \|Lf\|_k < 1\}$. By the assumption that $L : (E, W_{k+r}) \longrightarrow (F, W_k)$ is continuous, there is a $\varepsilon > 0$ such that U contains the open $\left\{f \in E : \|f\|_{k+r} < \varepsilon\right\}$. Now if $g \in E$ and $\delta > 0$ are arbitrary, define $f = \varepsilon g/(\|g\|_{k+r} + \delta)$. Then $\|f\|_{k+r} < \delta$, and we have

$$1 > \left\| L\left(\frac{\varepsilon g}{\|g\|_{k+r} + \delta}\right)\right\|_k = \frac{\varepsilon}{\|g\|_{k+r} + \delta} \|Lg\|_k .$$

Hence $\|Lg\|_k \leq \frac{1}{\varepsilon}(\|g\|_{k+r} + \delta)$, and we may take $\delta \to 0$ to obtain the required estimate. The converse statement is again trivial. $\square$

Remark 2.2.14. *Later on, in particular when we look at the proof of the Nash-Moser theorem, we often encounter long series of estimates in which there occur repeatedly increased constants. We take the habit from Hamilton to denote these constants with a 'C' throughout, giving no reference to the fact that these are consecutive and different estimates. If indices occur at all, they indicate the parameters on which the C's depend.*

2.2.2 Tame linear subspaces and quotients

Let F be a Fréchet space. If $j : E \to F$ is continuous linear injection of Fréchet spaces, then by completeness of E, the image is closed. Moreover, by the open mapping theorem for Fréchet spaces, j is a topological linear isomorphism from E onto a closed subspace of F. A (topological) linear subspace of F is hence naturally defined as a Fréchet space E together with a continuous linear inclusion $E \hookrightarrow F$. In the category **TameL** of tame linear maps, this notion needs to be refined slightly more.

Definition 2.2.15 (Tame linear susbspace)**.** *Let F be a graded Fréchet space. An r-tame linear subspace of F, denoted $E \leq_r F$, is a graded Fréchet space E together with an r-tame linear inclusion $E \hookrightarrow F$ of base* 0.

Every closed linear subspace E of F can be seen as a 0-tame linear subspace of F by restricting the seminorms to E. Note that an r-tame linear subspace $E \leq_r F$ isn't necessarily an 0-tame linear subspace, or an s-tame linear subspace with $s < r$. One can, however always define a new grading on E by

$$\|e\|'_k := \|i(e)\|_k \,,$$

but these seminorms might behave differently. In particular, redefining a grading like this defines a graded space that isn't 0-tame isomorphic to E.

Definition/Proposition 2.2.16 (Quotients)**.** *Let F be a graded space, and $E \leq_r F$ an r-tame linear subspace. Then the quotient E/F is a graded Fréchet space with seminorms*

$$\|f + E\|_k := \inf \left\{ \|f + e\|_k : \quad e \in E \right\}.$$

The projection $\pi : F \to F/E$ is an 0-tame linear map of base 0*. It has the property that, if $L : E \to G$ is an r-tame map of base b with $E \subseteq \ker L$, then there is a unique r-tame linear map $\tilde{L} : F/E \to G$ of base b such that*

$$\begin{tikzcd} F \arrow[rr, "L"] \arrow[dr, "\pi"'] & & G \\ & E/F. \arrow[ur, "\tilde{L}"'] & \end{tikzcd}$$

commutes.

Proof. It is clear that the seminorms defined for F/E are in fact a grading of seminorms. That π is 0-tame of base 0 follows from the fact that $0 \in E$. Next we will show that F/E is complete.

Let $\{f_n + E\}_{n\in\mathbb{N}}$ be a Cauchy sequence in F/E. By taking an appropriate subsequence we may assume that

$$\|f_n - f_{n-1} + E\|_n < 2^{-n},$$

so that in particular

$$\|f_n - f_{n-1} + E\|_k \leq \|f_n - f_{n-1} + E\|_n < 2^{-n}$$

for every $k \leq n$. We define a sequence $\{e_n\} \subseteq E$ recursively. Let $e_0 = 0$, and choose for every $n \geq 1$ an element $e_n \in E$ such that

$$\|(f_n + e_n) - (f_{n-1} + e_{n-1})\|_n \leq \|f_n - f_{n-1} + E\|_n + 2^{-n} \leq 2 \cdot 2^{-n}.$$

Then the sequence $\{f_n + e_n\}$ is Cauchy in F. For suppose that $k \in \mathbb{N}$, $l \in \mathbb{N}$, and $n \geq k$. Then we compute

$$\begin{aligned}
\|(f_{n+l} + e_{n+l}) - (f_n + e_n)\|_k &\leq \sum_{i=1}^{l} \|(f_{n+i} + e_{n+i}) - (f_{n+i-1} + e_{n+i-1})\|_n \\
&\leq \sum_{i=1}^{l} 2 \cdot 2^{-n-l+1} \\
&\leq 2^{1-n} \sum_{l=1}^{\infty} 2^{-l} \leq 2^{1-n},
\end{aligned}$$

which tends to 0 as $n \to \infty$. Since F is complete, there exists an element $f \in F$ such that $\{f_n + e_n\}$ converges to f. By the continuity of π, the sequence $\{f_n + E\}$ converges to $f + E$ in F/E. Now note that if a subsequence of a Cauchy sequence converges to f, then so does the original sequence.

Finally, we should check that $\tilde{L}$ is r-tame of base b. This follows directly from

$$\left\|\tilde{L}(f + E)\right\|_k = \|L(f + e)\|_k \leq C \|f + e\|_k$$

for every $e \in E$, since E lies in the kernel of L. $\square$

2.2.3 The direct sum

We define the direct sum of graded Fréchet spaces.

Definition/Proposition 2.2.17 (Direct sum)**.** *Let E_1 and E_2 be two graded Fréchet spaces with gradings $\left\{\|-\|_k^1\right\}$ and $\left\{\|-\|_k^2\right\}$ respectively. We define the direct sum of E_1 and E_2 as the coproduct $E_1 \oplus E_2$ in $\boldsymbol{TameL}_0$.*

Hence it is a graded Fréchet space $E_1 \oplus E_2$ together with 0-tame linear inclusions

$$
\begin{aligned}
&i_1 : E_1 \to E_1 \oplus E_2\\
&i_2 : E_2 \to E_2 \oplus E_2
\end{aligned}
$$

such that if $a : E_1 \to F$ and $b : E_2 \to F$ are 0-tame linear maps, then there is a unique 0-tame linear map

$$a + b : E_1 \oplus E_2 \to F$$

such that $(a + b) \circ i_1 = a$ and $(a + b) \circ i_2 = b$.

The direct sum is given by the vector space $E_1 \oplus E_2 := E_1 \times E_2$ equipped with the grading

$$\|e_1 \oplus e_2\|_k := \|e_1\|_k^1 + \|e_2\|_k^2 .$$

The graded Fréchet space $E_1 \oplus E_2$ is actually the biproduct in $\boldsymbol{TameL}_0$. Moreover, it satisfies nice estimates in $\boldsymbol{TameL}$, as specified in the proof below.

Proof. The uniqueness of $E_1 \oplus E_2$ up to 0-tame linear isomorphisms follows directly from the usual categorical argument.

It is directly clear that the $\|-\|_k$ define a grading on $E_1 \oplus E_2$, and that $E_1 \oplus E_2$ is Hausdorff and complete.

Let $a : E_1 \to F$ and $b : E_2 \to F$ be an r-tame and s-tame linear map of base b and c respectively. Then define the map also the categorical product, hence it is the biproduct in the category of graded Fréchet with tame maps.

$$a + b : E_1 \oplus E_2 \to F, \quad (a + b)(e_1 \oplus e_2) = a(e_1) + b(e_2).$$

Then $a + b$ is $\max(r, s)$-tame of base $\max(b, c)$, for we can make estimates of the form

$$
\begin{aligned}
\|(a + b)(e_1 \oplus e_2)\|_k &= \|a(e_1) + b(e_2)\|_k\\
&\leq \|a(e_1)\|_k + \|b(e_2)\|\\
&\leq C(\|e_1\|_{k+r} + \|e_2\|_{k+s}\\
&\leq C(\|e_1 \oplus e_2\|_{k+\max(r,s)}
\end{aligned}
$$

for all $k \geq \max(b, c)$.

On the other hand, let $a : E \to E_1$ and $b : E \to E_2$ be an r-tame and s-tame linear map of base b and c respectively. Then define the map

$$(a, b) : E \to E_1 \oplus E_2, \quad (a, b)e = a(e) \oplus b(e).$$

Then $a \oplus b$ is $\max(r,s)$-tame of base $\max(b,c)$, for we can make estimates of the form

$$\begin{aligned}\|(a,b)e\|_k &= \|a(e) \oplus b(e)\|_k \\ &= \|a(e)\|_k + \|b(e)\|_k \\ &\leq C(\|e\|_{k+r} + \|e\|_{k+s}) \\ &\leq C\,\|e\|_{k+\max(r,s)}\end{aligned}$$

for all $k \geq \max(b,c)$. $\square$

Remark 2.2.18. *Equivalently, we could have equipped $E_1 \oplus E_2$ with the seminorms*

$$\|(e_1,e_2)\|_k = \max(\|e_1\|_k^1, \|e_2\|_k^2),$$

since $\max(r,s) \leq r+s \leq 2\max(r,s)$ *for any $r,s \in \mathbb{R}_{>0}$.*

Remark 2.2.19. *Note that a tamely equivalent direct sum $E_1 \oplus' E_2$ in **TameL** might fail to be 0-tamely equivalent to the direct sum defined above. Therefore the direct sum we work with is always the 0-tame direct sum defined here.*

Let $E_{1,2}$ and $F_{1,2}$ be graded Fréchet spaces, and let $a : E_1 \to F_1$ and $b : E_2 \to F_2$ be an r-tame and s-tame linear map of base b and c respectively. Then by the previous lemma the map

$$a \oplus b : E_1 \oplus E_2 \to F_1 \oplus F_2, \quad (a \oplus b)(e_1 \oplus e_2) = a(e_1) \oplus b(e_2)$$

is $\max(r,s)$-tame linear of base $\max(b,c)$. Conversely, a and b are r-tame linear of base b if $a \oplus b$ is.

The interchange map is worth mentioning at this point. The graded Fréchet spaces E_1 and E_2 give rise to four 0-tame linear inclusions, namely two into $E_1 \oplus E_2$ and two into $E_2 \oplus E_1$. Since the tame direct sum is also the coproduct of 0-tame maps, it follows that there is a 0-tame linear isomorphism

$$\tau : E_1 \oplus E_2 \longrightarrow E_2 \oplus E_1.$$

Next we will discuss the tame direct summands of graded Fréchet spaces. Note that a tame linear subspace of a graded Fréchet space F isn't necessarily a tame direct summand of F.

Definition/Proposition 2.2.20. *Let E and F be two graded Fréchet spaces. E is a tame direct summand of F if there are tame linear maps*

$$E \overset{i}{\hookrightarrow} F \overset{p}{\to} E$$

of degre r and degree s, and of base b and base s, respectively, such that $p \circ i = id$. One can define r-tame direct summands by stipulating that $r = s$.

In this case, there is a 0-tame linear subspace $E' \hookrightarrow F$ such that F is $\max(r,s)$*-tame isomorphic to $E \oplus E'$.*

Proof. Let $E' := \ker(p)$ be equipped with the grading induced by F. Since p is continuous, E' is closed subspace of F, hence it is a graded Fréchet space. The inclusion $j : E' \hookrightarrow F$ is by definition 0-tame linear of base 0, and the projection $\mathrm{id} - p : F \to E'$ is s-tame linear of base c. We conclude that the map

$$i + j : E \oplus E' \to F$$

is r-tame linear of base b, and its inverse

$$(p, \mathrm{id} - p) : F \to E \oplus E'$$

is s-tame linear of base c. □

Sometimes one might wish to measure tameness separately in either component, for example to simplify estimates. The definition below can be extended to arbitrary finite direct sums of graded Fréchet spaces.

Definition 2.2.21. *Let U and V be open subsets in graded Fréchet spaces, F a graded Fréchet space, and $P : U \times V \to F$ a continous map. Then P is (r,s)-tame of base b if: For every $n \geq b$ there is a constant $C_n > 0$ such that*

$$\|P(f,g)\|_n \leq C(1 + \|f\|_{n+r} + \|g\|_{n+s})$$

for all $(f,g) \in U \times V$. △

Remark 2.2.22. *This definition appears not to be consistent with definition 2.2.6 on page 13, where an open cover enters. One can give a definition of tameness of degree r and base b like the definition above, without introducing a cover, to give a definition of 'global' tameness. Then one wants to use this definition locally for tame Fréchet manifolds. This gives back an extended definition of tameness for open subsets of graded Fréchet spaces, as in definition 2.2.6. With the latter definition it makes less sense to speak of degree and base globally. Even when one can find an open cover which provides a bound on the degrees and bases corresponding to open subsets of the cover, the constants C^i in the tameness estimates might still depend on these open subsets. We mainly encounter definition 2.2.21 in the proof of the Nash-Moser theorem. Here we are allowed to restrict to just one open subset in the cover such that we can utilize the degree of a tame map. On the other hand, the notion of tameness should be such that it applies to the general setting of continuous maps between tame Fréchet manifolds.*

The following proves to be useful in such situations. Its proof is completely analogous to lemma 2.2.13 on page 15, see [Ham82b] if more details are necessary.

Lemma 2.2.23. *Let U be an open subset in a graded Fréchet space, E and F graded Fréchet spaces, and*

$$P : U \times E \longrightarrow F$$

a continuous map that is linear in the second component. If P is (r,s)-tame of base b, then for every $n \geq b$ there is a constant $C_n > 0$ such that

$$\|P(f)g\|_n \leq C_n(\|g\|_{n+s} + \|f\|_{n+r}\|g\|_{b+s}),$$

for all $f \in U$, and $g \in E$.

The above lemma obviously extends to arbitrary multi-linear tame maps. We will write down the bilinear case explicitely, since we use it repeatedly in the proof of the Nash-Moser theorem.

Lemma 2.2.24. *Let U be an open subset in a graded Fréchet space, E_1, E_2, and F graded Fréchet spaces, and*

$$P : U \times E_1 \times E_2 \to F$$

a continuous map that is bilinear in the last two components. If P is (r, s_1, s_2)-tame of base b, then for every $n \geq b$ there is a constant $C_n > 0$ such that

$$\|P(f)\{g,h\}\|_n \leq C_n \left(\|g\|_{n+s_1} \|h\|_{b+s_2} + \|h\|_{n+s_2} \|g\|_{b+s_1} + \|f\|_{n+r} \|g\|_{b+s_1} \|h\|_{b+s_2} \right)$$

for all $(f, g, h) \in U \times E_1 \times E_2$.

2.3 Examples of graded Fréchet spaces

This section introduces the main examples of graded Fréchet spaces. Most results proven here are used in later chapters, since these graded Fréchet spaces are the local models of most tame Fréchet manifolds. Let $E \to M$ be a vector bundle over a compact base. The prototype graded Fréchet space is the space of smooth sections,

$$\Gamma_M E = \{\sigma : M \to E : p \circ \sigma = \mathrm{id}\}.$$

The vector bundles over M form a full subcategory of the category $\mathbf{Bund}_M$ of bundles over M, that is, the surjective submersions onto M. The morphisms $\mathbf{Bund}_M(E, E')$ are the (non-linear) smooth bundle maps

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "q"] \\ & M. & \end{tikzcd}$$

They give rise to smooth 0-tame maps by composition on the left, that is, they give a smooth tame map

$$\Gamma_M f = f_* : \Gamma_M E \longrightarrow \Gamma_M E',$$

hence we may view Γ_M as a functor $\mathbf{Vect}_M \longrightarrow \mathbf{Graded}$, from the category of vector bundles over M with bundle maps to the graded Fréchet spaces with smooth tame maps. Moreover, if f is a *vector* bundle map, in the sense that it is linear on each of the fibers, these maps f_* are 0-tame linear.

2.3.1 Sections of a vector bundle

The following is the most important class of examples. Let M be a *compact* smooth manifold and $E \xrightarrow{p} M$ a vector bundle over M. The space $\Gamma_M E$ of smooth sections is a graded Fréchet space in the following way. It will be the generic local model of the spaces $C^\infty(M, N)$ in the sense that each connected component has a $\Gamma_M E$ as local model; which are not necessarily isomorphic on different connected components.

Suppose that (U, φ) is a chart on M that admits a local trivialization

$$\psi : E_U \to U \times \mathbb{R}^n$$

of the vector bundle E. Let V be a relatively compact, open subset such that $\bar{V} \subseteq U$, and $\varphi(U)$ is an open neighborhood of the origin. To simplify notation we will identify V with its image $\varphi(V)$. Every section $\sigma \in \Gamma_M E$ can locally be represented by a function

$$(\mathrm{id}, \tilde{\sigma}) = \psi \circ \sigma \circ \varphi^{-1}|_{\overline{V}} : \overline{V} \to \overline{V} \times \mathbb{R}^n.$$

Now one may define seminorms on $\Gamma_M E$ by choosing a finite cover of such charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and corresponding local trivializations ψ_α such that the relatively compact subsets V_α

cover all of M. The definition of the semi-norms becomes

$$\|\sigma\|_k = \sum_{j=0}^{k} \max_{\alpha\in A} \sup_{x\in\overline{V_\alpha}} \left\|D^j\tilde{\sigma}(x)\right\|,$$

where the norms on the right-hand-side are the norms induced on the space of j-linear maps induced by the Euclidean norms.

Alternatively one can use the partial derivatives of the components of $\tilde{\sigma}$. For multi-indexes $\gamma, \delta \in \mathbb{N}^n$ we use the common conventions

$$\begin{aligned} \frac{\partial^\gamma}{\partial x^\gamma} = \partial^\gamma &= \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}; \\ |\gamma| &= \sum_{i=1}^{n} \gamma_i; \\ (\gamma+\delta)_i &= \gamma_i + \delta_i. \end{aligned}$$

Note that the local representative $\tilde{\sigma}$ is actually defined on the closure of V, which is compact, so that the supremum is always finite. The seminorms are now defined by

$$\|\sigma\|_k = \sum_{|\gamma|\le k} \sum_{1\le i\le n} \max_\alpha \sup_{x\in\overline{V_\alpha}} |\partial^\gamma \tilde{\sigma}_i(x)|.$$

Here we have written $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$. In words, we take the supremum over all partial derivatives up to order k for all components of $\tilde{\sigma}$.

Alternatively, choose Riemannian metrics on E and TM respectively and connections ∇ and $\bar{\nabla}$ respectively. Let

$$\Sigma^k(M,E) := \Gamma(M, S^k(T^*M)\otimes E)$$

denote the space of symmetric k-forms on M with values in V. Then define a $\mathbb{R}$-linear differential operator

$$D : \Sigma^k(M,E) \to \Sigma^{k+1}(M,E)$$

in the following way:

$$D(\omega)(M_0,\ldots,M_k) = \sum\nolimits_i \nabla_{M_i}\omega(\ldots,\hat{M}_i,\ldots) - \sum\nolimits_{i<j} \omega([M_i,M_j]_+,\ldots,\hat{M}_i,\ldots,\hat{M}_j,\ldots),$$

where $[M,N]_+ = \bar{\nabla}_M N + \bar{\nabla}_N M$ and the circumflex indicates omission of the indicated vector field. Locally we obtain an expression of the form

$$\begin{aligned} D^k\sigma = &\sum_{i,I} \frac{\partial^k \sigma_i}{\partial x_{i_1}\ldots\partial x_{i_k}} dx_{i_1}\otimes\ldots\otimes dx_{i_k}\otimes v_i \\ &+ \sum_{l<k,I,J} A_{i,j}^{I,J} \frac{\partial^l \sigma_j}{\partial x_{i_1}\ldots\partial x_{i_l}} dx_{i_1}\otimes\ldots\otimes dx_{i_k}\otimes v_i, \end{aligned}$$

where the I run through $\mathbf{n}^k = \{1, \dots, n\}^k$ and the J through $\mathbf{n}^l$. The $A^{I,J}_{i,j}$ are smooth functions further left unspecified, as the local formula really isn't relevant to the construction. Now define semi-norms on $\Gamma_M E$ by

$$\|\sigma\|_k := \sum_{j=0}^{k} \sup_{x \in M} \left| D^j \sigma(x) \right|,$$

where the absolute values indicate the norms on $(S^k(T^*M) \otimes E)_x$ induced by the metrics.

Proposition 2.3.1. *Let $E \to M$ be a vector bundle over a compact base. The space of sections $\Gamma_M E$ is graded Fréchet space with any of the above equivalent gradings. Moreover, these are independent of the choices made in the sense that different choices give 0-tame equivalent graded Fréchet spaces.*

Proof. The Hausdorffness is obvious, since the supremum norm $\|-\|_0$ is already a norm. As for completeness, take a Cauchy sequence $\{\sigma_n\}$. Locally on a ball V, it converges to a smooth map $\sigma : V \to \mathbb{R}^n$, and this convergence is uniform for all partial derivatives. For this we just need the usual argument that the point-wise limit of a uniform converging sequence of continuous maps is continuous; it is, after all, sufficent to check this locally. These maps coincide on the intersections, hence collate to a smooth section $\sigma \in \Gamma_M E$ to which the sequence converges.

For the second part of the lemma it is sufficient to check that for any two charts (U, φ) and (U', φ'), and respective local trivializations ψ and ψ', there is an estimate

$$\max_{|\alpha| \le k} \max_i \sup\nolimits_{x \in \overline{V \cap V'}} \left| \partial^\alpha \psi_i \sigma \varphi^{-1}(x) \right| \le C \max_{|\beta| \le k} \max_j \sup\nolimits_{y \in \overline{V \cap V'}} \left| \partial^\beta \psi'_j \sigma \varphi'^{-1}(y) \right|$$

for some constant $C = C(U, U') > 0$. For if $\{(U, \varphi, \psi)\}$ and $\{(U', \varphi', \psi')\}$ are two choices of coverings, then one can take the maximum of the constants $C(U, U')$ over all U and U', since these are only finite in number.

Fix some $x \in \varphi(U \cap U')$ and let $y = \varphi' \varphi^{-1}(x) \in \varphi'(U \cap U')$ be the corresponding coordinate in the other chart. Let $dx_i = d_x x_i$ denote the unit vector in i-th direction at x. Note that

$$T_x(\varphi' \varphi^{-1}) dx_i = \sum_j c^i_j(x) dy_j$$

for some smooth functions c^i_j. Moreover, introduce the notation

$$dx^\alpha = dx_1^{\alpha_1} \dots dx_m^{\alpha_m}$$

so that the partial derivative can be written as

$$\partial^\alpha \tilde{\sigma}_i(x) = (T_x^{|\alpha|} \tilde{\sigma}_i) dx^\alpha.$$

Observe the trivial identities $\psi_i = \psi_i \psi'^{-1} \psi'$ and likewise $\varphi^{-1} = \varphi'^{-1} \varphi' \varphi^{-1}$. We now have the estimates

$$\begin{aligned}
|\partial^\alpha \tilde\sigma_i(x)| &= \left|T_y^{|\alpha|}(\psi_i\psi'^{-1}\psi'\sigma\varphi'^{-1})T_x^{|\alpha|}(\varphi'\varphi^{-1})dx^\alpha\right| \\
&\le C(x)\max_{|\beta|=|\alpha|}\left|T_y^{|\alpha|}(\psi_i\psi'^{-1}\psi'\sigma\varphi'^{-1})dy^\beta\right| \\
&= C(x)\max_{|\beta|=|\alpha|}\left|T^{|\alpha|}_{\psi'\sigma\varphi'^{-1}(y)}(\psi_i\psi'^{-1})T_y^{|\alpha|}(\psi'\sigma\varphi'^{-1})dy^\beta\right| \\
&\le C_2(x)\max_{|\beta|=|\alpha|}\max_j\left|T_y^{|\alpha|}(\psi'_j\sigma\varphi'^{-1})dy^\beta\right| \\
&= C(x)\max_{|\beta|=|\alpha|}\max_j\left|\partial^\beta\tilde\sigma'(y)\right|.
\end{aligned}$$

Here the constants $C(x)$ depend continuously on $x \in \varphi(U \cap U')$. Hence these constants have a bound $C(x) \le C$ on the compact set $V \cap \varphi(U')$. $\square$

Remark 2.3.2. *The construction of the norms on $\Gamma_M E$ works equally well for compact manifolds M with boundary. As it turns out, it is sometimes useful to work in such a setting. Luckily, there are no significant complications in the proofs if M has boundary.*

All of the above fails if M is not compact. Even for the trivial line bundle, where we have $\Gamma_M E = C^\infty(M)$, there are many functions which are not bounded by the 0-th seminorm. The construction above fails at chosing the cover $\{V_\alpha\}_{\alpha\in A}$. One can cover M with relatively compact open sets V_α. One can find a finite cover $\{U_\alpha\}$ for any manifold M; this can be proven using dimension theory. But, clearly, one cannot find both at once if M is not compact.

A common solution is to define seminorms $\|-\|_{K,k}$ for every compact subset $K \subseteq M$ in the same way as above, but by only taking the supremum over all $x \in K$. This defines a Fréchet topology; it is already defined by a choosing a countable exhaustion $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq M$ of M in the sense that all the K_i are compact, $K_i \subseteq K_{i+1}^\circ$, and $M = \cup_i K_i$. This actually allows one to define a *graded* Fréchet space, by taking $\|-\|_k := \|-\|_{K_k,k}$, but this cannot be done in a cannonical way. The graded Fréchet spaces defined by choosing different compact exhaustions are not tamely isomorphic. For define a new compact exhaustion $K'_i = K_{2i}$, and define the corresponding grading $\|-\|'_k = \|-\|_{K'_k,k}$. Then one can at best obtain estimates of the form

$$\|f\|'_k \le C\,\|f\|_{2k}.$$

2.3.2 Jet bundles and the Whitney C^∞ topology

Let $E \to M$ be a vector bundle over a compact base. We will spend some time giving a alternative description of the topology on $\Gamma_M E$. This will aid us with identifying open subsets, and proving continuity of maps. There is a natural topology on $\Gamma_M E$ known as the Whitney C^∞-topology, see for example [Mat69]. We will give its definition for a vector bundle $E \to M$ with possibly non-compact base, and show it coincides with the topology induced by the C^k-norms whenever M *is* compact.

We begin by introducing jet bundles. For now assume that M and N are possibly non-compact manifolds. For every $k \in \mathbb{N}$ and $x \in M$ two smooth maps $f : U_1 \to N$ and $g : U_2 \to N$, defined on open subsets $x \in U_i \subseteq M$, are said to be *k-tangential* if $y = f(x) = g(x)$, and there exist charts (U, φ) around x and (V, ψ) around y such that

$$D^j(\psi \circ f \circ \varphi^{-1})(0) = D^j(\psi \circ f \circ \varphi^{-1})(0)$$

for all $1 \leq j \leq k$. A *k-jet* of M into N with source x and target y is an equivalence class of smooth maps $U \to N$ with $x \in U$. In particular, given a smooth map $f : M \to N$, the k-jet of f at x is denoted by $j^k(f)(x)$. Let $J^k(M,N)_{x,y}$ denote the set of all k-jets with source x and target y, $J^k(M,N)_x = \cup_{y \in N} J^k(M,N)_{x,y}$ the k-jets with source x, and $J^k(M,N) = \cup_{x \in M} J^k(M,N)_x$ the set of all k-jets of M into N. There is an obvious map

$$\pi : J^k(M,N) \to M \times N,$$

projecting onto the source and target of a k-jet. We will show that this makes $J^k(M,N)$ a fiber bundle over $M \times N$. Moreover, the obvious maps $\pi_M : J^k(M,N) \to M$ and $\pi_N : J^k(M,N) \to N$, and for $l \leq k$ the map

$$\pi^k_l : J^k(M,N) \longrightarrow J^l(M,N)$$

that forgets the last $(k-l)$-derivatives are fiber bundles. In particular note that $J^0(M,N) = M \times N$, so that $\pi = \pi^k_0$.

Let U be an open subset in $\mathbb{R}^m$ and V an open subset in $\mathbb{R}^n$. One easily makes the identification

$$J^k(U,V) = U \times V \times S^1(\mathbb{R}^m)^* \otimes \mathbb{R}^n \times \ldots \times S^k(\mathbb{R}^m)^* \otimes \mathbb{R}^n,$$

where the $S^j(\mathbb{R}^m)^* \otimes \mathbb{R}^n$ are the spaces of symmetric multi-linear maps, by choosing a representative f of $z \in J^k(U,V)_x$ and evaluating the first k derivatives in the point x,

$$(x, f(x), D^1 f(x), \ldots, D^k f(x)).$$

This is independent of the representative, and defines a bijection. This identifies $J^k(U,V)$ as a finite dimensional manifold.

Suppose that M' and N' is a second pair of manifolds, with $x' \in M'$ and $y' \in N'$. Let φ be a diffeomorphism from an open neighborhood of x' onto an open neighborhood of x, and, likewise, ψ a diffeomorphism from an open subset around y onto an open subset around y'. The former map induces a bijection

$$\varphi^* : J^k(M,N)_{x,y} \longrightarrow J^k(M',N)_{x',y}$$

by $\varphi^*(j^k(f)(x)) = j^k(f \circ \varphi)(x)$, and the latter a bijection

$$\psi_* : J^k(M,N)_{x,y} \longrightarrow J^k(M,N')_{x,y'}$$

by $\psi_*(j^k(f)(x)) = j^k(\psi \circ f)(x)$. Both maps are obviously independent of the chosen representatives, hence well-defined. Now if (U,φ) is a chart around x and (V,ψ) is a chart around y, then the composition

$$\psi_*(\varphi^{-1})^* : J^k(M,N)_{U,V} \longrightarrow J^k(\varphi(U),\psi(V))$$

defines a chart for $J^k(M,N)$. It is easy to check that these charts coincide and all the maps mentioned above are fiber bundles.

Moreover, for any smooth map $f : N \to N'$, the map $f_* : J^k(M,N) \to J^k(M,N')$ defined by composition on the left is a smooth fiber bundle map. In particular we can identify $J^1(\mathbb{R},N)_0$ with the tangent space TN and $f_* : TN \to TN'$ is just the tangent map of f.

We are now ready to define the Whitney C^∞ topology of $C^\infty(M,N)$.

Definition 2.3.3 (Whitney C^∞ topology)**.** *Let M and N be two manifolds. For every $k \in \mathbb{N}$ and open subset U of $J^k(M,N)$, define*

$$M(U) := \left\{ f \in C^\infty(M,N) : j^k(f)(M) \subseteq U \right\}.$$

Then $M(U) \cap M(V) = M(U \cap V)$, so the $M(U)$ form a basis of topology. This corresponding topology is the Whitney C^k topology *on $C^\infty(M,N)$. Whenever no confusion arises, both the topology and this basis of topology are denoted by W_k.*

Since $W_k \subseteq W_l$ for every pair $k \leq l$, the union $W_\infty = \cup_k W_k$ is also the basis of a topology. The corresponding topology is the Whitney C^∞ topology *on $C^\infty(M,N)$.* △

For every $k \in \mathbb{N}$, the Whitney C^k topology can actually be defined on the space $C^k(M,N)$ of k-differentiable mappings. Of course, one needs to adjust the definition of the jet bundles accordingly to account for all the C^k-maps. In particular, the topology W_0 on $C^0(M,N)$ contains the compact-open topology: for every compact set $K \subseteq M$ and and open set $U \subseteq N$, the sets

$$M(K,U) = \left\{ f \in C^0(M,N) : f(K) \subseteq U \right\}$$

form a basis of the compact-open topology on $C^0(M,N)$. Each of these sets $M(K,U)$ is open with respect to the W_0-topology. Namely, the set $V = K \times U \cup \pi_1^{-1}(M-K)$, with $\pi_1 : M \times N \to M$ the projection, is clearly open in $M \times N$, and we have $M(V) = M(K,U)$.

The compact-open topology and the W_0-topology are obviously equivalent if M is compact. However, if M is not compact, then the W_0-topolgy is strictly finer then the compact open topology. One can for example take $M = N = \mathbb{R}$. The set

$$A := \left\{ f \in C^0(M,N) : |f(x)| < e^{-x^2} \quad \forall x \in \mathbb{R} \right\}$$

is an open subset with respect to the W_0-topology. However it cannot be open in the compact-open topology. The sequence of constant maps $\{f_k\}_{k\in\mathbb{N}}$ defined by $f_k(x) = \frac{1}{k}$

for $x \in \mathbb{R}$ is fully contained in the complement of the set A. Yet f_k tends to the constant zero map with respect to the compact-open topology while the zero map is contained in A.

We will give an alternative description of the topologies W_k, by describing a local basis around an arbitrary member of $C^\infty(M, N)$. Its similarities to the Fréchet topology on $\Gamma_M E$ are unmistaken.

Let (U, φ) and (V, ψ) be charts of M and N, K a compact subset of U, $\epsilon > 0$, and $f : M \to N$ a smooth map such that $f(K) \subseteq V$. Let $N(f, K, \varphi, \psi, \epsilon)$ denote the set of all smooth maps $g : M \to N$ such that $g(K) \subseteq V$ and

$$\left\|\psi \circ f \circ \varphi^{-1} - \psi \circ g \circ \varphi^{-1}\right\|_{\varphi(K),k} < \epsilon,$$

where $\|-\|_{\varphi(K),k}$ is the regular C^k-norm, but only taking the supremum over the $x \in K$. More generally, let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is a cover by charts of M, $\{(V_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ a cover by charts of N, $K = \{K_\alpha\}$ a locally finite cover of M by compact subsets such that $K_\alpha \subseteq U_\alpha$, and $\epsilon = \{\epsilon_\alpha > 0\}$ a family of constants. Let $f : M \to N$ be a smooth map such that $f(K_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$, then set

$$N(f, K, \varphi, \psi, \epsilon) := \bigcap\nolimits_{\alpha\in A} N(f, K_\alpha, \varphi_\alpha, \psi_\alpha, \epsilon_\alpha).$$

Lemma 2.3.4. *Let M and N be two manifolds. For any $k \in \mathbb{N}$, the collection $\{N(f, K, \varphi, \psi, \epsilon)\}_\epsilon$ running over all families of positive numbers $\epsilon = \{\epsilon_\alpha > 0\}$ is a basis of the W_k topology around f.*

Proof. Let $\pi : J^k(M, N) \to M$ denote the projection. It is straightforward to see that for each $\alpha \in A$ there exists an open subset U_α of $\pi^{-1}(K_\alpha)$ such that

$$N(f, K_\alpha, \varphi_\alpha, \psi_\alpha, \epsilon_\alpha) = \left\{g \in C^\infty(M, N) : j^k(g)(K_\alpha) \subseteq U_\alpha\right\}.$$

The intersection

$$W = \cap_{\alpha\in A} \left(W_\alpha \cup \pi^{-1}(N - K_\alpha)\right)$$

is open, since the family $\{K_\alpha\}_{\alpha\in A}$ is locally finite. The equality

$$N(f, K, \varphi, \psi, \epsilon) = M(W)$$

folows directly. The converse is even simpler. Let $U \subseteq J^k(M, N)$ be an open subset such that $f \in M(U)$. For every $\alpha \in A$ we must find a constant $\epsilon_\alpha > 0$ such that

$$N(f, K_\alpha, \varphi_\alpha, \psi_\alpha, \epsilon_\alpha) \subseteq \left\{g \in C^\infty(M, N) : j^k(g)(K_\alpha) \subseteq U\right\}.$$

But since K_α is compact, and the fact that $J^k(M, N)$ is trivial over any product of charts $U_\alpha \times V_\alpha$ this can be done easily. $\square$

Let $E \to M$ be a vector bundle and consider again the space of sections $\Gamma_M E$. If M is not compact, the topology W_∞ seems to be too fine to allow a grading of seminorms. In fact, $\Gamma_M E$ isn't even a $\mathbb{R}$-linear topological vector space if it is equipped with the Whitney

C^∞ topology. For suppose that $\sigma \in \Gamma_M E$ is an arbitrary section. Then the sequence $\sigma_n = \frac{1}{n}\sigma$ should converge to the zero section. If we choose a vector bundle metric g on E, then we can define an open subset

$$U = \{e_x \in E : d(0_x, e_x) < \varphi(x) d(0_x, \sigma(x))\}$$

of E, where $\varphi : M \to \mathbb{R}$ a smooth map, $\varphi > 0$, that tends to zero at infinity. Now $M(U)$ defines an open neighborhood of the zero section in $\Gamma_M E$, yet none of the $\frac{1}{n}\sigma$ lie in $M(U)$.

Remark 2.3.5. *Let $E \to M$ be a vector bundle with a compact base. As for any fiber bundle one can also define the k-th jet bundle $J^k(E)$ by only considering the k-jets of* sections *of E. This is a submanifold of $J^k(M, E)$, hence the topologies on $\Gamma_M E$ induced by $J^k(E)$ and $J^k(M, E)$ coincide.*

Lemma 2.3.6. *Let $E \to M$ be a vector bundle over a compact manifold. Then the Fréchet topology on $\Gamma_M E$ from proposition 2.3.1 on page 25 coincides with the Whitney C^∞ topology. Moreover, if $U \subseteq E$ is an open, then*

$$M(U) = \{\sigma \in \Gamma_M E : \sigma(M) \subseteq U\}.$$

Proof. This follows directly from lemma 2.3.4 on the previous page by using local trivializations of $E \to M$. □

2.3.3 Basic properties of $\Gamma_M E$

The following lemma contains useful observations about graded Fréchet spaces of the form $\Gamma_M E$.

Lemma 2.3.7. *Let $E_1 \to M$ and $E_2 \to M$ be two vector bundles over the same compact base.*

- *If $E_1 \simeq E_2$ are isomorphic vector bundles than $\Gamma_M E_1$ and $\Gamma_M E_2$ are 0-tamely linear isomorphic.*
- *The natural map*

  $$\Gamma_M(E_1 \oplus E_2) \simeq \Gamma_M E_1 \oplus \Gamma_M E_2$$

  is a 0-tame linear isomorphism.
- *If E_1 is a sub vector bundle of E_2, then $\Gamma_M E_1$ is a 0-tame direct summand of $\Gamma_M E_2$.*

Proof. The first statement follows directly from the definitions.

For the second statement one can choose local trivializations of the Whitney sum $E_1 \oplus E_2$ such that for

$$\psi : (E_1 \oplus E_2)|_U \to U \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$$

the first k_1 coordinates trivialize E_1 and the latter k_2 trivialize E_2. Hence this specifies a grading for which

$$\|-\|_k^{E_1 \oplus E_2} = \|-\|_k^{E_1} + \|-\|_k^{E_2}.$$

For the third statement one can choose a vector bundle metric on E_2. Then the decomposition $E_2 = E_1 \oplus E_1^\perp$ implies that $\Gamma_M E_2$ is 0-tamely isomorphic to $\Gamma_M E_1 \oplus \Gamma_M E_1^\perp$. ☐

Remark 2.3.8. *Every vector bundle on a second countable Hausdorff manifold M is the direct summand of a trivial vector bundle $M \times \mathbb{R}^d$ for sufficiently high dimension $d \in \mathbb{N}$. In particular, if $E \to M$ is a vector bundle with compact base, then $\Gamma_M E$ is a 0-tame subspace of*

$$\Gamma_M(M \times \mathbb{R}^d) = C^\infty(M, \mathbb{R}^d) \simeq C^\infty(M)^d.$$

Hence one can define the graded Fréchet space structure on $\Gamma_M E$ by starting from the structure on $C^\infty(M)$.

This can be used in combination with lemma 2.3.7 to simplify some estimates. Suppose that M and N are compact manifolds and $f : M \to N$ is a smooth map. One can consider Γ_N as a functor between the category of vector bundles on N with vector bundle maps and the category of graded Fréchet spaces with tame linear maps. Likewise, one can consider $\Gamma_M f^*$ as a functor on the category of vector bundles on N with vector bundle maps by first pulling back along f and then applying Γ_M. Now suppose that $\alpha : \Gamma_M f^* \to \Gamma_N$ is a *natural transformation, which in particular implies that every component α_E is a tame linear map. If $E \to N$ is a vector bundle, then the tameness estimates for*

$$\alpha_E : \Gamma_M f^* E \to \Gamma_N E$$

follow directly from the estimates for $\alpha_{M\times\mathbb{R}} : C^\infty(M) \to C^\infty(N)$. This for example happens in lemma 2.3.12 on page 38.

The vector space $C^\infty(M) = \Gamma_M(M \times \mathbb{R})$ hence also plays an important role in the category of graded Fréchet spaces. It is a smooth 0-tame ring in the sense that point-wise addition and multiplication are 0-tame bilinear maps. We will state this in a more general result.

Lemma 2.3.9. *Let $E \to M$ be a vector bundle over a compact base manifold, then $\Gamma_M E$ is a 0-tame $C^\infty(M)$-module in the sense that point-wise addition*

$$+_E : \Gamma_M E \oplus \Gamma_E \longrightarrow \Gamma_M E, \quad (\sigma + \tau)(x) = \sigma(x) + \tau(x)$$

and point-wise multiplication

$$\cdot_E : C^\infty(M) \oplus \Gamma_M E \longrightarrow \Gamma_M E, \quad (f \cdot \sigma)(x) = f(x)\sigma(x)$$

are 0-tame bilinear maps of base 0.

Proof. The addition is 0-tame bilinear by the definition of the tame direct sum, and the fact that $\Gamma_M E$ is a topological vector space. Note that this is a slightly stronger statement than just saying that addition is continuous, as it depends on the choice of grading on the direct sum.

For the product, note that there exists a fiber-wise product

$$\mu : (M \times \mathbb{R}) \oplus E \longrightarrow E : (f, e) \mapsto f \cdot e,$$

for all $f \in \mathbb{R}$, $e \in E_x$, $x \in M$. This map is a smooth fiber-preserving map. Hence, by proposition 2.3.11 on page 34, the map

$$\mu_* : \Gamma_M((M \times \mathbb{R}) \oplus E) \to \Gamma_M E$$

defined by left-composition by μ is a 0-tame bilinear map. Now note that $\Gamma_M((M\times\mathbb{R})\oplus E)$ is 0-tame isomorphic to $C^\infty(M) \oplus \Gamma_M E$. □

The following lemma is trivial, yet it is still worth mentioning. Recall that a compact region $R \subseteq M$ is the closure of a relatively compact open submanifold, such that R has a smooth boundary. As noted in remark 2.3.2 on page 26 the spaces sections $\Gamma_R E$, where $E \to R$ is a vector bundle, are also graded Fréchet spaces. The norms are defined in exactly the same way as for compact manifolds without boundary.

Lemma 2.3.10. *Let $E \to M$ be a vector bundle over a compact base manifold and $N \subseteq M$ either a closed submanifold or a compact region in M. Then the restriction map*

$$\rho : \Gamma_M E \to \Gamma_N E|_N$$

is 0-tame linear.

In fact, one can show that if $f : N \to M$ is a smooth map with N compact as well, then the map

$$f^* : \Gamma_M E \to \Gamma_N(f^* E)$$

defined by composing a section on the right is a 0-tame linear map as well. We will not prove this, as it will be a consequence of proposition 3.2.16 on page 72.

2.3.4 Left composition by a fiber preserving map

We will prove the smooth tameness of a the collection of useful non-linear maps between graded Fréchet spaces of the form $\Gamma_M E$. Consider two vector bundles $E \to M$ and $E' \to M$ with the same compact base. Let $\mathbf{Bund}_M(E, E')$ be the set of bundle maps $E \to E'$ over M, that is, the smooth maps $E \xrightarrow{f} E'$ such that

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

commutes; in other words, these are smooth maps that map fibers into fibers over the same base point, but are *not* necessarily linear when restricted to a fiber. Such bundle maps induce a particularly useful class of smooth tame maps, since they will function as the coordinate charts of all of our examples of Fréchet manifolds.

In the statement of the following lemma we identify the sections $\nu \in \Gamma_M E$ with the vertical vector fields $X \in \mathcal{X}^{\text{vert}}(E)$ that are constant on the fibers of E via the formula

$$X_e = \frac{d}{dt}\bigg|_{t=0} (e + t\nu_m),$$

for $e \in E_m$, and $m \in M$.

Let $U \subseteq E$ be an open subset. Recall from definition 2.3.3 on page 28 and lemma 2.3.6 on page 30 that the set $M(U) \subseteq \Gamma_M E$ is defined by

$$M(U) := \{\sigma \in \Gamma_M E : \sigma(M) \subseteq U\}.$$

Proposition 2.3.11. *Let $E \to M$ and $E' \to M$ be vector bundles over the same compact base, and $f : U \to E'$ a fiber preserving map defined on an open subset $U \subseteq E$. Then the map*

$$f_* : M(U) \to \Gamma_M E' : \sigma \mapsto f \circ \sigma$$

is a smooth 0-tame map. Its tangent map at $\sigma \in M(U)$ is given by

$$T_\sigma f_* : \Gamma_M E \to \Gamma_M E' : \nu \mapsto Tf \circ \nu.$$

If f is a vector bundle map instead, then the map f_* is 0-tame linear.

Proof. Without much loss of generality we may assume that $U = E$.

First we will show that f_* is a continuous map. The induced map

$$\tilde{f} : J^k(E) \to J^k(F)$$

defined by pushing forward k-jets is smooth. Hence any open in $U \subseteq J^k(E)$ gives rise to an open subset $\tilde{f}^{-1}(U) \subseteq J^k(E)$ and the sets

$$M(\tilde{f}^{-1}(U)) = \left\{ g \in \Gamma_M E : j^k(g)(M) \subseteq \tilde{f}^{-1}(U) \right\}$$

and $f_*^{-1}(M(U))$ coincide. This shows that the preimage $f_*^{-1}(M(U))$ is a W_k-open, and thus that f_* is continuous.

Next we will show that f_* is a smooth map. Around any point $x \in M$ one can choose an open subset $U \subseteq M$ such that there are local trivializations

$$\begin{aligned} \psi &: E_U \xrightarrow{\cong} U \times \mathbb{R}^m, \\ \psi' &: E'_U \xrightarrow{\cong} U \times \mathbb{R}^n \end{aligned}$$

of both vector bundles. The bundle map f is now locally represented by

$$\psi' \circ f \circ \psi^{-1} = (id, \tilde{f}) : U \times \mathbb{R}^m \to U \times \mathbb{R}^n,$$

where $\tilde{f} : U \times \mathbb{R}^m \to \mathbb{R}^n$ is the appropriate map. Since the latter is a smooth map, there exists a continuous map

$$\tilde{l} : U \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n,$$

linear in the last component, such that

$$\tilde{f}(x, y_1) - \tilde{f}(x, y_0) = \tilde{l}(x, y_0, y_1)(y_1 - y_0).$$

It is a well-known principle that such a map can be defined as

$$\tilde{l}(x, y_0, y_1)z = \int_0^1 D\tilde{f}\left(x, y_0 + t(y_1 - y_0)\right) z dt$$

and that it satisfies $D\tilde{f}(x, y)z = \tilde{l}(x, y, y)z$. In the global picture this defines a continuous bundle map

$$l_U = \psi'^{-1} \circ (id, \tilde{l}) \circ (\psi, \psi, \psi) : E_U \oplus E_U \oplus E_U \to E'_U$$

that is a vector bundle map, that is, linear on each fiber, in the last summand. The open subsets U cover M and by choosing a partition of unity $\{\chi_U\}$ with respect to a locally finite subcover $\{U\}$ we obtain a global continuous bundle map

$$\begin{aligned} l : E \oplus E \oplus E &\longrightarrow E' \\ (y_0, y_1, z) &\mapsto \sum_{p(z) \in U} \chi_U\left(p(z)\right) l_U(y_0, y_1)z. \end{aligned}$$

Each summand $(\chi_U \circ p) l_U$ defines a continuous map $E_U \oplus E_U \oplus E_U \to E'_U$ that vanishes on the fibers above the boundary of U. Hence it extends by zero to a continuous bundle map on the entire vector bundle. For every point $x \in M$, the map l is a finite sum of these bundle maps on the fibers above an open neighborhood of x, hence l is also continuous. If $\Delta_E : E \to E \oplus E$ is the diagonal map of E, then the composition

$$l \circ (\Delta \times \mathrm{id}) : E \oplus E \to E'$$

is given by $l(y, y)z = Df(y)z$, hence it is actually a smooth bundle map.

Moreover, it has the property

$$\begin{aligned} l(y_0, y_0)(y_1 - y_0) &= \sum \chi_U(x) l_U(y_0, y_1)(y_1 - y_0) \\ &= \sum \chi_U \left(f(y_1) - f(y_0)\right) \\ &= f(y_1) - f(y_0) \end{aligned}$$

for every pair $y_0, y_1 \in \mathbb{R}^m$. Hence we found a continuous bundle operator map

$$\begin{aligned} l_* : \Gamma_M E \times \Gamma_M E \times \Gamma_M E \simeq \Gamma_M(E \oplus E \oplus E) &\longrightarrow \Gamma_M E', \\ (\sigma_0, \sigma_1, \zeta) &\mapsto l \circ (\sigma_0, \sigma_1, \zeta) \end{aligned}$$

for which $f_*(\sigma_1) - f_*(\sigma_0) = l_*(\sigma_0, \sigma_1)\,(\sigma_1 - \sigma_0)$. This implies that f_* is continuous differentiable with derivative given by

$$\left(Df_*(\sigma)\nu\right)(x) = l(\sigma(x), \sigma(x))\nu(x) = T_{\sigma(x)} f \nu(x).$$

Since the derivative $Df_* = \left(l \circ (\Delta_E \times \mathrm{id})\right)_*$, with $\Delta_E : E \to E \oplus E$ the diagonal map of E, is itself a bundle operator map, we conclude that f_* is smooth. Note that for this part of the proof the compactness of M isn't directly necessary.

Next we proof the tameness of f_* following the proof of Hamilton [Ham82b]. Note that this proves that all its derivatives are tame as well, since these are also given by composition on the left. Recall that all seminorms on $\Gamma_M E$ and $\Gamma_M E'$ are of the form

$$\sum_{j=1}^{k} \max_U \sup_{x \in V} \left\| D^j \tilde{\sigma} \right\|,$$

hence it sufficient to check tameness in local coordinates and then to take the maximum over a finite cover of local trivializations. Hence fix a local trivialization $(U, \varphi, \psi, \psi')$ and an open subset $V \subseteq U$ with its closure $\bar{V} \subseteq U$ compact.

Given a fixed section $\sigma \in \Gamma_M E$, let N be an open neighborhood of the image of σ with compact closure. Then the local representatives

$$\tilde{f} : U \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

of f and all their partial derivatives $\frac{\partial^\beta}{\partial x^\beta} \frac{\partial^\gamma}{\partial y^\gamma} \tilde{f}(x, y)$ are bounded when restricted to N. Let $M(\bar{V}, N)$ be the set of sections $\tau \in \Gamma_M E$ for which $\tau(\bar{V}) \subseteq N$. Then $M(\bar{V}, N)$ is an open neighborhood of σ. For any $\tau \in M(\bar{V}, N)$ we let

$$\tilde{\tau} : \bar{V} \longrightarrow \mathbb{R}^m$$

denote its local representative in relation to $(U, \varphi, \psi, \psi')$.

By the Leibniz-rule we have

$$\partial^\alpha \tilde{f}(x, \tilde{\sigma}(x)) = \sum \frac{\partial^\beta}{\partial x^\beta} \frac{\partial^\gamma}{\partial y^\gamma} \tilde{f}(x, \tilde{\tau}(x)) \prod_{i=1}^{m} \prod_{k=1}^{\gamma_i} \frac{\partial^{\gamma^{i,k}}}{dx^{\gamma^{i,k}}} \tilde{\tau}_i(x)$$

where $\tilde{\tau}_i$ is the i-th component function of $\tilde{\tau}$ and the sum runs over all multi-indices

$$\beta + \sum_{i,k} \gamma^{i,k} = \alpha.$$

We can choose a constant $C_0 > 0$ such that $\|\tau\|_0 \le C$ and such that

$$\left\| \frac{\partial^\beta}{\partial x^\beta} \frac{\partial^\gamma}{\partial y^\gamma} \tilde{f}(x, \tilde{\tau}(x)) \right\|_0 \le C_0$$

for all $\tau \in M(\bar{V}, N)$ and all indices as described above. So we get an estimate

$$\left\| \frac{d^\alpha}{dx^\alpha} \tilde{f}(x, \tilde{\tau}(x)) \right\|_0 \le C \sum \|\tau\|_{i_1} \cdots \|\tau\|_{i_k}$$

where the sum runs over $i_1 + \ldots + i_k \le |\alpha|$. Now by the interpolation estimates (which we will prove in the next chapter),

$$\|\tau\|_i^n \le C \|\tau\|_n^i \|\tau\|_0^{n-i} \le C \|\tau\|_n^i ,$$

we obtain the estimate $\|\tau\|_i \le C \|\tau\|_n^{i/n}$ for all $i \le n$. Which leads us to the required estimate,

$$\begin{aligned}
\|f_*(\tau)\|_n &\le \sum_{|\alpha| \le n} \max_U \sup_{x \in \bar{V}} \left\| \partial^\alpha \tilde{f}(x, \tilde{\tau}(x)) \right\| \\
&\le C(1 + \sum \|\tau\|_{i_1} \cdots \|\tau\|_{i_k}) \\
&\le C(1 + \sum \|\tau\|_n^{\frac{i_1 + \ldots + i_k}{n}}) \\
&\le C(1 + \|\tau\|_n)
\end{aligned}$$

for all $\tau \in M(\bar{V}, N)$ and $n \in \mathbb{N}$. Here we use that $x^\theta \le 1 + x$ for all $\theta \in [0, 1]$ and $x \ge 0$. $\square$

The results can be summarized as follows. Let M be a compact manifold. Then by the above proposition we can consider Γ_M as a covariant functor

$$\Gamma_M : \mathbf{Vect}_M \longrightarrow \mathbf{TameL}_0,$$

where $\mathbf{Vect}_M$ is the category of vector bundles over M with vector bundle maps, and $\mathbf{TameL}_0$ was the category of graded Fréchet spaces with 0-tame linear maps. Alternatively we may consider it as a covariant functor from the category of vector bundles over M with bundle maps to the category $\mathbf{TameS}_0$.

In the case of the former, Γ_M preserves the biproduct of $\mathbf{Vect}_M$. Moreover, Γ_M maps a subbundle $E \le F \to M$ to a 0-tame direct summand $\Gamma_M E$ of $\Gamma_M F$. Since every vector bundle is the direct summand of a trivial vector bundle, Γ_M maps $\mathbf{Vect}_M$ into the category of 0-tame projective modules over $C^\infty(M)$.

2.3.5 Integration of sections

In this section we show the tameness of integration. Let M and N be compact manifolds and $f : M \times N \to \mathbb{R}$ a smooth function. Then one can define a smooth function $I(f) : N \to \mathbb{R}$ by taking the integral $\int_M f(x, y)\, dx$ for every $y \in N$. The resulting map $I : C^\infty(M \times N) \to C^\infty(N)$ is proven to be tame linear. In fact, this is done in a slightly more general setting. This will be needed later on an application of the Nash-Moser theorem, in proving the tameness of certain 'homotopy operators' in sections 5.2.1 on page 112 and 5.2.2 on page 115.

Suppose that M is a compact manifold of dimension m. Recall that the density line bundle DM of a manifold M is the bundle whose fiber above $x \in M$ is given by the functions $\theta_x : \wedge^n T_x M \to \mathbb{R}$ satisfying

$$\theta_x(cv) = |c|\, \theta_x(v), \quad \forall c \in \mathbb{R}, v \in \wedge^n T_x M.$$

A density on M is a smooth section of DM. Given a diffeomorphism $\varphi : M \to N$ between manifolds, every multivector $v \in \Gamma_N(\wedge^n TN)$ can be pulled back along φ. By conjugation, every density can be pushed forward; we will write $\varphi_*\theta$ for the push-forward of θ.

A density θ is called positive if each of the θ_x is a strictly positive function. For example, the normalized Haar measure $d\mu$ on a Lie group G is a normalized positive density. It can be defined by choosing a G-invariant Riemannian metric g on G.

Any oriented Riemannian manifold M has a cannonical volume form given by

$$vol(g) = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n,$$

where the $dx^1, \ldots, dx^n$ form a basis of $\Omega^1(M)$, and $|g|_x$ is the absolute value of the determinant of $g_x : T_x^* M \to T_x M$ for every $x \in M$. Most importantly, the absolute value of $vol(g)$ defines a positive density $d\mu = |vol(g)|$ on M. Since M is compact, it can be normalized by dividing it by the integral $\int_M 1 d\mu$.

Lemma 2.3.12. *Let M be a compact manifold and θ a density on M. Let $E \to N$ be a vector bundle over a compact base and $\pi^* E \to M \times N$ the pullback bundle along the projection $M \times N \xrightarrow{\pi} N$. Then the integration map*

$$I_\theta : \Gamma_{M\times N}(\pi^* E) \to \Gamma_N E, \quad I_\theta(f)(x) = \int_M f(-, x)\, \theta$$

is a 0-tame linear map.

Proof. Let us first give a more precise description of what it means to integrate a section $f \in \Gamma_{M\times N}(\pi^* E)$. We will make some peculiar choices, such that this description of integration to aid us in the tameness estimates. Choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of M, and a cover $\{K_\alpha\}$ of M by precompact open subsets such that $\bar{K}_\alpha \subseteq U_\alpha$ for every $\alpha \in A$. This can be done such that A is finite. Moreover, choose a partition of unity $\{\chi_\alpha\}$ subordinate to $\{K_\alpha\}$. Also choose an atlas $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ of N that locally trivializes E by vector bundle maps

$$(\mathrm{proj}_E, \kappa_\beta) : E|_{V_\beta} \longrightarrow V_\beta \times \mathbb{R}^k.$$

Let $\{L_\beta\}$ be a cover of N by precompact open subsets with $\bar{L}_\beta \subseteq V_\beta$ for all $\beta \in B$, and $\{\rho_\beta\}$ a partition of unity subordinate to this cover. B can be chosen finite as well. Note that π^*E trivializes over $M \times V_\beta$ by the obvious map

$$\pi^*E|_{M\times V_\beta} \to M \times V_\beta \times E', \quad \tilde{\kappa}_\beta(e) = (x, v, \kappa_\beta(v, e)),$$

where $e \in (\pi^*E)_{x,v} = E_v$.

For every α and β define a smooth map $f_{\alpha,\beta} : U_\alpha \times V_\beta \to \mathbb{R}^k$,

$$f_{\alpha,\beta} := \chi_\alpha \circ \varphi_\alpha^{-1} \cdot \kappa_\beta \circ (\mathrm{id}_{V_\beta} \times (f \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{V_\beta})).$$

This map has support in $\varphi_\alpha(K_\alpha) \times V_\beta$. The push-forward $\varphi_{\alpha*}\theta$ defines a density on $\varphi_\alpha(U_\alpha)$, so that we can surely integrate $f_{\alpha,\beta}$ over $\varphi_\alpha(K_\alpha)$. This gives a smooth map $V_\beta \to \mathbb{R}^k$. By multiplying this map by the partition function ρ_β, we may extend it to a section of E. Summarized, we define integration of f by

$$I_\theta(f) = \sum\nolimits_{\alpha,\beta} \rho_\beta \cdot \kappa_\beta^{-1} \circ \int_{\varphi(K_\alpha)} f_{\alpha,\beta}\, \varphi_{\alpha*}\theta.$$

The integral denotes Lebesgue integration on $\varphi_\alpha(K_\alpha)$. One can check that this definition of I_θ doesn't depend on the choices made. Moreover, it has the property that

$$I_\theta(f)(n) = \int_M f(-, n)\, \theta, \quad \forall n \in N,$$

where the right-hand-side is defined as one usually does. From the above description it is directly obvious that $I_\theta(f)$ is a *smooth* section of E.

Suppose that $F \to N$ is another vector bundle over N. Denote integration by

$$I_E : \Gamma_{M\times N}(\pi^*E) \to \Gamma_N E,$$

and similarly for F and $E \oplus F$. From the linearity of the integrals and the κ_β, we deduce that, if we choose the correct charts for $I_{E\oplus F}$, we have

$$I_{E\oplus F} = I_E + I_F.$$

This implies that I_E is tame linear if $I_{E\oplus F}$ is. For if $E \overset{i}{\hookrightarrow} E \oplus F \xrightarrow{p} E$ are the inclusion and projection of vector bundles, then $I_E = p_* \circ I_{E\oplus F} \circ i_*$. The maps i_* and p_* defined by left composition are known to be tame, hence it suffices to check that $I_{E\oplus F}$ is tame. Conversely, $I_{E\oplus F}$ is tame linear if both I_E and I_F are. This follows from the computation

$$\begin{aligned}
\|I_{E\oplus F}(e + f)\|_k^{E\oplus F} &= \|I_E(e) + I_F(f)\|_k^{E\oplus F} \\
&\leq \|I_E(e)\|_k^{E\oplus F} + \|I_F(f)\|_k^{E\oplus F} \\
&\leq C \left(\|I_E(e)\|_k^E + \|I_F(f)\|_k^F \right) \\
&\leq C \left(\|e\|_{k+r}^E + \|f\|_{k+s}^F \right) \leq C\, \|e + f\|_{k+t}^{E\oplus F}.
\end{aligned}$$

Since every vector bundle is the direct summand of a trivial one, we conclude that we may assume that E is the trivial line bundle, and rid ourselves of all the local trivializations κ_α in the definition of the integral. In other words, I is given by

$$I_\theta : C^\infty(M \times N) \to C^\infty(N), \quad I_\theta(f) = \sum\nolimits_\alpha \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1} \cdot f \circ (\varphi_\alpha^{-1} \times \mathrm{id})\, \varphi_{\alpha*}\theta.$$

Recall that the C^k-norms on $C^\infty(N)$ are given by

$$\|f\|_k = \sum\nolimits_{|\gamma|\le k} \max_\beta \sup_{y\in\psi_\beta(\bar{L}_\beta)} \left|\partial^\gamma (f \circ \psi_\beta^{-1})(y)\right|,$$

where $\gamma \in \mathbb{N}^n$ with $n = \dim(N)$. Likewise, the C^k-norms on $C^\infty(M \times N)$ are defined by

$$\|f\|_k = \sum\nolimits_{|\gamma|\le k} \max_{\alpha,\beta} \sup_{(x,y)\in\bar{K}_\alpha\times\bar{L}_\beta} \left|\partial^\gamma (f \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1}))(x,y)\right|,$$

where now $\gamma \in \mathbb{N}^{m+n}$ with $m = \dim(M)$. From now on $\gamma \in \mathbb{N}^n$ denotes a multi-index, which can be seen as $\gamma \in \mathbb{N}^{m+n}$ by being 0 in the first m entries. We have, for $f \in C^\infty(M \times N)$,

$$\|I_\theta(f)\|_k = \sum\nolimits_{|\gamma|\le k} \max_\beta \sup_{\psi_\beta(\bar{L}_\beta)} \left|\partial^\gamma \sum\nolimits_\alpha \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1} \cdot f \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1})\, \varphi_{\alpha*}\theta\right|.$$

For each of the summands of the inner sum we can take the differential into the integral, and, since $\chi_\alpha \circ \varphi_\alpha^{-1}$ is compactly supported in $\varphi(K_\alpha)$ make the estimates

$$\begin{aligned}
&\left|\partial^\gamma \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1} \cdot f \circ (\varphi_\alpha \times \psi_\beta^{-1}) \varphi_{\alpha*}\theta\right| \\
&\le \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1} \cdot \left|\partial^\gamma f \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1})\right| \varphi_{\alpha*}\theta \\
&\le \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1}\, \varphi_{\alpha*}\theta \cdot \sup_{\varphi_\alpha(\bar{K}_\alpha)} \left|\partial^\gamma f \circ (\varphi_\alpha^{-1} \times \psi_\alpha^{-1})\right| \\
&\le C \sup_{\varphi_\alpha(\bar{K}_\alpha)} \left|\partial^\gamma f \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1})\right|.
\end{aligned}$$

So that we can estimate

$$\|I_\theta(f)\|_k \le C \sum\nolimits_{|\gamma|\le k} \max_{\alpha,\beta} \sup_{\varphi_\alpha(\bar{K}_\alpha)\times\psi_\beta(\bar{L}_\beta)} \left|\partial^\gamma f \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1})\right| \le C \|f\|_k. \qquad \square$$

Suppose that $B \xrightarrow{p} M$ is a compact fiber bundle. In the same spirit as the density bundle of a manifold, one can associate a vertical density bundle $D^{\mathrm{vert}}B \to B$ to B. Let $k = \dim(B) - \dim(M)$ denote the dimension of the fiber of B. Then the fiber at a base point $y \in B$ consists of the functions θ_y on $\wedge^k T_y^{\mathrm{vert}} B$ such that

$$\theta_y(cv) = |c|\, \theta_y(v), \quad \forall c \in \mathbb{R}, v \in \wedge^k T_y^{\mathrm{vert}} B.$$

A vertical density θ on B is a section of $D^{\mathrm{vert}}B \to B$. We call θ positive if the function θ_y is strictly positive for every $y \in B$. θ can be seen as a smooth family of positive densities on the fibers of B, simply by restricting θ to these fibers.

Lemma 2.3.13. *Let $B \xrightarrow{p} M$ be a compact fiber bundle, $E \to M$ a vector bundle over M, and θ a positive vertical density of $B \to M$. Then integration defines a tame linear map*

$$I_\theta : \Gamma_B(p^*E) \longrightarrow \Gamma_M E, \quad I(f)(m) = \int_{B_m} f\, \theta_{B_m}.$$

Proof. First note that, by arguments similar as in lemma 2.3.12 on page 38, we may assume without loss of generality that E is the trivial line bundle over E. Hence we will prove that the integration map

$$I_\theta : C^\infty(B) \to C^\infty(M), \quad I(f)(m) = \int_{B_m} f\, \theta_{B_m}$$

is tame linear. We will procede in a similar way as the lemma above. We give an explicit description of how to perform the integration, and then we deduce the tameness estimates from this.

Let F denote the fiber of B. Choose an atlas $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ of M such that it also trivializes B with fiber bundle maps

$$\kappa_\beta : B|_{V_\beta} \xrightarrow{\simeq} F \times V_\beta.$$

Moreover, choose a cover $\{L_\beta\}$ of M by precompact open subsets such that $\bar{L}_\beta \subseteq V_\beta$ for every $\beta \in B$, and let $\{\rho_\beta\}$ be a partition of unity subordinate to this cover. Likewise, choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of the fiber F, a cover $\{K_\alpha\}$ of F be precompact open subsets with $\bar{K}_\alpha \Subset U_\alpha$ for all $\alpha \in A$, and a partition of unity $\{\chi_\alpha\}$ subordinate to this cover. Both index sets A and B can be chosen finite, since both F and M are compact.

For every $\beta \in B$ one can push forward the vertical density θ along κ_β to obtain a vertical density $\kappa_{\beta *}\theta$ on the trivial bundle $F \times V_\beta$. Fix a positive density θ^F on the fiber F, then it defines a vertical density θ^F on $F \times V_\beta$ by $\theta^F(f, n) = \theta^F(f)$. It is a vertical density that is constant in the horizontal direction. The vertical density bundle is one-dimensional, and both vertical densities are positive, hence there exists a positive smooth map $g_\beta : F \times V_\beta \to \mathbb{R}$ such that

$$\kappa_{\beta *}\theta = g_\beta \cdot \theta^F.$$

The smooth function $f \circ \kappa_\beta^{-1} \cdot g_\beta$ defined on $F \times V_\beta$ can now be integrated along F using the density θ^F. Recall that this is given by the expression

$$\sum\nolimits_\alpha \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1} \cdot (f \circ \kappa_\beta^{-1} \cdot g_\beta) \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{V_\beta})\, \varphi_{\alpha *}\theta^F,$$

and defines a smooth map on V_β. One can multiply it with a partition function ρ_β, and extend it by zero to a smooth function on M. To summarize, the integration of f is given by

$$I_\theta(f) = \sum\nolimits_{\alpha,\beta} \rho_\beta \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha \circ \varphi_\alpha^{-1} \cdot (f \circ \kappa_\beta^{-1} \cdot g_\beta) \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{V_\beta})\, \varphi_{\alpha *}\theta^F.$$

One should check that this expression doesn't depend on the many choices that were made. Note that, because of the partition of unity $\{\rho_\beta\}$ and the change of variables rule for integration, this definition of I_θ satisfies

$$I_\theta(f)(m) = \int_{B_m} f\,\theta_{B_m}.$$

We are ready to look at the tameness estimates. Recall that the C^k-norms on $C^\infty(M)$ are given by

$$\|f\|_k = \sum\nolimits_{|\delta|\le k} \max_{\beta\in B} \sup_{m\in\psi_\beta(\bar{L}_\beta)} \left|\partial^\delta f\circ\psi_\beta^{-1}(m)\right|,$$

where $\delta\in\mathbb{N}^{\dim(M)}$. On the other hand, the map

$$\varphi_{\alpha,\beta,\gamma} := (\varphi_\alpha\times\psi_\beta)\circ\kappa_\gamma : \kappa_\gamma^{-1}(U_\alpha\times(V_\beta\cap V_\gamma)) \longrightarrow \varphi_\alpha(U_\alpha)\times\psi_\gamma(V_\beta\cap V_\gamma)\subseteq\mathbb{R}^{\dim(B)}$$

gives a chart of B for all $\alpha\in A$ and $\beta,\gamma\in B$. The open $K_{\alpha,\beta,\gamma} := \kappa_\beta^{-1}(K_\alpha\times(L_\beta\cap L_\gamma))$ is precompact, the closure $\bar{K}_{\alpha,\beta,\gamma}$ lies in the domain of the chart, and the entire family $\{K_{\alpha,\beta,\gamma}\}$ covers B. Hence the C^k-norms can be computed as

$$\|f\|_k = \sum\nolimits_{|\delta|\le k} \max_{\alpha,\beta,\gamma} \sup_{g\in\bar{K}_{\alpha,\beta,\gamma}} \left|\partial^\delta f\circ\kappa_\gamma^{-1}\circ(\varphi_\alpha^{-1}\times\psi_\beta^{-1})\right|,$$

where now $\delta\in\mathbb{N}^{\dim(B)}$. Hence we have

$$\begin{aligned}\|I(f)\|_k &\le \sum\nolimits_{|\delta|\le k} \max_{\beta} \sup_{\psi_\beta(\bar{L}_\beta)} \left|\partial^\delta \sum_{\alpha,\gamma}\rho_\gamma \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha\circ\varphi_\alpha^{-1}\cdot f\circ\varphi_{\alpha,\beta,\gamma}^{-1}\cdot g_\gamma\circ(\varphi_\alpha^{-1}\times\psi_\beta^{-1})\,\varphi_{\alpha*}\theta^F\right|\\ &\le \sum\nolimits_{|\delta|\le k} \max_{\alpha,\beta,\gamma} \sup_{\psi_\beta(\bar{L}_\beta)} \int_{\varphi_\alpha(K_\alpha)} \chi_\alpha\circ\varphi_\alpha^{-1}\cdot\left|\partial^\delta\rho_\gamma\cdot f\circ\varphi_{\alpha,\beta,\gamma}^{-1}\cdot g_\gamma\circ(\varphi_\alpha^{-1}\times\psi_\beta^{-1})\right|\,\varphi_{\alpha*}\theta^F,\end{aligned}$$

by interchanging integration and differentiation, and taking the absolute value inside the integral sign. Then by the Leibniz rule we have

$$\begin{aligned}&\left|\partial^\delta\rho_\gamma\cdot f\circ\varphi_{\alpha,\beta,\gamma}^{-1}\cdot g_\gamma\circ(\varphi_\alpha^{-1}\times\psi_\beta^{-1})\right|\\ &\le \sum_{\delta_1\le\delta_2\le\delta} \partial^{\delta_1}\rho_\gamma\cdot\left|\partial^{\delta_2-\delta_1} f\circ\varphi_{\alpha,\beta,\gamma}^{-1}\right|\cdot\left|\partial^{\delta-\delta_2} g_\gamma\circ(\varphi_\alpha^{-1}\times\psi_\beta^{-1})\right|\\ &\le C\sum_{\delta'\le\delta}\left|\partial^{\delta'} f\circ\varphi_{\alpha,\beta,\gamma}^{-1}\right|,\end{aligned}$$

since ρ_γ is compactly supported, and g_γ is a smooth map on $\kappa_\gamma^{-1}(\bar{K}_{\alpha,\beta,\gamma})$, hence their derivatives are bounded. The integrals $\int_{\varphi_\alpha(K_\alpha)}\chi_\alpha\circ\varphi_\alpha^{-1}\,\varphi_{\alpha*}\theta^F$ are also bounded by some constant. We conclude that

$$\|I(f)\|_k \le C\sum\nolimits_{|\delta|\le k}\sum\nolimits_{\delta'\le\delta} \max_{\alpha,\beta,\gamma} \sup_{g\in\varphi_{\alpha,\beta,\gamma}(K_{\alpha,\beta,\gamma})} \left|\partial^{\delta'} f\circ\varphi_{\alpha,\beta,\gamma}^{-1}\right| \le C\,\|f\|_k. \qquad \square$$

Remark 2.3.14. *In the setting of the lemma above, right-composition by $p : B\to M$ defines a 0-tame linear map*

$$p^* : \Gamma_M E \longrightarrow \Gamma_B(p^*E)$$

such that $I_\theta\circ p = id$. This follows from lemma 3.2.16 on page 72 below. Hence by lemma 2.2.20 on page 20, $\Gamma_M E$ is a tame direct summand of $\Gamma_B(p^*E)$, with tame compliment given by the smooth sections whose integral vanishes.

2.3.6 Concrete examples of graded Fréchet spaces

A particular examples of graded Fréchet spaces that come to mind are the vector fields $\mathcal{X}(M) = \Gamma_M(TM)$, or anti-symmetric k-vectors $\mathcal{X}^k(M) = \Gamma_M(\wedge^k TM)$, and the smooth k-forms $\Omega^k(M) = \Gamma_M(\wedge^k T^*M)$ on M.

The differential defines a 1-tame linear map sending a smooth function to a 1-form,

$$d : C^\infty(M) \longrightarrow \Omega^1(M).$$

It is easily seen to be a 1-tame linear map, as it locally just involves taking the first derivative of f. Alternatively, one can apply proposition 3.2.16 on page 72 by considering the differential as $d = i_* \circ j^1$. Here $j^1 : C^\infty(M) \to \Gamma_M J^k(M, \mathbb{R})$ is the map that sends a function to its first jet. The map i_* is the left-composition by the vector bundle map $i : J^1(M, \mathbb{R}) \to T^*M$.

On the other hand, the composition

$$\mathbf{com} : \Omega^1(M) \times \mathcal{X}(M) \mapsto C^\infty(M)$$

is 0-tame bilinear, see proposition 3.2.16 on page 72. Together they define a smooth tame map

$$\mathcal{X}(M) \times C^\infty(M) \longrightarrow C^\infty(M) : (v, f) \mapsto df(v)$$

that realizes the vector fields as tame linear maps $C^\infty(M) \to C^\infty(M)$; of course, these are the derivations. One can now see that the commutator bracket

$$[-,-] : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M) : (v, w) \mapsto v \circ w - w \circ v$$

is tame linear as well. Hence the vector fields form a *tame Lie algebra* $(\mathcal{X}(M), [-,-])$. It is the Lie algebra of the tame Lie group $\mathbf{Diff}(M)$ of diffeomorphisms.

Parts of this generalize to arbitrary vector bundles $E \to M$. A connection ∇ defines covariant derivative

$$d_\nabla : \Gamma_M E \longrightarrow \Gamma_M(E^* \otimes E)$$

by the usual Koszul-formula, and this map is 1-tame linear as well. This will recur in the chapter on applications.

The above are all examples of differential operators. Let $E \to M$ and $F \to M$ be two vector bundles over a compact base. A differential operator of order k can be seen as the composition of the k-th jet

$$j^k : \Gamma_M E \to \Gamma_M J^k(E),$$

which maps a section σ to its k-th jet $j^k\sigma : M \to J^k(E)$, and a 0-tame linear map $\Gamma_M J^k(E) \to \Gamma_M F$. It is straightforward to see that j^k should be a k-tame linear map, so that the differential operator of order k is k-tame as well.

2.4 Smoothing operators and interpolation estimates

In some of the proofs above we have encountered certain *interpolation estimates*, and have used them without proof. Recall that, given a graded Fréchet space F, these are the estimates of the form

$$\|f\|_m^{n-l} \leq C \|f\|_n^{m-l} \|f\|_l^{n-m}$$

with $l \leq m \leq n$ and $C > 0$ a constant depending only on l, m and n. Such estimates turn out to be a useful tool, but one cannot expect them to hold for any graded Fréchet space.

In this chapter we prove that interpolation estimates hold for all our examples of graded Fréchet spaces. In fact, we introduce the concept of smoothing operators as defined in [Ham82b, Nas56] and show that smoothing operators imply interpolation estimates. These smoothing operators are a even more useful tool than the interpolation estimates and are essential to the proof of the Nash-Moser theorem. We will show that all our examples allow for these smoothing operators.

2.4.1 The definition

Definition 2.4.1 (Smoothing operators). *Let F be a graded Fréchet space. A smoothing operator on F is a family of linear maps $\{S_t : F \to F\}_{t>1}$ with two integers $\beta \geq 0$, the base, and $\delta \geq 0$, the defect, such that*

1. *For every $n \geq \beta$ and $r \geq 0$ there is a $C > 0$, depending on n and r, such that the estimate*

   $$\|S_t f\|_{n+r} \leq C t^{r+\delta} \|f\|_n \, ,$$

   holds for all $f \in F$;

2. *For every $n \geq \beta$ and $r \geq \delta$ there is a $C > 0$, depending on n and r, such that we can estimate*

   $$\|f - S_t f\|_n \leq C t^{-r+\delta} \|f\|_{n+r} \, .$$

If such a family exists we say that the graded Fréchet space F allows smoothing operators. When the defect vanishes, $\delta = 0$, we will call the smoothing operator strict. △

In a sense, the maps S_t are tame linear of degree $-\infty$, but the constant in the estimate increases exponentially in r. Also note that the second estimate implies that $S_t f \to f$ as $t \to \infty$; f is approximated by the better behaved, *'smoother'*, points $S_t f$ in F.

Let us sketch the motivation behind smoothing operators as how they were introduced by Nash [Nas56]. Recall that the inverse function theorem holds for smooth maps between Banach spaces without problem. Suppose we wish to prove the inverse function theorem in the setting of Banach spaces, that is, to solve an equation $P(f) = g$ with $f, g \in B$.

Assume that $DP(f_0)$ is invertible for some $f_0 \in B$, then so this is the derivative $DP(f)$ for f near f_0. We may define a smooth map R by

$$R(f,g) := f - DP(f)^{-1}(P(f)+g)$$

for f near f_0 and g near $P(f_0)$. We can now make an estimate of the form

$$\|R(f_1,g) - R(f_2,g)\| \leq \theta \|f_1 - f_2\|, \quad 0 < \theta < 1,$$

for f_1 and f_2 near f_0, using the Taylor formula with integral remainder. This allows us to apply the contraction mapping principle to the sequence $f_n = R(f_{n-1}, g)$. Finally, one needs to check that the resulting inverse P^{-1} is again a smooth map. This trick is essentially the Newton-Raphson method applied to the map $f \mapsto -P(f) + g$. This method is also similar to proving the existence of solutions for ODEs, if one works with the Banach spaces $C^k[0,1]$ of k-differentiable functions, hence one might also call it Picard iteration.

Suppose we wish to apply this method to our situation. From maps between graded Fréchet spaces we can at best expect an estimate of the form

$$\|R(f_1,g) - R(f_2,g)\|_n \leq \theta \|f_1 - f_2\|_{n+r},$$

and subsequent iteration causes the norm index to tend to infinity. This means the estimates are insufficient to use the contraction mapping principle. Hence there is no easy proof to the inverse function theorem for graded Fréchet spaces even if we assume the map is smooth tame. Nash and Moser referred to this phenomenon as *'loss of derivatives'*.

Remark 2.4.2. *Smoothing operators appear often in the literature, but often differ slightly in their definition. In Nash [Nas56] the family S_t depends smoothly on $t > 1$ and there is an additional estimate of the form*

$$\left\|\frac{d}{dt} S_t f\right\|_{n+r} \leq C t^{r-1} \|f\|_n .$$

In combination with the point-wise convergence $S_t \to id$ as $t \to \infty$, this implies property (2).

In [Mos66] the smoothing operators depend on an additional parameter $l \in \mathbb{N}$, writing $S_t^l : F \to F$, and the estimates are of the form

$$\begin{aligned} \left\|S_t^l f\right\|_{n+r} &\leq C t^r \|f\|_n, \quad \text{for } r \geq 0, \\ \left\|f - S_t^l f\right\|_n &\leq C t^{-r} \|f\|_{n+r}, \quad \text{for } 0 \leq k \leq \min(l,n). \end{aligned}$$

The extra parameter leads to an easier proof of existence, but makes iteration processes more cumbersome, as there is yet an extra variable to keep track of. The Nash-Moser inverse function theorem works equally well with this definition.

Conn [Con85] uses yet another variation. In his proof the iteration process deals with smooth functions on closed balls $\bar{B}_r \subseteq \mathbb{R}^n$ of radius $0 < r < 1$ around the origin. The smoothing operators are maps

$$S_t^l : C^\infty(\bar{B}_R) \longrightarrow C^\infty(\bar{B}_r)$$

for $1 > R > r > 0$ and $t \geq 1/(R-r)$, and are otherwise as described above. These variations seem to be part of the same phenomenon, although the version used in Conn allows one to work with a large family of graded Fréchet spaces.

2.4.2 Basic properties and interpolation estimates

Definition/Proposition 2.4.3 (Interpolation estimates)**.** *If a graded Fréchet space F allows smoothing operators, then the interpolation estimates hold: for all $\beta \le l \le m \le n$ with $n - m \ge \delta$ there is a constant $C > 0$, dependent on l, m and n, such that*

$$\|f\|_m^{n-l} \le C \|f\|_n^{m-l+\delta} \|f\|_l^{n-m-\delta}, \quad \forall f \in F.$$

Proof. Note that if $\|f\|_l = \|f\|_n$ the estimate is trivial, so we may assume strict inequality, that is,

$$\|f\|_l < \|f\|_n .$$

By the assumption $l \ge \beta$ the previous lemma implies that $\|f\|_l \ne 0$. Since we assumed that $n - m \ge \delta$,

$$\|f\|_m \le \|S_t f\|_m + \|f - S_t f\|_m \le C(t^{m-l+\delta} \|f\|_l + t^{m-n+\delta} \|f\|_n).$$

Now choose t such that these two summands are equal, this is when

$$t^{n-l} = \|f\|_n / \|f\|_l > 1.$$

In particular, $t > 1$, hence such a choice is allowed. This leads to the estimate

$$\|f\|_m^{n-l} \le C t^{(m-l+\delta)(n-l)} \|f\|_l^{n-l} \le C \|f\|_n^{m-l+\delta} \|f\|_l^{n-m-\delta},$$

as required. □

The smoothing operators that appear in all of our examples are of the strict kind; this suggests one could remove the defect δ from the definition. Yet in the abstract setting of graded Fréchet spaces and tame manifolds it is more natural to allow a strictly positive defect. This is best illustrated by the following lemma.

Lemma 2.4.4. *Let E and F be graded Fréchet spaces which are tamely linear isomorphic. Then E allows smoothing operators if and only if F does.*

If the spaces are 0-tamely linear isomorphic, then the one allows strict *smoothing operators if and only if the other does.*

Proof. Let $S_t : F \to F$ be a smoothing operator for F with base β and defect δ. There exists a tame linear isomorphism $\varphi : E \to F$. Hence there is a $s \ge 0$ so that for all $n \ge b$ there is a $C > 0$ such that we have estimates

$$\|\varphi e\|_n \le C \|e\|_{n+s}, \quad \left\|\varphi^{-1} f\right\|_n \le C \|f\|_{n+s}$$

for all $e \in E$ and $f \in F$. Now choose a new base $\beta' \ge \max(\beta + s, b)$ and $\delta' \ge \delta + 2s$. Then for all $n \ge \beta'$, and $r \ge 0$,

$$\left\|\varphi^{-1} S_t \varphi e\right\|_{n+r}' \le C \|S_t \varphi e\|_{n+r+s} \le C t^{r+\delta+2s} \|\varphi e\|_{n-s} \le C t^{r+\delta'} \|e\|_n ,$$

and for all $n \ge \beta'$, $r \ge \delta'$,

$$\left\|e - \varphi^{-1} S_t \varphi e\right\|_n \le C \|\varphi e - S_t \varphi e\|_{n+s} \le C t^{-r+\delta+2s} \|\varphi e\|_{n+r-s} \le C t^{-r+\delta'} \|e\|_{n+r}$$

give the required estimates. If $s = 0$ one can take $\delta' = \delta$; this proves the second statement. □

The above lemma motivates why we keep track of the degree of tame maps between graded Fréchet spaces and, later, tame Fréchet manifolds.

Remark 2.4.5. *It seems likely that all natural (linear) isomorphisms between the relevant graded Fréchet spaces are* 0*-tame. Moreover, the transition maps of all our examples of smooth manifolds are smooth* 0*-tame. Hence it might really be sufficient to work with strict smoothing operators.*

The following lemma is useful in finding smoothing operators on graded Fréchet spaces. Recall that E is a tame direct summand of F if there is a linear subspace E' of F such that $F \simeq E \oplus E'$. In particular, there exist tame linear maps $i : E \to F$, the inclusion, and $p : F \to E$, the projection, so that $p \circ i = \mathrm{id}$. Conversely, if such maps exist then $E' := \ker(p)$ defines a tame compliment to E; the inclusion $E' \hookrightarrow F$ and projection $(\mathrm{id} - i \circ p)$ of E' are also tame.

Lemma 2.4.6. *A tame direct summand E of F allows smoothing operators if F does. Moreover, if i and p are* 0*-tame, E allows strict smoothing operators if F does.*

Proof. Let $S_t : F \to F$ be a smoothing operator for F. The new grading on E defined by inclusion,

$$\|e\|'_n = \|i(e)\|_n \,,$$

is tamely equivalent to the original grading on E and the composition $p \circ S_t \circ i$ defines a smoothing operator for this grading. Namely, take $\delta' = \delta + s$, where $s \geq 0$ is the degree of $i \circ p$, then for $n \geq \beta$ and $r \geq 0$ we have

$$\begin{aligned} \|pS_tie\|'_{n+r} &= \|ipS_tie\|_{n+r} \\ &\leq C \|S_tie\|_{n+r+s} \\ &\leq Ct^{r+\delta+s} \|ie\|_n \\ &\leq Ct^{r+\delta'} \|e\|'_n \end{aligned}$$

and for $n \geq \beta$ and $r \geq \delta'$ we have

$$\begin{aligned} \|e - pS_tie\|'_n &= \|ipie - ipS_tie\|_n \\ &\leq C \|ie - S_tie\|_{n+s} \\ &\leq Ct^{-r+\delta+s} \|ie\|_{n+r} \\ &\leq Ct^{-r+\delta'} \|e\|'_{n+r} \,. \end{aligned}$$

In particular, if the degree of $i \circ p$ is zero, then we may take $\delta' = \delta$. If, in addition, i and p are of degree zero, then $\|{-}\|_n$ is 0-tamely equivalent to the original grading on E. This proves the second statement. □

Remark 2.4.7. *In [Ham82b] Hamilton defines the notion of a tame Fréchet space. It is a graded Fréchet space that is the tame summand of $\Sigma(B)$. Here $\Sigma(B)$ is the space of all sequences $f = \{f_k\}$ in a Banach space B such that the semi-norms*

$$\|f\|_n = \sum_{k=0}^{\infty} e^{nk} \|f_k\|$$

are finite. This is a graded Fréchet space, and it is particularly easy to check that $\Sigma(B)$ allows smoothing operators. For let $s : \mathbb{R} \to \mathbb{R}_+$ such that $s(r) = 0$ if $r \leq 0$ and $s(r) = 1$ if $r \geq 1$. Then one defines smoothing operators by

$$(S_t f)_k = s(t-k) f_k,$$

one simply cuts off all terms f_k in the sequence for $k \geq t$. Hamilton then argues that the above spaces, for which we constructed smoothing operators, are tame direct summands of some $\Sigma(B)$.

The following property of smoothing operators tells us a lot about whether a graded Fréchet space allows smoothing operators.

Lemma 2.4.8. *If a graded Fréchet space F allows smoothing operators of base β, then the seminorms $\|-\|_n$ are norms for all $n \geq \beta$.*

Proof. Suppose that $\|f\|_n = 0$ with $n \geq \beta$, then $\|S_t f\|_{n+r} = 0$ for all $r \geq 0$. Now the convergence $S_t f \to f$ as $t \to \infty$ implies that $\|f\|_{n+r} = 0$ for all $r \geq 0$, hence also $f = 0$ by the assumed Hausdorffness. $\square$

2.4.3 Existence of smoothing operators on $\Gamma_M E$

Let us now look at the existence of smoothing operators for the graded Fréchet spaces $\Gamma_M E$.

Proposition 2.4.9. *Let $E \to M$ be a vector bundle with a compact base. Then $\Gamma_M E$ allows smoothing operators.*

The approach is to first construct a particular example and then deduce the general result. Note that for $K \subseteq \mathbb{R}^d$ a compact set the space $C^\infty_K(\mathbb{R}^d)$ of smooth maps $\mathbb{R}^d \to \mathbb{R}^d$ with support in K is a graded Fréchet space with the usual C^k-norms. The Schwarz space $\mathcal{S}(\mathbb{R}^d)$ can be equipped with the same grading of C^k-norms. We will construct 'smoothing operators'

$$S_t : C^\infty_K(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$$

in the sense that the S_t are linear maps satisfying the two necessary estimates. This will be the starting point for producing actual smoothing operators on graded Fréchet spaces.

Remark 2.4.10. *Note that the C^k-norms aren't well-defined on $C^\infty(\mathbb{R}^d)$ since $\mathbb{R}^d$ is not compact. One needs to work in at least the linear subspace of all smooth functions for which it and all its derivatives are bounded. This is in particular true for all Schwartz functions. The smoothing operators are constructed using the Fourier transform and consequently take values in the Schwartz spaces. Although one could take a bigger codomain for the S_t above, the current codomain suffices while $C^\infty(\mathbb{R}^d)$ does not.*

Lemma 2.4.11. *For every compact set $K \subseteq \mathbb{R}^d$ the graded Fréchet space $C^\infty_K(\mathbb{R}^d)$ allows strict smoothing operators in the sense described above.*

Proof. Define a smooth function with compact support $\varphi : \mathbb{R} \to \mathbb{R}$ with

$$\varphi(x) = 1, \quad \text{for } x \leq 1,$$
$$\varphi(x) = 0, \quad \text{for } x \geq 2,$$

and φ monotone decreasing on $[1, 2]$. One could, for example, take

$$\varphi(x) = e^{\frac{e^{1/(1-x)}}{x-2}}$$

on the interval $1 \leq x \leq 2$. We then define maps $\chi_t : \mathbb{R}^d \to \mathbb{R}$ by designating their fourier transforms as

$$\hat{\chi}_t(\xi) = \varphi(\|\xi\| / t)$$

for all $t > 0$ and $\xi \in \mathbb{R}^d$. χ_t is a Schwartz function, it and its derivatives vanish at infinity faster than any given polynomial on the coordinates of $\mathbb{R}^d$. As t tends to infinity, χ_t becomes more concentrated at the origin, but its integral remains constant. This follows from the following, easily derived, formula

$$\begin{aligned}\chi_t(x) &= \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x\rangle} \varphi(\|\xi\| / t) d\xi \\ &= t^d \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, tx\rangle} \varphi(\|\xi\|) d\xi \\ &= t^d \chi_1(tx).\end{aligned}$$

From the above formula we also obtain

$$\partial^\alpha \chi_t(x) = t^{d+|\alpha|}\chi_1^{(\alpha)}(tx),$$

where $\chi_1^{(\alpha)}(tx)$ indicates the partial derivative $\frac{\partial^\alpha}{\partial y^\alpha}\chi_1(y)$ at the point $y = tx$. Now define the smoothing operators $S_t : C_K^\infty(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ by convolution with χ_t,

$$S_t f(x) = (\chi_t * f)(x) = \int_{\mathbb{R}^d} \chi_t(y) f(x-y)dy.$$

The equalities above imply

$$\begin{aligned}\partial^\alpha(\chi_t * f)(x) &= (\partial^\alpha \chi_t) * f(x)\\ &= t^{d+|\alpha|}\int_{\mathbb{R}^d}\chi_1^{(\alpha)}(ty)f(x-y)dy\\ &= t^{|\alpha|}\int_{\mathbb{R}^d}\chi_1^{(\alpha)}(y)f(x-y/t)dy,\end{aligned}$$

hence from

$$|\partial^\alpha(\chi_t * f)(x)| \leq t^{|\alpha|}\left\|f\right\|_0 \int_{\mathbb{R}^d}\left|\chi_1^{(\alpha)}(y)\right| dy \leq Ct^{|\alpha|}\left\|f\right\|_0$$

we can conclude that

$$\left\|S_t f\right\|_r \leq Ct^r \left\|f\right\|_0.$$

The observation that $\partial^\alpha(\chi_t * f) = \chi_t * (\partial^\alpha f)$ now completes the first strict smoothing estimate by applying the above to $\partial^\alpha f$ instead.

Note that χ_t is smooth in the parameter $t > 0$. We obtain an expression for the derivative via its Fourier transform,

$$\widehat{\frac{d}{dt}\chi_t}(\xi) = \frac{d}{dt}\widehat{\chi_t}(\xi) = \frac{d}{dt}\varphi(\left\|\xi\right\|/t) = \frac{\left\|\xi\right\|}{t^2}\varphi'(\left\|\xi\right\|/t),$$

where φ' is the derivative of $\varphi : \mathbb{R} \to \mathbb{R}$; it satisfies

$$\varphi'(u) = 0, \quad \text{for } u < 1 \text{ or } u > 2.$$

By replacing φ with $\psi(u) = u\varphi'(u)$ in the computation above, we also deduce the equation

$$\frac{d}{dt}\chi_t(x) = t^{d-1}\left(\left.\frac{d}{dt}\right|_{t=1}\chi_t\right)(tx).$$

We can compute the convolution of $\frac{d}{dt}\chi_t$ with $f \in C_K^\infty(\mathbb{R}^d)$; first we make a change of variables $tx \mapsto x$, and then we apply partial integration in the y-variable, using that f is compactly supported, to obtain

$$\begin{aligned}
\frac{d}{dt}\chi_t * f(x) &= t^{d-1}\left(\frac{d}{dt}\bigg|_{t=1}\chi_t\right)(tx) * f(x)\\
&= t^{-1}\left(\frac{d}{dt}\bigg|_{t=1}\chi_t\right)(x) * f(x/t)\\
&= t^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(e^{-2\pi i\langle\xi,y\rangle}\,\|\xi\|\,\varphi'(\|\xi\|)\right)f(x-y/t)d\xi dy\\
&= t^{-1-|\alpha|}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left((-2\pi i)^{-|\alpha|}e^{-2\pi i\langle\xi,y\rangle}\frac{\|\xi\|}{\xi^\alpha}\varphi'(\|\xi\|)\right)\partial^\alpha f(x-y/t)dyd\xi\\
&= t^{-1-|\alpha|}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(-2\pi i)^{-|\alpha|}e^{-2\pi i\langle\xi,y\rangle}\frac{\|\xi\|}{\xi^\alpha}\varphi'(\|\xi\|)d\xi\right)\partial^\alpha f(x-y/t)dy.
\end{aligned}$$

The inner integral is well-defined since φ' is compactly supported and vanishes at a neighborhood of the origin. It can be bounded by some constant $C>0$ independent of $y\in\mathbb{R}^d$ since $\left|(-i)^{-|\alpha|}e^{-2\pi i\langle\xi,y\rangle}\right|\leq 1$. Hence we obtain an estimate of the form

$$\begin{aligned}
\left|\frac{d}{dt}\chi_t * f(x)\right| &\leq t^{-1-|\alpha|}\int_{\mathbb{R}^d}C\left|\partial^\alpha f(x-y/t)\right|dy\\
&\leq t^{-1-|\alpha|}\int_{y\in K}Cdy\sup_{x\in\mathbb{R}^d}\left|\partial^\alpha f(x)\right|\\
&\leq Ct^{-1-|\alpha|}\sup_{x\in\mathbb{R}^d}\left|\partial^\alpha f(x)\right|.
\end{aligned}$$

By applying the above estimates to the partial derivatives $\partial^\beta f(x)$ of f instead, we obtain

$$\left\|\frac{d}{dt}S_tf\right\|_k = \left\|\frac{d}{dt}\chi_t * f\right\|_k \leq Ct^{-1-r}\left\|f\right\|_{k+r}.$$

To obtain the second estimate for smoothing operators. Note that

$$S_tf = \chi_t * f \to f$$

as $t\to\infty$ uniformly for all $f\in C_K^\infty(\mathbb{R}^d)$, hence

$$f - S_tf = \int_t^\infty \frac{d}{ds}S_sfds.$$

So we conclude that, for $t>1$,

$$\begin{aligned}
\left\|if - S_tf\right\|_k &= \left\|\int_t^\infty\frac{d}{ds}S_sfds\right\|_k\\
&\leq \int_t^\infty\left\|\frac{d}{ds}S_sf\right\|_k ds\\
&\leq C\int_t^\infty s^{-1-r}\left\|f\right\|_{k+r}ds\\
&\leq Ct^{-r}\left\|f\right\|_{k+r},
\end{aligned}$$

where $i: C_K^\infty(\mathbb{R}^d)\hookrightarrow\mathcal{S}(\mathbb{R}^d)$ is the inclusion. $\square$

Suppose that M is a closed compact manifold. We will first show that $C^\infty(M,\mathbb{R}^n)$ allows smoothing operators. Note that

$$C^\infty(M,\mathbb{R}^n) = C^\infty(M) \oplus \ldots \oplus C^\infty(M)$$

is a 0-tame direct summand, hence it suffices to construct smoothing operators on $C^\infty(M)$. For this, embed M into some Euclidean space $\mathbb{R}^d$ such that it lies in the open unit ball $B_1(0)$. Now define a linear map

$$\varepsilon : C^\infty(M) \to C^\infty_{\bar{B}_1(0)}(\mathbb{R}^d)$$

as follows. Choose a tubular neighborhood of M in $\mathbb{R}^d$, which will remain fixed during the construction. The tubular neighborhood may be taken small enough such that it still lies inside $B_1(0)$. Also choose a (fixed) smooth bump function $\chi : \mathbb{R}^d \to \mathbb{R}$ that is 1 on an open neighborhood of M and vanishes outside the tubular neighborhood around M. To define ε, first extend the maps $f \in C^\infty(M)$ to be constant along the fibers. Then cut it off with the bump function and extend it by zero to a smooth map on $\mathbb{R}^d$.

The map described above is 0-tame linear. We've already seen that multiplication by a smooth function is 0-tame, while both smooth extensions are easily bounded by the original map.

On the other hand, the restriction map

$$\rho : \mathcal{S}(\mathbb{R}^d) \longrightarrow C^\infty(M)$$

is 0-tame linear for trivial reasons. Moreover, via the 0-tame inclusion, both spaces use the same semi-norms,

$$i : C^\infty_{\bar{B}_1(0)}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$$

we have that the composition $\rho i \varepsilon = \mathrm{id}$ is the identity. Hence $C^\infty(M)$ allows strict smoothing operators of the form

$$\rho \circ S_t \circ \varepsilon : C^\infty(M) \longrightarrow C^\infty(M).$$

Finally, any vector bundle $E \to M$ is a summand of a trivial vector bundle,

$$M \times \mathbb{R}^n = E \oplus F,$$

for a large enough $n \in \mathbb{N}$. We obtain $\Gamma_M E$ as a 0-tame direct summand

$$C^\infty(M,\mathbb{R}^n) = \Gamma_M E \oplus \Gamma_M F,$$

hence it also allows strict smoothing operators. This completes the proof of proposition 2.4.9 on page 49.

2.4.4 Smoothing operators when M has a boundary

In [Ham82b] a method is given to obtain smoothing operators for a compact manifold with boundary; it goes along the following lines. If M is a compact manifold with boundary, its double $M^{\#}$, that is, the manifold consisting of two, originally disconnected, copies of M glued smoothly along the boundary, is a compact manifold without boundary. The vector bundle $E \to M$ naturally extends to one on $E \to M^{\#}$, and $\Gamma_{M^{\#}}E$ allows smoothing operators. We will now construct tame linear maps

$$\Gamma_M E \xrightarrow{i} \Gamma_{M^{\#}} E \xrightarrow{p} \Gamma_M E,$$

with $p \circ i = \mathrm{id}$, so that $\Gamma_M E$ can be seen as a direct summand of $\Gamma_{M^{\#}}E$, and hence also allows smoothing operators. The choice of p is obvious; it should be the restriction map to the compact region M of $M^{\#}$, and it is already shown to be tame linear.

For the map i we have to describe a uniform method of extending a smooth section $\sigma \in \Gamma_M E$ to one on $M^{\#}$. We can first define this extension in local coordinates and then patch it together with a partition of unity. After all, we can cover M with local trivializations $\{(U, \varphi, \psi)\}$ whose closures are compact regions, say open balls, and which are the restriction of some covering $\{U\}$ on $M^{\#}$. Then the simultanuous restriction

$$\Gamma_M E \longrightarrow \bigoplus\nolimits_{U \cap M} \Gamma_{\bar{U}} E_{\bar{U}},$$

is a tame linear map. And so is multiplication with a partition function subordinate to $\{U\}$ and the subsequent summing of all sections. For the latter we actually use that the sections, after multiplying with the partition functions, lie in $\Gamma_{\bar{U},0}E$, the graded Fréchet space of smooth sections on $\bar{U}$ that vanish at the boundary. Those sections extend to $\Gamma_{M^{\#}}E$ by zero, and this extension is obviously tame linear. Only after this can the sections be summed, and the sum

$$\sum : \bigoplus\nolimits_U \Gamma_{M^{\#}} E \longrightarrow \Gamma_{M^{\#}} E$$

is obviously a tame linear map. We now only need to describe how to locally extend a section beyond the boundary of M.

In local coordinates we end up with a smooth map $\tilde{\sigma} : \mathbb{R}_+ \times \mathbb{R}^{n-1} \to \mathbb{R}^k$. It extends to a smooth map on $\mathbb{R}^n$ by defining

$$\tilde{\sigma}(-x, y) = \int_0^\infty \varphi(t)\tilde{\sigma}(tx, y)\, dt, \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}^{n-1},$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$\int_0^\infty t^n \varphi(t)\, dt = (-1)^n$$

for all $n \in \mathbb{N}$. A typical example of such a map is

$$\varphi(t) = \frac{e^{2\sqrt{2}}}{\pi(1+t)} e^{-(t^{1/4}+t^{-1/4})} \sin(t^{1/4} - t^{-1/4}).$$

The integration converges since φ is a Schwartz function, and a simple estimate of the form

$$\left|\partial^\alpha \int_0^\infty \varphi(t)(tx,y)\,dt\right| \leq \int_0^\infty \varphi(t)\,dt \sup_{(x,y)\in\mathbb{R}_+\times\mathbb{R}^{n-1}} |f(x,y)|$$

shows that such an extension by integration gives a tame linear map. The identity $p \circ i = \mathrm{id}$ is now obvious, hence we have produced smoothing operators on $\Gamma_M E$.

Chapter 3

Tame manifolds and examples

This chapter consists of two parts. First we discuss the basic definitions of tame Fréchets manifolds, which boils down to a Hausdorff manifold whose local model is a graded Fréchet space, and for which all transition maps are smooth tame. Next we discuss the different examples of tame manifolds that are directly related to differential geometry, and prove tameness conditions for several associated maps. Although the Nash-Moser theorem is essentially a local result, many sets of geometric objects of interest, such as foliations, and the like, lie in certain Fréchet manifolds, and it is conceptually clearer to describe the entire manifold instead of just particular neighborhoods.

3.1 Tame Fréchet manifolds

We have discussed the definition of Fréchet spaces, the direct consequences and some basic examples. Moreover, we have defined differentiability for maps between Fréchet spaces. This gives rise to the notion of a Fréchet manifold.

Definition 3.1.1. *A* Fréchet manifold *is a Hausdorff space $\mathcal{M}$ with an atlas of coordinate charts*

$$\{(U_i, \varphi_i : U_i \to F_i) : i \in I\},$$

where each F_i is a Fréchet space and the transition functions $\varphi_j \circ \varphi_i^{-1}$ are diffeomorphisms $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$. As usual, an equivalence class of atlases can be represented with a maximal atlas, and we assume that a Fréchet manifold is equipped with its maximal atlas. The notion of smooth maps between Fréchet manifolds is also analogous to the finite dimensional case.

*Let **FrMfd** denote the category of Fréchet manifolds with smooth maps. The finite dimensional manifolds are a* full *subcategory of **FrMfd**.* △

Note that we do not assume second countability, as is usual with finite dimensional manifolds. This is necessary, since Fréchet spaces are already not always second countable; this would be equivalent to seperability. As a consequence, **FrMfd** also contains the, often pathological, examples of non-second countable manifolds found in the finite dimensional case.

Remark 3.1.2. *Note that $\mathcal{M}$ has a fixed local model, up to isomorphism, on any of its connected components since the derivative at $m \in \mathcal{M}$,*

$$D(\varphi_j \varphi_i^{-1})(m) : F_i \to F_j,$$

is a continuous linear isomorphism for every $m \in \varphi_i(U_i \cap U_j)$. In contrast with the conventions of finite dimensional manifolds, we do allow the local model to vary on different connected components. One of the main examples of Fréchet manifolds will exhibit this property.

Next we wish to extend our definition of a Fréchet manifold to include the concepts of grading and tame maps.

Definition 3.1.3 (Tame Fréchet manifolds)**.** *A* tame *(Fréchet) manifold is a Hausdorff space $\mathcal{M}$ with an atlas of charts*

$$\{(U_i : \varphi_i : U_i \to F_i), i \in I\}$$

such that the F_i are graded Fréchet spaces and the transition functions are tame diffeomorphisms. We assume a tame manifold is equipped with its maximal tame atlas.

A smooth tame map between tame manifolds is defined in the same manner as a smooth tame map between graded Fréchet spaces. That is, a map $P : \mathcal{M} \to \mathcal{N}$ is tame *if for every $m \in \mathcal{M}$ there are open neighborhoods $m \in U \subseteq \mathcal{M}$ and $P(m) \in V \subseteq \mathcal{N}$ contained in charts such that: there is a degree $r_m \in \mathcal{N}$, a base $b_m \in \mathcal{N}$ and for every $k \geq b_m$ a constant $C = C_{m,k} > 0$ such that*

$$\|P(m')\|_k \leq C(1 + \|m'\|_{k+r_m}), \quad \forall m' \in U.$$

The map P is smooth tame *if it and all its derivatives are tame. Let* ***Tame*** *denote the category of tame manifolds with smooth tame maps.*

Likewise, a 0-tame manifold *has an atlas with only smooth 0-tame transition maps, that is, all transition maps*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are smooth, 0-tame and all their derivatives are 0-tame as well. A 0-tame manifold is assumed to be equipped with its maximal 0-tame atlas.

Let ***0Tame*** *denote the full subcategory of 0-tame manifolds. In this category it makes sense to speak of the degree r_m of a map $P : \mathcal{M} \to \mathcal{N}$ in a point $m \in \mathcal{M}$. There might, however, be no common bound $r_m \leq r$ for all $m \in \mathcal{M}$.* △

An *(embedded) submanifold* $\mathcal{N}$ of $\mathcal{M}$ is a subset that can be covered with an atlas of the form

$$\{(U_i, \varphi : U_i \to E_i \oplus F_i)\}$$

of $\mathcal{M}$. Here we impose that the local model $E_i \oplus F_i$ is the direct sum of two graded Fréchet spaces, and $\varphi_i(\mathcal{N} \cap U_i) = \varphi(U_i) \cap E_i \times \{0\}$, the submanifold corresponds to the first summand. This is the usual notion of embedded submanifold for finite dimensional manifolds. Note that one must enforce that the E_i are direct summands of F_i, since not every tame linear subspace is a direct summand. Note that this definition also works in **Tame** and **0Tame**, except that the described atlas should be tame respectively 0-tame. From this the following is evident.

Corollary 3.1.4. *The product $\mathcal{M} \times \mathcal{N}$ of two (0-tame) Fréchet manifolds is a (0-tame) Fréchet manifold. Both components $\mathcal{M}$ and $\mathcal{N}$ are embedded (0-tamely) into $\mathcal{M} \times \mathcal{N}$.*

There are some difficulties with defining injective immersions as submanifolds, due to the lack of a 'immersion theorem'. One can only work with the conclusion of the usual immersion theorem. As such, an *immersed submanifold* is a Fréchet manifold $\mathcal{N}$ together with a smooth injective map $i : \mathcal{N} \to \mathcal{M}$ such that for every $n \in \mathcal{N}$ there is an open subset $n \in V \subseteq \mathcal{N}$ and a chart $(U_i \to E_i \oplus F_i)$ as above so that

$$\varphi_i(i(V) \cap U_i) \subseteq E_i$$

is an open subset of $E_i \subseteq E_i \oplus F_i$.

The same difficulty arises with defining submersive maps. Hence define a *submersion* as a smooth map $p : \mathcal{N} \to \mathcal{M}$ such that for every $n \in \mathcal{N}$ there is a chart $(U, \varphi : U \to E \oplus F)$ around n, and there is a chart $(V, \psi : V \to E)$ around $p(n)$ such that $p(U) = V$, and the square

$$\begin{tikzcd} U \arrow[r, "\varphi"'] \arrow[d, "p"] & E \oplus F \arrow[d, "pr_E"] \\ V \arrow[r, "\psi"'] & E \end{tikzcd}$$

commutes. Here $pr_E : E \oplus F \to E$ is the projection onto the first component, which is a continuous linear map since F_i is closed in $E_i \oplus F_i$. In **Tame** and **0Tame** the direct sum should be tame and 0-tame respectively. Note that the derivative of a submersion is everywhere surjective, but that the converse doesn't always hold. The condition that $\varphi_i(U_i)$ is a closed subspace of $\psi_i(p^{-1}U_i)$ is, moreover, a weaker condition than the described splitting in a direct sum.

We will also refer to submersions as *bundles* over $\mathcal{M}$ with total space $\mathcal{N}$. Then a *bundle map* from $p : \mathcal{N} \to \mathcal{M}$ to $p' : \mathcal{N}' \to \mathcal{M}'$ is then a pair of smooth maps $P_1 : \mathcal{N} \to \mathbb{N}'$ and $P_0 : \mathcal{N} \to N'$ satisfying the usual bundle map condition

$$\begin{tikzcd} \mathcal{N} \arrow[r, "P_1"] \arrow[d, "p"] & \mathcal{N}' \arrow[d, "p'"] \\ \mathcal{M} \arrow[r, "P_0"] & \mathcal{M}'. \end{tikzcd}$$

More specified, a Fréchet *fiber bundle* with fiber F is a bundle $p : \mathcal{N} \to \mathcal{M}$ for which there is an open covering $\{U_i\}$ of $\mathcal{M}$ and for every i a diffeomorphism $\psi_i : \mathcal{N}_{U_i} \to U_i \times F$, such that the transition maps are of the form

$$\psi_j \circ \psi_i = (\mathrm{id}, \psi_{ij}) : U_i \cap U_j \times F \to U_i \cap U_j \times F,$$

as usual. In **Tame** and **0Tame** these transition maps should be tame respectively 0-tame.

A Fréchet *vector bundle* with fiber $\mathcal{F}$ are defined in the same spirit as fiber bundles. Namely, it is a fiber bundle $p : \mathcal{N} \to \mathcal{M}$ such that its fiber is a fixed Fréchet space $\mathcal{F}$ and the transition maps $\psi_{ij} : \mathcal{F} \to \mathcal{F}$ are continuous linear isomorphisms. A particular example is the tangent bundle of a tame Fréchet manifold. It is the usual set of velocities of curves.

Definition 3.1.5 (Tangent bundle)**.** *Let $\mathcal{M}$ be a tame Fréchet manifold and $p \in \mathcal{M}$ a fixed point. Two smooth curves*

$$\sigma, \gamma \in C^\infty((-\varepsilon, \varepsilon), \mathcal{M}), \quad \sigma(0) = \gamma(0) = p,$$

through p are equivalent if there exists a tame chart (U, φ) of $\mathcal{M}$ around p such that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \varphi \circ \sigma = \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi \circ \gamma.$$

In this case, by the chain rule, the above equality holds if φ is replaced by any other chart (V, ψ) around p. The set $T_p\mathcal{M}$ of equivalence classes of smooth curves through p is the tangent space of $\mathcal{M}$ at p. The tangent bundle is, as a set, the disjoint union of all $T_p\mathcal{M}$ with $p \in \mathcal{M}$.

Every chart (U, φ) around the point $p \in \mathcal{M}$ endows $T_p\mathcal{M}$ with the structure of a graded Fréchet space, and these structures are all tame linear isomorphic. The above definition can also be formulated for 0-tame manifolds; in which case the graded Fréchet structure on $T_p\mathcal{M}$ is uniquely determined by a chart (U, φ) up to a 0-tame linear isomorphism.

Corollary 3.1.6. *The tangent space $T\mathcal{M}$ is a vector bundle over the Fréchet manifold $\mathcal{M}$. If in addition $\mathcal{M}$ is tame (0-tame) then $T\mathcal{M}$ is tame (0-tame).*

Proof. To better illustrate the 0-tameness of the tangent bundle, we will give a short proof. Recall how the tangent bundle is defined: There is map $C^\infty((-1,1), \mathcal{M}) \times C^\infty(\mathcal{M}) \to \mathbb{R}$ defined by

$$\langle \sigma_t, f \rangle = \left.\frac{d}{dt}\right|_{t=0} f \circ \sigma_t.$$

Two smooth curves σ_t and $\tilde{\sigma}_t$ with $\sigma_0 = \tilde{\sigma}_0$ are equivalent if

$$\langle \sigma_t, f \rangle = \langle \tilde{\sigma}_t, f \rangle, \quad \forall f \in C^\infty(\mathcal{M})$$

and the tangent space at σ_0 is the set of equivalence classes. Any chart $(U, \varphi : U \to F)$ at σ_0 induces a map

$$TU \to \varphi(U) \times F : [\sigma_t] \mapsto (\varphi(\sigma_0), \left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \sigma_t).$$

And for any other chart (V, ψ) the transition function

$$\varphi(U \cap V) \times F \to \psi(U \cap V) \times F$$

sends the vector $\frac{d}{dt}\big|_{t=0} (\varphi \circ \sigma_t)$ to $\frac{d}{dt}\big|_{t=0} (\psi \circ \sigma_t) = D_{\varphi(\sigma_0)}(\psi\varphi^{-1}) \frac{d}{dt}\big|_{t=0} (\varphi \circ \sigma_t)$. Hence the transition function is just $T(\psi\varphi^{-1})$, which is tame (0-tame) if $\psi\varphi^{-1}$ is tame (0-tame). $\square$

A *Fréchet Lie group* can be defined as expected as a group object in the category of Fréchet manifolds with smooth maps. More concretely, it is a Fréchet manifold $\mathcal{G}$ together with a specified element $e \in \mathcal{G}$ and two maps $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, multiplication, and $i : \mathcal{G} \to \mathcal{G}$, inversion, satisfying the usual group axioms. Of course, for a tame Lie group the corresponding maps need to be smooth tame. The most prominent example of a tame Lie group is the space of diffeomorphisms of a compact manifold. Likewise, a tame Lie algebra is a graded Fréchet space $\mathfrak{g}$ with a tame bilinear Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. The main example of a tame Lie algebra is the space of vector field $\mathcal{X}(M)$ on a compact manifold M, with the usual commutator bracket.

3.2 Examples of tame Fréchet manifolds

With the basic definitions out of the way, we may have a closer look at the quintessential examples of tame manifolds. First we have a direct generalisation of the fact that $\Gamma_M E$, with E a vector bundle with compact base, is a graded Fréchet space. Namely, the smooth sections of any surjective submersion $p : B \to M$ with compact codomain form a tame manifold. Its local model is of the form $\Gamma_M E$, hence it locally allows smoothing operators.

As a consequence, the spaces of maps $C^\infty(M, N)$ are tame manifolds, and allow smoothing operators, if M is compact. Of particular interest is the open subspace $\mathbf{Diff}(M)$ of diffeomorphisms and the fact that this forms a tame Lie group.

In the last sections we extend our list of examples, in particular to include the spaces of bundle maps between any pair of surjective submersions. This gives a more flexible way of identifying tame manifolds and maps.

3.2.1 A tubular neighborhood lemma

Let $B \xrightarrow{p} M$ be a surjective submersion. Let $\sigma \in \Gamma_M B$ be a section of B. We will prove the existence of a tubular neighborhood around the image of σ in B such that the exponent map preserves the fibers of p as described in the paragraphs below.

The vertical bundle $T^{\mathrm{vert}}B$ of B is the linear subbundle of TB defined by

$$T^{\mathrm{vert}}_y B = \ker(d_y p) = T_y B_{p(y)}$$

for every $y \in B$. Vectors in $T^{\mathrm{vert}}B$ are called vertical vectors.

Let $A \subseteq B$ be any submanifold of B. With NA we denote the normal bundle of A in B. It is given by the quotient

$$N_x A = T_x B / T_x A$$

for every $x \in A$. The zero section $z \in \Gamma_A NA$ gives a canonical embedding $z(A)$ of A into NA. By choosing a Riemmannian metric on B, the tubular neighborhood theorem gives an open subset $U \subseteq NA$ around A, an open subset $V \subseteq B$ around A, and a diffeomorphism

$$\exp : U \to V.$$

This gives the open V around A the structure of a vector bundle on A by stipulating that exp should be an isomorphism of vector bundles. This construction is known as a tubular neighborhood, and exp is called an exponent map.

Let $\sigma \in \Gamma_M B$ be a section of B. The image $\sigma(M)$ of σ describes M as an embedded submanifold of B, and we tacitly identify $\sigma(M) = M$. We wish to described a particular type of tubular neighborhood of $\sigma(M)$ in B. A regular tubular neighborhood gives a smooth map

$$\exp_\sigma : U_\sigma \to \nu(\sigma) \subseteq B$$

with $U_\sigma \subseteq N\sigma(M)$ and $\nu(\sigma)$ an open subset of B around $\sigma(M)$. The restriction of p is already a surjective submersion, since $\nu(\sigma)$ contains $\sigma(M)$. The exponent map completely ignores this, and defines an alternative bundle structure on $\nu(\sigma)$. In this section we define a tubular neighborhood such that $\exp_\sigma$ does preserve the fibers. The essential part of the proof is that the vertical bundle $T^{\text{vert}}B$ restricted to $M = \sigma(M)$ is isomorphic to the normal bundle of $\sigma(M)$. This follows from the observation that $\sigma(M) \cap B_x = \{\sigma(x)\}$ is just a single point for all $x \in M$.

Lemma 3.2.1 (Vertical tubular neighborhoods)**.** *Let $B \xrightarrow{p} M$ be a surjective submersion, and $\sigma \in \Gamma_M B$ a section of B. Then there exists an open subset $\nu(\sigma) \subseteq B$ around the image $\sigma(M)$, an open subset $U_\sigma \subseteq \sigma^* T^{vert} B$ around M, and a fiber preserving diffeomorphism*

$$\begin{tikzcd} U_\sigma \arrow[rr, "\exp_\sigma"] \arrow[dr] & & \nu(\sigma) \arrow[dl, "p"] \\ & M. & \end{tikzcd}$$

More generally, let $A \subseteq B \xrightarrow{p} M$ be a subbundle of B and g a Riemannian metric on B. Let $(TA)^\perp$ denote the orthogonal complement to TA in $TB|_A$, such that $(TA)^\perp \simeq NA$. Then there exists an open subset $\nu(A) \subseteq B$ around A, an open subset $U_A \subseteq (TA)^\perp \cap T^{vert}B|_A$ around A, and a fiber preserving diffeomorphism

$$\begin{tikzcd} U_A \arrow[rr, "\exp_A"] \arrow[dr] & & \nu(A) \arrow[dl, "p"] \\ & M, & \end{tikzcd}$$

where the diagonal arrow on the left-hand-side is the composition $U_A \to A \to M$.

Proof. We will only do the first part and give a sketch of the second part.

First assume that $B = M \times F$ is a trivial bundle. Choose a Riemannian metric g on B, and let g_F be its restriction to $TF \otimes TF$. Let $\pi_F : M \times F \to F$ denote the projection onto F. Let $U' \subseteq TF$ an open subset around F in TF such that the exponent map

$$\exp : U' \to F$$

induced by g_F is a diffeomorphism onto its image whenever it is restricted to a fiber $U'_x = U' \cap T_x F$, $x \in F$. Let $\sigma \in C^\infty(M, F)$ be a section of B seen as a smooth map $M \to F$. Define an open subset around its graph by

$$U := (\pi_F^* U')|_{\text{graph}_\sigma} \subseteq T^{\text{vert}} B|_{\text{graph}_\sigma}.$$

If we identify $M = \text{graph}_\sigma \subseteq M \times F$, then the required map is given by

$$\exp_\sigma := (\text{id}, \exp \circ \pi_F)|_{\text{graph}_\sigma} \subseteq T^{\text{vert}} B|_{\text{graph}_\sigma}.$$

Now assume that $B \xrightarrow{p} M$ is an arbitrary surjective submersion, and $\sigma \in \Gamma_M B$ is a section of B. Choose a Riemannian metric g on B. Cover the image $\sigma(M)$ by open subsets W_α for which $V_\alpha := p(W_\alpha)$ is open, and there are fiber preserving diffeomorphisms

$$
\begin{tikzcd}
W_\alpha \arrow[rr, "\psi_\alpha"] \arrow[dr, "p"'] & & V_\alpha \times F_\alpha \arrow[dl, "\mathrm{pr}_1"] \\
& V_\alpha. &
\end{tikzcd}
$$

Moreover, assume that every V_α contains a relatively compact open K_α, with $\bar{K}_\alpha \subseteq V_\alpha$, and that the family $\{K_\alpha\}$ still form a locally finite cover of M.

For every index α let g_α be the Riemannian metric on $V_\alpha \times F_\alpha$ such that ψ_α is an isometry. Let

$$\sigma_\alpha = \psi_\alpha \circ \sigma|_{U_\alpha}$$

be the local description of σ. Apply the previous paragraph to σ_α and g_α to obtain an exponent map

$$\exp_\alpha : U_\alpha \to \nu_\alpha \subseteq V_\alpha \times F_\alpha$$

with $U_\alpha \subseteq T^{\text{vert}}(V_\alpha \times F_\alpha)$ an open subset around $V_\alpha \times F_\alpha$, and $\nu_\alpha = \exp_\alpha(V_\alpha)$ an open subset around the image $\sigma_\alpha(V_\alpha)$.

Let d denote the distance metric on M introduced by g. Now choose a smooth map $\varepsilon : M \to \mathbb{R}_{>0}$ such that the open

$$\nu(\sigma) := \{b \in B : d(b, \sigma(p(b))) < \varepsilon(p(b))\} \subseteq B$$

satisfies

$$\nu(\sigma) \subseteq \bigcup\nolimits_\alpha \psi_\alpha^{-1}(\nu_\alpha),$$

and for every $m \in M$

$$\nu(\sigma)_m \subseteq \bigcap \left\{\psi_\alpha^{-1}(\nu_\alpha)|_{K_\alpha} : K_\alpha \text{ contains } m\right\}. \tag{3.2.1}$$

The local finiteness of $\{K_\alpha\}$ ensures that the right-hand-side of (3.2.1). The relative compactness of the K_α ensures that ε can be chosen nowhere zero.

For every index α define

$$U'_\alpha := T\psi_\alpha \circ \exp_\alpha^{-1} \circ \psi_\alpha^{-1} (\nu(\sigma)|_{K_\alpha}) \subseteq T^{\text{vert}}_{K_\alpha} B$$

to obtain a fiber preserving diffeomorphism

$$\exp'_\alpha := \psi_\alpha \circ \exp_\alpha \circ T\psi_\alpha^{-1} : U'_\alpha \to \nu(\sigma)|_{K_\alpha}$$

over K_α. Let β be another index such that $K_\alpha \cap K_\beta$ is non-empty. Condition (3.2.1) on $\nu(\sigma)$ ensures that the map

$$\psi_\beta \circ \psi_\alpha^{-1} : (K_\alpha \times F_\alpha) \cap \psi_\alpha(\nu(\sigma)|_{K_\beta}) \to (K_\beta \times F_\beta) \cap \psi_\beta(\nu(\sigma)|_{K_\alpha})$$

is an isometry on each of the fibers over $K_\alpha \cap K_\beta$. Hence the $\exp'_\alpha$ coincide on the intersections of the K_α, and they glue together to the desired fiber preserving diffeomoprhism

$$\exp_\sigma : U_\sigma \to \nu(\sigma).$$

For the second statement, one takes a cover $\{W_k\}$ of A with the analoguous properties instead. The roles of the V_α, interpreted as $V_\alpha = W_\alpha \cap \sigma(M)$, are replaced by the intersections $W_\alpha \cap A$. Moreover, use the metric g on B to define a normal bundle

$$(TA)^\perp \subseteq TB|_A,$$

and work with $T^{\text{vert}}B|_A \cap (TA)^\perp$ instead of $T^{\text{vert}}B$. □

Remark 3.2.2. *The first statement of the lemma gives the submersion $p : B \to M$ 'locally' around $\sigma(M)$ the structure of a vector bundle. It is now clear how this will lead to charts for $\Gamma_M B$. Similarly, the second statement gives $p : B \to M$ 'locally' around A the structure of a tower*

$$E \to A \to M$$

of a vector bundle over a submersion.

3.2.2 Smooth maps and sections of a submersion

We will discuss the main examples of tame manifolds. Let $B \xrightarrow{p} M$ be a surjective submersion with compact codomain. We will show that the space of sections $\Gamma_M B$ is a 0-tame manifold. Let N be another finite dimensional manifold. By considering the trivial bundle $M \times N \to M$, we deduce that $C^\infty(M, N) = \Gamma_M(M \times N)$ is a 0-tame manifold as well.

Recall that the vertical bundle $T^{\mathrm{vert}}B \to B$ of B is defined as the subbundle of TB whose fibers are

$$T_y^{\mathrm{vert}}B = \ker(d_y p)$$

for all $y \in B$. Suppose that $E = B \to M$ is a vector bundle over M, and let $z \in \Gamma_M E$ denote the zero section. If one considers

$$T^{\mathrm{vert}}E \xrightarrow{\pi_E} E \to M$$

as a fibered manifold over M, then each for each $m \in M$ we have a decomposition

$$\begin{aligned}(T^{\mathrm{vert}}E)_m &= \left\{(e, X) : e \in E_m, X \in T_e^{\mathrm{vert}}E\right\} \\ &\simeq E_m \oplus T_{z(m)}E_m \simeq E_m \oplus E_m.\end{aligned}$$

This decomposition is natural, and hence induces a decomposition

$$T^{\mathrm{vert}}E \simeq E \times_M E = E \oplus E,$$

in the following way. Let $e \in E$, and let

$$\tau_e : E \to E : f \mapsto f - e$$

denote linear translation by e. Its tangent map $T\tau_e : TE \to TE$ depends smoothly on $e \in E$, since it is just a partial derivative of the map

$$\tau : E \times E \to E : (e, f) \mapsto f - e.$$

Hence the map

$$T^{\mathrm{vert}}E \to E \oplus z^* T^{\mathrm{vert}}E \simeq E \oplus E : v \mapsto (\pi_E(v), T\tau_{\pi_E(v)}v)$$

is the desired isomorphism of manifolds fibered over M.

Recall the definition of the Whitney C^∞ topology W_∞ from definiton 2.3.3 on page 28. We assume that $\Gamma_M B$ is equipped with this topology. As mentioned in remark 2.3.2 on page 26 before, one can work with a manifold M with boundary as well.

Remark 3.2.3. *Let $B \xrightarrow{p} M$ be a surjective submersion with compact codomain. The vertical bundle $T^{vert}B \xrightarrow{\pi_B} B$ is a vector bundle over the total space B. On the other hand the composition*

$$T^{vert}B \xrightarrow{\pi_B} B \xrightarrow{p} M$$

is again a surjective submersion with compact codomain. In the proposition below we show that the space of sections $\Gamma_M B$ is a tame manifold. Consequently, so is the space

$$\Gamma_M(T^{vert}B) := \left\{\sigma \in C^\infty(M, T^{vert}B) : p \circ \pi_B \circ \sigma = id\right\}$$

of smooth section of $T^{vert}B$ over M. It comes with the obvious map

$$(\pi_B)_* : \Gamma_M(T^{vert}B) \to \Gamma_M B$$

that sends $\sigma \in \Gamma_M(T^{vert}B)$ to $\pi_B \circ \sigma$.

Proposition 3.2.4. *Let $B \xrightarrow{p} M$ be a surjective submersion with compact codomain. The set of smooth sections $\Gamma_M B$ with the W_∞-topology can be given the structure of a 0-tame manifold. Its tangent bundle is 0-tame isomorphic to the 0-tame vector bundle*

$$T\Gamma_M B \simeq \Gamma_M(T^{vert}B),$$

whose fibers are $\Gamma_M(T^{vert}B)_\sigma = \Gamma_M(\sigma^* T^{vert}B)$ for $\sigma \in \Gamma_M B$.

Proof. Fix a section $\sigma \in \Gamma_M B$, and let

$$\sigma^* T^{\mathrm{vert}} B \supseteq U_\sigma \xrightarrow{\exp_\sigma} \nu(\sigma) \subseteq B$$

be a vertical tubular neighborhood around the image $\sigma(M)$ in B, as defined in lemma 3.2.1 on page 60. As the domain of a coordinate chart around σ we take the set of sections that stay $\nu(\sigma)$ close to the image of σ, namely the W_∞-open

$$M(\nu(\sigma)) := \{\tau \in \Gamma_M B : \tau(M) \subseteq \nu(\sigma)\}.$$

We will call the map

$$\varphi_\sigma := (\exp_\sigma^{-1})_* : M(\nu_\sigma) \to \Gamma_M(U_\sigma)$$

defined by left-composition by $\exp_\sigma^{-1}$ a *typical chart* around σ on $\Gamma_M B$. Clearly, φ_σ is a homeomorphism if the space $\Gamma_M(U_\sigma)$ is also equipped with the Whitney C^∞ topology. The latter is an open neighborhood of the zero section in the graded Fréchet spaces $\Gamma_M(\sigma T^{\mathrm{vert}} B)$.

Let $\tau \in \Gamma_M B$ be a second sections such that $\nu(\sigma) \cap \nu(\tau)$. Then the sets $\exp_\sigma^{-1}(\nu(\sigma) \cap \nu(\tau))$ and $\exp_\tau^{-1}(\nu(\sigma) \cap \nu(\tau))$ are open in U_σ and U_τ respectively. This gives rise to a fiber preserving map

$$\exp_\tau^{-1} \circ \exp_\sigma : \exp_\sigma^{-1}(\nu(\sigma) \cap \nu(\tau)) \longrightarrow \exp_\tau^{-1}(\nu(\sigma) \cap \nu(\tau)).$$

This describes the transition function

$$\varphi_\tau \circ \varphi_\sigma^{-1} = \left(\exp_\tau^{-1} \circ \exp_\sigma\right)_* : \varphi_\sigma(M(\nu(\sigma) \cap \nu(\tau)) \longrightarrow \varphi_\tau(\nu(\sigma) \cap \nu(\tau)).$$

as the left-composition by a fiber preserving map. In lemma 2.3.11 on page 34 we have proven that such a map is smooth 0-tame. This shows that the typical charts form an atlas for $\Gamma_M B$.

For Hausdorffness, fix two different sections $\sigma, \tau \in \Gamma_M B$ and fix a Riemannian metric on B. Let $r = \max_{x \in M} d(\sigma(x), \tau(x))$ be the maximal point-wise distance between σ and τ. Here d is the metric on B induced by the Riemannian metric. We can now find vertical tubular neighborhoods $\nu(\sigma)$ of $\sigma(M)$ and $\nu(\tau)$ of $\tau(M)$ with the extra restriction that

$$d(\sigma'(x), \tau(x)) < \frac{1}{4} r$$

for every $\sigma' \in M(\nu(\sigma))$, and similarly for $\nu(\tau)$. It is now easy to see that the corresponding open subsets in $M(\nu(\sigma))$ and $M(\nu(\tau))$ are disjoint.

We will now compute the tangent bundle of $\Gamma_M B$. Let $\sigma_0 \in \Gamma_M B$ be a fixed section. Any smooth path $\sigma_t : (-\varepsilon, \varepsilon) \to \Gamma_M B$ through σ_0 can also be seen as a smooth map

$$\sigma : (-\varepsilon, \varepsilon) \times M \to B,$$

for which $\sigma(t, x) \in B_x$ for all $x \in M$ and t. Hence its partial derivative at $t = 0$ gives a smooth map

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \sigma : M \to TB$$

for which

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \sigma(x) \in T_{\sigma_0(x)} B_x = \left(\sigma^* T^{\text{vert}} B\right)_x ;$$

It defines a section of $\sigma_0^* T^{\text{vert}} B$. This in turn defines a natural map

$$T_{\sigma_0} \Gamma_M B \to \Gamma_M \left(\sigma_0^* T^{\text{vert}} B\right) : \left.\frac{d}{dt}\right|_{t=0} \sigma_t \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} \sigma,$$

which is clearly linear and bijective. This describes the tangent space of $\Gamma_M B$ at σ_0 as the graded Fréchet space $\Gamma_M \sigma_0^* T^{\text{vert}} B$. We will show that there is actually a 0-tame isomorphism of 0-tame vector bundles $T\Gamma_M B \simeq \Gamma_M T^{\text{vert}} B$, as described in the statement of the lemma.

Let $\sigma \in \Gamma_M B$ be a fixed section, and fix a vertical tubular neighborhood $\exp_\sigma : U_\sigma \to \nu(\sigma)$ around $\sigma(M)$. The tangent map of the chart φ_σ induces a 0-tame isomorphism

$$T(\exp_\sigma^{-1})_* : TM(\nu(\sigma)) \xrightarrow{\simeq} T\Gamma_M U_\sigma \simeq \Gamma_M U_\sigma \oplus \Gamma_M \sigma^* T^{\text{vert}} B.$$

Let $z \in \Gamma_M(U_\sigma)$ denote the zero section of $\sigma^* T^{\text{vert}} B$. By the remarks prior to this lemma, there is a natural isomorphism between vector bundles

$$T^{\text{vert}} U_\sigma \simeq U_\sigma \oplus z^* T^{\text{vert}} U_\sigma \simeq U_\sigma \oplus \sigma^* T^{\text{vert}} B.$$

This induces a natural 0-tame isomorphism

$$\Gamma_M U_\sigma \oplus \Gamma_M \sigma^* T^{\text{vert}} B \simeq \Gamma_M T^{\text{vert}} U_\sigma.$$

Hence the tangent bundle of $\Gamma_M B$ is 0-tame isomorphic to $\Gamma_M T^{\text{vert}} B$. $\square$

Remark 3.2.5. *Note that we can identify the smooth curves into $\Gamma_M B$ as*

$$C^\infty([-1,1],\Gamma_M B) \longrightarrow \Gamma_{[-1,1]\times M}(\pi^* B) : \gamma_t \mapsto \tilde{\gamma}(t,x) = \gamma_t(x),$$

where $\pi_M : [-1,1]\times M \to M$ is the projection onto M. Moreover, it maps the derivative along t to the partial derivative along t in the sense that

$$\begin{tikzcd} C^\infty([-1,1],\Gamma_M B) \arrow[r, "\sim"] \arrow[d, "\frac{d}{dt}\big|_{t=0}"'] & \Gamma_{[-1,1]\times M}(\pi^* B) \\ T\Gamma_M B \arrow[r, "\sim"] & \Gamma_M(T^{vert}B), \arrow[u, "\frac{\partial}{\partial t}\big|_{t=0}"'] \end{tikzcd}$$

commutes. It follows that a continuous map $P : \Gamma_M A \to \Gamma_N B$ is C^1-differentiable if there exists a continuous vector bundle map $TP : \Gamma_M(T^{vert}A) \to \Gamma_N(T^{vert}B)$ such that

$$\begin{tikzcd} \Gamma_{[-1,1]\times M}(\pi^* B) \arrow[r, "P_*"] \arrow[d, "\frac{\partial}{\partial t}\big|_{t=0}"'] & \Gamma_{[-1,1]\times M}(\pi^* B) \\ \Gamma_M(T^{vert}A) \arrow[r, "TP"] & \Gamma_N(T^{vert}B), \arrow[u, "\frac{\partial}{\partial t}\big|_{t=0}"'] \end{tikzcd}$$

commutes. Likewise, one can make the analoguous statement for differentiability at a point. This can be helpful when computing derivatives.

Corollary 3.2.6. *Let M and N be manifolds with M compact. The set of smooth maps $C^\infty(M,N)$ equipped with the W_∞-topology has the structure of a 0-tame manifold. Its tangent bundle is 0-tame isomorphic to the 0-tame vector bundle*

$$TC^\infty(M,N) \simeq C^\infty(M,TN)$$

whose fibers are $C^\infty(M,TN)_f = \Gamma_M(f^*TN)$ for every $f \in C^\infty(M,N)$.

Proof. Given two manifolds M and N with M compact, there is a natural bijection between $C^\infty(M,N)$ and the smooth sections of the bundle

$$M \times N \to M$$

by identifying a map f with its graph

$$(\mathrm{id}, f) : M \to M \times N.$$

It is straightforward to check that this identification is a homeomorphism if

$$\Gamma_M(M\times N) \subseteq C^\infty(M, M\times N)$$

is equipped with the Whitney C^∞ topology. Obviously we have

$$T^{\mathrm{vert}}(M\times N) = M \times TN,$$

and $\Gamma_M((\mathrm{id},f)^*(M\times TN)) = \Gamma_M f^*TN$. $\square$

3.2.3 Sections of $E \to B \to M$

In the previous section we observed that the tangent bundle of $\Gamma_M B$ is 0-tamely ismorphic to $\Gamma_M(T^{\text{vert}}B)$. The latter 0-tame manifold can be seen as a bundle of sets

$$\Gamma_M(T^{\text{vert}}B) \xrightarrow{\pi_{B*}} \Gamma_M(B)$$

whose fibers are $\Gamma_M(T^{\text{vert}}B)_\sigma = \Gamma_M(\sigma^* T^{\text{vert}}B)$ for all $\sigma \in \Gamma_M B$. Each of these fibers is a graded Fréchet space but we did not yet show that the whole naturally has the structure of a 0-tame vector bundle. More generally, we can look at a fiber bundle $B \to M$ together with a vector bundle $E \to B$ and show that $\Gamma_M E$ is a 0-tame vector bundle over $\Gamma_M B$ with its fiber at $\sigma \in \Gamma_M B$ naturally isomorphic to $\Gamma_M \sigma^* E$.

Lemma 3.2.7. *Let $B \xrightarrow{p} M$ be a surjective submersion with compact codomain, and $E \xrightarrow{\pi} B$ a vector bundle over B. Then the map*

$$\Gamma_M E \xrightarrow{\pi_*} \Gamma_M B,$$

whose fibers are $(\Gamma_M E)_\sigma = \Gamma_M \sigma^* E$ for all $\sigma \in \Gamma_M B$, forms a 0-tame vector bundle.

Proof. Let $\sigma \in \Gamma_M B$ be fixed. We will first construct a map that trivializes E along (parts of) the fibers of B in a neighborhood of the image of σ. We may cover the image $\sigma(M)$ with finitely many local trivializations $(U_\alpha, \varphi_\alpha)$ of E, with

$$\varphi_\alpha = (q, \psi_\alpha) : E|_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times E_{\sigma(m)}.$$

Define a partition of unity $\{\chi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$ that is constant on the fibers of B. This can be done by first choosing a partition of unity on $\sigma(M)$ subordinate to $\{U_\alpha|_M\}$. Now let $U = \cup_\alpha U_\alpha$ and define a map on $E|_U$ by

$$\varphi = (q, \psi) : E|_U \to U \times_M \sigma^* E : \quad e \mapsto \left(q(e), \sum_\alpha \chi_\alpha(q(e)) \psi_\alpha(e) \right).$$

It is a smooth vector bundle map and it has an obvious inverse. Note that for any $m \in M$, the space $U \times_M \sigma^* E$ seen as a vetor bundle over U trivializes when restricted to $U|_{\sigma(m)}$.

Note that a section of $U \times_M \sigma^* E \to M$ is just an element of $\Gamma_M U \times \Gamma_M \sigma^* E$ and that this identification is tame linear. Any section $\nu \in \Gamma_M \tau^* E$ above $\tau \in \Gamma_M U$ can be seen as a smooth map $M \to E$ and is mapped to such a section $\varphi \circ \nu \in \Gamma_M(U \times_M \sigma^* E)$ by composition on the left. This map φ_*, composition on the left with φ, is hence a valid candidate for a local trivialization of $\Gamma_M E \to \Gamma_M B$. One only needs to verify that the transition maps are tame linear.

Let $\sigma_{1,2} \in \Gamma_M B$ be two fixed sections, $U_{12} = U_1 \cap U_2$, and $\varphi_i : E_{U_i} \to U_i \times_M \sigma_i^* E$ the respective maps defined above. Assume that $\Gamma_M U_{12}$ is non-empty, then the transition map is a map

$$(\mathrm{id}, \rho) = \varphi_2^* \circ (\varphi_1^{-1})^* : \Gamma_M U_{12} \times \Gamma_M \sigma_1^* E \to \Gamma_M U_{12} \times \Gamma_M \sigma_2^* E$$

defined by sending a pair (τ, ν) to $\rho(\nu) = \varphi_2 \circ \varphi_1^{-1} \circ \nu$, since the to middle maps, induced by identifying an element $\Gamma_M \sigma_i^* E$ with a map $M \to E$ over σ_i, cancel out. This map ρ is a vector bundle operator, hence it is tame linear. $\square$

The following is a nice application of the above lemma. Let $A \to M$ and $B \to N$ be two finite rank vector bundles, for which M is a compact manifold. We wish to consider vector bundle maps between these to bundles. Recall that the smooth maps $M \to N$ can be regarded as the space of sections $\Gamma_M(M \times N)$.

One can define a vector bundle

$$\mathbf{Hom}(A, B) \to M \times N$$

whose fiber over a point $(m, n) \in M \times N$ is the space of natural transformations $L : A_m \to B_n$. This defines a tower of bundles $Hom(A, B) \to M \times N \to M$ as in lemma 3.2.7 on the previous page. Given a smooth map $f : M \to N$, the vector bundle maps $A \to B$ with base map f are exactly the sections $\Gamma_M f^* \mathbf{Hom}(A, B)$: these form a smooth family of linear maps $A_m \to B_{f(m)}$. Hence the total space $\Gamma_M(\mathbf{Hom}(A, B))$ is the smooth tame manifold of all vector bundle maps $A \to B$.

Corollary 3.2.8. *Let $A \to M$ and $B \to N$ be two vector bundles over compact bases, and let*

$$\boldsymbol{Hom}(A, B) \to M \times N$$

denote the vector as defined above. Then

$$\Gamma_M \boldsymbol{Hom}(A, B) \to C^\infty(M, N)$$

is the 0-tame vector bundle of vector bundle maps $A \to B$.

This approach doesn't seem to extend to the space of fiber bundle maps between to arbitrary fiber bundles in any straightforward way. Perhaps a direct proof is possible, but I haven't succeeded in doing this so far.

3.2.4 Basic properties of $\Gamma_M B$

Lemma 3.2.9. *Let $A \to M$ and $B \to M$ be surjective submersions with the same compact codomain, and let $A \times_M B$ be the fibered product. Then the natural map*

$$\Gamma_M(A \times_M B) \simeq \Gamma_M A \times \Gamma_M B$$

is a 0-tame diffeomorphism.

Proof. For any pair $(\sigma, \tau) \in \Gamma_M A \times \Gamma_M B$ we have the pull-back

$$\begin{tikzcd}
M \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\tau"] \arrow[ddr, bend right, "\sigma"'] & & \\
& A \times_M B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] \arrow[dr, phantom, "p.b."] & B \arrow[d, "q"] \\
& A \arrow[r, "p"] & M.
\end{tikzcd}$$

The obvious map

$$\Gamma_M(A \times_M B) \longrightarrow \Gamma_M A \times \Gamma_M B$$

that sends $\rho \in \Gamma_M(A \times_M B)$ to $(\pi_{A*}(\rho), \pi_{B^*}(\rho))$ is a smooth tame map since π_{A*} and π_{B*} are. It is in fact a bijection of sets. We will see that it is also locally a tame diffeomorphism, hence that the inverse is smooth tame as well.

For any pair of sections $(\sigma, \tau) \in \Gamma_M A \times \Gamma_M B$, choose tubular neighborhoods along the fibers of A and B respectively, and exponent maps

$$\exp_\sigma : E_\sigma \xrightarrow{\simeq} \nu(\sigma),$$
$$\exp_\tau : E_\tau \xrightarrow{\simeq} \nu(\tau)$$

as done before. They also induce fiber preserving diffeomorphism

$$\exp_{(\sigma,\tau)} = \exp_\sigma \times \exp_\tau : E_\sigma \oplus E_\tau \xrightarrow{\simeq} \nu(\sigma) \times_M \nu(\tau),$$

with inverse $\exp_\sigma^{-1} \oplus \exp_\tau^{-1}$. Now $\nu(\sigma) \times_M \nu(\tau)$ is an open neighborhood of the graph of (σ, τ) seen as a section of $A \times_M B$. Hence the set $M(\nu(\sigma) \times_M \nu(\tau))$ of sections take values in this neighborhood forms an open subset around (σ, τ) in $\Gamma(A \times_M B)$. The map

$$\left(\exp_{(\sigma,\tau)}^{-1}\right)_* : M(\nu(\sigma) \times_M \nu(\tau)) \to \Gamma_M(E_\sigma \oplus E_\tau)$$

is a chart of $\Gamma_M(A \times_M B)$ around (σ, τ). Hence the natural bijection is locally just the 0-tame linear isomorphism

$$\Gamma_M E_\sigma \oplus \Gamma_M E_\tau \simeq \Gamma_M(E_\sigma \oplus E_\tau).$$

This completes the proof of the lemma. $\square$

Corollary 3.2.10. *Let M, N and P be manifolds with M compact. Then the natural map*

$$C^\infty(M, N \times P) \simeq C^\infty(M, N) \times C^\infty(M, P)$$

is a 0-tame diffeomorphism.

Let $B \to M$ be a surjective submersion, then a subbundle $A \subseteq B$ of B is an embedded manifold such that $p|_A : A \to M$ is still a surjective submersion.

Lemma 3.2.11. *Let $B \xrightarrow{p} M$ be a surjective submersion with compact codomain, and $A \subseteq B$ a subbundle of B. Then the inclusion*

$$\Gamma_M A \subseteq \Gamma_M B$$

is a 0-tamely embedded submanifold.

Proof. Choose a vertical tubular neighborhood $\exp_A : U_A \to \nu(A) \subseteq B$ around A in B, as in lemma 3.2.1 on page 60. By the remark below the lemma, we may consider $\nu(A) \to A$ as vector bundle over the surjective submersion $A \to M$. Hence by lemma 3.2.7 on page 67 we have a 0-tame vector bundle

$$\Gamma_M \nu(A) \longrightarrow \Gamma_M A.$$

In particular, $\Gamma_M A$ is an embedded submanifold of $\Gamma_M \nu(A)$. The latter is an open submanifold of $\Gamma_M B$. $\square$

Corollary 3.2.12. *Let M and P be manifolds with M compact, and $N \subseteq P$ an embedded submanifold. Then*

$$C^\infty(M, N) \subseteq C^\infty(M, P)$$

is a 0-tamely embedded submanifold.

Corollary 3.2.13. *Let $p : B \to M$ be a bundle over a compact base manifold, then*

$$\Gamma_M B \subseteq C^\infty(M, B)$$

is a 0-tamely embedded submanifold.

Proof. Note that B can be seen as a subbundle of the trivial bundle $M \times B$ via

$$\begin{tikzcd} B \arrow[rr, "{(p, \mathrm{id})}"] \arrow[dr, "p"'] & & M \times B \arrow[dl, "\mathrm{pr}_1"] \\ & M. & \end{tikzcd}$$

This completes the corollary. $\square$

Suppose that $A \xrightarrow{p} M$ is a surjective submersion with B compact. Then p is a proper submersion, hence it is a fiber bundle by the Ehresmann theorem. If $B \xrightarrow{q} M$ is another surjective submersion, then one can consider the space of sections $\Gamma_A(p^*B)$. The sections $\Gamma_M B$ are included in $\Gamma_A(p^*B)$ as the sections $A \to p^*B$ that are constant along the fibers of A. In other words, those are included via the map

$$p^* : \Gamma_M B \longrightarrow \Gamma_A(p^*B)$$

defined by right-composition by $p : A \to M$.

Lemma 3.2.14. *Let $A \xrightarrow{p} M$ be a compact fiber bundle, and $B \xrightarrow{q} M$ a surjective submersion. Then $\Gamma_M B$ is a 0-tamely embedded submanifold of $\Gamma_A(p^*B)$.*

Proof. Let $\sigma \in \Gamma_M B$ be a fixed section, and choose a vertical tubular neighborhood $\exp_\sigma : U_\sigma \to \nu(\sigma)$ around $\sigma(M)$, as in lemma 3.2.1 on page 60. The pull-back of $\exp_\sigma$ along p, that is, the map

$$p^* \exp_\sigma : p^*U_\sigma \to p^*\nu(\sigma)$$

defines a vertical tubular neighborhood around the image $\sigma \circ p(A)$ of the section $\sigma \circ p \in \Gamma_A(p^*B)$. Now right composition by p restricts to a smooth 0-tame map, see proposition 3.2.16 on the next page,

$$p^* : M(\nu(\sigma)) \longrightarrow M(p^*\nu(\sigma)),$$

where $M(p^*\nu(\sigma))$ indicates the set of all $\tau \in \Gamma_A(p^*B)$ such that $\tau(A) \subseteq p^*\nu(\sigma)$. The local representation of p^* along the charts induced by the vertical tubular neighborhoods is just the map

$$p^* : \Gamma_M(\sigma^*T^{\mathrm{vert}}B) \longrightarrow \Gamma_A((\sigma \circ p)^*T^{\mathrm{vert}}B)$$

restricted to the open $M(U_\sigma)$ in $\Gamma_M(\sigma^*T^{\mathrm{vert}}B)$. Let θ be a normalized positive vertical density on A. By lemma 2.3.13 on page 41, where B in the lemma becomes A and E becomes $\sigma^*T^{\mathrm{vert}}B$, there exists a tame linear integration map

$$I_\theta : \Gamma_A((\sigma \circ p)^*T^{\mathrm{vert}}B) \to \Gamma_M(\sigma^*T^{\mathrm{vert}}B).$$

We have

$$I_\theta \circ p^*(\tau) = \tau, \quad \forall \tau \in \Gamma_M(\sigma^*T^{\mathrm{vert}}B),$$

since θ is chosen to be normalized. The map p^* hence embeds $\Gamma_M(\sigma^*T^{\mathrm{vert}}B)$ as a 0-tame linear subspace into

$$\Gamma_A((\sigma \circ p)^*T^{\mathrm{vert}}B) \simeq \Gamma_A(\sigma^*T^{\mathrm{vert}}p^*B)$$

with its tame linear compliment given by $\ker(I_\theta)$. □

Corollary 3.2.15. *Let M, N, and P be manifolds, with M and N compact. The projection $\pi_N : M \times N \to N$ defines a 0-tame embedding of $C^\infty(N, P)$ into $C^\infty(M \times N, P)$ as the smooth maps $M \times N \to P$ that are constant in M.*

3.2.5 Composition

In this section we discuss show that the composition maps is smooth tame. Then we look at the diffeomorphisms $\mathbf{Diff}(M)$ as an open submanifold of $C^\infty(M,M)$, and, lastly, we show that taking the inverse is also smooth tame. These results are revisited in the next section, when we look at bundle maps instead of just regular smooth maps.

Proposition 3.2.16. *Let M, N and O be manifolds of finite dimension, with M and N compact. Then the composition map*

$$\boldsymbol{com} : C^\infty(N,O) \times C^\infty(M,N) \to C^\infty(M,O)$$

is a smooth tame map of degree 0*. Its tangent map at (f,g) is given by*

$$\begin{aligned} T_{(f,g)}com : \Gamma(f^*TO) \times \Gamma(g^*TN) &\to \Gamma((fg)^*TO), \\ (\varphi,\gamma) &\mapsto \varphi \circ g + Tf \circ \gamma, \end{aligned}$$

and hence is tame linear of degree 1 *in the first factor and degree* 0 *in the second. Higher order tangent maps $T^k com$ are of similar form, and are tame of up to degree k in each factor.*

Proof. For the smoothness of the composition map, fix two smooth maps $f : N \to O$ and $g : M \to N$, and write $h = f \circ g$ henceforth. Choose a tubular neighborhood $\nu(h)$ of the graph of h. Its fiber $\nu(h)_x \subseteq O$ is an open subset around $h(x)$ for every $x \in M$.

Any tubular neighborhood $\nu(f) \subseteq N \times O$ defines a fiber bundle $M \times \nu(f) \to M \times N$, which can be restricted to a tubular neighborhood $\nu(g) \subseteq M \times N$ of choice. This defines an open subbundle

$$(M \times \nu(f))_{\nu(g)} \to \nu(g) \to M$$

of $M \times N \times O \to M$. Its fiber at any point $x \in M$ is the restricted tubular neighborhood $\nu(f)$ restricted to the open $\nu(g)_x \subseteq N$.

On the other hand, $\nu(h)$ defines an open subbundle $\tilde{\nu}(h)$ whose fiber at $x \in M$ is given by

$$\tilde{\nu}(h)_x = N \times \nu(h)_x \subseteq N \times O.$$

If we can choose $\nu(f)$ and $\nu(g)$ such that $(M \times \nu(f))_{\nu(g)}$ is contained in $\tilde{\nu}(h)$, which implies that for every $x \in M$

$$\nu(f)_y \subseteq \nu(h)_x, \forall y \in \nu(g)_x,$$

we have succeeded in restricting the composition map to typical charts, that is,

$$\mathbf{com} : U_{\nu(f)} \times U_{\nu(g)} \to U_{\nu(h)}.$$

This can actually be done only assuming that g is proper (apparently without assuming that M and N are compact). One can cover N with open subsets $y \in V$ such that there is a natural local trivialization

$$(f^*TO)_V \xrightarrow{\simeq} V \times T_{f(y)}O.$$

In fact, one can choose open subsets $y \in V'$ such that $\bar{V}' \subseteq V$, $\bar{V}'$ is compact, and the V' still cover N. In particular the V' cover N, hence the $g^{-1}V'$ cover M. The above map gives a local trivialization

$$(h^*TO)_{g^{-1}(\bar{V}')} \xrightarrow{\simeq} g^{-1}(\bar{V}') \times T_{f(y)}O.$$

Now surely the $g^{-1}(\bar{V}')$ are mapped in V by g. And since the former are compact, it is possible to choose a tubular neighborhood $\nu(g)$ such that

$$\nu(g)_{g^{-1}(\bar{V}')} \subseteq g^{-1}(\bar{V}') \times V.$$

Hence we now have two inclusions of bundles over $g^{-1}(\bar{V}')$, namely,

- $\left((M \times f^*TO)_{\nu(g)}\right)_{g^{-1}(\bar{V}')} \subseteq (M \times f^*TO)_{g^{-1}(\bar{V}')\times V} \simeq g^{-1}(\bar{V}') \times V \times T_{f(y)}O,$
- $\tilde{\nu}(h)_{g^{-1}(\bar{V}')} \hookrightarrow N \times (h^*TO)_{g^{-1}(\bar{V}')} \simeq g^{-1}(\bar{V}') \times N \times T_{f(y)}O,$

and the second gives a open neighborhood of the zero-section of h^*TO. Now we must choose a tubular neighborhood $\nu(f) \subseteq N \times O$ such that if we include

$$M \times \nu(f) \hookrightarrow M \times f^*TO,$$

its image under these identifications is contained in the image of the second inclusion of bundles. This can be done since $g^{-1}(\bar{V}')$ is compact.

One can compute the tangent map of the composition map directly. For any pair of vectors

$$(\varphi, \gamma) \in T_f C^\infty(N, O) \times T_g C^\infty(M, N)$$

one can choose representations of the form

- $\varphi = \frac{d}{dt}\big|_{t=0} f_t$, with $f_t : (-1,1) \to C^\infty(N,O)$ and $f_0 = f$,
- $\gamma = \frac{d}{dt}\big|_{t=0} g_t$, with $g_t : (-1,1) \to C^\infty(M,N)$ and $g_0 = g$.

Now the composite can be seen as a map of the form

$$f_t \circ g_t : (-1,1) \times M \to O$$

and its partial derivative at $t = 0$ can be computed by the usual formula

$$\frac{\partial}{\partial t}\bigg|_{t=0} f_t \circ g_t = \frac{\partial}{\partial t}\bigg|_{t=0} f_t \circ g_0 + \frac{\partial}{\partial t}\bigg|_{t=0} f_0 \circ g_t.$$

Hence the tangent map at (f,g) can be computed as

$$\begin{aligned} T_{(f,g)}com\,\{\varphi,\gamma\} &= \frac{d}{dt}\bigg|_{t=0} com(f_t, g_t) = \frac{\partial}{\partial t}\bigg|_{t=0} f_t \circ g_t \\ &= \frac{d}{dt}\bigg|_{t=0} f_t \circ g_0 + Tf_0 \circ \frac{d}{dt}\bigg|_{t=0} g_t \\ &= \varphi \circ g + Tf \circ \gamma. \end{aligned}$$

Now for the tameness of the composition map perform the following process.

- Consider all products of charts $V_y \times V_z \subseteq N \times O$ such that the closure $\bar{V}_y \times \bar{V}_z$ is compact and contained in the tubular neighborhood around f. Moreover, assure that $\bar{V}_y \subseteq f^{-1}(V_z)$. There are many such charts, since $V_y \times V_z$ can be taken arbitrarily small, and the V_y cover N.
- Next consider all products of charts $V_x \times V_y \subseteq M \times N$, where V_y is one of the above, and the closure $\bar{V}_x \times \bar{V}_O$ is compact and contained in the tubular neighborhood around g. Also assure that $\bar{V}_x \subseteq g^{-1}(V_y)$.
- Choose finitely many of the triples (V_x, V_y, V_z) such that the $V_x \times V_z$ cover the graph of h and the closure $\bar{V}_x \times \bar{V}_z$ are contained in the tubular neighborhood of h.

For a pair (V_y, V_z) we have an open subset given by

$$W(\bar{V}_y, V_z) := \left\{ f' \in C^\infty(N, O) : f'(\bar{V}_y) \subseteq V_z \right\}.$$

And since there are only finitely many such pairs, this gives an open neighborhood around f of the form

$$W_f = \bigcap W(\bar{V}_y, V_z) \cap U_{\nu(f)}.$$

Likewise, we have open neighborhoods W_g of g and W_h of h. The composition map now restricts to a map

$$\mathbf{com} : W_f \times W_g \to W_h.$$

Now the seminorms on W_h may be computed as

$$\|g'\|_k = \sum_{j=1}^{k} \max_{V_x} \sup_{x \in V_x} \left\| D^j h(x) \right\|$$

if the charts are taken small enough. If we restrict the open subsets W_f and W_g even further such that $\|f'\|_1, \|g'\| < c$ for a constant $c > 0$, $f' \in W_f$ and $g' \in W_g$, the argument is reduced to the following lemma, c.f. [Ham82b].

As for the tameness of the tangent maps, note that it is combination of compositions and derivatives. Hence one can give similar arguments. □

Lemma 3.2.17. *Let $g : U \to V$ and $f : V \to W$ be two smooth maps between bounded open subsets in Euclidian spaces such that they extend to smooth maps on the closures $\bar{U}$ and $\bar{V}$ respectively. Assume that that there is a constant $k > 0$ such that $\|f\|_1, \|g\|_1 < k$. Then for every $n \geq 1$ there is a constant $C_{k,n} > 0$ such that*

$$\|f \circ g\|_n \leq C_{k,n}(1 + \|f\|_n + \|g\|_n).$$

Proof. For $n \geq 1$, repeated application of the chain rule gives

$$D^n fg(x) = \sum_{k=1}^{n} \sum_{j_1 + \ldots + j_k = k} c_{k,j_1,\ldots,j_k} D^k_{g(x)} f \left(D^{j_1}_x g, \ldots, D^{j_k}_x g \right),$$

where the $c_{k,j_1,\ldots,j_k} > 0$ are suitable constants. Hence we may estimate

$$\|D^n f \circ g\|_0 \leq C \sum_{k=1}^{n} \sum_{j_1+\ldots+j_k=k} \|f\|_k \|g\|_{j_1} \cdots \|g\|_{j_k}.$$

By the interpolation estimates

$$\|f\|_k^{n-1} \leq C \|f\|_1^{n-k} \|f\|_n^{k-1},$$
$$\|f\|_j^{n-1} \leq C \|g\|_1^{n-j} \|g\|_n^{j-1}$$

each of the terms in the double summation can be estimated by

$$\begin{aligned}
\|f\|_k \|g\|_{j_1} \cdots \|g\|_{j_k} &\leq C \|f\|_1^{(n-k)/(n-1)} \|f\|_n^{(k-1)/(n-1)} \|g\|_1^{(kn-n)/(n-1)} \|g\|_n^{(n-k)/(n-1)} \\
&\leq C \|g\|_1^{k-1} \left(\|f\|_1^{(n-k)/(n-1)} \|f\|_n^{(k-1)/(n-1)} \|g\|_1^{(k-1)/(n-1)} \|g\|_n^{(n-k)/(n-1)}\right) \\
&\leq C \|g\|_1^{k-1} (\|g\|_1 \|f\|_n)^{(k-1)/(n-1)} (\|f\|_1 \|g\|_n)^{(n-k)/(n-1)} \\
&\leq C \|g\|_1^{k-1} (\|g\|_1 \|f\|_n + \|f\|_1 \|g\|_n).
\end{aligned}$$

Hence in general we obtain an estimate

$$\|D^n f \circ g\|_0 \leq C \|g\|_1^{k-1} (\|g\|_1 \|f\|_n + \|f\|_1 \|g\|_n) \leq C(1 + \|f\|_n + \|g\|_n),$$

since we assume that the $\|f\|_1$ and $\|g\|_1$ are bounded by a constant. Since the maps f and g are bounded, we also have $\|f \circ g\|_0 < C$ and the lemma follows. $\square$

Suppose that M, N and O are arbitrary manifolds, not necessarily compact. In [Mat69], Mather proves that the composition map is continuous as a map

$$C^k(N, O) \times C^k_p(M, N) \longrightarrow C^k(M, O)$$

with the appropriate W_k-topologies for all $k \in \mathbb{N}$, and hence also for the case $k = \infty$. Here $C^k_p(M, N)$ denotes the space of proper maps $M \to N$. In the case that M and N are compact, this provides an alternative proof of the tameness of composition.

Remark 3.2.18. *It is necessary to restrict the composition map to the proper maps $M \to N$. For suppose that $\overline{p} : M \to N$ denotes the constant map with value $p \in N$. Let $g_n : N \to O$ be a sequence of smooth maps that converge to g in $C^\infty(M, N)$. Then if M is not compact the sequence $g_n \circ \overline{p}$ doesn't converge to $g \circ \overline{p}$. For one can construct an open neighborhood U of $M \times \{g(p)\}$ in $M \times O$ such that the graph of no constant function other than $g \circ \overline{p}$ lies in U. Obviously this problem is mood as soon as M is compact or we restrict to proper maps.*

For the following corollary let $I = [0, 1]$ be the unit interval, and let $\pi_M : M \times I \to M$ denote the projection on the first component.

Lemma 3.2.19. *Let $B \to M$ be a bundle over a compact manifold M, and $\sigma \in \Gamma_M B$ a fixed section. Then there is an open neighborhood $U \subseteq \Gamma_M B$ of σ such that every section $\tau \in U$ is smoothly path-connected to σ, and there is a smooth tame map*

$$\gamma_\sigma : U \longrightarrow \Gamma_{M\times I}(\pi_M^* B)$$

that sends each $\tau \in U$ to such a path and σ to the constant path.

If M is not compact, then γ_* is still continuous with respect to the W_k-topologies on both sides for every $k \in \mathbb{N} \cup \{\infty\}$.

Proof. Since $\Gamma_M B$ is a closed submanifold of $C^\infty(M, B)$, it is sufficient to prove the analogous statement for the latter Fréchet manifold and show that γ_σ maps sections to paths of sections.

For any manifold B one can choose a geodesic $\gamma : V \times I \to B$, that is, one can find an open neighborhood of the diagonal δ_B in $B \times B$ and a smooth map $\gamma : V \times I \to B$ such that $\gamma(x, y, 0) = x$, $\gamma(x, y, 1) = y$ and $\gamma(x, x, t) = x$ for all $(x, y) \in V$ and $t \in I$. For example, choose a Riemannian metric on B, and define γ by following the geodesics of the corresponding Levi-Cevita connection.

Now let U be the set $\{\tau \in C^\infty(M, B) : (\sigma(m), \tau(m)) \in V \forall m \in M\}$. It is straightforward to see that this set is open in $C^\infty(M, B)$. Define γ_σ by

$$\gamma_\sigma(\tau)(m, t) = \gamma(\sigma(m), \tau(m), t), \quad \forall (m, t) \in M \times I.$$

This is map is defined by composition on the left by γ and composition on the right by the map $(\delta, \mathrm{id}) : M \times I \to M \times M \times I$ that sends (m, t) to (m, m, t). By proposition 3.2.16 on page 72, this map is smooth tame.

To ensure that γ_σ maps into the space $\Gamma_{M\times I}(\pi_M^* B)$ it suffices to make sure that if x and y lie in the same fiber B_m, then so does $\gamma(x, y, t)$ for all $t \in I$. This condition can easily be satisfied.

For the last statement of the lemma we only need to ensure that the map (δ, id) is proper, but this holds trivially. $\square$

3.2.6 The group of diffeomorphisms

The diffeomorphism group $\mathbf{Diff}(M)$ is an open submanifold of $C^\infty(M,M)$. We will recall the proof from Guillemin [MG73] here for the sake of completeness. In fact, we prove the following slightly adjusted result, which we will use in the next section. In what follows we say that a smooth map $f \in C^\infty(M \times N, N)$ is a diffeomorphism in the N-variable if

$$f(x,-) : N \longrightarrow N \in \mathbf{Diff}(N,N)$$

for all $x \in M$. We aim to prove the following about such maps.

Proposition 3.2.20. *Let M and N be compact manifolds. Then the subset of maps $f \in C^\infty(M \times N, N)$ that are diffeomorphisms in the N-variable lies open.*

If O is another manifold of finite dimension and $\dim(N) \leq \dim(O)$ then consider the set of smooth maps $f \in C^\infty(M \times N, O)$ where

$$f(x,-) : N \to O \in \mathbf{Imm}(N,O)$$

is an immersion for all $x \in M$. Call such a map f an immersion, or immersive, in the N-variable. We will show that this set lies open in $C^\infty(M \times N, O)$ as a first step towards proving the above proposition. For if $O = N$ is compact, any injective immersion $N \to O$ is a diffeomorphism.

Lemma 3.2.21. *Let M,N and O be manifolds of finite dimension, with M and N compact, and assume* $\dim(N) \leq \dim(O)$. *Then the subset of maps $f \in C^\infty(M \times N, O)$ for which*

$$f(x,-) \in \boldsymbol{Imm}(N,O), \quad \forall x \in M$$

lies open.

Proof. Let $\mathbb{R}^{n\times m}$ be the space of $n \times m$-matrices, topologized as Euclidian space, where $m = \dim(N)$ and $n = \dim(O)$. Then the set of maximal rank matrices lies open in $\mathbb{R}^{n\times m}$. For let $\mathbf{k} = (k_1, \dots, k_{n-m})$ denote a $(n-m)$-tuple of distinct integers with $1 \leq k_i \leq m$. Suppose that, for a particular matrix $A \in \mathbb{R}^{n\times m}$, we let $A_\mathbf{k}$ denote the matrix obtained by omitting rows k_1 through k_{n-m}. Then the map $p_\mathbf{k} : \mathbb{R}^{n\times m} \to \mathbb{R}$ defined by $p_\mathbf{k}(A) = \det(A_\mathbf{k})$ is polynomial in the coordinates of A, hence continuous. Now the set of maximal rank matrices is given by

$$\cup_\mathbf{k} p_\mathbf{k}^{-1}(\mathbb{R} - \{0\})$$

and hence is open.

Let (U,φ) and (V,ψ) be charts of $M \times N$ and O respectively. Let $k = \dim(M)$ be the dimension of M, and $U' = \varphi(U) \subseteq \mathbb{R}_+ \times \mathbb{R}^{k+m-1}$ and $V' = \psi(V) \subseteq \mathbb{R}^n$ be the local domains. Then a typical chart of the first jet bundle $J^1(M \times N, O)$ is of the form

$$\Psi_{U,V} = \psi_*(\varphi^{-1})^* : J^1(U,V) \xrightarrow{\cong} J^1(U',V') \cong U' \times V' \times \mathbb{R}^{n\times(k+m)}.$$

Elements of $\mathbb{R}^{n\times(k+m)}$ can be written as $(A|B)$ with $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{n\times m}$, separating the first k columns from the latter n. The set $\mathcal{O}'$ of matrices for which B is of maximal rank is open, hence so is $U' \times V' \times \mathcal{O}'$, and it defines an open subset of $J^1(U,V)$. In turn, the union of all such open subsets over all typical charts defines an open subset $\mathcal{O} \subseteq J^1(M \times N, O)$. It is easy to see that this open is independent of the chosen covering of charts; maximality of matrices is invariant under conjugation.

A smooth map $f : M \times N \to O$ is immersive in the N-variable if and only if its first jet maps into O, that is, if

$$j^1(f)(x) \in \mathcal{O}, \quad \forall x \in M,$$

hence the subset is a typical open for the compact-open topology on $C^\infty(M \times N, O)$. $\square$

Lemma 3.2.22. *Let M, N and O be manifolds of finite dimension, with M and N compact, and assume* $\dim(N) \leq \dim(O)$. *Suppose that $f \in C^\infty(M \times N, O)$ is an immersion in the N-variable. Let $(x_0, y_0) \in M \times N$ be arbitrary points, then there are open neighborhoods*

$$(x_0, y_0) \in U^1 \times U^2 \subseteq M \times N$$

and

$$f \in W \subseteq C^\infty(M \times N, O)$$

such that $g(x, -)|_{U^2}$ is an injective immersion whenever $g \in W$ and $x \in U^1$.

Proof. Adopt the notation $f(x) = f(x, -) : N \to O$ for all $x \in M$, so that the notation is less cumbersome.

Begin by choosing open subsets $x_0 \in \mathcal{U}^1 \subseteq M$, $y_0 \in \mathcal{U}^2 \subseteq N$ and $f(x_0, y_0) \in V_f \subseteq O$, together with chartings $(\mathcal{U}^2, \varphi)$ and (V_f, ψ). Now choose smaller open subsets $U^i \subseteq \mathcal{U}^i$ with compact closures $\bar{U}^i$ still contained in $\mathcal{U}^i$, for $i = 1, 2$. We may assume that the open subsets are chosen such that $f(\bar{U}^1 \times \bar{U}^2) \subseteq V_f$ and $\varphi(\bar{U}^2)$ is convex.

For any map $g \in C^\infty(M \times N, O)$ such that $g(\bar{U}^1 \times \bar{U}^2) \subseteq V_f$ let

$$\tilde{g} = \psi \circ g \circ (1 \times \varphi^{-1}) : \bar{U}^1 \times \varphi(\bar{U}^2) \to \mathbb{R}^n$$

define its 'local representative'. Then for every $x \in U^1$ we define a constant $k(x) > 0$ by the expression

$$k(x) := \min_{y \in \varphi(\bar{U}^2)} \inf_{\|v\|=1} \left\| (d_y \tilde{f}(x)) v \right\|.$$

Note that $k(x) > 0$ because $d_y \tilde{f}(x)$ is injective for all y and $\varphi(\bar{U}^2)$ is compact. Moreover, the map $k : x \mapsto k(x)$ is easily seen to be continuous.

Now let W be the set of all $g \in C^\infty(M \times N, O)$ such that

- $g(\bar{U}^1 \times \bar{U}^2) \subseteq V_f$
- $\sup_{\|v\|=1} \left\| d_y \left(d_{\varphi y_0} \tilde{f}(x) - \tilde{g}(x) \right) v \right\| < \frac{k(x)}{2} \quad \forall y \in \varphi(\bar{U}^2),\ x \in U^1$

The first is an open C^0 condition because $\bar{U}^1 \times \bar{U}^2$ is compact. The second can be rephrased in a way similar to the former, and is an open C^1 condition since $\varphi(\bar{U}^2)$ is compact as well. Hence W is an open neighborhood of $f \in C^\infty(M \times N, O)$.

Now let $g \in W$ and $x \in U^1$ be arbitrary. Then for all $y_1, y_2 \in \varphi(\bar{U}^2)$ we have the estimate

$$\begin{aligned}\left\| d_{\varphi y_0} \tilde{f}(x, -)(y_1 - y_2) \right\| \leq & \; \|\tilde{g}(x, y_1) - \tilde{g}(x, y_2)\| \\ & + \left\| \left(d_{\varphi y_0} \tilde{f}(x) y_1 - \tilde{g}(x, y_1) \right) - \left(d_{\varphi y_0} \tilde{f}(x) y_2 - \tilde{g}(x, y_2) \right) \right\|\end{aligned}$$

and by the first order Taylor formula with integral remainder there is some $y \in [y_1, y_2] \subseteq \varphi(\bar{U}^2)$ such that the final term is bounded by

$$\sup_{\|v\|=1} \left\| d_y (d_{\varphi y_0} \tilde{f}(x) - \tilde{g}(x)) v \right\| \|y_1 - y_2\| < \frac{k(x)}{2} \|y_1 - y_2\| .$$

Hence from the estimate

$$\|\tilde{g}(x, y_1) - \tilde{g}(x, y_2)\| \geq \left\| d_{\varphi y_0} \tilde{f}(x)(y_1 - y_2) \right\| - \frac{k(x)}{2} \|y_1 - y_2\| > \frac{k(x)}{2} \|y_1 - y_2\|$$

we conclude that $g(x)$ is injective on $\bar{U}^2$ for all $x \in U^1$. Since we already shown that the immersions in the N-variable form an open subset, we have completed the proof. $\square$

Proposition 3.2.23. *Let M, N and O be manifolds of finite dimension, with M and N compact, and assume* $\dim(N) \leq \dim(O)$. *Then the subset of $f \in C^\infty(M \times N, O)$ such that*

$$f(x, -) : N \to O$$

is an injective *immersion for all $x \in M$ lies open in $C^\infty(M \times N, O)$.*

Proof. Fix an $f \in C^\infty(M \times N, O)$ as described above. Now choose finitely many points $(x_0, y_0) \in M \times N$ such that the $U^1_{(x_0,y_0)} \times U^2_{(x_0,y_0)} = U^1 \times U^2$ from the previous lemma cover $M \times N$. Let $W_{(x_0,y_0)}$ be the corresponding open $W \subseteq C^\infty(M \times N, O)$. Then define two open subsets

$$\begin{aligned} W &:= \cap_{(x_0,y_0)} W_{(x_0,y_0)}, \\ V &:= \cup_{(x_0,y_0)} U^2_{(x_0,y_0)} \times U^2_{(x_0,y_0)}. \end{aligned}$$

Now the map $g(x, -)$ is an immersion for every $g \in W$ and $x \in M$. Note that the diagonal Δ of $N \times N$ is fully contained in V. Moreover, if $(y_1, y_2) \in V - \Delta$ and $x \in M$, then $(y_1, y_2) \in U^2_{(x_0,y_0)} \times U^2_{(x_0,y_0)}$ and $x \in U^1_{(x_0,y_0)}$ for some pair $(x_0, y_0) \in M \times N$. This holds because there should be a pair (x_0, y_0) such that $(x, y_i) \in U^1_{(x_0,y_0)} \times U^2_{(x_0,y_0)}$ for both $i = 1$ and $i = 2$, and $(y_1, y_2) \in V$ implies that these pairs should coincide. Since $y_1 \neq y_2$, we conclude that $g(x, y_1) \neq g(x, y_2)$.

We complete the proof by contradiction. Suppose there is no suitable open neighborhood of f. Since we already saw that $C^\infty(M, \mathbf{Imm}(N, O))$ is open, the failure must lie with the injectivity condition. Take a sequence $g_n \in W$ converging uniformly to f such that for every $n \in \mathbb{N}$ there are triples $(x_n, y_n, \eta_n) \in M \times (N \times N - \Delta)$ with

$$g_n(x_n, y_n) = g_n(x_n, \eta_n).$$

Since $g_n \in W$, we must have $(y_n, \eta_n) \in N \times N - V$ by the arguments above. The set $M \times (N \times N - V)$ is compact, it is closed in $M \times N \times N$, hence we may assume without loss of generality that $x_n \to x$, $y_n \to y$ and $\eta_n \to \eta$ converge. Clearly $(y, \eta) \in N \times N - V$ still holds, so in particular $y \neq \eta$. Yet by the uniform convergence we still have $f(x, y) = f(x, \eta)$, which gives a contradiction. $\square$

Corollary 3.2.24. *Let M be a compact manifold. The space of diffeomorphisms $\boldsymbol{Diff}(M)$ is an open submanifold of $C^\infty(M, M)$. Its tangent map at the identity of M is 0-tame isomorphic to*

$$T_{id}\boldsymbol{Diff}(M) \simeq \mathcal{X}(M),$$

the space of vector fields on M.

Proof. We have just shown that $\mathbf{Diff}(M)$ is an open subset of $C^\infty(M, M)$ with respect to the Whitney-C^∞ topology and that this topology and the Fréchet topology on $C^\infty(M, M)$ coincide. Furthermore, the tangent space at the identity of M is canonically 0-tame isomorphic to $\Gamma_M(\mathrm{id}^*TM) = \Gamma_M(TM) = \mathcal{X}(M)$, as was proven in proposition 3.2.4 on page 64. $\square$

3.2.7 Inversion

We will now show that the map that sends a diffeomorphism to its inverse is a smooth tame map.

Proposition 3.2.25. *Let M be a compact manifold then the inversion map*

$$\boldsymbol{inv} : \boldsymbol{Diff}(M) \to \boldsymbol{Diff}(M)$$

is a smooth tame map of degree 0*. Its tangent map at $f \in \boldsymbol{Diff}(M)$ is given by*

$$\begin{aligned} T_f\boldsymbol{inv} : \Gamma(f^*TM) &\to \Gamma((f^{-1})^*TM), \\ \varphi &\mapsto -Tf^{-1} \circ \varphi \circ f^{-1}. \end{aligned}$$

Moreover, it is 0*-tame in an open neighborhood of the identity map.*

Proof. There are three things to prove here. We need to show that **inv** is a continuous map, **inv** is a smooth map with the tangent map as described above, and all derivatives of **inv** are tame maps.

Let us begin by showing that **inv** is a continuous map. Note that this is necessary because we have only defined differentiability for continuous maps between Fréchet manifolds. Recall that if $U \subseteq J^k(M, M)$ is an open subset of the k-th jet bundle, then

$$M(U) = \left\{ f \in \mathbf{Diff}(M) : j^k(f)(M) \subseteq U \right\}$$

defines an open subset of $\mathbf{Diff}(M)$, and the family of all such open subsets forms a basis of the topology on $\mathbf{Diff}(M)$. Hence it is sufficient to check that the inverse image of such an $M(U)$ along **inv** is again an open subset of $\mathbf{Diff}(M)$.

Suppose that $k = 0$, then we have $J^0(M, M) = M \times M$ as a trivial bundle over M. For any $f \in \mathbf{Diff}(M)$, the 0-th jet $j^0(f)$ corresponds to $\mathrm{graph}_f \subseteq M \times M$, the graph of f. The interchange map,

$$\tau : M \times M \to M \times M : (x, y) \mapsto (y, x),$$

has the property that $\mathrm{graph}_{f^{-1}} = \tau(\mathrm{graph}_f)$ for every $f \in \mathbf{Diff}(M)$. Suppose that $U \subseteq M \times M$ is an open subset . Then $\mathrm{graph}_{f^{-1}} \subseteq U$ if and only if $\mathrm{graph}_f \subseteq \tau^{-1}(U)$, because τ is a homeomorphism. This implies that

$$\mathbf{inv}^{-1}(M(U)) = \left\{ f \in \mathbf{Diff}(M) : \mathrm{graph}_{f^{-1}} \subseteq U \right\}$$

is an open subset of $\mathbf{Diff}(M)$. Hence **inv** is continuous with respect to the Whitney C^0 topologies on both the domain and codomain.

Next we will show that **inv** is a smooth map. Fix a diffeomorphism $f \in \mathbf{Diff}(M)$. We have already shown that the composition map **com**, and both $f_* = \mathbf{com}(f, -)$ and $f^* = \mathbf{com}(-, f)$ are smooth tame maps. Since we can always write

$$\mathbf{inv}(g) = g^{-1} = f^{-1} \circ (g \circ f^{-1})^{-1} = (f^{-1})_* \circ \mathbf{inv} \circ (f^{-1})^*(g),$$

it is therefore sufficient to prove differentiability at the identity element $\mathrm{id} \in \mathbf{Diff}(M)$ in order to prove that **inv** is C^1. For the higher derivatives we note that the first derivative is again a combination of compositions and inversions.

Let $v \in T_{\mathrm{id}}\mathbf{Diff}(M) = \mathcal{X}(M)$ be a tangent vector at id. Its flow $\varphi : (-1,1) \times M \to M$ is complete, because M is compact. In particular, it satisfies $\varphi_0 = \mathrm{id}$ and $\frac{\partial}{\partial t}\big|_{t=0} \varphi_t = v$, hence it can be seen as a smooth curve in $\mathbf{Diff}(M)$ that represents the tangent vector $v \in T_{\mathrm{id}}\mathbf{Diff}(M)$. It follows that

$$\frac{d}{dt}\bigg|_{t=0} \mathbf{inv}(\varphi_t) = \frac{\partial}{\partial t}\bigg|_{t=0} \varphi_t^{-1} = -v,$$

since

$$0 = \frac{\partial}{\partial t}\bigg|_{t=0} \mathrm{id} = \frac{\partial}{\partial t}\bigg|_{t=0} \varphi_t \circ \varphi_t^{-1} = v + \frac{\partial}{\partial t}\bigg|_{t=0} \varphi_t^{-1},$$

where we use the chain-rule and that $\varphi_0 = \varphi_0^{-1} = \mathrm{id}$. Hence $\mathbf{inv}$ is differentiable at id and $(T_{\mathrm{id}}\mathbf{inv})v = -v$ by remark 3.2.5 on page 66. Recall from proposition 3.2.16 on page 72 what the tangent maps of $(f^{-1})^*$ and $(f^{-1})_*$ are. For any $f \in \mathbf{Diff}(M)$ and tangent vector $v \in \Gamma_M(f^*TM)$ we have

$$\begin{aligned}(T_f\mathbf{inv})v &= T_f((f^{-1})_* \circ \mathbf{inv} \circ (f^{-1})^*)v \\ &= T_{\mathrm{id}}\mathbf{com}(f^{-1}, -) \circ T_{\mathrm{id}}\mathbf{inv} \circ T_f\mathbf{com}(-, f^{-1})v \\ &= -Tf^{-1} \circ v \circ f^{-1},\end{aligned}$$

hence $\mathbf{inv}$ is also differentiable at f. The tangent map of $\mathbf{inv}$ is a combination of compositions, inversions, and differentials, hence it is again a differentiable map. The same observation holds for higher order tangent maps of $\mathbf{inv}$. We conclude that $\mathbf{inv}$ is a smooth map.

Finally we will show that $\mathbf{inv}$ is actually a smooth tame map. Because the tangent map of $\mathbf{inv}$ up to any order is again a combination of compositions, inversions, and differentials, it is sufficient to check that $\mathbf{inv}$ is a tame map. By the translation trick

$$g^{-1} = f^{-1} \circ (g \circ f^{-1})^{-1},$$

it is sufficient to check that $\mathbf{inv}$ is tame in an open neighborhood of the identity. We will do this by reducing the proof to a lemma by Hamilton.

Let m be the dimension of M, and let $B_r \subseteq \mathbb{R}^m$ denote the open ball of radius $r \in \mathbb{R}_{\geq 0}$ around the origin. Cover M by finitely many charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that $\varphi_\alpha(U_\alpha) = B_3$ for all $\alpha \in A$, and that the open subsets $U_\alpha^1 := \varphi_\alpha^{-1}(B_1)$ still cover M. Also define $U_\alpha^2 := \varphi_\alpha^{-1}(B_2)$ for $\alpha \in A$. We will first show that there is an open neighborhood V of the identity in which every f satisfies

$$U_\alpha^1 \subseteq f(U_\alpha^2) \subseteq U_\alpha. \tag{3.2.2}$$

Note that the set A is finite, and the set

$$M(\bar{U}_\alpha^2, U_\alpha) = \left\{ f \in \mathbf{Diff}(M) : f(\bar{U}_\alpha^2) \subseteq U_\alpha \right\}$$

is open in $\mathbf{Diff}(M)$ for every $\alpha \in A$. Hence the set $\cap_{\alpha \in A} M(\bar{U}_\alpha^2, U_\alpha)$ is open as well. This enforces the right-hand-side of (3.2.2). The continuity of $\mathbf{inv}$ implies that also the set

$$\begin{aligned}\mathbf{inv}^{-1}(M(\bar{U}_\alpha^1, U_\alpha^2)) &= \left\{ f \in \mathbf{Diff}(M) : f^{-1}(\bar{U}_\alpha^1) \subseteq U_\alpha^2 \right\} \\ &= \left\{ f \in \mathbf{Diff}(M) : \bar{U}_\alpha^1 \subseteq f(U_\alpha^2) \right\}\end{aligned}$$

lies open in $\mathbf{Diff}(M)$. Hence the set $\cap_{\alpha\in A}\mathbf{inv}^{-1}(M(\bar{U}^1_\alpha, U^2_\alpha))$ is open as well, and this enforces the left-hand-side of (3.2.2). The definition of V is now obvious.

Secondly we will choose two charts of $\mathbf{Diff}(M)$ around the identity and give descriptions of the corresponding gradings to clarify why we want to prove lemma 3.2.26 below. We begin by choosing a chart in the codomain of $\mathbf{inv}$. Let $E \xrightarrow{\simeq} \nu$ be a tubular neighborhood around the diagonal Δ along the fibers of $\mathrm{pr}_1 : M \times M \to M$. Take this tubular neighborhood small enough such that $\nu \subseteq \cup_{\alpha\in A} U^1_\alpha \times U^2_\alpha$. Then the U_α provide local trivializations of E, such that $E_{U_\alpha} \hookrightarrow U_\alpha \times U^2_\alpha$, for all $\alpha \in A$. If $g \in \Gamma_M\nu$, then let $g^\alpha : B_1 \to B_2$ denote the local representative of $g|_{U^1_\alpha}$ along all the trivializations and charts. Since the family $\{U^1_\alpha\}$ already covers M, the C^k-norms on $\Gamma_M\nu$ are of the form

$$\|g\|_k = \max_\alpha \|g^\alpha\|_k = \sum_{|\beta|\leq k}\sum_{1\leq i\leq m} \max_\alpha \sup_{x\in\bar{B}_1} \left|\partial^\beta g^\alpha_i(x)\right|$$

for every $g \in \Gamma_M\nu$.

Likewise, let $\tilde{E} \xrightarrow{\simeq} \tilde{\nu}$ be a tubular neighborhood around Δ along the fibers of $M\times M$. Moreover, assume that $\tilde{\nu} \subseteq V \cap \mathbf{inv}^{-1}(\Gamma_M\nu)$, so that all $f \in \Gamma_M\tilde{\nu}$ satisfy (3.2.2), and $\mathbf{inv}$ maps $\Gamma_M\tilde{\nu}$ into $\Gamma_M\nu$. Let $f^\alpha : B_2 \to B_3$ denote the local representative of $f|_{U^2_\alpha}$ along all the trivializations and charts. Then the C^k-norms on $\Gamma_M\tilde{\nu}$ are of the form

$$\|f\|_k = \max_\alpha \|f^\alpha\|_k = \sum_{|\beta|\leq k}\sum_{1\leq i\leq m} \max_\alpha \sup_{x\in\bar{B}_2} \left|\partial^\beta f^\alpha_i(x)\right|.$$

To make tameness estimates it is now sufficient to compare $\|f^\alpha\|_k$ with $\|(f^\alpha)^{-1}\|_k$. □

The following lemma and its proof are copied directly from [Ham82b], except for a correction in the estimates, and a few comments and clarifications of the arguments. I have included the proof for the sake of completeness.

Lemma 3.2.26 (Hamilton)**.** *Let $B_r \subseteq \mathbb{R}^m$ denote an open ball of radius $r \in \mathbb{R}_{\geq 0}$ around the origin. Let $f : B_2 \to B_3$ be a smooth map wich extends to a smooth map $f : U \to \mathbb{R}^m$ on an open subset U of the closure $\bar{B}_2$. If $\varepsilon > 0$ is sufficiently small, and if $\|f - id\|_1 < \varepsilon$, then $f^{-1} : B_1 \to B_2$ exists, and for every $k \in \mathbb{N}$ there is a constant $C > 0$ independent of f such that*

$$\left\|f^{-1}\right\|_k \leq C(1 + \|f\|_k)$$

for all such f.

Proof. In the previous proposition we already showed that the inverse of f exists and maps B_1 into B_2 if f is C^0-close to the identity. We will write $g = f^{-1}$, so that $D_x g = (D_{f^{-1}(x)}f)^{-1}$. This implies that if $\varepsilon > 0$ is sufficiently small, there is constant $C > 0$ independent of g such that

$$\|Dg\|_0 \leq C.$$

Here we have used that we can make estimates

$$\|Df\|_0 \leq C(1 + \|Df - \mathrm{id}\|_0) \leq C(1 + \|f - \mathrm{id}\|_1) \leq C,$$

and the fact that $\|A\| = \|A^{-1}\|$ for any linear map $A : \mathbb{R}^m \to \mathbb{R}^m$.

For $n \geq 2$ we have $D^n(f \circ g) = D^n \mathrm{id} = 0$, hence we obtain an equation

$$0 = D_x^n(f \circ g) = D_{g(x)} f \circ D_x^n g + \sum_{k=2}^{n} \sum_{j_1+\ldots+j_k=n} c_{k,j_1,\ldots,j_k} D_{g(x)}^k f(D_x^{j_1} g, \ldots, D_x^{j_k} g),$$

where the $c_{k,j_1,\ldots,j_k}$ are suitable constants. We can solve this for $D_x^n g$ to get the equation

$$D_x^n g = -D_x g \sum_{k=2}^{n} \sum_{j_1+\ldots+j_k=n} c_{k,j_1,\ldots,j_k} D_{g(x)}^k f(D_x^{j_1} g, \ldots, D_x^{j_k} g).$$

This leads to the estimate

$$\|D^n g\|_0 \leq C \sum_{k=2}^{n} \sum_{j_1+\ldots+j_k=n} \|f\|_k \|g\|_{j_1} \cdots \|g\|_{j_k}.$$

By interpolation we have estimates

$$\|f\|_k^{n-1} \leq C \|f\|_1^{n-k} \|f\|_n^{k-1},$$
$$\|g\|_j^{n-2} \leq C \|g\|_1^{n-j-1} \|g\|_{n-1}^{j-1}.$$

Moreover, we can estimate $\|f\|_1, \|g\|_1 \leq C$, and $k \geq 2$ and $j_1 + \ldots + j_k = n$ implies that $j_1, \ldots, j_k \leq n-1$. This leads to the estimate

$$\|D^n g\|_0 \leq C \sum_{k=2}^{n} \|f\|_n^{(k-1)/(n-1)} \|g\|_{n-1}^{(n-k)/(n-2)}.$$

We complete the proof by induction on n. For $n = 1$ the lemma holds. Now suppose that $n \geq 2$ and

$$\|g\|_{n-1} \leq C(1 + \|f\|_{n-1}).$$

Then by the interpolation estimate

$$\|f\|_{n-1}^{n-1} \leq \|f\|_n^{n-2} \|f\|_1,$$

we have that

$$(1 + \|f\|_{n-1}) \leq C(1 + \|f\|_n)^{(n-2)/(n-1)}$$

if we ensure that $\|f\|_1 \leq C$. Hence we need to take $C \geq 1 + \varepsilon$. This implies that

$$\|D^n g\|_0 \leq C \sum_{k=2}^{n} \|f\|_n^{(k-1)/(n-1)} (1 + \|f\|_n)^{(n-k)/(n-1)} \leq C(1 + \|f\|_n).$$

The one in $(1 + \|f\|_n)$ on the right-hand-side is necessary. we cannot estimate

$$\|f\|_n^{(k-1)/(n-1)} \leq \|f\|_n,$$

because $k-1$ might be smaller than $n-1$, but we can make an estimate

$$\|f\|_n^{(k-1)/(n-1)} \leq C(1 + \|f\|_n),$$

and continue our estimates using this. Now use that

$$\|g\|_n \leq \|g\|_{n-1} + \|D^n g\|_0 \leq C(2 + \|f\|_{n-1} + \|f\|_n) \leq C(1 + \|f\|_n). \qquad \square$$

Remark 3.2.27. *At the end of the proof of proposition 3.2.25 on page 81 we apply lemma 3.2.26 on page 83 once for every* f^α*. In the lemma we impose an open condition* $\|f^\alpha - id\|_1 < \varepsilon_\alpha$ *for every* $\alpha \in A$*. Since the index set* A *can be chosen finite, these conditions define an open subset* $\{f \in V : \|f^\alpha - id\|_1 < \varepsilon_\alpha, \forall \alpha \in A\}$ *of* V*.*

Recall that the composition map is *not* 0-tame, its tangent map is at best tame of degree 1 in the first component. With the translation trick used here it cannot be avoided that g appears once in both components of the composition map, hence this proof doesn't imply that the inversion map is 0-tame. Moreover, the formula for the tangent map suggests that it is highly unlikely that the inversion map is 0-tame, even though it is 0-tame in a neighborhood of the identity map.

We finish this section by summarizing a part of the results in a corollary.

Corollary 3.2.28. *Let* M *be a compact manifold, then the diffeomorphism group* $\boldsymbol{Diff}(M)$ *is a tame Lie group. Its tangent space at the identity is the graded Fréchet space* $\mathcal{X}(M)$ *of vector fields on* M*.*

3.2.8 Bundle maps $A \to B$

This section provides more examples of tame manifolds, smooth tame maps, and, in addition, examples of tame fiber bundles. It generalizes the results from the previous section to maps between fiber bundles. We will do this in the two steps described below.

First we will consider two bundles $A \to M$ and $B \to M$ over the same base manifold. We will typically assume that A, and hence also M, is compact. Then we consider bundle maps of the form

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

in other words, the smooth maps $f : A \to B$ that map fibers into fibers over the same base point. These are the usual arrows in the category $\mathbf{Bund}_M$ of bundles over M, hence we will denote the set by $\mathbf{Bund}_M(p,q)$ or if no ambiguity arises by $\mathbf{Bund}_M(A,B)$. We will show that it naturally becomes a tame manifold and the natural maps remain smooth tame as well. In particular we have the open submanifold

$$\mathbf{Diff}_M p = \mathbf{Diff}_M A$$

of fiber preserving diffeomorphisms over M and show it is also a tame Lie group.

As to be expected, the previous section returns to us as the special case where $M = \{*\}$ is the one-point set and to sections of a fiber bundle via the bundle maps

$$\Gamma_M A = \mathbf{Bund}_M(M, A).$$

We show that $\mathbf{Bund}_M(A,B)$ is a closed submanifold of $C^\infty(A,B)$; in a sense they are not more general than smooth maps.

We begin with two bundles $A \xrightarrow{p} M$ and $B \xrightarrow{q} M$ over a compact manifold M and assume that A is compact. There are at least three equivalent descriptions of the space of bundle maps. The first is just the subspace of $C^\infty(A,B)$ of smooth maps

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

as described above. For the second description, consider the pull-back bundle p^*B as in the diagram below,

$$\begin{tikzcd} p^*B \arrow[r, "p'"] \arrow[d] \arrow[dr, phantom, "p.b."] & B \arrow[d, two heads, "q"] \\ A \arrow[r, two heads, "p"] & M. \end{tikzcd}$$

The fibers of p^*B are of the form $p^*B_a = B_{p(a)}$ hence a smooth section of p^*B also defines a smooth bundle map via composition on the left with p',

$$p'_* : \Gamma_A(p^*B) \longrightarrow \mathbf{Bund}_M(A,B) : \sigma \mapsto p' \circ \sigma.$$

This is just the restriction of the map $p'_* : C^\infty(A, p^*B) \longrightarrow C^\infty(A,B)$ to the closed submanifold $\Gamma_A(p^*B)$, hence it is continuous if we induce $\mathbf{Bund}_M(A,B)$ with the subset topology.

On the other hand, a bundle map $f : A \to B$ defines a smooth section $\sigma_f \in \Gamma_A(p^*B)$ simply by defining $\sigma_f(a) := f(a) \in B_{p(a)}$. This defines a map

$$\sigma_{(-)} : \mathbf{Bund}_M(A,B) \longrightarrow \Gamma_A(p^*B) : f \mapsto \sigma_f,$$

which is clearly the algebraic inverse of p'_*. It is straightforward to check that this maps is also continuous. Hence $\mathbf{Bund}_M(A,B)$ may be equipped with the usual smooth tame structure on $\Gamma_A(p^*B)$. We should make certain that if $\mathbf{Bund}_M(A,B)$ can be considered as a submanifold of $C^\infty(A,B)$, this coincides with the smooth tame structure defined above.

Proposition 3.2.29. *Let $A \xrightarrow{p} M$ and $B \xrightarrow{q} M$ be bundles over the same base manifold and assume that A is compact. Then the space $\boldsymbol{Bund}_M(A,B) \simeq \Gamma_A(p^*B)$ of bundle maps is a tame submanifold of $C^\infty(A,B)$. Its tangent space at $f \in \boldsymbol{Bund}_M(A,B)$ is given by*

$$T_f \boldsymbol{Bund}_M(A,B) = \Gamma_A(f^*T^{vert}p^*B).$$

Proof. We only need to check that p'_* embeds $\Gamma_A(p^*B)$ into $C^\infty(A,B)$. The local model at f of $\Gamma_A(p^*B)$ is given by $\Gamma_A(f^*T^{\mathrm{vert}}B)$. We have an inclusion of vector bundles

$$T^{\mathrm{vert}}p^*B \hookrightarrow p'^*T^{\mathrm{vert}}B \hookrightarrow p'^*TB$$

defined by

$$\begin{tikzcd}
T^{\mathrm{vert}}p^*B \arrow[dr, dashed, tail, "\exists!"] \arrow[drr, bend left, "Tp'"] \arrow[ddr, bend right, "\pi_{p^*B}"'] & & \\
& p'^*T^{\mathrm{vert}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & T^{\mathrm{vert}}B \arrow[d, "\pi_B"] \\
& p^*A \arrow[r, "p'"] & B,
\end{tikzcd}$$

where Tp' maps between vertical vectors into vertical vectors since $dq \circ Tp' = Tp \circ dq'$, and $p' \circ \pi_{p^*B} = \pi_B \circ Tp'$ is obvious. The fact that this map is injective is easily checked by writing down its explicit formula.

Note that the local model of $C^\infty(A,B)$ at $p'_*(f) = p'f$ is just $\Gamma_A((p'f)^*TB)$. The above gives an inclusion of vector bundles

$$\varphi : f^*T^{\mathrm{vert}}p^*B \hookrightarrow (p'f)^*TB.$$

The composition on the left map φ_* coincides, up to some isomorphisms, with the map

$$\Gamma(f^*T^{\mathrm{vert}}p^*B) \longrightarrow \Gamma((p'f)^*TB)$$

induced by p'_* and the chosen charts by virtue of it being defined as a universal arrow.

Now one can choose a metric g on TB and choose an orthocomplement $E \subseteq (p'f)^*TB$ of $\varphi(f^*T^{\mathrm{vert}}p^*B)$. This gives a tame direct sum

$$\Gamma_A((p'f)^*TB) \simeq \Gamma_A(\varphi^*(f^*T^{\mathrm{vert}}p^*B)) \oplus \Gamma(E). \qquad \square$$

In section 5.1 on page 97 on the stability of mappings we will expand this setting further by considering bundle maps between two bundles $A \to M$ and $B \to N$ with possibly distinct base manifolds.

Suppose that B is compact as well and $C \to M$ is a third bundle over M, not necessarily compact. Then by the above it immediately follows that the fiber-wise composition map

$$\mathbf{com} : \mathbf{Bund}_M(B, C) \times \mathbf{Bund}_M(A, B) \to \mathbf{Bund}_M(A, C)$$

is a smooth tame map of degree 0. Its tangent map at (f, g) is given by the familiar formula

$$T_{(f,g)}\mathbf{com} : (\varphi, \gamma) \mapsto \varphi \circ g + Tf \circ \gamma.$$

The space of invertible bundle maps of $A \to M$ is of course just the intersection $\mathbf{Diff}_M A = \mathbf{Bund}_M(A, A) \cap \mathbf{Diff} A$, and hence is an open submanifold of the space of all bundle maps. Its tangent space at the identity is the space of vertical vector fields of A, commonly denoted as $\mathcal{X}^{\mathrm{vert}}(A)$. Lastly, the inverse of any bundle map preserves fibers itself, so the restriction of the inversion map,

$$\mathbf{inv} : \mathbf{Diff}_M A \longrightarrow \mathbf{Diff}_M A,$$

is a smooth tame map of degree 0. Its tangent map at f is given by

$$T_f\mathbf{inv} : \eta \mapsto -Tf^{-1} \circ \eta \circ f^{-1},$$

where, as with the tangent of the composition map, one must interpret η as a tangent vector of $C^\infty(A, B)$ at $p'_*(f)$.

The third description begins by defining a tame fiber bundle over M whose smooth sections will describe the bundle maps $A \to B$. Each of its fibers consists of the space of smooth maps $A_x \to B_x$. In defining this bundle there is no need to assume M is compact. We do need that p is a proper map such that each of the fibers is a tame manifold. Then p is automatically a fiber bundle by the Ehresmann theorem, and for the construction of $C^\infty(p, q)$ we will assume that q is a fiber bundle as well.

Lemma 3.2.30. *Let $A \xrightarrow{p} M$ and $B \xrightarrow{q} M$ be fiber bundles over the same base manifold M and let p be* proper. *Then the bundle $C^\infty(p, q) \to M$ whose fiber at $x \in M$ is given by*

$$C^\infty(p, q)_x := C^\infty(A_x, B_x)$$

is a tame fiber bundle over M.

Proof. For an arbitrary point $x \in M$ one can choose an open neighborhood $U \subseteq M$ and two local trivializations

$$\psi_A : A_U \xrightarrow{\cong} U \times A_x,$$
$$\psi_B : B_U \xrightarrow{\cong} U \times B_x,$$

so that we obtain a trivialization of the large bundle,

$$\Psi : C^\infty(p,q)_U \longrightarrow U \times C^\infty(A_x, B_x),$$

defined as follows. If $y \in U$ then $f \in C^\infty(A_y, B_y)$ is send by Ψ to

$$\left(\psi_B|_{B_y}\right) \circ f \circ \left(\psi_A|_{A_y}\right)^{-1} \in C^\infty(A_x, B_x).$$

Any other open $U' \subseteq M$ around a point $x' \in M$ with corresponding maps ψ'_A, ψ'_B and Ψ' yields a transition map

$$\Psi' \circ \Psi^{-1} : U \cap U' \times C^\infty(A_x, B_x) \longrightarrow U \cap U' \times C^\infty(A_{x'}, B_{x'}).$$

Fix a point $y \in U \subseteq U'$, then $\Psi'\Psi^{-1}(y, -)$ maps a function $f \in C^\infty(A_x, B_x)$ to

$$\Psi' \circ \Psi^{-1}(y, f) = \left(y, (\psi'_B|_{B_y}) \circ (\psi_B|_{B_y})^{-1} \circ f \circ (\psi_A|_{A_y}) \circ (\psi'_A|_{A_y})^{-1}\right),$$

which is just a combination of pre- and post composition by certain maps. Hence $\Psi'\Psi^{-1}$ is smooth tame in the second component.

Now fix a function $f \in C^\infty(A_x, B_x)$ and vary the first component of the transition function. If we write

$$(p_1, \varphi_A) = \psi_A(\psi'_A)^{-1} : U \cap U' \times A'_x \longrightarrow U \cap U' \times A_x,$$
$$(p_1, \varphi_B) = \psi'_B \psi_B^{-1} : U \cap U' \times B_x \longrightarrow U \cap U' \times B'_x,$$

then the φ_A and φ_B define smooth tame maps

$$\tilde{\varphi}_A : U \cap U' \longrightarrow C^\infty(A'_x, A_x) : y \mapsto \varphi_A(y, -),$$
$$\tilde{\varphi}_B : U \cap U' \longrightarrow C^\infty(B_x, B'_x) : y \mapsto \varphi_B(y, -),$$

respectively. Now $\Psi'\Psi^{-1}(-, f)$ is the smooth tame map $U \cap U' \longrightarrow U \cap U' \times C^\infty(A'_x, B'_x)$ whose first component is the identity on $U \cap U'$ and whose second component is given by

$$\Psi'\Psi^{-1}(-, f) = \mathbf{com} \circ (\mathrm{id} \times \mathbf{com}(f, -)) \circ (\tilde{\varphi}_B, \tilde{\varphi}_A) \circ \Delta_{U \cap U'}.$$

Hence the transition maps are smooth tame. □

Assume once again that M is compact, so that we are working with two compact (fiber) bundles A and B over M. A bundle map $f : A \to B$ gives rise to a section τ_f of $C^\infty(p,q)$ by restricting f to the appropriate fiber of A, that is, by the formula $\tau_f(x) = f|_{A_x} : A_x \to B_x$. This section is in fact smooth, for consider an open neighborhood $x \in U \subseteq M$ on which both A and B trivialize, as in lemma 3.2.30 on the facing page. the trivialization of f,

$$\begin{tikzcd}
U \times A_x \arrow[r, "\tilde{f}"] & U \times B_x \\
A_U \arrow[u, phantom] \arrow[r, "f"] & B_U \arrow[u, "\wr"']
\end{tikzcd}$$

$$\begin{array}{ccc} U \times A_x & \xrightarrow{\tilde{f}} & U \times B_x \\ \downarrow \wr & & \uparrow \wr \\ A_U & \xrightarrow{f} & B_U, \end{array}$$

defines a smooth map $U \to C^\infty(A_x, B_x) : y \mapsto \tilde{f}(y, -)$. The latter map is the local trivialization of τ_f, hence τ_f does indeed define an element in $\Gamma_M C^\infty(p, q)$. We also need to show that $\tau_{(-)} : f \mapsto \tau_f$ is a continuous map if $\Gamma_M C^\infty(p, q)$ is equipped with the compact open topology. Its inverse is the map

$$f_{(-)} : \Gamma_M C^\infty(p, q) \longrightarrow \mathbf{Bund}_M(A, B) : \tau \mapsto f_\tau,$$

where $f_\tau(a) = \tau(p(a))(a)$, and this map is continuous for the same reason. We conclude that the natural maps between $\mathbf{Bund}_M(A, B)$ and $\Gamma_M C^\infty(A, B)$ give homeomorphisms, hence the smooth tame structure on $\mathbf{Bund}_M(A, B)$ carries over.

It is perhaps possible to prove that $\Gamma_M C^\infty(p, q)$ is a tame manifold in a more direct, and possibly more natural, way, although this might actually result in reordening the above construction. On the other hand, it does not seem likely that $\Gamma_M \mathcal{B}$ can be given a tame Fréchet structure for any tame fiber bundle over M. The charts for $\Gamma_M B$, with B a finite rank fiber bundle, where constructed via a choice of Riemannian metric and the resulting exponent map. Following this approach, one has to construct a Riemannian metric on $\mathcal{B}$ and a exponent map, which is typically defined using the existence and uniqueness of solutions to ODEs. Both constructions are non-trivial for Fréchet spaces. However, in this case it is not necessary, and we might as well avoid it; such manifolds would become even more cumbersome to work with.

This discription does have at least one advantage: it suggests that the tangent space should be of the somewhat more conceptual form

$$T_\tau \Gamma_M C^\infty(p, q) = \Gamma_M(\tau^* T^{\mathrm{vert}} C^\infty(p, q)).$$

In particular, this heuristic will lead to the following, rather nice, description of the tangent space of $\mathbf{Diff}_M(A)$. Let $\mathcal{X}(p) \to M$ be the vector bundle whose fiber at $x \in M$ is the Fréchet space of vector fields on the fiber A_x, $\mathcal{X}(p)_x = \mathcal{X}(A_x)$. Then the tangent space of $\mathbf{Diff}_M(A)$ at the identity should be the space of sections $\Gamma_M \mathcal{X}(p)$.

Proposition 3.2.31. *Let $A \xrightarrow{p} M$ be a proper fiber bundle over a base manifold M with boundary. Then the bundle $\mathcal{X}(p) \to M$ whose fibers at $x \in M$ are given by*

$$\mathcal{X}(p)_x = \mathcal{X}(A_x),$$

the vector fields on the fiber A_x, forms a 0-tame vector bundle.

Proof. The proof is essentially analogous to lemma 3.2.30 on page 88 with the addition of checking that the transition maps are linear. □

Now the space of vertical vector fields is easily seen to be homeomorphic to $\Gamma_M \mathcal{X}(p)$ by the map that sends $v \in \mathcal{X}^{\mathrm{vert}}(A)$ to $\sigma_v(x) = v|_{A_x}$. In fact, there is a natural Fréchet grading on $\Gamma_M \mathcal{X}(p)$ which makes the above map a tame linear isomorphism. Going into more detail on this grading, however, doesn't seem to add anything to the discussion at this point.

Chapter 4

The Nash-Moser theorem

In this chapter we state the Nash-Moser inverse function theorem and a variation, the Nash-Moser theorem for non-linear chain complexes. We will prove that the latter implies the inverse function theorem, yet a converse is not so easily obtained. Chapter 5 treats several applications where mainly the version for chain complexes is applied. It is only in chapter 6 that we get to proving this theorem.

Theorem 4.0.32 (Nash-Moser-Hamilton)**.** *Let $P : \mathcal{M} \to \mathcal{N}$ be a smooth tame map between tame manifolds with smoothing operators. If there is a smooth tame vector bundle map*

$$VP : P^*T\mathcal{N} \to T\mathcal{M}$$

that is the point-wise inverse of the derivative of P in the sense that for every $x \in \mathcal{M}$ we have

$$D_xP \circ VP_x = id_{T_{P(x)}\mathcal{N}}$$
$$VP_x \circ D_xP = id_{T_x\mathcal{M}},$$

then P is locally invertible in the sense that around every $x_b \in \mathcal{M}$ there is an open neighborhood $x_b \in \mathcal{U} \subseteq \mathcal{M}$ on which P has a smooth tame inverse P^{-1}. Naturally, in this case the derivative of the inverse is $D(P^{-1}) = VP$ on $\mathcal{U}$.

Before stating the version for non-linear chain complexes, we will first introduce the needed terminology. A *tame non-linear complex* is a triple of smooth tame maps

$$\mathcal{M} \xrightarrow{P} \mathcal{N} \underset{S}{\overset{R}{\rightrightarrows}} \mathcal{O}$$

such that $R \circ P = S \circ P$. Such a complex is *tame exact* at a point $x_0 \in \mathcal{M}$ if there exist neighborhoods $x_b \in \mathcal{U} \subseteq \mathcal{M}$ and $P(x_b) \in \mathcal{V} \subseteq \mathcal{N}$ and a smooth tame map

$$Q : \mathcal{V} \to \mathcal{U}$$

such that if $y \in \mathcal{V}$ satisfies $R(y) = S(y)$ then $PQ(y) = y$. In other words, Q is a local right inverse of P when restricted to the equalizer set of R and S,

$$\mathrm{eq}(R, S) = \{y \in \mathcal{N} : R(y) = S(y)\}\,.$$

Such a set is generally not a submanifold of $\mathcal{N}$, but the smooth tameness of Q still makes sense on open subsets of $\mathcal{N}$ around points in $\mathrm{eq}(R, S)$. The chain complex is *locally tame exact* if it is tame exact at every point of $\mathcal{M}$ and *tame exact* if the map Q can be chosen globally. Clearly, tame exact implies locally tame exact.

Note that R and S both coincide on the image of P, yet their derivatives might still differ. The defect can be measured by

$$\delta_x R : P^*T\mathcal{N} \longrightarrow (RP)^*T\mathcal{O} = D_{P(x)}R - D_{P(x)}S$$

since both maps on the right map into the same fiber of $(RP)^*T\mathcal{O}$. The *linearization* of a tame non-linear complex is the sequence of vector bundle maps

$$T\mathcal{M} \xrightarrow{DP} P^*T\mathcal{N} \xrightarrow{\delta R} (RP)^*T\mathcal{O}.$$

It satisfies the usual condition $\delta R \circ DP = 0$ of linear chain complexes.

Such a linear chain complex is *tame exact* if there are smooth tame vector bundle maps

$$(RP)^*T\mathcal{O} \xrightarrow{VR} P^*T\mathcal{N} \xrightarrow{VP} T\mathcal{M}$$

satisfying the homotopy relation

$$DP \circ VP + VR \circ \delta R = \mathrm{id}_{P^*T\mathcal{N}}.$$

The maps VP and VR will be called homotopy operators.

The version for non-linear chain complexes can now be formulated as follows.

Theorem 4.0.33 (Hamilton)**.** *A tame non-linear complex is locally tame exact if its linearization is tame exact.*

Corollary 4.0.34. *The inverse function theorem follows from the non-linear chain complexes theorem.*

Proof. Suppose that $P : \mathcal{M} \to \mathcal{N}$ satisfies the hypothesis of the Nash-Moser inverse function theorem. Take $\mathcal{O} = \mathcal{N}$ and let $R = S$ both be the identity on $\mathcal{N}$ to obtain the chain complex

$$\mathcal{M} \xrightarrow{P} \mathcal{N} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathcal{N}.$$

The equality $R(y) = S(y)$ is trivially satisfied for all $y \in \mathcal{N}$ and we are allowed to take $VR = \mathrm{id}_{T\mathcal{N}}$ such that

$$DP \circ VP + VR \circ \delta R = \mathrm{id}_{T\mathcal{N}} + \mathrm{id}_{T\mathcal{N}} \circ 0 = \mathrm{id}_{T\mathcal{N}}.$$

Hence there are open subsets $x_b \in \mathcal{U} \subseteq \mathcal{M}$ and $P(x_b) \in \mathcal{V} \subseteq \mathcal{N}$ and a smooth tame map $Q : \mathcal{V} \to \mathcal{U}$ so that $P \circ Q = \mathrm{id}$ on $\mathcal{V}$, that is, P has locally a right inverse.

Now consider the chain complex

$$\mathcal{V} \xrightarrow{Q} \mathcal{U} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathcal{U}.$$

Let $VQ : Q^*T\mathcal{U} \to T\mathcal{V}$ be defined by

$$VQ_y := D_{Q(y)}P$$

and let VR be the identity map of $Q^*T\mathcal{U}$. VQ is a smooth tame vector bundle map over $\mathcal{V}$. Then we have

$$DQ \circ VQ + VR \circ \delta R = DQ \circ VQ = \mathrm{id}_{Q^*T\mathcal{U}},$$

since the identity $D_{Q(y)}P \circ D_y Q = \mathrm{id}$ from above implies that

$$V_{Q(y)}P = V_{Q(y)}P \circ D_{Q(y)}P \circ D_y Q = D_y Q,$$

so that

$$D_y Q \circ V_y Q = V_{Q(y)}P \circ D_{Q(y)}P = \mathrm{id}_{Q^*T\mathcal{U}}.$$

Hence there are open subsets $P(x_b) = y_b \in \tilde{\mathcal{V}} \subseteq \mathcal{V}$ and $Q(y_b) = x_b \in \tilde{\mathcal{U}} \subseteq \mathcal{U}$ and a smooth tame map $\tilde{Q} : \tilde{\mathcal{U}} \to \tilde{\mathcal{V}}$ so that $Q \circ \tilde{Q} = \mathrm{id}$ on $\tilde{\mathcal{U}}$.

By restricting the open neighborhoods we conclude that $Q : \mathcal{U} \to \mathcal{V}$ has a smooth tame right inverse $\tilde{Q}$ and is injective, hence it has an inverse. We deduce that $\tilde{Q} = PQ\tilde{Q} = P$ on $\tilde{\mathcal{U}}$, hence P is locally invertible. □

4.1 A version with group actions

Let $\mathcal{G}$ be a tame Lie group and let $\mathcal{M}$ and $\mathcal{N}$ be two tame manifolds, such that $\mathcal{M}$,$\mathcal{N}$, and $\mathcal{G}$ all allow smoothing operators. Suppose that $\mathcal{G}$ acts smooth tamely on the left on both $\mathcal{M}$ and $\mathcal{N}$ in the sense that the maps

$$\varphi : \mathcal{G} \times \mathcal{M} \to \mathcal{M},$$
$$\psi : \mathcal{G} \times \mathcal{N} \to \mathcal{N}$$

are smooth tame. Now let $R, S : \mathcal{M} \to \mathcal{N}$ be two smooth tame maps equivariant under the actions of $\mathcal{G}$.

Suppose that $x \in \mathcal{M}$ satisfies $y = R(x) = S(x)$, then by equivariance any element x' of the orbit $\mathcal{G}(x)$ through x satisfies $R(x') = S(x')$. Define a smooth tame map

$$P : \mathcal{G} \longrightarrow \mathcal{M} : g \mapsto g \cdot x$$

and consider the resulting non-linear chain complex (P, R, S). We wish to apply the Nash-Moser theorem to this situation. This will provide conditions under which the orbit $\mathcal{G}(x)$ lies open in the equalizer set $\mathrm{eq}(R, S)$ induced with the subset topology.

Assume that the linearization at the unit $e \in \mathcal{G}$, which we define as

$$\mathfrak{g} = T_e\mathcal{G} \xrightarrow{d_0} T_x\mathcal{M} \xrightarrow{d_1} T_y\mathcal{N}$$

with $d_0 = T_eP$ and $d_1 = T_xR - T_xS$, splits tamely in the sense that there are tame linear maps

$$T_y\mathcal{N} \xrightarrow{h_1} T_x\mathcal{M} \xrightarrow{h_0} \mathfrak{g}$$

such that the homotopy relation

$$d_0 \circ h_0 + h_1 \circ d_1 = \mathrm{id}_{T_x\mathcal{M}}$$

is satisfied. We wish to prove the hypothesis of the non-linear chain complexes theorem from this assumption.

Note that the action $\varphi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ induces a smooth tame map

$$\varphi_* : \mathcal{G} \times T_x\mathcal{M} \longrightarrow P^*T\mathcal{M}$$

by taking the tangent map at $x \in \mathcal{M}$ in the second component. It maps a pair (g, v), with $g \in \mathcal{G}$ and $v \in T_x\mathcal{M}$, to the vector $T\varphi(g)v \in T_{\varphi(g)x}\mathcal{M} = T_{P(g)}\mathcal{M}$ and hence is a vector bundle map over $\mathcal{G}$. Its inverse, the map

$$\varphi^* : P^*T\mathcal{M} \longrightarrow \mathcal{G} \times T_x\mathcal{M}$$

defined by sending $v \in T_{P(g)}\mathcal{M}$ to the pair $(g, T\varphi(g^{-1})v)$ is easily seen to be a smooth tame map as well. For recall the maps associated to the pull-back bundle $P^*T\mathcal{M}$, as indicated in the pull-back square

$$\begin{tikzcd} P^*T\mathcal{M} \arrow[r, "Q"] \arrow[d, "q_{\mathcal{M}}"'] \arrow[dr, phantom, "p.b."] & T\mathcal{M} \arrow[d, "\pi_{\mathcal{M}}"] \\ \mathcal{G} \arrow[r, "P"] & \mathcal{M} \end{tikzcd}$$

Then φ^* is the composition of smooth tame maps

$$\varphi^* = (\mathrm{id} \times T^{\mathcal{M}}\varphi) \circ (\mathrm{id} \times i \times \mathrm{id}) \circ (\Delta_{\mathcal{G}} \times \mathrm{id}) \circ \langle q_{\mathcal{M}}, Q\rangle,$$

which takes values in the tame submanifold $\mathcal{G} \times T_x\mathcal{M}$ of $\mathcal{G} \times T\mathcal{M}$. Hence the vector bundle $P^*T\mathcal{M}$ is tamely isomorphic to the trivial bundle $\mathcal{G} \times T_x\mathcal{M}$ over $\mathcal{G}$.

The same can be said about the multiplication $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, seen as a left action $L = m$ of $\mathcal{G}$ on itself, and the action $\psi : \mathcal{G} \times \mathcal{N} \to \mathcal{N}$. We obtain that $T\mathcal{G} \simeq \mathcal{G} \times \mathfrak{g}$ and $(RP)^*T\mathcal{N} \simeq \mathcal{G} \times T_y\mathcal{N}$ tamely.

This allows us to define vector bundle maps over $\mathcal{G}$ by

$$\begin{aligned} VP : P^*T\mathcal{M} &\longrightarrow T\mathcal{G}, \quad VP = L_* \circ h_0 \circ \varphi^*, \\ VR : (RP)^*T\mathcal{N} &\longrightarrow P^*T\mathcal{M}, \quad VR = \varphi_* \circ h_1 \circ \psi^*. \end{aligned}$$

These maps satisfy the identities

$$\begin{aligned} T_gP \circ V_gP &= T_gP \circ T_eL(g) \circ h_0 \circ T_{P(g)}\varphi(g^{-1}) \\ &= T_x\varphi(g) \circ T_eP \circ h_0 \circ T_{P(g)}\varphi(g^{-1}) \\ &= T_x\varphi(g) \circ d_0h_0 \circ T_{P(g)}\varphi(g^{-1}), \\ V_gR \circ \delta_gR &= T_x\varphi(g) \circ h_1 \circ T_{RP(g)}\psi(g^{-1}) \circ \delta_gR \\ &= T_x\varphi(g) \circ h_1d_1 \circ T_{P(g)}\varphi(g^{-1}), \end{aligned}$$

so that the required homotopy relation is satisfied,

$$\begin{aligned} T_g P \circ VP(g) + VR(g) \circ \delta R(g) &= T_x\varphi(g) \circ (d_0 h_0 + h_1 d_1) \circ T_{P(g)}\varphi(g^{-1}) \\ &= \mathrm{id}_{T_{P(g)}\mathcal{M}} \end{aligned}$$

for every $g \in \mathcal{G}$. Hence the hypothesis of the non-linear chain complex theorem are satisfied. The results of this section can be summarized in the following theorem.

Theorem 4.1.1. *Let $\mathcal{G}$ be a tame Lie group acting tamely on the left on two tame manifolds $\mathcal{M}$ and $\mathcal{N}$, all of which allow smoothing operators. Let $R, S : \mathcal{M} \to \mathcal{N}$ be two smooth tame equivariant maps and let*

$$P : \mathcal{G} \to \mathcal{M} : g \mapsto g \cdot x$$

be the action on $x \in \mathcal{M}$. If the linear exact sequence

$$\mathfrak{g} \xrightarrow{T_e P} T_x\mathcal{M} \xrightarrow{T_x R - T_x S} T_y\mathcal{N}$$

splits tamely, then there are open neighborhoods $e \in \mathcal{U} \subseteq \mathcal{G}$ and $x \in \mathcal{V} \subseteq \mathcal{M}$ and a smooth tame map $g : \mathcal{V} \to \mathcal{U}$ such that $g_y \cdot x = y$ whenever $R(y) = S(y)$. In particular, the orbit $\mathcal{G}(x)$ of $x \in \mathcal{M}$ forms an open subset of

$$eq(R, S) = \{y \in \mathcal{M} : R(y) = S(y)\}\,.$$

Chapter 5

Applications

This chapter discusses some applications of the Nash-Moser theorem, in particular of the version with group actions. Most, if not all, applications in the literature follow the concept

infinitesimal stability $\Rightarrow$ stability.

Typically one studies a collection of geometric objects on a fixed compact manifold M whose defining property can be expressed by an algebraic relation. As an example, regular foliations are the distributions that are involutive, or group actions are maps $a : G \times M \to M$ satisfying the associativity condition $(gh) \cdot m = g \cdot h \cdot m$. Next one lets a group of diffeomorphisms act on a fixed object; for example, the pull-back of distributions, or conjugation of group actions; which yields an equivalence relation on the geometric objects. The conclusion of the Nash-Moser theorem then gives a statement of the form *(stability)*

'The orbit of said object lies open in the space of all such geometric objects,'

while the hypothesis of the theorem gives a technical condition on the fixed object and the base manifold. Lastly, one looks for more natural conditions under which the hypothesis is satisfied *(infinitesimal stability)*.

Most of these applications concern global results. Yet it does seem likely, for example from Conn's proof [Con85] of the normal form of Poisson form around a singular point, that the theory can also be fitted to resolve questions about *local* stability: the deformation of germs at a point in M.

This approach has one major drawback. In all applications one has to carefully check all smooth tameness conditions are met. This often makes the Nash-Moser theorem a cumbersome technical tool. This thesis aims to give an overview of applications and examples such that it becomes more readily applicable.

5.1 Stable maps

Let M and N be manifolds of finite dimension with M compact and consider the space of smooth maps $C^\infty(M,N)$. First we introduce the concept of stability for smooth maps, as given in [MG73, Mat69]. The product of diffeomorphism groups $\mathbf{Diff}(M) \times \mathbf{Diff}(N)$ acts on the smooth maps $M \to N$ by a change of coordinates, namely

$$\mathbf{Diff}(M) \times \mathbf{Diff}(N) \times C^\infty(M,N) \longrightarrow C^\infty(M,N) : (\varphi, \psi, f) \mapsto \psi \circ f \circ \varphi^{-1}.$$

Two smooth maps f and g are called *equivalent* if there are φ and ψ such that

$$g = \psi \circ f \circ \varphi^{-1}.$$

A smooth map f is called *stable* if there is an neighborhood of f in $C^\infty(M,N)$ in which all functions are equivalent to f. This is clearly equivalent to saying that the orbit of f under the action of the topological group $\mathbf{Diff}(M) \times \mathbf{Diff}(N)$ lies open in the smooth functions.

Proposition 5.1.1. *Let M and N be manifolds, with M compact, then $f : M \to N$ is stable if it is infinitesimally stable.*

Clearly not every smooth map is stable. The change of coordinates preserves the rank of the tangent map $T_{\varphi^{-1}(x)}(\psi f \varphi^{-1})$ at every point $x \in M$. As a particularly simple example, consider the map

$$f : [-1,1] \to \mathbb{R} : x \mapsto x^3$$

and approximate it with the one-parameter family of maps $f_t : x \mapsto x^3 - tx$ with the parameter $t \in [0,1]$. None of the f_t for $t > 0$ is equivalent to f since the number of critical points differs from f.

Mather gave the following conditions for infinitesimal stability. A smooth map $f : M \to N$ is *infinitesimally stable* if for every $w \in \Gamma_M(f^*TN)$ there are $u \in \mathcal{X}(M)$ and $v \in \mathcal{X}(N)$ such that

$$w = Tf \circ u + v \circ f.$$

Guillemin and Golubitsky mention that the statement of the theorem was motivated by heuristics with Fréchet manifolds, without knowledge of an inverse function theorem. Nonetheless, Mather's proof is more direct, without mention of Fréchet manifolds, and in addition works for any proper map $f : M \to N$.

The Nash-Moser theorem obviously cannot be applied directly to this setting, as $\mathbf{Diff}(N)$ is not a tame manifold, hence let us assume that N is compact as well. The spaces $\mathbf{Diff}(M)_{\mathrm{id}} \times \mathbf{Diff}(N)_{\mathrm{id}}$, the connected component at the identity, and $C^\infty(M,N)$ are now all smooth tame manifolds. Fix a smooth map $f \in C^\infty(M,N)$ and define the map

$$\begin{tikzcd}
\mathbf{Diff}(M)_{\mathrm{id}} \times \mathbf{Diff}(N)_{\mathrm{id}} \arrow[rr, "P"] \arrow[d, "(\mathbf{inv} \times \mathrm{id})"'] & & C^\infty(M,N) \\
\mathbf{Diff}(M)_{\mathrm{id}} \times \mathbf{Diff}(N)_{\mathrm{id}} \arrow[r, "(f_* \times \mathrm{id})"'] & C^\infty(M,N) \times \mathbf{Diff}(N)_{\mathrm{id}} \arrow[r, "\tau"'] & \mathbf{Diff}(N)_{\mathrm{id}} \times C^\infty(M,N), \arrow[u, "\mathbf{com}"']
\end{tikzcd}$$

in other words,

$$P = \mathbf{com} \circ \tau \circ (\mathbf{com}(f, -) \times \mathrm{id}) \circ (\mathbf{inv} \times \mathrm{id}),$$

where τ is the interchange map and **com** and **inv** are composition and inversion. Moreover, tangent map

$$T_{(\mathrm{id},\mathrm{id})}P : \mathcal{X}(M) \times \mathcal{X}(N) \longrightarrow \Gamma_M(f^*TN)$$

is given by

$$\begin{aligned} T_{(\mathrm{id},\mathrm{id})}P(u,v) &= T_{(\mathrm{id},f)}\mathbf{com} \circ T_{(\mathrm{id},f)}\tau \circ (T_{\mathrm{id}}\mathbf{com}(f,-) \times \mathrm{id})(-u,v) \\ &= T_{(\mathrm{id},f)}\mathbf{com} \circ T_{(f,\mathrm{id})}\tau(-Tf \circ u, v) \\ &= v \circ f - Tf \circ u. \end{aligned}$$

In other words, let $\varphi_t : (-1,1) \times M \to M$ and $\psi_t : (-1,1) \times N \to N$ be smooth curves of diffeomorphisms, with $\varphi_0 = \mathrm{id}$ and $\psi_0 = \mathrm{id}$, representing the vector fields $u \in \mathcal{X}(M)$ and $v \in \mathcal{X}(N)$ respectively. Then we may compute the tangent map of P at $(\mathrm{id}, \mathrm{id})$ as

$$\begin{aligned} T_{(\mathrm{id},\mathrm{id})}P(u,v) &= \left.\frac{d}{dt}\right|_{t=0} (\psi_t \circ f \circ \varphi_t^{-1}) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \psi_t\right) \circ f + Tf \circ \left(\left.\frac{d}{dt}\right|_{t=0} \varphi_t^{-1}\right) \\ &= v \circ f - Tf \circ u. \end{aligned}$$

From the Nash-Moser theorem with group actions applied to the chain complex

$$\mathbf{Diff}(M) \times \mathbf{Diff}(N) \xrightarrow{P} C^\infty(M,N) \rightrightarrows \{*\}$$

we conclude that f is stable if there exists a tame linear right inverse to the map $T_{(\mathrm{id},\mathrm{id})}P$ described above. We arrive at the following proposition.

Proposition 5.1.2. *Let M and N be compact manifolds. Then $f \in C^\infty(M,N)$ is stable, and the maps $\varphi : g \mapsto \varphi_g$ and $\psi : g \mapsto \psi_g$ such that $g = \psi_g \circ f \circ \varphi_g^{-1}$ are smooth, if and only if for every $w \in \Gamma_M(f^*TN)$ there are $u \in \mathcal{X}(M)$ and $v \in \mathcal{X}(N)$ such that*

$$w = v \circ f + Tf \circ u.$$

Proof. We must show that the above hypothesis can produce a suitable tame linear map $w \mapsto (u,v)$. Choose a finite covering $\{U_\alpha\}_{\alpha \in A}$ of M by local trivializations of f^*TN, and let $z_1^\alpha, \ldots, z_n^\alpha$ be the corresponding frame of $f^*TN|_U$ for every $\alpha \in A$. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$ whose square root is still differentiable. The $\sqrt{\chi_\alpha} z_i^\alpha$ extend by zero to the whole of M, and for every $\alpha \in A$ and $1 \leq i \leq n$ we find a pair $u_i^\alpha \in \mathcal{X}(M)$ and $v_i^\alpha \in \mathcal{X}(N)$ such that

$$\sqrt{\chi_\alpha} z_i^\alpha = v_i^\alpha \circ f + Tf \circ u_i^\alpha.$$

Now for an arbitrary $w \in \Gamma_M(f^*TN)$, we have that $w|_{U_\alpha} = \sum_i w_\alpha^i z_i^U$ for certain smooth functions $w_\alpha^i : U_\alpha \to \mathbb{R}$, and we may define $u = \sum_{\alpha,i} \sqrt{\chi_\alpha} w_\alpha^i u_i^\alpha$ and $v = \sum_{\alpha,i} \sqrt{\chi_\alpha} w_\alpha^i v_i^\alpha$. Then the map $w \mapsto (u,v)$ is clearly tame linear, and

$$v \circ f + Tf \circ u = \sum\nolimits_{\alpha,i} \sqrt{\chi_\alpha} w_\alpha^i (v_i^\alpha \circ f + Tf \circ u_i^\alpha) = \sum\nolimits_i \sum\nolimits_\alpha \chi_\alpha w_\alpha^i z_i^\alpha = w.$$

For the converse, let $w \in \Gamma_M(f^*TN) = T_f C^\infty(M,N)$ be a vector at f. Choose a representative $f_t : (-1,1) \times M \to N$ of w, that is, a smooth map such that $f_0 = f$ and $\frac{\partial}{\partial t}\big|_{t=0} f_t = w$. Then the compositions φ_{f_t} and ψ_{f_t}, where we see φ as the map $g \mapsto \varphi_g$ and likewise for ψ, are smooth maps and represent the required vectors $u \in \mathcal{X}(M)$ and $v \in \mathcal{X}(N)$, respectively. □

With this approach to stability it is unavoidable that M should be chosen compact, but it is desirable to at least prove that the restriction on N can be dropped. This can be done as follows. Let N be non-compact and $f : M \to N$ smooth. Since the image $f(M)$ is compact, we may choose a compact region R, a compact submanifold of codimension 0 with smooth boundary, such that $f(M) \subseteq R^\circ$. Moreover, suppose that $f : M \to N$ is infinitesimally stable, then so is $f : M \to R$. The set $M(M, R^\circ)$ of smooth maps $M \to N$ that map into the interior of R form an open neighborhood of f in both $C^\infty(M,N)$ and $C^\infty(M,R)$. For g near enough to f there exist diffeomorphisms $\varphi : M \to M$ and $\tilde{\psi} : R \to R$ such that $g = \psi \circ f \circ \varphi^{-1}$. In fact, by assuming that g is smoothly path connected to f, which we can since $C^\infty(M,R)$ is locally path connected, we can find a smooth curve $\tilde{\psi}_t : [0,1] \times R \to R$ of diffeomorphisms such that $\tilde{\psi}_1 = \tilde{\psi}$. Choose a smooth bump function on N that is constantly 1 in an open neigborhood of $f(M)$ and vanishes beyond the boundary of R. Let $\tilde{v} \in \mathcal{X}(R)$ be the infinitesimal generator of $\tilde{\psi}_t$. Then one can cut $\tilde{v}$ off with the bump function and extend it by zero to a $v \in \mathcal{X}(N)$. Since it is supported in a compact set, it has a global flow $\psi_t : N \to N$, and we see that $g = \psi_1 \circ f \circ \varphi^{-1}$. Hence we conclude that $f : M \to N$ is stable.

A few additional details of Mather's approach are worth mentioning. Aside from working with non-compact manifolds, he obtains a slightly more general *'infinitesimal stability implies stability'* theorem. It takes the form of the proposition below. Note that in [Mat69] he assumes that all manifolds may be manifolds with corner. Manifolds are allowed to locally look like a quadrant in $\mathbb{R}^n$ defined by a finite set of linearly independent linear inequalities $l_1 \geq 0, \ldots, l_k \geq 0$. The spaces of smooth maps $M \to N$ are equipped with the Whitney C^∞-topology described earlier.

Proposition 5.1.3. *Let M and N be manifolds, and let $f : M \to N$ be a proper smooth map. Then f is infinitesimally stable in the sense that for every $w \in \Gamma_M(f^*TN)$ there are $u \in \mathcal{X}(M)$ and $v \in \mathcal{X}(N)$ such that*

$$w = Tf \circ u + v \circ f,$$

if and only if f is stable in the sense that for every smooth $g : M \to N$ close to f there are $\varphi_g \in \boldsymbol{Diff}(M)$ and $\psi_g \in \boldsymbol{Diff}(N)$ such that

$$g = \psi_g \circ f \circ \varphi_g.$$

Moreover, the mappings $g \mapsto \varphi_g$ and $g \mapsto \psi_g$ are continuous for the W_∞ topology.

5.1.1 More stable maps

Note that submersions are examples of stable maps. For if $f : M \to N$ is a submersion between with M a compact manifold. Then its tangent map Tf is locally a projection $U_\alpha \times \mathbb{R}^m \to U_\alpha \times \mathbb{R}^n \times \{0\}$. Hence there exists a smooth bundle map $u_\alpha : f^*TN|_{U_\alpha} \to TM|_{U_\alpha}$ that serves as a right inverse of $Tf|_{U_\alpha}$. Now the u_α can be patched together with a partition of unity to a right inverse $u : f^*TN \to TM$ of Tf. The corresponding tame linear map $u_* : \Gamma_M f^*TN \to \mathcal{X}(M)$ is a right inverse of Tf_*. Hence by the Nash-Moser theorem there exists an open neighborhood $V \subseteq C^\infty(M, N)$ of f and a smooth tame map $\varphi : V \to \mathbf{Diff}(M, N)$ such that

$$g = f \circ \varphi_g, \quad \forall g \in V.$$

Recall that the submersions form an open subset of all smooth maps $M \to N$, and the space of diffeomorphisms $\mathbf{Diff}(M)$ acts on the submersions by multiplication on the right. Hence in particular this discussion shows that the orbits of this action lie open in $C^\infty(M, N)$. Moreover, it leads to the following observation.

Proposition 5.1.4. *Let M and N be compact manifolds, and $f : M \to N$ a submersion. Then for every $g : N \to N$ close to the identity there is a φ_g such that*

$$g \circ f = f \circ \varphi_g,$$

and $\varphi_{id} = id$. Moreover, the map $g \mapsto \varphi_g$ is smooth tame.

Proof. Let V and φ be as in the discussion above. Composition on the right by f defines a smooth tame map

$$\mathbf{com}(-, f) : C^\infty(N, N) \to C^\infty(M, N).$$

Hence we may choose an open $U \subseteq C^\infty(N, N)$ around id_N small enough such that $\mathbf{com}(U, f) \subseteq V$. Then the composed map $\tilde{\varphi} = \varphi \circ \mathbf{com}(-, f)$ does the job. $\square$

Michor [Mic84] gives some additional applications of the Nash-Moser theorem to spaces of smooth maps. I would like to highlight one particular example. Consider a compact fiber bundle $B \xrightarrow{p} M$. The space of fiber-preserving diffeomorphisms $\mathbf{Diff}_M(B)$ is a tame Fréchet submanifold of all diffeomorphisms $\mathbf{Diff}(B)$ by proposition 3.2.29 on page 87. It acts smooth tamely on the left on $\Gamma_M B$ by the map

$$P : \mathbf{Diff}_M(B) \times \Gamma_M B \longrightarrow \Gamma_M B, \quad P(\varphi, \sigma) = \varphi \circ \sigma.$$

Recall that for $\varphi \in \mathbf{Diff}_M(B)$, the tangent space of $\mathbf{Diff}_M(B)$ at φ is given by $T_\varphi \mathbf{Diff}_M(B) = \Gamma_B(\varphi^* T^{\mathrm{vert}} p^* B)$. In particular, the tangent space at the identity is just the space $\mathcal{X}^{\mathrm{vert}}(B)$ of vertical vector fields on the total space. For $\sigma \in \Gamma_M B$ a fixed section, consider the map

$$P_\sigma : \mathbf{Diff}_M(B) \longrightarrow \Gamma_M B, \quad P_\sigma(\varphi) = \varphi \circ \sigma.$$

Its tangent map at the identity is the map

$$T_{\mathrm{id}} P_\sigma : \mathcal{X}^{\mathrm{vert}}(B) \longrightarrow \Gamma_M(\sigma^* T^{\mathrm{vert}} B), \quad T_{\mathrm{id}} P_\sigma(\nu) = \nu \circ \sigma = \sigma^*(\nu).$$

Note that there is a natural vector bundle isomorphism $\sigma^* T^{\text{vert}} B \simeq T^{\text{vert}} B|_{\sigma(M)}$. Any section of $T^{\text{vert}} B|_{\sigma(M)}$ can be extended to B by a partition of unity, and the resulting extension map $\Gamma_{\sigma(M)}(T^{\text{vert}} B|_{\sigma(M)}) \longrightarrow \Gamma_B T^{\text{vert}} B$ can be chosen to be tame linear. This defines a tame linear right inverse to $T_{\text{id}} P_\sigma$. Hence we have shown the following proposition, which, in particular, implies that the orbits of $\mathbf{Diff}_M(B)$ lie open in $\Gamma_M B$. Michor then concludes that σ is stable in the space $\Gamma_M B$ in the sense that for every nearby section τ there is a fiber preserving diffeomorphism $\psi_\tau : B \to B$ such that $\tau = \psi_\tau \circ \sigma$. However, with a little extra work we can give a better result.

Proposition 5.1.5. *Let $B \xrightarrow{p} M$ be a compact fiber bundle, then every section σ of B is stable in $C^\infty(N, B)$ in the sense that for every smooth map $\tau : N \to B$ close to σ there is a diffeomorphism $\psi_\tau : B \to B$ such that*

$$\tau = \psi_\tau \circ \sigma,$$

and the mapping $\psi : \tau \mapsto \psi_\tau$ is smooth tame. The diffeomorphism ψ_τ is fiber-preserving if τ is a section of B, and $\psi_\sigma = id$.

Proof. Let $\sigma \in \Gamma_M B$ be a section of B. By proposition 5.1.4 on the preceding page there is an open neighborhood $V \subseteq \mathbf{Diff}(M)$ around the identity, and a smooth tame map $\varphi : V \to \mathbf{Diff}(B)$ such that $g \circ p = p \circ \varphi_g$ for all $g \in V$. Since $p \circ \sigma = \text{id}$, there is an open neighborhood $U \subseteq C^\infty(M, B)$ of σ such that p_* maps into the diffeomorphisms of M. The map $\mathbf{inv} \circ p_*$ is smooth tame as well, hence the open U can be chosen small enough such that $(p \circ \tau)^{-1} \in V$ for every $\tau \in U$. Then for every $\tau \in U$ we have

$$p \circ \varphi_{(p\circ\tau)^{-1}} \circ \tau = (p \circ \tau)^{-1} \circ p \circ \tau = \text{id}.$$

Hence we can define a smooth tame map

$$\tilde{\varphi} : U \longrightarrow \mathbf{Diff}(B) : \tau \mapsto \varphi_{(p\circ\tau)^{-1}}$$

that approximates any $\tau \in V$ by a section $\tilde{\varphi}_\tau \circ \tau$ of B. Note that $\tilde{\varphi}_\tau = \text{id}$ whenever $p \circ \tau = \text{id}$.

By the discussion above this proposition there is an open neighborhood $W \subseteq \Gamma_M B$ around σ and a smooth tame map $\tilde{\psi} : W \to \mathbf{Diff}_M(B)$ such that $\tau = \tilde{\psi}_\tau \circ \sigma$ for all $\tau \in W$. Now choose U small enough such that $\tilde{\varphi}$ maps into W. It is Then

$$\psi : U \to \mathbf{Diff}(B) : \tau \mapsto \tilde{\psi}_{\tilde{\varphi}_\tau \circ \tau} \circ \tilde{\varphi}_\tau$$

is the desired map. □

As another example of stable maps, a map $M \to \mathbb{R}$ is stable if and only if it is a Morse function whose critical points all have distinct values. Hence in a sense the theory of stable mappings generalizes Morse theory. For a overview of the classification of stable mappings and related results we refer to [MG73].

5.1.2 Bundle maps as a tame fiber bundle

We will apply the result from the previous section to the setting of bundles maps between surjective submersions, as mentioned in section 3.2.8 on page 86. Suppose that $A \xrightarrow{p} M$ and $B \xrightarrow{q} N$ are surjective submersions with A compact. We wish to consider the obvious map

$$\mathbf{Bund}(A, B) \to C^\infty(M, N)$$

which associates to a bundle map $A \to B$ its base map $M \to N$ as either a tame Fréchet fiber bundle or at least a surjective submersion between Fréchet manifolds. To the author this seems to be the most general and flexible setting for applying the Nash-Moser arguments to questions of rigidity in differential geometry. It doesn't seem to be true in general but at least the following can be said. First consider the smooth tame map

$$q_* : C^\infty(A, B) \to C^\infty(A, N)$$

defined by left composition by q. Its fiber above a smooth map $f : M \to N$ can be cannonically identified with the tame Fréchet manifold $\Gamma_A(f^*B)$. Think of it as a set-theoretical bundle over $C^\infty(A, N)$ with fibers $C^\infty(A, B)_f = \Gamma_A(f^*B)$. It is the space of all sections of B along smooth maps $A \to N$.

Proposition 5.1.6. *Let $A \xrightarrow{p} M$ and $B \xrightarrow{q} N$ be submersions with A compact. The restriction $C^\infty(A, B)|_{\boldsymbol{Subm}(A,N)}$ to the space of submersion $A \to N$ is a tame Fréchet fiber bundle. Moreover, if B is compact as well, then its restriction $C^\infty(A, B)|_{\boldsymbol{Stab}(A,N)}$ to the stable mappings is a tame Fréchet fiber bundle.*

Proof. Let $f : A \to N$ be a submersion. Then there is an open neighborhood $U \subseteq C^\infty(A, N)$ of f and a smooth tame map $\varphi : U \to \mathbf{Diff}(A)$ such that $g = f \circ \varphi_g^{-1}$ for all $g \in U$. This means that if $F \in C^\infty(A, B)$ is a bundle map with base map $g \in U$, then $F \circ \varphi_g$ is a bundle map with base map f. Hence the map

$$C^\infty(A, B)|_U \longrightarrow U \times \Gamma_A(f^*B) : F \mapsto (q_*F, F \circ \varphi_{q_*F})$$

is a smooth tame map. Its inverse is given by sending a pair $(g, F) \in U \times \Gamma_A(f^*B)$ to $F \circ \varphi_g^{-1}$.

Now suppose that B is also compact, and $f : A \to N$ be a stable map. Let $\varphi : U \to \mathbf{Diff}(A)$ and $\psi : U \to \mathbf{Diff}(N)$ be the corresponding maps. Since q is a submersion, there exists an open $V \subseteq C^\infty(B, N)$ around q and a smooth map $\rho : V \to \mathbf{Diff}(B)$ such that $r = q \circ \rho_r$ for all $r \in V$. The open $U \subseteq C^\infty(A, N)$ can be chosen small enough such that $\mathbf{com}(-, q) \circ \psi$ maps U into V. Define a map

$$\tilde{\psi} : U \longrightarrow \mathbf{Diff}(B), \quad \tilde{\psi}_g = \rho_{\psi_g \circ q},$$

so that $\psi_g \circ q = q \circ \rho_g$ for all $g \in U$. Observe that

$$q = q \circ \tilde{\psi}_g \circ \tilde{\psi}_g^{-1} = \psi_g \circ q \circ \tilde{\psi}_g^{-1}$$

implies that $\psi_g^{-1} \circ q = q \circ \tilde{\psi}_g^{-1}$. Now if $F : A \to B$ is a bundle map with base map $g \in U$, then we have

$$q \circ \tilde{\psi}_g^{-1} \circ F \circ \varphi_g = \psi_g^{-1} \circ q \circ F \circ \varphi_g = \psi_g^{-1} \circ g \circ \varphi_g = f.$$

Hence we may define a tame diffeomorphism

$$C^\infty(A, B)|_U \to U \times \Gamma_A(f^*B) : F \mapsto (q_*F, \tilde{\psi}_{q_*F}^{-1} \circ F \circ \varphi_{q_*F}),$$

whose inverse is given by sending a pair $(g, F) \in U \times \Gamma_A(f^*B)$ to $\tilde{\psi}_g \circ F \circ \varphi_g^{-1}$. $\square$

If $M = N$, then we recover the bundle maps $\mathbf{Bund}_M(A, B)$ over M as the fiber $C^\infty(A, B)_p$. It is a submanifold of $C^\infty(A, B)$.

The surjective submersion $p : A \to M$ defines a smooth tame map

$$p^* : C^\infty(M, N) \to C^\infty(A, N)$$

whose image is the set of all smooth maps $A \to N$ which are constant along the fibers of A. Clearly this map p^* is a bijection onto its image. We can restrict the set-theoretical bundle $C^\infty(A, B) \to C^\infty(A, N)$ to the image $p^*C^\infty(M, N)$ to obtain the space of bundle maps $A \to B$. We wish to consider it as a (set-theoretical) bundle

$$\mathbf{Bund}(A, B) := C^\infty(A, B)|_{p^*C^\infty(M,N)} \to C^\infty(M, N)$$

by identifying $C^\infty(M, N)$ with its image.

Proposition 5.1.7. *Let $A \xrightarrow{p} M$ and $B \xrightarrow{q} N$ be surjective submersions with A compact. The space of bundle maps $A \to B$ with submersions as base maps form a tame Fréchet fiber bundle*

$$\boldsymbol{Bund}(A, B)|_{\boldsymbol{Subm}(M,N)} \longrightarrow \boldsymbol{Subm}(M, N).$$

If B is compact as well, we obtain a tame Fréchet bundle

$$\boldsymbol{Bund}(A, B)|_{\boldsymbol{Stab}(M,N)} \longrightarrow \boldsymbol{Stab}(M, N)$$

above the stable maps $M \to N$ instead.

Proof. The proof is identical to proposition 5.1.6 on the preceding page. We will only prove the second statement.

Therefore suppose that B is compact, and let $f : M \to N$ be a stable map. Let $\varphi : U \to \mathbf{Diff}(M)$ and $\psi : U \to \mathbf{Diff}(N)$ denote the smooth tame maps such that $g = \psi_g \circ f \circ \varphi_g^{-1}$ for all $g \in U$. Since p is a submersion we may find an open V around p and a smooth tame map $\rho : V \to \mathbf{Diff}(A)$ such that $r = p \circ \rho_r$ for all $r \in V$. Similarly, we find a smooth tame map $\rho' : V' \to \mathbf{Diff}(B)$ corresponding to q.

Note that φ and ψ map f to the identity. Moreover, both $\mathbf{com}(-, p) \circ \varphi$ and $\mathbf{com}(-, q) \circ \psi$ are smooth tame maps, hence we may choose U small enough such that they map U into V and V' respectively. This allows us to define smooth tame maps that lift φ and ψ to diffeomorphisms of the total spaces A and B. These maps are defined by

$$\tilde{\varphi} : U \longrightarrow \mathbf{Diff}(A) : g \mapsto \rho_{\varphi_g \circ p},$$
$$\tilde{\psi} : U \longrightarrow \mathbf{Diff}(B) : g \mapsto \rho'_{\psi_g \circ q}.$$

We have that $\varphi_g \circ p = p \circ \tilde{\varphi}_g$ for all $g \in U$, and by

$$p = p \circ \tilde{\varphi}_g \circ \tilde{\varphi}_g^{-1} = \varphi_g \circ p \circ \tilde{\varphi}_g^{-1},$$

we have that $\varphi_g^{-1} \circ p = p \circ \tilde{\varphi}_g^{-1}$ for all $g \in U$. The analoguous statements hold for $\tilde{\psi}$. For any bundle map $G : A \to B$ with base map $g \in U$ we have that

$$\begin{aligned} q \circ \tilde{\psi}_g^{-1} \circ G \circ \tilde{\varphi}_g &= \psi_g^{-1} \circ q \circ G \circ \tilde{\varphi}_g \\ &= \psi_g^{-1} \circ g \circ p \circ \tilde{\varphi}_g \\ &= \psi_g^{-1} \circ g \circ \varphi_g \circ p = f \circ p, \end{aligned}$$

so that $\tilde{\psi}_g^{-1} \circ G \circ \tilde{\varphi}_g$ is a bundle map with base map f. This allows us to define a local trivialization

$$\mathbf{Bund}(A, B)|_U \longrightarrow U \times \Gamma_A((f \circ p)^* B), \quad G \mapsto (q^* G, \tilde{\psi}_{q^* G}^{-1} \circ G \circ \tilde{\varphi}_{q^* G}). \qquad \square$$

Let $C \to P$ be another surjective submersion, and assume that B is compact. Note that the composition of bundle maps defines a smooth tame map

$$\mathbf{com} : \mathbf{Bund}(B, C)|_{\mathbf{Subm}(N,P)} \times \mathbf{Bund}(A, B)|_{\mathbf{Subm}(M,N)} \longrightarrow \mathbf{Bund}(A, C)|_{\mathbf{Subm}(M,P)}.$$

It is just the restriction of the composition map between smooth maps $A \to B$ and $B \to C$. Likewise, diffeomorphisms are submersions, hence we may consider the inversion of bundle maps,

$$\mathbf{inv} : \mathbf{Diff}(A, B)|_{\mathbf{Diff}(M,N)} \longrightarrow \mathbf{Diff}(B, A)|_{\mathbf{Diff}(N,M)},$$

as a smooth tame map.

5.2 Stability of groupoid actions

Let G be a Lie group and N a manifold. Recall that a left action of G on N is a smooth map $a : G \times N \to N$, for which we will write $g \cdot m = a(g)(n) = a(g,n)$, such that the usual

- $(gh) \cdot n = g \cdot (h \cdot n)$,
- $1 \cdot n = n$,

holds for all $g, h \in G$ and $n \in N$. Alternatively, one could consider them as smooth group homomorphisms $G \to \mathbf{Diff}(N)$, and this is the description we will be using. The Lie group actions form a subset $\mathcal{A}(G,N)$ of the space of all maps $C^\infty(G \times N, N)$.

Two actions a and b are said to be equivalent if they are conjugate by a diffeomorphism of N, that is, there exists a diffeomorphism φ of N such that

$$b(g) = \varphi \circ a(g) \circ \varphi^{-1}, \quad \forall g \in G.$$

An action a is stable if there is a neighborhood of a in $\mathcal{A}(G,N)$, with the induced topology of $C^\infty(G \times N, N)$, in which all actions are equivalent to a. In [Pal61] it is proven that all actions are stable given that G and N are both compact. In [RP63] a counterargument is given for $N = \mathbb{R}^n$, hence compactness of N is necessary. It should be noted that this counterargument only shows that not all actions $G \times N \to N$ can be stable.

We will prove the stability of group actions using Nash-Noser arguments. An adaptation of Mather's proof, or a careful search for a possible tame Fréchet manifold structure on $C^\infty_p(G \times N, N)$, might allow one to prove the stability of actions of a non-compact group under some extended conditions. We haven't been successful with this so far.

In fact, we extend the result to actions of compact Lie groupoids on a compact manifold with a fixed moment map. Recall the definition of a Lie groupoid. A Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of

- two manifolds $\mathcal{G}$, arrow space, and M, object space;
- two surjective submersions $s, t : \mathcal{G} \to M$, the source and target map, respectively;
- a smooth multiplications map $m : \mathcal{G} \times_{s,t} \mathcal{G} \to \mathcal{G}$ on the manifold of composable arrows $\mathcal{G} \times_{s,t} \mathcal{G} = \{(g,h) \in \mathcal{G} \times \mathcal{G} : s(g) = t(h)\}$;
- a smooth $u : M \to \mathcal{G}$ of both s and t indicating the unit elements, we identify $u(M)$ with M;
- a smooth inversion map $i : \mathcal{G} \to \mathcal{G}$.

These maps should satisfy the usual diagrams for an internal groupoid object. The only difference between a Lie groupoid and an arbitrary internal groupoid object in **Mfd** is that the source and target map of the latter do not need to be submersions. For the latter it is, for example, sufficient if they are transversal.

A Lie group G can be seen as the Lie groupoid $G \rightrightarrows \{*\}$ over the one-point set. A Lie groupoid is called proper if the map $(s,t) : \mathcal{G} \to M \times M$ is proper. In particular, compact Lie groups are proper Lie groupoids, and, as a general principle, whatever holds for compact Lie groups has an analogue for proper Lie groupoids. Although one can prove that the differential cohomology of a groupoid action, which we define in the next section, vanishes for any proper groupoid, the Nash-Moser argument presented here requires us to work with a compact Lie groupoid instead. It is as of yet unclear to us whether actions of proper Lie groupoids are stable.

Let $\mathcal{G} = \mathcal{G} \rightrightarrows M$ be a Lie groupoid and $j : N \to M$ a surjective submersion, which is usually called the moment map of the action. We will write

$$\mathcal{G}^{(k)} = \mathcal{G} \times_{s,t} \ldots \times_{s,t} \mathcal{G}$$

for the set of k-tuples $\mathbf{g} = (g_1, \ldots, g_k)$ of composable arrows,

$$\begin{tikzcd} \underset{t(g_1)}{\bullet} & \bullet \arrow[l, bend right, "g_1"'] & \bullet \arrow[l, bend right, "g_2"'] \arrow[r, dotted, no head] & \bullet & \underset{s(g_k).}{\bullet} \arrow[l, bend right, "g_k"'] \end{tikzcd}$$

Each $\mathcal{G}^{(k)}$ has a source and target map, given by $s(\mathbf{g}) = s(g_k)$ and $t(\mathbf{g}) = t(g_1)$ respectively. Correspondingly, the fibered product $\mathcal{G}^{(k)} \times_{s,j} N$ is the set of all $(\mathbf{g}, n)$ with $s(\mathbf{g}) = j(n)$. By convention we have $\mathcal{G}^{(0)} = M$.

An action of $\mathcal{G}$ on N with moment map j is a smooth map

$$a : \mathcal{G} \times_{s,j} N \to N,$$

for which we will write any of the three notations $g \cdot n = a(g)n = a(g, n)$, satisfying

- $j(g \cdot n) = t(g)$,
- $(gh) \cdot n = g \cdot h \cdot n$,
- $u(m) \cdot n = n$,

for all $(g, h) \in \mathcal{G}^{(2)}$ and $n \in N_m$. Let $\mathcal{A}(\mathcal{G}, N, j)$ denote the set of Lie groupoid actions with fixed moment map j; they form a subset of $C^\infty(\mathcal{G} \times_{s,j} N, N)$. In particular, we retreive group actions if we take $M = \{*\}$, and $j : N \to \{*\}$ the unique map.

Our notion of stability of a Lie groupoid action is defined as follows. For any groupoid action a and any arrow $g \in \mathcal{G}$, the expression $a(g)$ defines a diffeomorphism $N_{s(g)} \longrightarrow N_{t(g)}$. Now two actions a and b are equivalent if there is a diffeomorphism $\varphi \in \mathbf{Diff}_M(N)$, that is, a bundle map

$$\begin{tikzcd} N \arrow[rr, "\varphi"] \arrow[dr, "j"'] & & N \arrow[dl, "j"] \\ & M & \end{tikzcd}$$

over M which is also a diffeomorphism, such that

$$b(g) = \varphi_{t(g)} \circ a(g) \circ \varphi_{s(g)}^{-1}, \quad \forall g \in \mathcal{G}.$$

As usual, an action a is stable if there is a neighborhood of a in $\mathcal{A}(\mathcal{G}, N, j)$ of only equivalent actions. In this section we prove the following proposition.

Theorem 5.2.1. *Let $\mathcal{G} \rightrightarrows M$ be a compact Lie groupoid, N a compact manifold, and $j : N \to M$ a surjective submersion. Then every Lie groupoid action a of $\mathcal{G}$ on N with moment map j is stable in the sense that if b is Lie groupoid action with moment map j near a, then there is a fiberpreserving diffeomorphism*

$$\varphi_b : N \to N$$

over M such that $b(g) = \varphi_{t(g)} \circ a \circ \varphi_{s(g)}$ for every $g \in G$. Moreover, the map $b \mapsto \varphi_g$ is a smooth tame map between tame manifolds.

As noted before, it is necessary that N is compact. But this doesn't seem to imply that $\mathcal{G}$ should be compact as well. One might be tempted to take the group G from [RP63] and form the action groupoid $G \ltimes \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to give a counter example for non-compact groupoids. However, the chosen action then becomes the trivial action over the moment map given by the identity $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. Since no other Lie groupoid action has the identity as moment map, this action is trivially stable. This leaves the issue whether actions of proper groupoids are stable still unsettled.

To apply a Nash-Noser argument, we will first rewrite the definition of groupoid actions somewhat. Note that the map $s^*N = \mathcal{G} \times_{s,j} N \xrightarrow{s_1} \mathcal{G}$, given by mapping (g, n), with $s(g) = n$, to g, is a compact fiber bundle over $\mathcal{G}$; it is a surjective submersion with compact domain. Likewise, there is a compact fiber bundle

$$t^*\mathcal{G} = \mathcal{G} \times_{t,j} N \xrightarrow{t_1} \mathcal{G},$$

sending (g, n), with $t(g) = j(n)$, to g. The first condition on a Lie groupoid action a, namely that $j(g \cdot n) = t(g)$, is equivalent to stating that a is a bundle map,

$$\begin{tikzcd} \mathcal{G} \times_{s,j} N \arrow[rr, "a"] \arrow[dr, "s_1"'] & & \mathcal{G} \times_{t,j} N \arrow[dl, "t_1"] \\ & \mathcal{G}. & \end{tikzcd}$$

It can be seen as a point in the tame manifold $\mathbf{Bund}_{\mathcal{G}}(s_1, t_1)$; even in $\mathbf{Diff}_{\mathcal{G}}(s_1, t_1)$, if one wants. This alternative definition induces the same topology on $\mathcal{A}(\mathcal{G}, N, j)$ as the topology induced by viewing Lie groupoid actions as smooth maps $\mathcal{G} \times_{s,j} N \to N$.

More generally, we introduce the bundles

$$s^*N = \mathcal{G}^{(k)} \times_{s,j} N \xrightarrow{s_k} \mathcal{G}^{(k)},$$
$$t^*N = \mathcal{G}^{(k)} \times_{t,j} N \xrightarrow{t_k} \mathcal{G}^{(k)}$$

and consider the manifolds $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s^*N, t^*N) = \mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$ of diffeomorphisms over $\mathcal{G}^{(k)}$, for all $k \geq 0$. In other words, they are the spaces of smooth families

$$\left\{ a(\mathbf{g}) : N_{s(g_k)} \xrightarrow{\simeq} N_{t(g_1)} \mid \mathbf{g} = (g_1, \ldots, g_k) \in \mathcal{G}^{(k)} \right\}.$$

Note that, by convention, for $k = 0$ we have

$$\mathbf{Diff}_M(s_0, t_0) = \mathbf{Diff}_M(N),$$

which is the space of diffeomorphisms of N that preserve fibers. These tame Fréchet manifolds, in particular for $k = 0, 1$ and 2, will occur in the non-linear chain complex of the Nash-Moser argument we are about to give. In the case of group actions these definitions lead to

$$\mathbf{Diff}_{G^k}(s_k, t_k) = \mathbf{Diff}_{G^k}(G^k \times N) = C^\infty(G^k, \mathbf{Diff}(N)),$$

and $\mathbf{Diff}_{G^0}(s_0, t_0) = \mathbf{Diff}(N)$, where G^k is just the k-fold Cartesian product of the group G.

Next we will describe the tangent spaces of the tame Fréchet manifolds $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$. Let a be a fixed groupoid action and let $m_k : \mathcal{G}^{(k)} \to \mathcal{G}$ denote the combined composition of k composable arrows. One can consider a as an element of $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$ by pulling it back along m_k, that is, one can identify it with the image of a along the map

$$m_k^* : \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \longrightarrow \mathbf{Diff}_{\mathcal{G}^{(k)}}(m_k^*(s_1), m_k^*(t_1)) \simeq \mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k),$$
$$m_k^* b(g_1, g_2, \ldots, g_k)n = b(g_1 \cdot g_2 \cdots g_k)n,$$

for every $b \in \mathbf{Diff}_{\mathcal{G}}(s_1, t_1)$, $n \in N$,and k-tuple $(g_1, \ldots, g_k)$ of composable arrows with $s(g_k) = j(n)$. Here $m_k^*(s_1)$ and $m_k^*(t_1)$ denote the pull-back of the bundles s_1 and t_1 along m_k. They are easily identified with s_k and t_k respectively, and left- and right- composition with the resulting isomorphisms yields a tame diffeomorphism between the tame Fréchet manifolds $\mathbf{Diff}_{\mathcal{G}^{(k)}}(m_k^*(s_1), m_k^*(t_1))$ and $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$. When there is no ambiguity, we will just write $a \in \mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$ instead of $m_k^* a$.

In turn, this allows us to define the tame diffeomorphism

$$\mathbf{com}(-, a^{-1}) : \mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k) \longrightarrow \mathbf{Diff}_{\mathcal{G}^{(k)}}(t_k, t_k),$$
$$\mathbf{com}(-, a^{-1})b(\mathbf{g}) = b(\mathbf{g})a(\mathbf{g})^{-1},$$

where $b \in \mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$, and $\mathbf{g}$ is a k-tuple of composable arrows. Its inverse is obviously given by the analogously defined map $\mathbf{com}(-, a)$. Its tangent map allows us to simplify the description of the tangent space at $a \in \mathbf{Bund}_{\mathcal{G}^{(k)}}(s_k, t_k)$ as follows. Let

$$\pi_N : T^{\mathrm{vert}} N \to N$$

denote the vertical tangent bundle over N and u the smooth section of the bundle

$$\mathcal{G}^{(k)} \times_{t,j} N \to t_k^*(\mathcal{G}^{(k)} \times_{t,j} N)$$

representing the identity $\mathrm{id} \in \mathbf{Bund}_{\mathcal{G}^{(k)}}(t_k, t_k)$. Then the tangent space of $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$ at a is computed as

$$\begin{aligned} T_a \mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k) &\xrightarrow{\simeq} T_{\mathrm{id}} \mathbf{Diff}_{\mathcal{G}^{(k)}}(t_k, t_k) \\ &= \Gamma_{\mathcal{G}^{(k)} \times_{t,j} N}(u^* T^{vert} t_k^*(\mathcal{G}^{(k)} \times_{t,j} N)) \\ &\simeq \Gamma_{\mathcal{G}^{(k)} \times_{t,j} N}(\mathcal{G}^{(k)} \times_{t,j\pi_N} T^{vert} N). \end{aligned}$$

An element v in the latter graded Fréchet space takes values $v(\mathbf{g}, n) \in T_n^{vert} N = T_n N_{j(n)}$; as such, they can be interpreted as smooth sections of the bundle

$$t^* \mathcal{X}(j) \to \mathcal{G}^{(k)}$$

whose fiber above $\mathbf{g} \in \mathcal{G}^{(k)}$ is the space of vector fields $\mathcal{X}(N_{t(\mathbf{g})})$ on the corresponding fiber of N. In words, they are the vertical vector fields of the bundle obtained by pulling $N \to M$ back along $t_k : \mathcal{G}^{(k)} \times_{t,j} N \to M$. This description of the tangent space is only relevant to us for $k = 0, 1$ and 2.

For group actions we obtain a simpler description of the tangent space,

$$\begin{aligned} T_a \mathbf{Diff}_{G^k}(G^k \times N) &\simeq \Gamma_{G^k \times N}(G^k \times TN) \\ &\simeq C^\infty(G^k, \mathcal{X}(N)), \end{aligned}$$

where $G^k \times TN \to G^k \times N$ maps $(\mathbf{g}, v)$ to $(\mathbf{g}, \pi_N(v))$. An element v in the tangent space really assigns a vector field $v(\mathbf{g}) \in \mathcal{X}(N)$ to every k-tuple $\mathbf{g}$ of group elements.

Recall that the group of fiber-preserving diffeomorphisms of the bundle $N \to M$ acts on $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$ via conjugation, namely by

$$(\varphi \cdot a)(\mathbf{g}) = \varphi_{t(\mathbf{g})} \circ a(\mathbf{g}) \circ \varphi^{-1}_{s(\mathbf{g})}, \quad \forall \mathbf{g} \in \mathcal{G}^{(k)}.$$

It is a smooth tame action, since it is the composition of a series of smooth tame maps. Hence also the map

$$P : \mathbf{Diff}_M(N) \longrightarrow \mathbf{Diff}_{\mathcal{G}}(s_1, t_1), \quad P(\varphi)(g) = \varphi_{t(g)} \circ a(g) \circ \varphi^{-1}_{s(g)}$$

is smooth tame. It will be the first map in the non-linear chain complex. Let

$$\tilde{P} = \mathbf{com}(-, a^{-1}) \circ P,$$

then the tangent map of $\tilde{P}$ at the identity is given by

$$\begin{aligned} T_{\mathrm{id}} \tilde{P} v &= T_{\mathrm{id}}(\mathbf{com}(-, a^{-1}) \mathbf{com}(\mathrm{id} \times \mathbf{com}(a, -))(\mathrm{id} \times \mathbf{inv})(t^* \times s^*)\Delta) v \\ &= T_{(\mathrm{id},\mathrm{id})}(\mathbf{com}(-, a^{-1}) \mathbf{com}(\mathrm{id} \times \mathbf{com}(a, -))(\mathrm{id} \times \mathbf{inv}))(t^* v, s^* v) \\ &= T_{(\mathrm{id},\mathrm{id})}(\mathbf{com}(-, a^{-1}) \mathbf{com}(\mathrm{id} \times \mathbf{com}(a, -)))(t^* v, -s^* v) \\ &= T_{(\mathrm{id},a)}(\mathbf{com}(-, a^{-1}) \mathbf{com})(t^* v, -Ta \circ s^* v) \\ &= T_a \mathbf{com}(-, a^{-1})(t^* v \circ a - Ta \circ s^* v) \\ &= t^* v - Ta \circ s^* v \circ a^{-1}. \end{aligned}$$

In other symbols, we have $T_{\mathrm{id}} \tilde{P} v(g) = v(t(g)) - a(g)_* v(s(g))$.

Next we define the maps which identify the groupoid actions out of all bundle maps $s_1 \to t_1$, that is, the maps R and S defined on $\mathbf{Diff}_{\mathcal{G}}(s_1, t_1)$ such that $\mathcal{A}(\mathcal{G}, N, j)$ is exactly the set of b such that $R(b) = S(b)$. Let

$$R, S : \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \longrightarrow \mathbf{Diff}_{\mathcal{G}^{(2)}}(s_2, t_2)$$

be defined by $R(b)(g, h) = b(gh)$, and $S(b)(g, h) = b(g)b(h)$. The second requirement for groupoid actions, that $(gh) \cdot n = g \cdot (h \cdot n)$, follows directly from $R(b) = S(b)$, while $u(m) \cdot n = n$ follows from taking $g = u(m)$ and $h = u(m)$ in turn. The equation $R(b) = S(b)$ simply expresses that b is a groupoid morphism between $\mathcal{G} \rightrightarrows M$ and the groupoid over

M whose arrows from $m \in M$ to $n \in M$ are the diffeomorphisms $\mathbf{Diff}(N_m, N_n)$. These maps are alternatively described by

$$R = m^*,$$
$$S = \mathbf{com} \circ (p_1^* \times p_2^*) \circ \Delta.$$

Here $p_i, m : \mathcal{G}_2 \to \mathcal{G}$ are the projections and multiplication, and their counterparts with an superscript asterisk are the induced maps

$$\begin{aligned} p_1^* &: \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \longrightarrow \mathbf{Diff}_{\mathcal{G}_2}(p_1^*(s_1), p_1^*(t_1)) \simeq \mathbf{Diff}_{\mathcal{G}_2}(s_2, p_1^*(t_1)), \\ p_2^* &: \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \longrightarrow \mathbf{Diff}_{\mathcal{G}_2}(p_2^*(s_1), p_2^*(t_1)) \simeq \mathbf{Diff}_{\mathcal{G}_2}(p_1^*(s_1), t_2), \\ m^* &: \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \longrightarrow \mathbf{Diff}_{\mathcal{G}_2}(s_2, t_2). \end{aligned}$$

Again we use the obvious fiber bundle isomorphisms $p_1^*(s_1) \simeq s_2$ and $t_2 \simeq p_2^*(t_1)$ to induce tame diffeomorphisms by left- and right-composition respectively. As we did with the map P, redefine the maps R and S as $\tilde{R} = \mathbf{com}(-, a^{-1}) \circ R \circ \mathbf{com}(-, a)$ and $\tilde{S}$ analogously. The tangent map of $\tilde{S}$ at the identity is computed as follows. Introduce the notation $a_i := p_i^* a$ and $v_i := p_i^* v$, then

$$\begin{aligned} T_{\mathrm{id}} \tilde{S} v &= T_{\mathrm{id}}(\mathbf{com}(-, a^{-1}) \mathbf{com}(p_1^* \times p_2^*) \Delta \mathbf{com}(-, a)) v \\ &= T_{(a,a)}(\mathbf{com}(-, a^{-1}) \mathbf{com}(p_1^* \times p_2^*))(va, va) \\ &= T_{(a_1,a_2)}(\mathbf{com}(-, a^{-1}) \mathbf{com})(v_1 a_1, v_2 a_2) \\ &= T_a \mathbf{com}(-, a^{-1})(v_1 a_1 a_2 + (T a_1) v_2 a_2) \\ &= v_1 + (T a_1) v_2 a_1^{-1}, \end{aligned}$$

where in the last step we have used that $(a_1 a_2)(g, h) = a(g) a(h) = a(gh) = a(g, h)$, since a is already a groupoid action. Hence we find

$$T_{\mathrm{id}} \tilde{R} v(g, h) = m^* v(g, h) = v(gh),$$
$$T_{\mathrm{id}} \tilde{S} v(g, h) = v(g) + a(g)_* v(h).$$

In conclusion, we wish to apply the Nash-Noser theorem to the non-linear chain complex

$$\mathbf{Diff}_M(N) \xrightarrow{P} \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \underset{S}{\overset{R}{\rightrightarrows}} \mathbf{Diff}_{\mathcal{G}^{(2)}}(s_2, t_2)$$

given by the maps P, R, and S described above. For this we must prove that the linear chain complex

$$T_{\mathrm{id}} \mathbf{Diff}_M(N) \xrightarrow{T_{\mathrm{id}} P} T_a \mathbf{Diff}_{\mathcal{G}}(s_1, t_1) \xrightarrow{T_a R - T_a S} T_a \mathbf{Diff}_{\mathcal{G}^{(2)}}(s_2, t_2)$$

splits tamely. Instead, by interposing the tame diffeomorphisms $\mathbf{com}(-, a^{-1})$ and $\mathbf{com}(-, a)$ repeatedly, we it suffices to prove that the linear chain complex

$$\Gamma_N(T^{vert} N) \xrightarrow{d_0} \Gamma_{\mathcal{G} \times_{t,j} N}(\mathcal{G} \times_{t, j\pi_N} T^{vert} N) \xrightarrow{d_1} \Gamma_{\mathcal{G}_2 \times_{t,j} N}(\mathcal{G}_2 \times_{t, j\pi_N} T^{vert} N)$$

given by

$$\begin{aligned} d_0 v(g) &= v(t(g)) - a(g)_* v(s(g)), \\ d_1 v(g,h) &= v(g) + a(g)_* v(h) - v(gh) \end{aligned}$$

splits tamely. For group actions this linear chain complex somewhat simplifies; namely, it becomes

$$\mathcal{X}(N) \xrightarrow{d_0} C^\infty(G, \mathcal{X}(N)) \xrightarrow{d_1} C^\infty(G^2, \mathcal{X}(N)), \tag{5.2.1}$$

with coboundary maps

$$\begin{aligned} d_0 v(g) &= v - a(g)_* v, \\ d_1 v(g,h) &= v(g) + a(g)_* v(h) - v(gh). \end{aligned}$$

This is the first part of the standard complex computing the differential cohomology of the Lie group G with coefficients in the smooth module of vector fields on N, as described in the next section.

5.2.1 Homotopy operators for group actions

We start by constructing the homotopy operators for group actions, as this is the classical example, and hence deserves special attention, and by far not all mathematicians are actually interested in Lie groupoids. Let G be a Lie group and (E, ρ) a (finite dimensional) representation of G. Then the space of k-cochains is by definition the space of smooth maps $G^k \to E$,

$$C^k(G; E) := C^\infty(G^k, E).$$

Define a coboundary map

$$C^k(G; E) \xrightarrow{d_k} C^{k+1}(G; E)$$

by means of the formula

$$dv(g_1, \dots, g_{k+1}) = (-1)^k v(g_1, \dots, g_k) - \rho(g_1) v(g_2, \dots, g_{k+1}) + \sum_{i=1}^{k} (-1)^{i+1} v(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}).$$

A simple computation shows that $d^2 = 0$. The *differential cohomology* of G with values in (E, ρ), written as $H^k(G; E)$ for $k \geq 0$, is the cohomology associated to this cochain complex. It is a well-known result that the cohomology groups $H^k(G; E)$ vanish for $k \geq 1$ if G is compact.

Our situation bears much similarity; the linearized chain complex 5.2.1 on the preceding page forms the first part of the cochain complex with values in the representation $(\mathcal{X}(N), a_*)$. Here the representation assigns to $g \in G$ the tame linear map

$$a(g)_* : \mathcal{X}(N) \to \mathcal{X}(N)$$

defined by pushing forward a vector field on N. Although this picture is conceptually preferrable for its similarity with the differential cohomology, it is more convenient to define the spaces $C^k(G; \mathcal{X}(N))$ of k-cochains as the graded Fréchet spaces $\Gamma_{G^k \times N}(G^k \times TN)$. For our application we require the first cohomology group $H^1(G; \mathcal{X}(N))$ not just to vanish; there should be a *tame* splitting of the cochain complex 5.2.1 on the previous page. We reconstruct the proof the vanishing differential cohomology and make the appropriate additions.

Begin by defining a linear map α that sends a $v \in \Gamma_{G^{k+1} \times N}(G^{k+1} \times TN)$ to the map described by

$$\alpha(v)(h, g_1, \dots, g_k, x) = (a(h)_* v(h^{-1}, g_1, \dots, g_k))x \in T_x N, \quad \forall (h, g_1, \dots, g_k) \in G^{k+1}, x \in N.$$

This can easily seen to be a tame linear map by considering it as a succession of tame linear maps. Let $i : G \to G$ denote the inversion map of G. First one pre-composes v with the map $i \times \mathrm{id} : (h, \mathbf{g}, x) \mapsto (h^{-1}, \mathbf{g}, x)$. Then one discerns that a defines two maps

$$a^{-1} : G^{k+1} \times N \longrightarrow G^{k+1} \times N : (h, \mathbf{g}, x) \mapsto (h, \mathbf{g}, a(h)^{-1} x),$$

and

$$Ta : G^{k+1} \times N \longrightarrow G^{k+1} \times TN : (h, \mathbf{g}, v) \mapsto (h, \mathbf{g}, Ta(h)v),$$

where the latter is actually a bundle map over a map $a : G^{k+1} \times N \to G^{k+1} \times N$ defined similarly to a^{-1}. Now regard $v \circ (i \times \mathrm{id})$ as a bundle map over $i \times \mathrm{id}$,

$$\begin{tikzcd} G^{k+1} \times N \arrow[r, "v \circ (i \times \mathrm{id})"] \arrow[d, equal] & G^{k+1} \times TN \arrow[d, two heads] \\ G^{k+1} \times N \arrow[r, "i \times \mathrm{id}"] & G^{k+1} \times N, \end{tikzcd}$$

and then pre-compose with a^{-1} and post-compose with Ta consecutively.

Let $d\mu$ denote a normalized Haar measure on G, for example [DK04]. The actual homotopy operators are then defined by sending $v \in \Gamma_{G^{k+1} \times N}(G^{k+1} \times TN)$ to

$$\begin{aligned} h_k(v)(\mathbf{g}, x) &= \int_G \alpha(v)(h, \mathbf{g}, x) d\mu(h) \\ &= \int_G a(h)_* v(h^{-1}, \mathbf{g}) x dh. \end{aligned}$$

The following computations show that the maps h_k are indeed the desired homotopy operators. We have

$$\begin{aligned} d_{k-1} h_{k-1}(v)(\mathbf{g}) =& a(g_1)_* \int a(h)_* v(h^{-1}, g_2, \ldots, g_k) x dh \\ &+ (-1)^k \int a(h)_* v(h^{-1}, g_1, \ldots, g_{k-1}) x dh \\ &+ \sum_{i=1}^{k-1} (-1)^i \int a(h)_* v(h^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_k) x dh, \\ h_k d_k(v)(\mathbf{g}) =& \int a(h)_* a(h^{-1})_* v(\mathbf{g}) x dh \\ &+ (-1)^{k+1} \int a(h)_* v(h^{-1}, g_1, \ldots, g_{k-1}) x dh \\ &\sum_{i=1}^{k} (-1)^{i+1} \int a(h)_* v(h^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_k) x dh \\ &- \int a(h)_* v(h^{-1} \mathbf{g}) x dh, \end{aligned}$$

and by linearity of integration and left invariance of the Haar measure we have

$$\begin{aligned} a(g_1)_* \int a(h)_* v(h^{-1}, g_2, \ldots, g_k) x dh = & \int a(g_1 h)_* v((g_1 h)^{-1} g_1, g_2, \ldots, g_k) x dh \\ &= \int a(h)_* v(h^{-1} g_1, g_2, \ldots, g_k) x dh, \end{aligned}$$

so that all terms except for $v(\mathbf{g})$ cancel each other out. This implies the homotopy relation

$$d_{k-1}h_{k-1} + h_k d_k = \mathrm{id}.$$

All that is left is to check that the process of integration is also a tame linear map. This follows from lemma 2.3.12 on page 38.

5.2.2 Homotopy operators for groupoid actions

Next we generalize the previous construction to the differential cohomology of a Lie groupoid action. The notion of differential cohomology extends to Lie groupoids. Let $\mathcal{G}$ be a Lie groupoid and (E, ρ) a representation of $\mathcal{G}$, that is, a finite rank vector bundle $E \xrightarrow{p} M$ and a groupoid action

$$\rho : \mathcal{G} \times_{s,p} E \to E$$

for which $\rho(g) : E_{s(g)} \to E_{t(g)}$ is a linear transformation for every arrow g. By slight abuse of notation, let $t : \mathcal{G}^{(k)} \to M$ denote the target of the final arrow of a k-tuple arrows. Then the k-cochains with values in (E, ρ) are defined as

$$C^k(\mathcal{G}; E) = \Gamma_{\mathcal{G}^{(k)}}(t^*E),$$

and the coboundary maps of the linear cochain complex

$$C^0(\mathcal{G}, E) \xrightarrow{d_0} C^1(\mathcal{G}, E) \xrightarrow{d_1} C^2(\mathcal{G}, E) \longrightarrow \dots$$

are defined by the same formula,

$$dv(g_1, \dots, g_{k+1}) = \rho(g_1)v(g_2, \dots, g_{k+1}) + (-1)^{k+1}v(g_1, \dots, g_k) \\ + \sum_{i=1}^{k}(-1)^i v(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}).$$

The resulting cohomology is called the differential cohomology of $\mathcal{G}$ with values in (E, ρ), see for example [Cra03]. The differential cohomology with values in E can be shown to vanish for proper groupoids in nearly the same way as in the group case. We will again adapt this construction to our situation.

We have described the tangent space $\mathbf{Diff}_{\mathcal{G}^{(k)}}(s_k, t_k)$ at id as the set of smooth sections of the vector bundle $t^*\mathcal{X}(j) \to \mathcal{G}^{(k)}$ whose fiber at a k-tuple $\mathbf{g}$ of composable arrows is the space of vector fields $\mathcal{X}(N_{t(\mathbf{g})}$. Hence in our case the k-cochains take values in the vector bundle $\mathcal{X}(j) \to M$, and the action a_* is given by pushing a vector field forward along $a(\mathbf{g})$,

$$a_*(g, v) = a(g)_* v, \quad \forall g \in \mathcal{G}, v \in \mathcal{X}(N_{s(g)}).$$

This describes the heuristic picture nicely, and clearifies how the differential cohomology of $\mathcal{G}$ appears as an obstruction for the stability of groupoid actions. However, we haven't developped the theory tame Fréchet vector bundles sufficiently to work with the space of sections of $\mathcal{X}(j) \to M$ directly. Instead we will work with the more technical definition of the tangent spaces,

$$T_{\mathrm{id}}\mathbf{Diff}_{\mathcal{G}^{(k)}} \simeq \Gamma_{\mathcal{G}^{(k)} \times_{t,j} N}(u^* T^{\mathrm{vert}} t_k^*(\mathcal{G}^{(k)} \times_{t,j} N)) \simeq \Gamma_{\mathcal{G}^{(k)} \times_{t,j} N}(\mathcal{G}^{(k)} \times_{t, j\pi_N} T^{\mathrm{vert}} N),$$

while keeping the heuristics of differential cohomology with values in $\mathcal{X}(j)$ in mind.

Since the formula's for the coboundary maps are identical to those of the Lie *group* case, one expects to be able to prove the tame vanishing of the cohomology using similar methods as well, that is, we want to somehow define smooth tame maps of the form

$$h_k(v)(\mathbf{g}, n) = \int_{\mathcal{G}} a(h)_* v(h^{-1}, \mathbf{g}) n \, dh.$$

Let $\pi : \mathcal{G} \times_t \mathcal{G}^{(k)} \times_{t,j} N \longrightarrow \mathcal{G}^{(k)} \times_{t,j} N$ denote the obvious projection. We begin by constructing a map

$$\alpha : C^{k+1}(\mathcal{G}; \mathcal{X}(j)) \longrightarrow \Gamma_{\mathcal{G}\times_t\mathcal{G}^{(k)}\times_{t,j}N}(\pi^*(\mathcal{G}^{(k)} \times_{t,j\pi_N} T^{\mathrm{vert}}N))$$

that sends $v \in C^{k+1}(\mathcal{G}; \mathcal{X}(j))$ to the integrand in the formula of h_k,

$$\alpha(v)(h, \mathbf{g}, n) = (a(h)_* v(h^{-1}, \mathbf{g}))n \in T_n^{\mathrm{vert}}N, \quad \forall (h, \mathbf{g}) \in \mathcal{G} \times_t \mathcal{G}^{(k)}, n \in N_{t(\mathbf{g})}.$$

The inversion map i of $\mathcal{G}$ gives rise to a smooth map

$$i \times \mathrm{id} : N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)} \longrightarrow \mathcal{G}_{k+1} \times_{t,j} N : (h, \mathbf{g}, n) \mapsto (h^{-1}, \mathbf{g}, n),$$

where $N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)}$ is the set of $(h, \mathbf{g}, n)$ such that $t(h) = t(\mathbf{g})$ and $j(n) = s(h)$. Precomposing a $(k+1)$-cochain $v \in C^{k+1}(\mathcal{G}; \mathcal{X}(j))$ yields a tame linear map. The resulting $v \circ (i \times \mathrm{id})$ can be interpreted as a section of $(i \times \mathrm{id})^*(\mathcal{G}_{k+1} \times_{t,j\pi_N} T^{\mathrm{vert}}N)$, and hence seeing it as a smooth bundle map

$$\begin{tikzcd}
N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)} \arrow[r, "v \circ (i \times \mathrm{id})"] \arrow[d, equal] & (i \times \mathrm{id})^*(\mathcal{G}_{k+1} \times_{t,j\pi_N} T^{\mathrm{vert}}N) \arrow[d] \\
N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)} \arrow[r, equal] & N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)}.
\end{tikzcd}$$

The fixed groupoid action a induces two particular smooth maps, namely

$$a^{-1} : \mathcal{G} \times_t \mathcal{G}^{(k)} \times_{t,j} N \longrightarrow N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)} : (h, \mathbf{g}, n) \mapsto (h, \mathbf{g}, a(h)^{-1}n),$$

and a smooth vector bundle map

$$\begin{tikzcd}
(i \times \mathrm{id})^*(\mathcal{G}_{k+1} \times_{t,j\pi_N} T^{\mathrm{vert}}N) \arrow[r, "Ta"] \arrow[d] & \pi^*(\mathcal{G}^{(k)} \times_{t,j\pi_N} T^{\mathrm{vert}}N) \arrow[d] \\
N \times_{j,s} \mathcal{G} \times_t \mathcal{G}^{(k)} \arrow[r, "a"] & \mathcal{G} \times_t \mathcal{G}^{(k)} \times_{t,j} N,
\end{tikzcd}$$

where

$$a(h, \mathbf{g}, n) = (h, \mathbf{g}, a(h)n), \quad \forall t(h) = t(\mathbf{g}), n \in N_{s(h)},$$

and

$$Ta(v) = Ta(h)v \in T_{a(h)n}N_{t(h)}, \quad \forall v \in T_n^{\mathrm{vert}}N = T_nN_{s(h)}.$$

The map α is finally obtained by pre-composing consecutively precomposing $v \circ (i \times \mathrm{id})$ with a^{-1} and post-composing with Ta, hence α is a tame linear map.

Next we need to define an integration map

$$I : \Gamma_{\mathcal{G}\times_t\mathcal{G}^{(k)}\times_{t,j}N}(\pi^*(\mathcal{G}^{(k)} \times_{t,j} N)) \longrightarrow \Gamma_{\mathcal{G}^{(k)}\times_{t,j}N}(\mathcal{G}^{(k)} \times_{t,j} N),$$

for which we need to introduce a smooth Haar system on $\mathcal{G}$, see for example [Cra03, Ren80]. A smooth normalized Haar system μ on $\mathcal{G}$, is a family $\mu = \{\mu_x : x \in M\}$ of Radon measures μ_x on the manifolds $\mathcal{G}(-, x) = t^{-1}(x)$ with the following properties:

1. *smoothness*: for any compactly supported function $f \in C_c^\infty(\mathcal{G})$, the formula

$$I_\mu(f)(x) = \int_{\mathcal{G}(-,x)} f(g) d\mu_x(g)$$

defines a smooth function $I_\mu(f)$ on M;

2. *left-invariance*: for every $g \in \mathcal{G}$, with $x = s(g)$ adn $y = t(g)$, and $f \in C_c^\infty(\mathcal{G}(-,y))$ we have

$$\int_{\mathcal{G}(-,x)} f(gh) d\mu_x(h) = \int_{\mathcal{G}(-,y)} f(h) d\mu_y(h).$$

3. *normalization*: $\int_{\mathcal{G}(-,x)} d\mu_x(g) = 1$ for every $x \in M$.

The construction of smooth normalized Haar systems can be found in the literature, for example [Ren80]), but we will recall it here since it is particularly simple in a smooth, Hausdorff and compact setting. Note that for proper Lie groupoids the normalization is replaced by a so-called cut-off function, see e.g [Cra03]. This is not necessary if the Lie groupoid is compact since then the function $f(g) = 1$ is compactly supported on $\mathcal{G}$.

For the construction of such a family $\mu = \{\mu_x\}$, consider a vector bundle metric g on the bundle $u^*T^t\mathcal{G}$, where $T^t\mathcal{G}$ denotes the target-vertical bundle of $\mathcal{G}$. Then by left translation we obtain a Riemannian metric g on the target-vertical bundle $T^t\mathcal{G} \to \mathcal{G}$ whose positive densities $\mu_x = \left|vol(g|_{\mathcal{G}(-,x)})\right|$ for $x \in M$ form a smooth left-invariant family of densities. If the groupoid $\mathcal{G}$ is compact, then in particular each of the t-fibers are compact, hence one can integrate the functions $1 : \mathcal{G} \to \mathbb{R}$ to obtain a smooth map $x \mapsto \int_{\mathcal{G}(-,x)} 1\, d\mu_x$. This allows one to find a normalized Haar system for $\mathcal{G}$.

The homotopy operators are now defined by

$$h_k(v)(\mathbf{g}, n) = \int_{\mathcal{G}(-,j(n))} \alpha(v)(h, \mathbf{g}, n)\, d\lambda_{j(n)}(h).$$

They satisfy the homotopy relation by the same computations as in the previous section, since these computations primarily rely on left-invariance, linearity, and normalization. Finally, the integration map is a tame linear map, as per lemma 2.3.13 on page 41.

5.3 Deformation theory of foliations

This section presents an overview of the unpublished paper [Ham82a], *'Deformation theory of foliations'*, as another demonstration of the concept 'infinitesimal stability implies stability'. This theory is far from complete, as it proves to be difficult to translate the rather technical hypothesis of the Nash-Moser theorem to elegant, tangible conditions. The notion of stability is again self-evident: a foliation, seen as a involutive distribution $B \subseteq TM$, is stable if nearby foliations B' are conjugate by a diffeomorphism $\varphi \in \mathbf{Diff}(M)_{\mathrm{id}}$ in the sense that

$$B_x = (T\varphi)B'_x\varphi^{-1} := \left\{ T\varphi(b) : b \in B'_{\varphi^{-1}(x)} \right\}, \quad \forall x \in M.$$

There are already some typical stability results for foliations, although global stability results are much harder to obtain. The celebrated Reeb stability theorem [Ree52] proves local stability in a neighborhood of a compact leaf L under the condition of a finite holonomy group. *Global* Reeb stability focuses on describing describing the foliation out of information on a single compact leaf. For example, if a codimension 1 foliation on a closed manifold contains a compact leaf with finite fundamental group, then all leaves are compact and have finite fundemental group. Such results generally fail if the codimension is larger than 1. For reference, we state the Reeb stability theorem in full detail.

Theorem 5.3.1 (Reeb)**.** *Let F be a foliation of codimension k on M and L a compact leaf with a finite holonomy group. Then there is a saturated neighborhood $U \subseteq M$ of L in which every leaf is compact and has a finite holonomy group. Moreover, there is a smooth retraction $p : U \to L$ such that for every leaf $L' \subseteq U$, the restriction*

$$p|_{L'} : L' \longrightarrow L$$

is a covering map with finitely many sheets and each fiber $p^{-1}(x)$, for $x \in L$, is homeomorphic to a k-dimensional disc transversal to the foliation. The neighborhood U with these properties may be taken arbitrarily small.

Thurston's [Thu74] condition of 'infinitesimal stability' is that the first leaf cohomology $H^1(L, \mathbb{R})$ vanishes, instead. Here we see again the concept that the vanishing of suitable cohomology groups expresses the impediment for stability.

The aim of Hamilton's paper was the following. Let $F \subseteq TM$ be a Hausdorff foliation, in the sense that the leaf space is Hausdorff, on a compact manifold M. Now if $H^1(L, \mathbb{R})$ vanishes for a generic leaf L, then the foliation F is stable. This is done via a Nash-Moser argument. We will reproduce some parts of the argument.

For completeness, let us begin by recalling some basic definitions. A *p-dimensional distribution* on a manifold M is a smooth sub-bundle $B \subseteq TM$ of fixed rank p. Equivalently, it has a fixed codimension $q = \dim(M) - p$. Such a distribution is called *integrable* if there exist local coordinates $x_1, \ldots, x_p, y_1, \ldots, y_q$ around every point $x \in M$ such that

$$B_x = span \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p} \right\}$$

A integrable distribution $B \subseteq TM$ determines an equivalence relation on M, where $x \simeq y$ if and only if $\varphi_1(x) = y$ with φ_t the (local) flow of some vector field $X \in \Gamma_M(B)$. Such an equivalence class $L_x = [x] \subseteq M$ is called a *leaf* of the foliation, in particular the leaf through $x \in M$, and carries the natural structure of an *immersed* submanifold such that $T_y L_x = B_y$ for all $y \in L_x$.

A *regular foliation* of codimension q on M is commonly defined as a covering of charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that the transition maps are of the form

$$\varphi_j \circ \varphi_i^{-1}(x, y) = (\psi_{ij}^1(x), \psi_{ij}^2(x, y)),$$

where $(x, y) \in \mathbb{R}^{p+q}$, $\psi_{ij}^1 : \mathbb{R}^p \to \mathbb{R}^p$ and $\psi_{ij}^2 : \mathbb{R}^{p+q} \to \mathbb{R}^q$. The connected component of U_i containing $x \in U_i$ is called the *plaque* of x. A *leaf* of the foliation is a connected subset $L \subseteq M$ which is a union of plaques, and maximal under these conditions. From this definition the 1-1 correspondence between regular foliations and integrable distributions and their notions of leaves is clear. In the remainder we will only use the definition of a foliation as a integrable distribution.

The famous Frobenius theorem [Fro77] identifies the foliations among distributions via an algebraic condition on the space of sections $\Gamma_M B$.

Theorem 5.3.2 (Frobenius)**.** *A distribution $B \subseteq TM$ is integrable if and only if it is involutive in the sense that*

$$[\Gamma_M B, \Gamma_M B] \subseteq \Gamma_M B,$$

where $[-,-]$ is the Lie bracket on $\mathcal{X}(M)$.

Next we lift this to the setting of tame Fréchet manifolds. Let

$$Gr_p(TM) \xrightarrow{\pi} M$$

be the Grasmannian bundle of p-planes in TM. Recall that its fibers are

$$\pi^{-1}(x) = \{B_x \leq T_x M \text{ a linear subspace of dimension } p\}$$

and that it carries the natural structure of a fiber bundle over M. Notice that a distribution corresponds to a section of the Grasmannian bundle, so we define the *space of distributions* as

$$\mathbf{Dist}_p(M) = \Gamma_M(Gr_p(TM)).$$

We have seen that if M is compact, this space is a tame manifold.

The *integrability bundle* is the vector bundle $\mathcal{I} \longrightarrow \mathbf{Dist}_p(M)$ whose fiber at a distribution $B \in \mathbf{Dist}_p(M)$ is given by

$$\mathcal{I}_B = \Omega^2(B; TM/B) := \Gamma_M(\wedge^2 B^* \otimes TM/B),$$

the space of 2-forms on B with values in the vector bundle $TM/B \to M$. Alternatively, define a vector bundle $I \to Gr_p(TM)$ over the total space of the Grasmannian bundle whose fiber at $B_x \in Gr_p(T_x M)$, with $x \in M$, is

$$I_{B_x} = \wedge^2 B_x^* \otimes T_x M / B_x,$$

then we may define $\mathcal{I}_B = \Gamma_M(B^*I)$. The idea behind the integrability bundle becomes clear by the following simple observation. Let $R : Dist_p(M) \to \mathcal{I}$ be the map defined by

$$R(B)(X,Y) = [X,Y] \mod B$$

for $X, Y \in \Gamma_M B$. Note that this is well-defined by the Leibniz-rule;

$$[fX,Y] = f[X,Y] + X(f)Y \cong f[X,Y] \mod B.$$

The Lie bracket modulo B defines a $C^\infty(M)$-bilinear map on $\Gamma_M B$ with values in TM/B, hence it corresponds to an element of $\mathcal{I}_B$. By the Frobenius theorem, a distribution B defines a regular foliation if and only if B satisfies the structure equation $R(B) = 0 \mod B$. Hence we may define the space of p-foliations by

$$\mathbf{Fol}_p(M) := \ker R \subseteq \mathbf{Dist}_p(M).$$

Of course, this integrability bundle is only useful to the method at hand if it is tame Fréchet.

Proposition 5.3.3. *The integrability bundle $\mathcal{I} \to \boldsymbol{Dist}_p(M)$ is a tame Fréchet vector bundle.*

Proof. This follows directly from the second description of $\mathcal{I}$ and lemma 3.2.7 on page 67. □

We will treat the smooth tameness of R later on. First we will consider the remaining components of the non-linear chain complex, to provide a complete picture before delving into further details.

The diffeomorphism group of M acts naturally on $\mathbf{Dist}_p(M)$ by push-forwards; namely by the map

$$\mathbf{Diff}(M) \times \mathbf{Dist}_p(M) \longrightarrow \mathbf{Dist}_p(M), \quad \varphi \cdot B = \varphi_* B = T\varphi \circ B \circ \varphi^{-1}.$$

Here the tangent map is a interpreted as a smooth bundle map

$$\begin{tikzcd} Gr_p(TM) \arrow[r, "T\varphi"] \arrow[d] & Gr_p(TM) \arrow[d] \\ M \arrow[r, "\varphi"] & M, \end{tikzcd}$$

so that the composition with a distribution $B \in \Gamma_M(Gr_p(TM))$ makes sense. It is obvious that pre-composition with φ^{-1} is necessary to obtain a distribution again. Note that if B is a foliation then so is $\varphi \cdot B$, since φ_* is a Lie algebra morphism:

$$[\varphi_* X, \varphi_* Y] = \varphi_*[X,Y], \quad X, Y \in \mathcal{X}(M).$$

Hence the action restricts to an action

$$\mathbf{Diff}(M) \times \mathbf{Fol}_p(M) \longrightarrow \mathbf{Fol}_p(M)$$

on foliations.

In addition, there is a natural action

$$\mathbf{Diff}(M) \times \mathcal{I} \longrightarrow \mathcal{I}$$

on the total space of the integrability bundle defined as follows. Note that, for a fixed distribution B, the push-forward also defines a map $\varphi_* : \Gamma_M B \to \Gamma_M(\varphi_* B)$ by pushing forward vector fields. Likewise, smooth maps $C^\infty(M)$ can be pushed forward by pre-composition with φ^{-1}, $f \mapsto f \circ \varphi^{-1}$. For a 2-form $\omega \in \mathcal{I}_B$ we define its action $\varphi \cdot \omega \in \mathcal{I}_{\varphi_* B}$ under φ by

$$\varphi \cdot \omega(X, Y) = \varphi_*(\omega(\varphi_*^{-1} X, \varphi_*^{-1} Y)).$$

Note that the zero section of $\mathcal{I}$ is equivariant, since

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "\varphi_*"] & \mathcal{I} \\
\mathbf{Dist}_p(M) \arrow[u] \arrow[r, "\varphi_*"] & \mathbf{Dist}_p(M) \arrow[u]
\end{tikzcd}$$

and so is the map $R : \mathbf{Dist}_p(M) \to \mathcal{I}$, since also φ_* preserves the bracket for every diffeomorphism φ of M.

Now fix a foliation B on M. We obtain a non-linear chain complex

$$\mathbf{Diff}(M) \xrightarrow{P_B} \mathbf{Dist}_p(M) \underset{z}{\overset{R}{\rightrightarrows}} \mathcal{I},$$

where $P_B(\varphi) = \varphi \cdot B$, R is as defined above, and z is the zero section. We are still required to prove the smooth tameness of all involved maps. At least we already recover the correct notion of stability: a foliation B is *stable* if there exists a neighborhood $U \subset \mathbf{Dist}_p(M)$ of B such that every foliation $B' \in U \cap \mathbf{Fol}_p(M)$ is conjugate to B.

5.3.1 Linearization of the complex

In this section we fix a foliation B on M. We are to determine the corresponding linear complex. To begin with, we saw that $T_{\mathrm{id}}\mathbf{Diff}(M) = \mathcal{X}(M)$ is the space of vector fields. For the second Fréchet manifold we have

$$T_B \mathbf{Dist}_p(M) = T_B \Gamma_M Gr_p(TM) \simeq \Gamma_M(B^* T^{\mathrm{vert}} Gr_p(TM)),$$

hence we should have a closer look at the vector bundle $B^* T^{\mathrm{vert}} Gr_P(TM) \to M$; our goal is to show that it is the bundle $Hom(B, V/B)$, whose fibers over $m \in M$ are the linear transformations $B_m \to V_m/B_m$. For this it is sufficient to consider the following situation.

Let V be a vector space, $Gr_p V$ its Grassmannian of dimension p, and $B \in Gr_p V$ a fixed linear subspace. Then we will show that $T_B Gr_p V \simeq Hom(B, V/B)$ by a natural identification. Consider the following map from the general linear space of V,

$$\varphi_B : GL(V) \longrightarrow Gr_p(V), \quad \varphi_B(L) = L(B),$$

where $L(B)$ is the image of B under L. Note that $\varphi_B(\mathrm{id}) = B$ and $T_{\mathrm{id}}GL(V) = Hom(V,V)$ is the space of linear endomorphisms, so that

$$T_{\mathrm{id}}\varphi_B : Hom(V,V) \longrightarrow T_B Gr_p V.$$

This linear map factors through the obvious map $Hom(V,V) \to Hom(B, V/B)$,

$$\begin{tikzcd} Hom(V,V) \arrow[d] \arrow[dr, "T_{\mathrm{id}}\varphi_B"] & \\ Hom(B,V/B) \arrow[r, "\sim"] & T_B Gr_p V. \end{tikzcd}$$

One can see this, for example, by choosing a basis $v_1, \ldots, v_n \in V$ such that $B = Span\{v_i | 1 \leq i \leq p\}$. The general linear group is identified with the space of invertible $(n \times n)$-matrices, and the map φ_B sends a matrix to the span of the first p column vectors,

$$\varphi_B(M) = Span\{Mv_i\}.$$

Take a smooth curve of invertible matrices M_t, with $M_0 = \mathrm{id}$, representing a tangent vector at the identity, that is, a linear transformation $L \in Hom(V,V)$. Then if $\{\nu^i\}$ is the basis dual to $\{v_i\}$, we may write

$$M_t = v_i \nu^i + t L_{ij} v_i \nu^j + O(t^2).$$

Here L_{ij} denotes the matrix entry in the i-th row and j-th column. The tangent map at the identity is now given by

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \varphi_B(M_t) &= \frac{d}{dt}\bigg|_{t=0} Span\left\{(v_i\nu^i + tL_{ij}v_i\nu^j)v_k | 1 \leq k \leq p\right\} \\ &= \frac{d}{dt}\bigg|_{t=0} Span\left\{v_k + \sum_i tL_{ik}v_i | 1 \leq k \leq p\right\}. \end{aligned}$$

Obviously, only the first p columns of L matter. Secondly, whenever the sum $\sum_i L_{ik}v_i$ lies in B, it doesn't contribute anything to changing the span, we only need to look at those terms with $p+1 \leq i \leq n$. This specifies the isomorphism $Hom(B, V/B) \simeq T_B Gr_p V$. It is in fact completely determined by $T_{\mathrm{id}}\varphi_B$ and the map $Hom(V,V) \to Hom(B, V/B)$. We conclude that the tangent space of $\mathbf{Dist}_p(M)$ at B is given by

$$T_B \mathbf{Dist}_p(M) \simeq \Gamma_M Hom(B, TM/B).$$

Next we compute the tangent map of P_B at the identity. First note that both actions of $\mathbf{Diff}(M)$, and hence also P_B, are smooth tame maps. Now let $\varphi_t : (-1,1) \times M \to M$ be a smooth curve of diffeomorphisms with $\varphi_0 = \mathrm{id}$, representing a vector field $v \in \mathcal{X}(M)$, and compute.

$$\frac{d}{dt}\bigg|_{t=0} P_B(\varphi_t) = \frac{d}{dt}\bigg|_{t=0} T\varphi_t \circ B \circ \varphi_t^{-1} = Tv \circ B - TB \circ v,$$

where TB denotes the tangent map of B seen as a section $M \to Gr_p(M)$. When we keep track of the identifications, we see that this is the map

$$\mathcal{X}(M) \to \Gamma_M Hom(B, TM/B), \quad T_{\mathrm{id}} P_B(v)(X) = v \circ X - X \circ v,$$

where $X \in \Gamma_M B$ and composition is by interpreting vector fields as derivations. Up to a minus sign, this is the 0-th deffirential of the deRham cohomology of B, defined in the next section.

Next, recall that $\mathcal{I} \xrightarrow{\pi} \mathbf{Dist}_p(M)$ is a tame Fréchet vector bundle. Let z denote its zero section. As with finite dimensional vector bundles, the tangent bundle along z, $z^*T\mathcal{I}$, naturally admits a splitting

$$z^*T\mathcal{I} \simeq \mathcal{I} \oplus T\mathbf{Dist}_p(M) \to \mathbf{Dist}_p(M).$$

Consider for a moment a vector bundle $E \xrightarrow{\pi} M$ with zero section $z : M \to E$. For any point $m \in M$, we have two maps

$$\begin{aligned} T_m z : T_m M &\longrightarrow T_m E, \\ T_{z(m)}\pi : T_{z(m)} E &\longrightarrow T_m M. \end{aligned}$$

For any vector $v \in T_{z(m)}E$, we have that

$$T_{z(m)}\pi(\mathrm{id} - T_m z \circ T_{z(m)}\pi)v = 0,$$

so that $(\mathrm{id} - T_m z \circ T_{z(m)}\pi)v$ is a vertical vector of E at m. Now the vertical bundle along z, $z^*T|E$, is isomorphic to E, since every vector space is cannonically identified with its tangent space at 0. Hence we can decompose $z^*TE \simeq E \oplus TM$. For $\mathcal{I}$ above, these isomorphisms are readily seen to be tame. We conclude that

$$(P_B^* z^* T\mathcal{I})_{\mathrm{id}} \simeq \mathcal{I}_B \oplus T_B \mathbf{Dist}_p(M) \simeq \mathcal{I}_B \oplus \Gamma_M(Hom(B, TM/B)).$$

Finally, we must show that R is a smooth tame section of $\mathcal{I}$ and compute the map $T_B R - T_B z : T_B \mathbf{Dist}_p(M) \to T_{z(B)}\mathcal{I}$, where R was the map measuring the involutivity of the distribution,

$$R : \mathbf{Dist}_p(M) \to \mathcal{I}, \quad R(B)(X, Y) = [X, Y] \mod B.$$

Note that both R and the zero section z are sections, so their tangent maps take the same values in the second component of $T_{z(B)}\mathcal{I} \simeq \mathcal{I}_B \oplus \Gamma_M Hom(B, TM/B)$. The difference $T_B R - T_B z$ is called the vertical differntial of R at B; we may consider it as a map

$$\partial^{\mathrm{vert}} R : \Omega^1(B; TM/B) \longrightarrow \Omega^2(B; TM/B),$$

where $\Omega^k(B; TM/B)$ denote the space of anti-symmetric k-forms on B with values in TM/B, that is, the graded Fréchet space $\Gamma_M(\wedge^k B^* \otimes TM/B)$.

5.3.2 The de-Rham complex

In what follows the foliation B is left fixed. The quotient bundle TM/B naturally holds a flat linear connection, usually called the Bott connection. It is the connection

$$\nabla = \nabla^B : \Gamma_M B \times \Gamma_M TM/B \to \Gamma_M TM/B$$

defined by $\nabla_X \overline{Y} = \overline{[X,Y]}$. It is easily seen to be $\mathbb{R}$-bilinear and, by the Leibniz-rule and the fact that $X \in \Gamma_M B$, $C^\infty(M)$-linear in the first coordinate. In the second entry it satisfies the Leibniz-rule,

$$\nabla_X f\overline{Y} = f\nabla_X \overline{Y} + L_X(f)\overline{Y}.$$

Finally, one can easily check that the flatness property

$$\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} = \nabla_{[X_1,X_2]}$$

follows directly from the Jacobi-identity for $[-,-]$.

The Bott-connection induces a covariant derivation

$$d_B : \Omega^\bullet(B, TM/B) \to \Omega^{\bullet+1}(B, TM/B)$$

defined by the usual Koszul formula

$$\begin{aligned} d_B\omega(X_0,\ldots,X_k) = &\sum_i (-1)^i \nabla_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) \\ &+ \sum_{i<j} (-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k), \end{aligned}$$

where the circumflex indicates leaving said section out of the formula. Here we have used the notation

$$\Omega^\bullet(B, TM/B) := \sum\nolimits_k \Gamma(\wedge^k B^* \otimes TM/B)$$

to indicate the exterior algebra of forms with values in TM/B. A simple computation show that $d_B^2 = 0$ if and only if ∇^B is a flat connection. The de-Rahm cohomology of B with values in the Bott representation on TM/B, its k^{th} group denoted by $H^k(B;TM/B)$, is the the cohomology of the cochain complex

$$\ldots \longrightarrow \Omega^k(B;TM/B) \xrightarrow{d_B} \Omega^{k+1}(B;TM/B) \longrightarrow \ldots.$$

A foliation $B \in Fol_p(M)$ will be called *infinitesimally stable* if its first cohomology group $H^1(B;TM/B)$ splits tamely. Recall that this means the existence of tame linear operators

$$\Omega^2(B, TM/B) \xrightarrow{h_1} \Omega^1(B, TM/B) \xrightarrow{h_0} \Gamma(TM/B)$$

that satisfy the homotopy relation $d_B \circ h_0 + h_1 \circ d_B = \mathrm{id}$. By the above we can conclude that a foliation is stable if it is infinitesimally stable. Although the vanishing of $H^1(B;TM/B)$ would have been an elegant condition for stability, its tame splitting is *not*. The existence of tame homotopy operators is generally not easy to check and typically depends on explicit constructions.

Hamilton [Ham82a] continues by giving sufficient conditions for the tame vanishing of the cohomology. We say that a smooth function $f : M \to \mathbb{R}$ is constant along the leaves of B if

$$df|_B = 0.$$

Consider a Riemannian metric g on M as a metric on the cotangent bundle TM. Then g is said to be a holonomy-invariant metric if for every two smooth functions $f, g : M \to \mathbb{R}$ constant along the leaves, the inner product

$$\langle df, dg \rangle \in \Omega^1(M)$$

is again constant along the leaves. Any Riemannian metric g on M induces inner products on the vector bundles $\wedge^k B^* \otimes TM/B$ and a volume form θ on M. Since M is compact, the inner product

$$\langle \varphi, \psi \rangle := \int_M \langle \varphi(x), \psi(x) \rangle \theta(x), \quad \forall \varphi, \psi \in \Omega^k(B, TM/B)$$

is well-defined and defines an L^2-norm on the Fréchet spaces $\Omega^k(B, TM/B)$. Define the adjoint d_B^* of the differential d_B by

$$\langle , d_B^* \varphi, \psi \rangle = \langle \varphi, d_B \psi \rangle, \quad \forall \varphi, \psi \in \Omega^k(B, TM/B).$$

Hamilton has managed to deduce the following requirements (without mention of technical terms such as tameness) for the stability of a foliation.

Theorem 5.3.4. *Let M be a compact manifold and B a regular foliation. If*

1. *there exists a holonomy-invariant foliation g on M;*

2. *there is a constant $C > 0$ such that the following estimates hold: for every $\varphi \in \Gamma_M Hom(B, TM/B)$ we have*

$$\|\varphi\|^{L^2} \leq C \left(\|d_B \varphi\|^{L^2} + \|d_B^* \varphi\|^{L^2} \right),$$

then the foliation B is stable in the sense that every regular distribution sufficiently close to B in $\boldsymbol{Dist}_p(M)$ is conjugate to B by a diffeomorphism.

In the case a compact manifold with a Hausdorff foliation the above requirements can be much simplified. First of all, every Hausdorff foliation B on a compact manifold M admits a holonomy-invariant Riemannian metric. Fiber bundles clearly admit holonomy-invariant Riemannian metrics, and one can use generic leaves and a suitable partition of unity to obtain one on all of M. Hamilton then derives the estimates in the theorem above from the requirement that $H^1(L; \mathbb{R}) = 0$ for a generic leaf L.

Theorem 5.3.5. *Let M be a connected, compact manifold and B a regular Hausdorff foliation. If*

$$H^1(L;\mathbb{R}) = 0$$

for a generic leaf L, then B is stable in the sense that every regular distribution sufficiently close to B in $\boldsymbol{Dist}_p(M)$ is conjugate to B by a diffeomorphism.

These conditions on M and the foliation B are very restrictive. For example, see [**?**, Ree52], let (M, B) is a compact, connected, transversely orientable, foliated manifold of codimension one. If there is a compact leaf L with $H^1(L;\mathbb{R}) = 0$, then either M is isomorphic to $L \times [0, 1]$ as a foliated product, if one allows M to have a boundary, or M is the total space of a fiber bundle $M \to S^1$ having the leaves of B as fibers.

Chapter 6

Proof of the Nash-Moser theorem

This chapter goes through the proof of the Nash-Moser theorem for non-linear complexes, as originally given in [Ham77]. The proof is somewhat simplified, by producing a large formula in the 'preliminary estimates' section, so that a peculiar repetition of arguments becomes unnecessary.

We start with a non-linear chain complex,

$$\mathcal{M} \xrightarrow{P} \mathcal{N} \underset{S}{\overset{R}{\rightrightarrows}} \mathcal{O},$$

satisfying the hypothesis of the Nash-Moser theorem: the tame manifolds allow smoothing operators, all maps are smooth tame, and there exist smooth tame maps VP and VR such that

$$D_xP \circ V_xP + V_xQ \circ (D_{P(x)}R - D_{P(x)}S) = \mathrm{id}, \quad \forall x \in \mathcal{M}$$

Since the theorem is of a local nature, we may replace $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ by graded Fréchet spaces E, F, and G that allow smoothing operators, respectively. Because of the now present additive structure, we may define $Q = R - S$. This leaves us in the following situation.

Let $P : E \to F$ and $Q : F \to G$ be smooth tame maps between tame Fréchet spaces that satisfy $Q \circ P(x) = 0$ for every $x \in E$. Note that P may be translated by $P(0)$ without affecting this property. Neither does such a translation affect any identity involving only derivatives of P and Q; nor the conclusion of the Nash-Moser theorem, since translation is an invertible map. This leads to the following local version of the Nash-Moser theorem for exact sequences.

Theorem 6.0.6. *Let $P : U \to V$ and $Q : V \to G$ be smooth tame maps between open subsets in Fréchet spaces that allow smoothing operators. Suppose that $Q \circ P = 0$, and $P(0) = 0$. Moreover, assume there are maps $VP : E \times F \to E \times E$ and $VQ : E \times G \to E \times F$ so that*

$$D_xP \circ V_xP + V_xQ \circ D_{P(x)}Q = id.$$

Then there are open neighborhoods $0 \in U \subseteq E$ and $0 \in V \subseteq F$ and a smooth tame map $R : V \to U$ such that $PR(y) = y$ whenever $Q(y) = y$.

In this section we give a general outline of the proof. Define a smooth tame map $\Gamma : E \times F \times G \to E \times F \times G$ by

$$\Gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - V_x P(P(x) - y) \\ y - V_x Q(Q(y)) \\ z - D_{P(x)} Q(P(x) - y) \end{pmatrix}. \tag{6.0.1}$$

In the remainder of this chapter we prove that there is a smooth tame projection $\pi : W \to E \times F \times G$, that is, a map such that $\pi^2 = \pi$, with the same fixed point set as Γ. Since

$$D_x P \circ V_x P(P(x) - y) + V_x Q \circ D_{P(x)} Q(P(x) - y) = P(x) - y$$

it is easy to see that the fixed point set of Γ is just the graph $Graph(P) \times G$. Writing $\pi = (\pi_1, \pi_2, \pi_3)$, we see that $P \circ \pi_1 = \pi_2$. Since $P(0) = 0$ we obtain open neighborhoods $0 \in U \subseteq E$ and $0 \in V \subseteq F$ and a smooth tame map $R : V \to U$ defined by

$$R(y) = \pi_1(0, y, 0).$$

Moreover, from the iterative definition of π we conclude that $\pi_2(x, y, z) = y$ whenever $Q(y) = 0$, hence $PR(y) = y$ whenever $Q(y) = 0$. This concludes the proof of the Nash-Moser theorem.

6.1 Near-projections

Let E be a Fréchet space and $U \subseteq E$ a convex open subset. As Hamilton, by a smooth tame projection we mean a smooth tame map $P : U \to U$ that satisfies $P^2 = P$. By Taylor's formula with integral remainder in $x \in U$ we have

$$P^2(x) = P(x) + D_x P(P(x) - x) + \Delta(x)(P(x) - x)^2,$$

where $\Delta : U \times E \times E \to E$ is the quadratic error

$$\Delta(x)(v, w) = \int_0^1 D^2 P(x + t(P(x) - x))(v, w) dt.$$

Hence the linear term in Taylor's formula is in fact quadratic in $P(x) - x$, that is,

$$D_x P(P(x) - x) + \Delta(x)(P(x) - x)^2 = 0.$$

This motivates the following definition.

Definition 6.1.1. *Suppose that $U \subseteq E$ is an open in a graded Fréchet space. A smooth tame map $G : U \to E$ is called a near projection if there exists a smooth tame map $\Lambda : U \times E \times E \to E$, bilinear in the last two coordinates, such that*

$$D_x G(G(x) - x) + \Lambda(x)(G(x) - x)^2 = 0. \qquad \triangle$$

Again by Taylor's formula with integral remainder this implies that

$$G^2(x) - G(x) = -\Lambda(x)(G(x) - x)^2 + \Delta(x)(G(x) - x)^2$$

whenever the composition is well-defined. For G to be a projection is equivalent to image of G being equal to its fixed point set $Fix(G) := \{x \in U : G(x) = x\}$. In the same line of reasoning, G being a near-projection means that the error of $G(x)$ lying in the fixed point set can be measured quadratically in the error of x lying the fixed point set.

Lemma 6.1.2. Γ *defined in 6.0.1 on the preceding page is a near-projection*

Proof. Writing

$$\begin{aligned}
\Delta x &= V_x P(P(x) - y), \\
\Delta y &= V_x Q(Q(y)), \\
\Delta z &= D_{P(x)} Q(P(x) - y),
\end{aligned}$$

note that $D_x P \Delta x + V_x Q \Delta z = P(x) - y$. Hence it suffices to show that $D_{(x,y,z)}\Gamma(\Delta x, \Delta y, \Delta z)$ is quadratic in Δx, Δy, Δz and $P(x) - y$. The remainder is a straightforward computation. □

Given such a near-projection G we would like to find a projection P with the same fixed point set. In the case of Banach spaces the proof is much easier. This proof will form a blueprint for the general case.

Theorem 6.1.3. *Let E be a Banach space, $U \subseteq E$ an open subset and $G : U \to E$ a near-projection. Then there is an open $Fix(G) \subseteq V \subseteq U$ and a smooth projection $P : V \to V$ with the same fixed point set as G.*

Proof. Fix an $x_b \in Fix(G)$ and let $\varepsilon > 0$ be small enough that $B_\varepsilon(x_b) \subset U$, $G(B_\varepsilon(x_b)) \subseteq U$, and there is a $C > 0$ so that

$$\left\|G^2(x) - G(x)\right\| = \left\|(\Delta - \Lambda)(x)(G(x) - x)^2\right\| \leq C \left\|G(x) - x\right\|^2$$

for all $x \in B_\varepsilon(x_b)$. Such estimates can be obtained because $\Delta - \Lambda$ from the discussion above is continuous and bilinear in the last two coordinates.

We aim to prove that the sequence $\{G^n(x)\}_{n\in\mathbb{N}}$ of repeated compositions is well-defined and converges uniformly for a small enough neighborhood of x_b. To this end, define

$$V_b := \{x \in U : \|x - x_b\| < \eta, \|G(x) - x\| < \eta\} \subset B_\varepsilon(x_b),$$

where $\eta < \frac{1}{3}\varepsilon$ and $\eta C = \theta < \frac{1}{2}$.

For all $x \in V_b$ we have $G(x) \in B_\varepsilon(x_b)$ and $\|G^2(x) - G(x)\| \leq C \|G(x) - x\|^2$. Now suppose that $G^k(x) \in B_\varepsilon(x_b)$ and

$$\left\|G^{k+1}(x) - G^k(x)\right\| \leq C \left\|G^k(x) - G^{k-1}(x)\right\|^2$$

for all $1 \leq k \leq n$ and $x \in V_b$. Then

$$\left\|G^{k+1}(x) - G^k(x)\right\| \leq C^{2^k-1} \left\|G(x) - x\right\|^{2^k} \leq \theta^{2^k-1}\eta.$$

Hence

$$\begin{aligned}\left\|G^{n+1}(x)-x_b\right\| &\leq \sum_{k=0}^{n}\left\|G^{k+1}(x)-G^k(x)\right\|+\|x-x_b\| \\ &\leq \sum_{k=0}^{n}\theta^{2^k-1}\eta+\eta \leq 3\eta<\varepsilon\end{aligned}$$

implies that $G^{n+1}(x) \in B_\varepsilon(x_b)$, so that $G^{n+2}(x)$ is defined, and $G^n(x) \in B_\varepsilon(x_b)$ gives

$$\left\|G^{n+2}(x)-G^{n+1}(x)\right\| \leq C\left\|G^{n+1}(x)-G^n(x)\right\|^2.$$

By induction $G^n(x) \in B_\varepsilon(x_b)$ and above estimate holds for all $x \in V_b$. Moreover, we have the estimate

$$\left\|G^{m+n}(x)-G^m(x)\right\| \leq \sum_{k=m}^{n-1}\left\|G^{k+1}(x)-G^k(x)\right\|<\sum_{k=m}^{n-1}\theta^{2^k-1}\eta \leq 2\theta^{2^m}\eta$$

for all $x \in V_b$ and $m, n \in \mathbb{N}$. Hence $\{G^n\}$ converges uniformly on V_b to a continuous map $P: V_b \to E$. Note that $P(x) \in Fix(G)$ and $Fix(G) \cap V_b \subset Fix(P)$ both hold trivially.

To prove that P is smooth note that $TG: U \times E \to E$ is also a near projection. For $x \in V_b$ fixed, $(P(x), 0)$ is a fixed point of TG. By the above, there are open subsets

$$V_\infty=\{y \in U:\|y-P(x)\|<\tilde{\eta},\|G(y)-y\|<\tilde{\eta}\} \quad \text{and}$$

$$W_\infty=\{(y, v) \in U \times E:\|y-P(x)\|<\tilde{\eta},\|v\|<\tilde{\eta},\|TG(y, v)-(y, v)\|<\tilde{\eta}\}$$

so that G^n converges to $P: V_\infty \to E$ and $T(G^n)=(TG)^n$ to a continuous map $Q: W_\infty \to E$ uniformly. Here we have taken the minimum over the two occurring values of $\tilde{\eta}$. $DG: V_\infty \times E \to E$ is linear in the second entry, hence so is $Q_2=pr_2 \circ Q$ whenever it is defined. For arbitrary $(y, v) \in V_\infty \times E$, $v \neq 0$, we have

$$\frac{\tilde{\eta}}{2\|v\|} DG(x) v \to Q(x) \frac{\tilde{\eta}}{2\|v\|} v,$$

so that Q extends linearly, and hence continuously, to $V_\infty \times E$. It is now easy to show that $P: V_\infty \to E$ is continuously differentiable and $TP=Q: V_\infty \to E$. Note that for this we had to shrink the domain of P.

Returning to our point $x \in V_b$, for $m \geq 1$ large enough we have $\|G^m(x)-P(x)\|<\tilde{\eta}$ and

$$\left\|G^{m+1}(x)-G^m(x)\right\|<\theta^{2^m-1}\eta<\tilde{\eta}.$$

Hence V_∞ is a neighborhood of $G^m(x)$ on which P is C^1 with $TP=Q$. In turn, $(G^m)^{-1}(V_\infty)$ is a neighborhood of x on which $P=P \circ G^m$ is C^1 with $TP=Q$. Since $x \in V_b$ is arbitrary, we conclude that $P: V_b \to E$ is smooth.

Running through the fixed points $x_b \in Fix(G)$ the maps $P: V_b \to E$ coincide on intersections, hence collate to a smooth map $P: V \to E$. It still satisfies $P(V) \subset Fix(G)$, $Fix(G) \cap V \subset Fix(P)$ and, additionally, $Fix(G) \subset V$. So it restricts to a smooth projection $P: V \to V$ with the same fixed point set as G. □

Theorem 6.1.4. *Let E be a tame Fréchet space, $U \subseteq E$ an open subset and $G : U \to E$ a smooth tame near-projection. Then there is an open $Fix(G) \subseteq \tilde{U} \subseteq U$ and a smooth tame projection $P : \tilde{U} \to \tilde{U}$ with the same fixed point set as G.*

The above proof obviously fails in the case of Fréchet spaces. Instead, we adjust the iteration by use of the smoothing operators $S_t : E \to E$. Let $t_0 \geq 3$ and $P_0 = id$ on E. Inductively we define

$$\begin{aligned} P_{n+1}x &= P_n x + S_{t_n}\left(G(P_n x) - P_n x\right), \\ t_{n+1} &= t_n^{3/2}. \end{aligned}$$

We hope to show that P_n converges to the desired smooth tame projection P.

6.2 Preliminary estimates

We begin by recalling some estimates in order to fix notation and relevant constants. Choose some $x_b \in Fix(G)$. Later we will see that it is sufficient to work in neighborhoods of every $x_b \in Fix(G)$.

For $\theta > 0$ sufficiently small there is a $b \geq 0$ such that for all $k \geq b$ there is a constant C_k, dependent only on k, such that for all $s \geq 0$,

$$x, y \in N := \left\{x \in U : \|x - x_b\|_{b+s} \leq 2\theta\right\},$$

and $v, w \in E$ the following tameness estimates hold.

$$\begin{aligned} \|G(x)\|_k &\leq C_k\left(1 + \|x\|_{k+s}\right), \\ \|DG(x)v\|_k &\leq C_k\left(\|v\|_{k+s} + \|x\|_{k+s}\|v\|_b\right), \\ \|\Delta(x)(v,w)\|_k &\leq C_k\left(\|v\|_{k+s}\|w\|_b + \|w\|_{k+s}\|v\|_b + \|x\|_{k+s}\|v\|_b\|w\|_b\right), \\ \|\Phi(x,y)(v,w)\|_k &\leq C_k\left(\|v\|_{k+s}\|w\|_b + \|w\|_{k+s}\|v\|_b + \left(\|x\|_{k+s} + \|y\|_{k+s}\right)\|v\|_b\|w\|_b\right), \end{aligned}$$

where $\Phi : U \times U \times E \times E \to E$ is the map defined by

$$\Phi(x,y)(v,w) = \int_0^1 D^2((1-t)x + ty)(v,w)dt.$$

In addition, the smoothing operators give constants $C_k > 0$ and estimates

$$\|S_t x\|_{k+s} \leq C_k t^{s+\delta}\|x\|_k$$

for all $k \geq b$ and $s \geq 0$ and

$$\|x - S_t x\|_k \leq C_k t^{-s+\delta}\|x\|_{k+s}$$

for all $k \geq b$ and $s \geq \delta$. Hence we are allowed to choose $s \geq \delta \geq 3$ and $k \geq b$ to obtain all the above estimates simultaneously. Moreover, by choosing $\theta > 0$ small enough and b large enough, and using the fact that all semi-norms are actually norms, we may ensure that N is a convex open.

Let us rewrite the iterative process as follows: Fix some $t_0 \geq 3$, and define $t_{n+1} = t_n^{3/2}$ and

$$P^0_{n+1}x = P^0_n x + \Delta_n x, \quad \Delta_n x = S_{t_n} Z_n x, \quad Z_n x = G(P^0_n x) - P^0_n x.$$

Subsequent steps of the iteration are well-defined if $P^0_n x$ lies in the domain of G. Some explanation of the notation is in place: The iterative process depends on the chosen value of t_0, whose dependence is indicated by the superscript in P^0_n. If we break up the iteration at some step $m \in \mathbb{N}$, we define the continued iteration P^m_n to be the $(n-m)^{th}$ iteration with starting t_m as starting t-value. This notation leads to the identity

$$P^0_n = P^m_n \circ P^0_m$$

whenever the composition is defined. The purpose of this practice lies in the observation that P^0_m is a smooth tame map on it's domain of definition; this will be useful in proving smoothness of the projection. We will make sure that all constants $C > 0$ from here on are independent of t_0 such that they hold equally well for P^m_n.

We may now write down a recursive formula for the difference $Z_n x$ using Taylor's formula with quadratic remainder.

$$\begin{aligned} Z_{n+1}x &= G(P^0_{n+1}x) - P^0_{n+1}x \\ &= Z_n x - \Delta_n x + DG(P^0_n x) Z_n x - DG(P^0_n x)(id - S_{t_n}) Z_n x + \Phi(P^0_n x, P^0_{n+1}x)(\Delta_n x)^2 \\ &= (id - DG(P^0_n x))(id - S_{t_n}) Z_n x - \Delta(P^0_n x)(Z_n x)^2 + \Phi(P^0_n x, P^0_{n+1}x)(\Delta_n x)^2 \end{aligned}$$

The idea of Hamilton's proof revolves around estimating the norm of the above equation to show that $Z_n x = G(P^0_n x) - P^0_n x$ tends to 0 fast enough. For this, we make some preliminary estimates. They involve parameters α and β which will be specified later on in the proof.

Lemma 6.2.1. *For all $k \geq b$, $\alpha \geq 0$ and $\beta \geq -1$ there is a $C_k > 0$ such that the following estimates hold.*

$$\begin{aligned} \|Z_{n+1}x\|_k \leq{}& C_k t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+2)s} (1 + \|Z_n x\|_b) + C_k t_n^{-\beta s} \|Z_n x\|_{k+\beta s} \\ &+ C_k t_n^{2s} \|Z_n x\|_b (t_n^s \|Z_n x\|_k + \|Z_n x\|_b (\|P^0_n x\|_{k+s} + \|P^0_{n+1}x\|_{k+s})) \end{aligned}$$

Proof. By the above we have

$$\begin{aligned} \|Z_{n+1}x\|_k \leq{}& \|(id - DG(P^0_n x))(id - S_{t_n}) Z_n x\|_k \\ &+ \|\Lambda(P^0_n x)(Z_n x)^2\|_k + \|\Phi(P^0_n x, P^0_{n+1}x)(\Delta_n x)^2\|_k \end{aligned}$$

The first term is estimated by

$$\begin{aligned} &\|(id - S_{t_n}) Z_n x\|_k + \|DG(P^0_n x)(id - S_{t_n}) Z_n x\|_k \\ &\leq C_k (t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+1)s} + \|(id - S_{t_n}) Z_n x\|_{k+s} + \|P^0_n x\|_{k+s} \|(id - S_{t_n}) Z_n x\|_b) \\ &\leq C_k (t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+1)s} + t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+2)s} + t_n^{-\beta s} \|P^0_n x\|_{k+s} \|Z_n x\|_{b+(1+\beta)s}) \\ &\leq C_k t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+2)s} + C_k t_n^{-\beta s} \|P^0_n x\|_{k+s} \|Z_n x\|_{b+\beta s}. \end{aligned}$$

Here the estimate for $\beta = -1$ doesn't follow from smoothing estimates for $id - S_{t_n}$, but but from S_{t_n} instead. Note that we can estimate

$$\begin{aligned}\|Z_n x\|_{k+s} &\leq \|S_{t_n} Z_n x\|_{k+s} + \|(id - S_{t_n}) Z_n x\|_{k+s} \\ &\leq C_k t_n^{2s} \|Z_n x\|_k + C_k t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+2)s},\end{aligned}$$

hence the Λ-term is bounded by

$$\begin{aligned}&C_k(\|Z_n x\|_{k+s} \|Z_n x\|_b + \left\|P_n^0 x\right\|_{k+s} \|Z_n x\|_b^2) \\ &\leq C_k t_n^{2s}(\|Z_n x\|_k + t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+2)s}) \|Z_n x\|_b + C_k \left\|P_n^0 x\right\|_{k+s} \|Z_n x\|_b^2\end{aligned}$$

Lastly, the Φ-term is estimated by

$$\begin{aligned}&C_k(\|S_{t_n} Z_n x\|_{k+s} \|S_{t_n} Z_n x\| b + (\left\|P_n^0 x\right\|_{k+s} + \left\|P_{n+1}^0 x\right\|_{k+s}) \|S_{t_n} Z_n x\|_b^2) \\ &\leq C_k t_n^{3s} \|Z_n x\|_k \|Z_n x\|_b + C_k t_n^{2s}(\left\|P_n^0 x\right\|_{k+s} + \left\|P_{n+1}^0 x\right\|_{k+s}) \|Z_n x\|_b^2,\end{aligned}$$

completing our estimates. □

Next we state a simple lemma allowing us to estimate $G(x) - x_b$ in terms of $x - x_b$ for $x \in N$, useful in ensuring that the iteration remains within certain bounds.

Lemma 6.2.2. $\|G(x) - x_b\|_k \leq C_k \|x - x_b\|_{k+s}$ *for all* $x \in N$*.*

Proof. Again by Taylor's formula we obtain

$$G(x) - x_b = G(x) - G(x_b) = \int_0^1 DG(tx + (1-t)x_b)(x - x_b)dt,$$

hence we can estimate

$$\begin{aligned}\|G(x) - x_b\|_k &\leq \int_0^1 \|DG(tx + (1-t)x_b)(x - x_b)\|_k \, dt \\ &\leq C_l(\|x - x_b\|_{k+s} + \|x - x + 0\|_b \int_0^1 \|tx + (1-t)x_b\|_{k+s} \, dt)\end{aligned}$$

and

$$\|tx + (1-t)x_b\|_{k+s} \leq \|x - x_b\|_{k+s} + \|x_b\|_{k+s} \leq \|x - x_b\|_{k+s} + C_k$$

together with the rather coarse estimates

$$\begin{aligned}\|x - x_b\|_b (\|x - x_b\|_{k+s} + C_k) &\leq (2\theta + C_k) \|x - x_b\|_{k+s} \\ &\leq C_k \|x - x_b\|_{k+s}\end{aligned}$$

completes the proof. □

6.3 The low-norm estimates

Lemma 6.3.1. *For all $k \geq b$ and there is a $C_k \geq 0$ such that for all $x \in N$*

$$\left\|P_n^0 x - x_b\right\|_{k+s} \leq C_k t_n^{5s} \left\|x - x_b\right\|_{k+s}$$

whenever x_n is defined. Moreover, we have the estimate

$$\left\|Z_n x\right\|_k \leq C_k t_n^{5s} \left\|x - x_b\right\|_{k+s}.$$

Proof. The case $n = 0$ is trivial. Now suppose that $P_m^0 x$ is defined for all $m \leq n$ and there are $A_{m,k}$ so that

$$\left\|P_n^0 x - x_b\right\|_{k+s} \leq A_{m,k} t_n^{5s} \left\|x - x_b\right\|_{k+s}.$$

Suppose in addition that $P_n^0 x \in N$, so that $P_{n+1}^0 x$ is defined, then

$$\begin{aligned}
\left\|P_{n+1}^0 x - x_b\right\|_{k+s} &\leq \left\|P_n^0 x - x_b\right\|_{k+s} + \left\|S_{t_n}(Z_n x)\right\|_{k+s} \\
\left\|S_{t_n}(Z_n x)\right\|_{k+s} &\leq C_k t^{s+\delta} \left\|Z_n x\right\|_k \\
&\leq C_k t_n^{2s} \left\|Z_n x\right\|_k \\
\left\|Z_n x\right\|_k &\leq \left\|G(P_n^0 x) - G(x_b)\right\|_k + \left\|P_n^0 x - x_b\right\|_k \\
&\leq C_k \left\|P_n^0 x - x_b\right\|_{k+s}
\end{aligned}$$

Thus we have $\left\|P_{n+1}^0 x - x_b\right\|_{k+s} \leq C_k A_{n,k} t_n^{7s} \left\|x - x_b\right\|_{k+s}$. Hence we find the estimate

$$\left\|P_{n+1}^0 x - x_b\right\|_{k+s} \leq A_{n+1,k} t_{n+1}^{5s} \left\|x - x_b\right\|_{k+s}$$

if we take

$$C_k A_{n,k} t_n^{7s} \leq A_{n+1,k} t_{n+1}^{5s}.$$

Since $t_{n+1} = t_n^{3/2}$ and $s \geq 2$ we have $t_n^{7s} t_{n+1}^{5s} = t_n^{-s/2} \leq t_n^{-1}$. $t_n^{-1} \to 0$ as $n \to \infty$ implies that we can choose the $A_{n,k}$ such that $A_{n,k} \to 0$, hence they are bounded from above by some larger $C_k > 0$.

Finally, note that

$$\left\|Z_n x\right\|_k \leq C_k \left\|P_n^0 x - x_b\right\|_{k+s} \leq C_k t_n^{5s} \left\|x - x_b\right\|_{k+s}.$$

□

Lemma 6.3.2. *We can choose $\varepsilon > 0$ and $\eta > 0$ sufficiently small such that for all $t_0 \geq 3$ and*

$$x \in V_b^0 := \left\{x \in N : \left\|x - x_b\right\|_{b+26s} < \eta, \qquad \left\|G(x) - x_b\right\|_b < \varepsilon t_0^{-12s}\right\}$$

we have $P_n^0 x$ defined for all $n \in \mathbb{N}$, $P_n^0 x \in N$ and the estimates

$$\left\|Z_n x\right\|_b \leq \varepsilon t_n^{-12s}, \qquad \left\|\Delta_n x\right\|_{b+s} \leq \theta t_n^{-10s}.$$

Proof. Suppose as induction hypothesis that, for all $m \leq n$, the iterative $P_m^0 x$ is well-defined and

$$\|Z_m x\|_b \leq \varepsilon t_m^{-12s}.$$

Then

$$\|\Delta_m x\|_{b+s} = \|S_{t_m} Z_m x\|_{b+s} \leq C t_m^{s+\delta} \|Z_m x\|_b \leq C\varepsilon t_m^{-10s},$$

hence

$$\|\Delta_m x\|_{b+s} \leq \theta t_m^{-10s}$$

if we take $C\varepsilon \leq \theta$, which is possible since $C > 0$ doesn't depend on n. Now choose $\eta \leq \theta$ sufficiently small such that

$$\|x - x_b\|_{b+s} \leq \|x - x_b\|_{b+26s} \leq \eta \leq \theta,$$

then

$$\begin{aligned} \left\|P_{n+1}^0 x - x_b\right\|_{b+s} &\leq \|x - x_b\|_{b+s} + \sum_{m=0}^{n} \|S_{t_m}(Z_m x)\|_{b+s} \\ &\leq \theta + \sum_{m=0}^{n} \theta t_j^{-10s} \leq 2\theta. \end{aligned}$$

Hence $P_{n+1}^0 x \in N$ and $P_{n+2} x$ is also well-defined. What remains is to prove that $P_{n+1}^0 x$ also satisfies the required estimates.

Apply the big estimate in lemma 6.2.1 with $k = b$ and $\alpha = \beta = 23$ to obtain

$$\begin{aligned} \|Z_{n+1} x\|_b \leq\ & C t_n^{-23s} \|Z_n x\|_{b+25s} (1 + \|Z_n x\|_b) + C t_n^{-23s} \|Z_n x\|_{b+24s} \\ & + C t_n^{3s} (1 + \left\|P_n^0 x\right\|_{b+s} + \left\|P_{n+1}^0 x\right\|_{b+s}) \|Z_n x\|_b^2 \end{aligned}$$

Now by the induction assumption we have $1 + \|Z_n x\|_b \leq C$ and

$$1 + \left\|P_n^0 x\right\|_{b+s} + \left\|P_{n+1}^0 x\right\|_{b+s} \leq 1 + 2\theta + \|x_b\|_{b+s} \leq C,$$

so we obtain

$$\begin{aligned} \|Z_{n+1} x\|_b &\leq C t_n^{-23s} \|Z_n x\|_{b+25s} + C t_n^{3s} \|Z_n x\|_b^2 \\ &\leq C t_n^{-23s} t_n^{5s} \|x - x_b\|_{b+26s} + C t_n^3 s \varepsilon^2 t_n^{-24s} \\ &\leq (C\eta + C\varepsilon^2) t_n^{-18s} \leq \varepsilon t_n^{-18s} \end{aligned}$$

by lemma 6.2.2 and the induction hypothesis. Here we have taken $C\varepsilon \leq \frac{1}{2}$ and $C\eta \leq \varepsilon/2$, which is possible since C doesn't depend on n. Finally, recall that $t_{n+1} = t_n^{3/2}$, hence $t_n^{-18s} = t_{n+1}^{-12s}$. $\square$

6.4 The high-norm estimates

Lemma 6.4.1. *With the same hypothesis as in lemma 6.3.2 we obtain for all $k \geq b$*

$$\begin{aligned} \left\|P_n^0 x\right\|_{k+s} &\leq C_k(1 + \|x\|_{k+19s}) \\ \|\Delta_n x\|_{k+s} &\leq C_k t_n^{-5s}(1 + \|x\|_{k+19s}) \\ \|Z_n x\|_k &\leq C_k t_n^{-7s}(1 + \|x\|_{k+19s}) \end{aligned}$$

Proof. Suppose as induction hypothesis that for all $0 \leq m \leq n$ we have estimates

$$\|Z_m x\|_k \leq A_m t_m^{-7s}(1 + \|x\|_{k+19s}),$$

with $1 \leq A_0 \leq A_1 \leq \ldots \leq A_n$ dependent on m. Then by the smoothing estimates we have

$$\|\Delta_m x\|_{k+s} = \|S_{t_m} Z_m x\|_{k+s} \leq C_k t_m^{2s} \|Z_m x\|_k \leq C_k A_m t_m^{-5s}(1 + \|x\|_{k+19s}),$$

which also shows that the second result follows directly from the third. Summing over all $m \leq n$ gives

$$\begin{aligned} \sum_{m=0}^{n} \|\Delta_m x\|_{k+s} &\leq \sum_{m=0}^{\infty} t_m^{-5s} C_k A_n (1 + \|x\|_{k+19s}) \\ &\leq C_k A_n (1 + \|x\|_{k+19s}), \end{aligned}$$

since the A_m are nondecreasing and $t_0 \geq 3$. The obvious estimates

$$\begin{aligned} \left\|P_n^0 x\right\|_{k+s} + \left\|P_{n+1}^0 x\right\|_{k+s} &\leq \sum_{m=0}^{n} -1 \|\Delta_m x\|_{k+s} + \|x\|_{k+s} + \sum_{m=0}^{n} \|\Delta_m x\|_{k+s} + \|x\|_{k+s} \\ &\leq C_k A_n (1 + \|x\|_{k+19s}) \end{aligned}$$

show that also the first result follows from the third. Apply the big estimate in lemma 6.2.1 for $\alpha = 16$ and $\beta = -1$ to obtain

$$\begin{aligned} \|Z_{n+1} x\|_k \leq\ & C_k t_n^{-16s} \|Z_n x\|_{k+18s} (1 + \|Z_n x\|_b) + C_k t_n^s \|Z_n x\|_b \\ & + C_k t_n^{2s} \|Z_n x\|_b (t_n^s \|Z_n x\|_k + \|Z_n x\|_b (\left\|P_n^0 x\right\|_{k+s} + \left\|P_{n+1}^0 x\right\|_{k+s})) \end{aligned}$$

Recall the low-norm estimates from lemma 6.3.2,

$$\|Z_n x\|_b \leq C t_n^{-12s},$$

and from lemma 6.3.1,

$$\|Z_n x\|_{k+18s} \leq C_k t_n^{5s} \|x - x_b\|_{k+19s} \leq C_k t_n^{5s}(1 + \|x\|_{k+19s})$$

and the induction hypothesis

$$\|Z_n x\|_k \leq A_n t_n^{-7s}(1 + \|x\|_{k+19s}).$$

We obtain

$$\begin{aligned}\|Z_{n+1}x\|_k &\leq C_k t_n^{(-16+5)s}(1+\|x\|_{k+19s}) + Ct_n^{-11s} \\ &\quad + C_k t_n^{-10s}(C_k A_n t_n^{(1-7)s}(1+\|x\|_{k+19s}) + C_k A_n t_n^{-12s}(1+\|x\|_{k+19s}) \\ &\leq C_k A_n t_n^{-11s}(1+\|x\|_{k+19s}),\end{aligned}$$

so we may estimate

$$\|Z_{n+1}x\|_k \leq A_{n+1} t_{n+1}^{-7s}(1+\|x\|_{k+19s})$$

if we take $C_k t_n^{-11s} A_n \leq t_n^{-\frac{21}{2}s} A_{n+1}$, or equivalently $A_{n+1} \geq C_k A_n t_n^{-s/2}$. As soon as $C_k t_n^{-s/2} \leq 1$, we may take $A_{n+1} = A_n$, hence the sequence A_n can be chosen bounded. Note that the A_n may also be chosen independent of t_0 since $t_0 \geq 3$ allows us to bound $t_n^{-s/2} \leq 3^{-(\frac{3}{2})^n s/2}$. This gives the desired third result and, as mentioned before, the other results follow. □

Hence on the set

$$V_b^0 = \left\{x \in U : \|x - x_b\|_{b+26s} < \eta, \|G(x) - x\|_b < \varepsilon t_0^{-12s}\right\}$$

the maps $P_n^0 : V_b^0 \to E$ are all well-defined smooth tame maps. The estimate

$$\|\Delta_n x\|_{k+s} \leq C_k t_n^{-5s}(1+\|x\|_{k+19s})$$

for all $k \geq b$ from the previous lemma gives

$$\begin{aligned}\|P_{m+n}x - P_m x\|_{k+s} &\leq \sum_{l=m}^{n} \|\Delta_l x\|_{k+s} \leq C_k t_m^{-s} \sum_{l=m}^{n} t_l^{-4s}(1+\|x\|_{k+19s}) \\ &\leq C_k t_m^{-s}(1+\|x\|_{k+19s}).\end{aligned}$$

The sequence $P_n^0 x$ is Cauchy for all $x \in V_b^0$ hence converges to some $P_\infty^0 x \in E$; this defines a map

$$P_\infty^0 : V_b^0 \to E.$$

Moreover, the function $1 + \|-\|_{k+19s}$ is bounded on every compact subset $K \subset V_b^0$. We obtain an estimate of the form

$$\|P_{m+n}x - P_m x\|_{k+s} \leq C_k^K t_m^{-s}.$$

Hence the sequence of maps

$$P_n^0 : K \to (E, \|-\|_{k+s}),$$

where $(E, \|-\|_{k+s})$ is has the topology given by the norm $\|-\|_{k+s}$ and the topology of K remains unchanged, converges uniformly to a continuous map

$$P_\infty^0 : K \to (E, \|-\|_{k+s}).$$

In particular, given a sequence $\{x^n\}_{n\in\mathbb{N}}$ with $x^n \to x \in V_b^0$, the set $K = \{x^n\} \cup \{x\} \subset V_b^0$ is compact, so $P_\infty^0 x^n \to P_\infty^0 x$. Hence the map

$$P_\infty^0 : V_b^0 \to (E, \|-\|_{k+s})$$

is continuous for every $k \geq b$. The open balls $B_{k+s}(x,r) = \left\{y \in V_b^0 : \|y-x\|_{k+s} < r\right\}$ form a basis of topology for E. We conclude that the map

$$P_\infty^0 : V_b^0 \to E$$

is continuous. Moreover, the estimate $\|P_n x\|_{k+s} \leq C_k(1 + \|x\|_{k+19s})$ proves that P_∞^0 is tame.

We also obtain some other properties of P_∞^0. The estimate

$$\|Z_n x\|_k \leq C_k t_n^{-7s}(1 + \|x\|_{k+19s})$$

shows that $G(P_\infty^0 x) = P_\infty^0 x$, so that

$$P_\infty^0(V_b^0) \subset Fix(G) \subset U.$$

Moreover, for $x \in V_b^0 \cap Fix(G)$ we have $P_n x = x$, so that

$$Fix(G) \cap V_b^0 \subset Fix(P_\infty^0).$$

It is only later that we conclude from this that P_∞^0 is a projection. First we prove the smooth tameness of P_∞^0.

6.5 Smooth tameness of P_∞^0

Lemma 6.5.1. *If $G : U \to E$ is a near-projection then so is its tangent map*

$$TG : U \times E \to E \times E.$$

Proof. For $(x,u) \in U \times E$ and $v, w \in E \times E$ define

$$\Psi(x,u)(v,w) = \begin{pmatrix} \Lambda(x)(v_1,w_1) \\ D\Lambda(x)(v_1,w_1)u + \Lambda(x)(v_1,w_2) + \Lambda(x)(v_2,w_1) \end{pmatrix},$$

where $D\Lambda(x)$ is the partial derivative of Λ to the first coordinate. It is clearly a smooth tame map and bilinear in $(v,w) \in E^2 \times E^2$. Since we have

$$DTG(x,u)(v_1,w_1) = \begin{pmatrix} DG(x)v_1 \\ D^2G(x)(v_1,u) + DG(x)w_1 \end{pmatrix}$$

for $(x,u) \in U \times E$ and $v_1, w_1 \in E$ we only have to check the second component. Now

$$DG(x)(G(x) - x) = -\Lambda(x)(G(x)-x)^2$$

implies, by linearity and bilinearity, that

$$\begin{aligned} D^2G(x)(G(x)-x,u) + DG(x)(DG(x)u - u) \\ = &- D\Lambda(x)(G(x)-x)^2 u \\ &- \Lambda(x)(G(x)-x, DG(x)u-u) \\ &- \Lambda(x)(DG(x)u-u, G(x)-x). \end{aligned}$$

Hence TG is a near-projection. $\square$

Our aim is to apply the previous results to TG and show that the resulting iteration converges to the tangent map of P^0_∞. For this we need smoothing operators $S_t : E \times E \to E \times E$, which we define by

$$S_t(x, v) = (S_t x, S_t v)$$

for all $(x, v) \in E \times E$. If $x_b \in Fix(G)$ then $(x_b, 0) \in U \times E$ is trivially a fixed point of TG. Hence we obtain constants $\tilde{b}, \tilde{s} \in \mathbb{N}$ and $\tilde{\eta}, \tilde{\varepsilon} > 0$ and an open neighborhood W^0_b of $(x_b, 0)$ defined by

$$W^0_b = \left\{(x, v) \in U \times E : \|x - x_b\|_{\tilde{b}+26\tilde{s}} < \tilde{\eta}, \|v\|_{\tilde{b}+26\tilde{s}}, \|TG(x, v) - (x, v)\|_{\tilde{b}} < \tilde{\varepsilon} t_0^{-12\tilde{s}}\right\}$$

on which we have a sequence of smooth tame maps

$$\tilde{P}^0_{n+1} = \tilde{P}^0_n + S_{t_n}(TG \circ \tilde{P}^0_n - \tilde{P}^0_n)$$

that converge uniformly on compact subsets to a continuous and tame map $\tilde{P}^0_\infty$.

We may prove that $\tilde{P}^0_n = TP^0_n$ for all $n \in \mathbb{N}$ by a simple induction argument. It trivially holds for $n = 0$, so assume it hold for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\tilde{P}^0_{n+1}(x, v) &= \tilde{P}^0_n(x, v) + S_{t_n}(TG(\tilde{P}^0_n(x, v)) - \tilde{P}^0_n(x, v)) \\
&= TP^0_n(x, v) + S_{t_n}(T(G \circ P^0_n)(x, v) - TP^0_n(x, v)) \\
&= (P^0_n x + S_{t_n}(G(P^0_n x) - P^0_n x), DP^0_n(x)v + S_{t_n}(D(G \circ P^0_n)(x)v - DP^0_n(x)v)) \\
&= (P^0_{n+1}(x), D(P^0_n + S_{t_n}(G \circ P^0_n - P^0_n))(x)v) = TP^0_{n+1}(x, v)
\end{aligned}$$

for all $(x, v) \in W^0_b$. In particular, $DP^0_n(x)v$ is linear in v and converges uniformly on compact subsets of W^0_b to the second component R^0_∞ of $\tilde{P}^0_\infty = (Q^0_\infty, R^0_\infty)$. The following lemma is a first step towards the smoothness of P^0_∞.

Lemma 6.5.2. *P^0_∞ is C^1 on $\tilde{V}^0_b$ and $DP^0_\infty = R^0_\infty$, where $\tilde{V}^0_b = pr_1(V^0_b \times \{0\} \cap W^0_b)$.*

Proof. Fix a point $y \in \tilde{V}^0_b$. Let $B^l_r(y) = \{x \in E : \|x - y\|_l\}$ denote the open ball of radius r around y induced by the l-norm. Then for large enough $l \geq b$ and small enough radius $r > 0$ the closure of the open square $B = B^l_r(y) \times B^l_r(0)$ lies completely in $W^0_b \cap (V^0_b \times E)$. By Taylor approximation we have

$$P^0_n(x + v) - P^0_n(x) = \int_0^1 DP^0_n(x + tv)vdt$$

for all $(x, v) \in B$. Since DP^0_n converges uniformly on the compact set $\{(x + thv, hv) : 0 \leq t \leq 1\}$, we have for all $(x, v) \in B^l_r(y) \times E$ and $h > 0$ with $hv \in B^l_r(0)$ that

$$P^0_\infty(x + hv) - P^0_\infty(x) = \int_0^1 R^0_\infty(x + thv)hvdt.$$

Note that DP^0_n converges point-wise on $B^l_r(y) \times E$. For suppose $(x, v) \in B^l_r(x) \times E$, with $v \neq 0$, then $\frac{r}{2\|v\|_l}v \in B^l_r(0)$, hence

$$\frac{r}{2\left\|v\right\|_l} DP^0_n(x)v \to Q^0_\infty(x) \frac{r}{2\left\|v\right\|_l} v$$

implies that

$$DP_n^0(x)v \to \frac{2\left\|v\right\|_l}{r} Q_\infty^0(x) \frac{r}{2\left\|v\right\|_l} v.$$

Hence we can extend the map Q_∞^0 to $B_r^l(x_b) \times E$ and it remains linear in the second entry. In particular we have $Q_\infty^0(x+thv)hv = hQ_\infty^0(x+thv)v$.

Now by continuity of Q_∞^0 on B we obtain

$$\frac{1}{h}(P_\infty^0(x+hv) - P_\infty^0(x)) \to R_\infty^0(x)v$$

as $h \to 0$. It is well-known that a linear map between Fréchet spaces is continuous if it is so in a neighborhood of the origin, hence P_∞^0 is C^1 on $B_r^l(y)$. The point $y \in \tilde{V}_b^0$ was chosen arbitrary, so the proof is complete. $\square$

Moreover, TP_∞^0 is tame on $W_b^0 \cap V_b^0 \times E$, as we have seen in the high norm estimates. Since DP_∞^0 is linear in the second entry, it directly follows that TP_∞^0 is continuous and tame on all of $\tilde{V}_b^0 \times E$. These arguments hold analogously for higher order derivatives T^kG.

The only remaining obstruction for the tame smoothness of P_∞^0 occurs if $\tilde{V}_b^0$ can become arbitrarily small as we take higher order derivatives. Hence we need to show that P_∞^0 is C^1 on all of V_b^0. For this, recall the definition of the maps P_n^m; they are just the $(n-m)^{th}$ iterative step with initial t-value equal to t_m, such that $P_n^0 = P_n^m \circ P_m^0$ wherever defined. Since we ensured that all our estimates do not depend on t_0, the above arguments hold equally well for P_∞^m.

Lemma 6.5.3. *$TP_\infty^0(x)v$ is well-defined for all $(x,v) \in V_b^0 \times E$. Moreover, it is continuous and tame.*

Proof. Fix a $t_0 \geq 3$ and some $x \in V_b^0$. By the same argument as above it is sufficient to check that TP_∞^0 is continuous and tame in some neighborhood of $(x,0)$.

Now note that

$$x_\infty := P_\infty^0(x)$$

is a fixed point of G. Hence there are constants $b_\infty, s_\infty \in \mathbb{N}$ and $\varepsilon_\infty, \eta_\infty$, and for every $m \in \mathbb{N}$ neighborhoods

$$V_\infty^m = \left\{ y \in U : \left\|y - x_\infty\right\|_{b_\infty + 26s_\infty} < \eta_\infty, \left\|G(y) - y\right\|_{b_\infty} < \varepsilon t_m^{-12s_\infty} \right\}$$

and W_∞^m of points $(y,v) \in U \times E$ with

$$\begin{aligned} \left\|y - x_\infty\right\|_{b_\infty + 26s_\infty} &\leq \eta_\infty, \\ \left\|v\right\|_{b_\infty + 26s_\infty} &\leq \eta_\infty, \\ \left\|TG(y,v) - (y,v)\right\|_{b_\infty} &\leq \varepsilon_\infty t_m^{-12s_\infty} \end{aligned}$$

as in the arguments so far. Originally the constants involved in V_∞^m may differ from those in W_∞^m but we are free to take the respective maxima and minima where necessary.

Then $TP_n^m(y)v$ is well-defined for all $n \geq m$ and converges uniformly to TP_∞^m on compact subsets of $\tilde{V}_\infty^m \times E$. We continue by showing that

$$(P_m^0 x, 0) \in \tilde{V}_\infty^m \times E$$

for m large enough.

For suppose this holds. Recall that $P_n^0 = P_n^m \circ P_m^0$ whenever the composition is defined. Hence $P_\infty^0 = P_\infty^m \circ P_m^0$ and by the chain rule

$$TP_\infty^0 = TP_\infty^m \circ TP_m^0.$$

TP_∞^m is tame and continuous on the neighborhood $\tilde{V}_\infty^m \times E$ of $TP_m^0(x, 0) = (P_m^0 x, 0)$ and so is TP_m^0 on its own domain of definition. Hence TP_∞^0 is tame and continuous on the open neighborhood $(TP_m^0)^{-1}(\tilde{V}_\infty^m \times E)$ of $(x, 0)$.

The $b_\infty, s_\infty, \varepsilon_\infty$ and η_∞ depend on TG and x_∞, but not on t_m. Since $P_m^0 x \to x_\infty$ as $m \to \infty$ we obtain two of the required inequalities. The second inequality for W_∞^0 is trivial for $v = 0$. For the last two inequalities, note that

$$TG(P_m^0 x, 0) = (G(P_m^0 x), 0),$$

hence it suffices to show that

$$\left\|G(P_m^0 x) - P_m^0 x\right\|_{b_\infty} \leq \varepsilon_\infty t_m^{-12 s_\infty}.$$

We claim that for all $c \in \mathbb{N}$ there is a constant $C_k(x) > 0$, possibly depending on k, c and x, such that

$$\left\|G(P_m^0 x) - P_m^0 x\right\|_k \leq C_k(x) t_m^{-c}$$

for all $m \in \mathbb{N}$, provided that $k \geq b$. We still have the freedom to choose b_∞ as high as we like, hence this requirement is easily met. From this it is clear that $(P_m^0 x, 0) \in \tilde{V}_\infty^m \times E$ for m large enough. □

The following addresses the claim made in the lemma above.

Lemma 6.5.4. *For all $x \in V_b^0$, $k \geq b$ and $c \in \mathbb{N}$ there is a constant $C(x) = C_{k,c}(x)$ such that for all $n \in \mathbb{N}$,*

$$\|Z_n x\|_k \leq C(x) t_n^{-c}.$$

Proof. We will proof the lemma by induction on c. By lemma 6.4.1 we have

$$\|Z_n x\|_k \leq C_k t_n^{-7s}(1 + \|x\|_{k+19s} \leq C(x) t_n^{-7s}.$$

Assume the required estimate holds for some c and all n. We are still free to choose s, and from now on we will assume $s \geq 3$. By the big estimate in lemma 6.2.1 we have

$$\begin{aligned}
\|Z_n x\|_k &\leq C_k t_n^{-\alpha s} \|Z_n x\|_{k+(\alpha+2)s} (1 + \|Z_n x\|_b) + C_k t_n^{-\beta s} \|Z_n x\|_{k+\beta s} \\
&\quad + C_k t_n^{2s} \|Z_n x\|_b \left(t_n^s \|Z_n x\|_k + \|Z_n x\|_b \left(\left\|P_n^0 x\right\|_{k+s} + \left\|P_{n+1}^0 x\right\|_{k+s}\right)\right) \\
&\leq C_{k,s,c,\alpha}(x) t_n^{-\alpha s - c}(1 + C_{k,s,c}(x) t_n^{-c}) + C_{k,s,c,\beta}(x) t_n^{-\beta s - c} \\
&\quad + C_{k,c}(x) t_n^{2s-c}(C_{k,c}(x) t_n^{s-c} + C_s(x) t_n^{-c}(\left\|P_n^0 x\right\|_{k+s} + \left\|P_{n+1}^0 x\right\|_{k+s})).
\end{aligned}$$

Recall that $\|P_n^0 x\|_{k+s} \leq C_k(1 + \|x\|_{k+19s}) \leq C_k(x)$, hence it is irrelevant for these estimates. Now take $\alpha = \beta \geq \frac{1}{2} + \frac{1}{2}c$ so that we obtain

$$\begin{aligned}
\|Z_n x\|_k &\leq C_{k,c,\alpha,\beta}(x) t_n^{-\frac{1}{2}s - \frac{3}{2}c} + C_{k,s,c}(x) t_n^{3s-2c} \\
&\leq C_{k,c,\alpha,\beta}(x) t_{n+1}^{-c-\frac{1}{3}s} + C_{k,s,c}(x) t_n^{3s-2c}.
\end{aligned}$$

For the second summand we have

$$C_{k,s,c}(x) t_n^{3s-2c} = C_{k,s,c} t_{n+1}^{2s - \frac{4}{3}c} \leq C_{k,s,c}(x) t_{n+1}^{-c-\frac{1}{3}s}$$

whenever $c \geq 7s$, and we've already seen that the estimates hold for $c = 7s$. Hence the estimate

$$\|Z_n x\|_k \leq C(x) t_n^{-c-\frac{1}{3}s}$$

holds for all $n \geq 1$. For $n = 0$ the estimate is trivial, since

$$\|Z_0 x\|_k = \|G(x) - x\|_k \leq C_k(x)$$

implies that we can choose the constant $C(x)$ large enough to account for $n = 0$. By assumption we have $s \geq 3$, hence this proves it also holds for $c + 1$ and completes the proof. Note that this argument is not based on induction on n, hence all estimates remain independent of n. $\square$

Around every fixed point $x_b \in Fix(G)$ we have found a neighborhood V_b^0 and a smooth tame map $P_\infty^0 : V_b^0 \to E$ such that $P_\infty^0(V_b^0) \subset Fix(G)$ and $Fix(G) \cap V_b^0 \subset Fix(P_\infty^0)$. By their definition these maps coincide on intersecting domains, hence they collate to a smooth tame map $P : V \to E$ with $P(V) \subset Fix(G)$ and $Fix(G) = Fix(G) \cap V \subset Fix(P)$. Clearly, this map defines a projection $P : V \to V$ with the same fixed point set as G.

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