CLASSIFYING SPACES

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ABSTRACT. Introduction to principal G-bundles and classifying spaces, constructions (Milnor's, simplicial techniques) of classifying spaces, short introduction to Quantum geometry.

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Preface

The notion of a fibre bundle is less than eighty years old, and by the year 1950, the definition of a fibre bundle had been clearly formulated, the homotopy classification of fibre bundles was achieved and the theory of characteristic classes of fibre bundles had been developed by several mathematicians. In 1955, Milnor gave a construction of a universal fibre bundle for any topological group.

The first goal of this thesis will be to give an introduction to bundles and classifying spaces, and to give some constructions of classifying spaces. Because of my background in physics, I have added an application of the Hopf bundle in quantum mechanics and I have added a section about quantum geometry. Although the application is dealing with at least third year bachelor level physics, I hope it to be understandable for mathematicians as well. This thesis should be understandable for final year bachelor students or beginning master students in mathematics, who followed an introduction course in topology.

I do not pretend to state anything original or new in this thesis, and it must be said that most of the propositions and proofs are already in the used literature. But although this is true, this bachelor thesis is not just a copy of literature already available. I made this introduction to the subject in my own way and brought it in the order I thought appropriate. I have tried to explain parts where most of the literature steps over and with that to make it more readable for final year bachelor students. I also tried to add many examples and explanation, which can make the thesis more fun to read.

I should also note that English is not my native language and that I am not yet used to writing papers in English. Because of that the number of grammatical errors could be a little high, and I hope I will be forgiven for that. I do think it is a good exercise for me to write in English.

I think this thesis deals with a beautiful part of mathematics, that is, a very abstract part in which many other parts of mathematics come together. Although this part of mathematics can sometimes be hard to understand, because it is so abstract, it does not have to be hard when explained well. I hope you will enjoy reading this.

First, I will define and explain Principal G-bundles, in the second section I will give an example of a bundle, the Hopf bundle, and an application of that bundle in quantum entanglement. In the third section I will explain and define classifying spaces. The fifth and sixth section provide two ways of constructing classifying spaces, using Milnor's construction and using simplicial techniques. In the final section, I will give a short and less formal introduction to quantum geometry.

1. Principal G-Bundles

The aim of this section is to introduce the notion of bundle and principal G-bundle. This entire thesis is centered around the notion of principal G-bundle, since a classifying space is a space that classifies (numerable) principal G-bundles in some way. After giving the definition of a principal G-bundle the definition will be illustrated with some examples and numberable principal G-bundles will be defined. The next chapter gives the Hopf bundle as example of a principal G-bundle, and gives an application of the Hopf bundle in Quantum Entanglement.

1.1. **Bundles.** We will start with the definition of a bundle. One may think of a bundle as a Base space B together with a fibre $F(b)$ at each point $b \in B$, such that the fibres are glued together continuously into a bigger space called the total space E.

Definition 1.1. A bundle ξ is the triple $\xi = (E, p, B)$ with E and B topological spaces and p an open (in the sense that it sends opens to opens), continuous and surjective map $p : E \to B$. E is called the **total space**, B is called the **base space** and p is called the **projection** from E into B. For each $b \in B$, $p^{-1}(b)$ is called the **fibre over** b.
We say that ξ is a **bundle with fibre** F, when for all $b \in B$, the fibre $p^{-1}(b)$ over b is homeomorphic to F.

Example 1.2. Take as base space the circle, $B = S^1$, and as total space E the cylinder $C = \{(h, \phi) | h \in \mathbb{R}, \phi \in S^1)\} \simeq \mathbb{R} \times S^1$. Define the projection $p : E \to B : (h, \phi) \to \phi$. Then $\xi = (E, p, B) = (\mathbb{R} \times S^1, (h, \phi) \to \phi, S^1)$ is a bundle with as fibre F the real line $\mathbb{R}$, that is, for each $b \in B$, $p^{-1}(b)$ is homeomorphic to the real line $\mathbb{R}$.

Example 1.3. The Trivial bundle is the product bundle $\xi = (E = B \times F, p, B)$ with fibre F where p is the trivial projection $p : B \times F \to B : (b, f) \to b$.

Definition 1.4. A **morphism between bundles** $(u, f) : (E_1, p_1, B_1) \to (E_2, p_2, B_2)$ is a pair of maps $u : E_1 \to E_2$ and $f : B_1 \to B_2$ such that $p_2 u = f p_1$.
Two bundles ξ_1, ξ_2 are **isomorphic** when there is an **isomorphism** $g : \xi_1 \to \xi_2$, that is, a morphism $g = (u, f) : \xi_1 \to \xi_2$ such that there exists a morphism $h = (u_2, f_2) : \xi_2 \to \xi_1$ such that $g \circ h = id_{\xi_2}$ and $h \circ g = id_{\xi_1}$, or more precise, $u \circ u_2 = id_{E_2}$, $u_2 \circ u = id_{E_1}$, $f \circ f_2 = id_{B_2}$ and $f_2 \circ f = id_{B_1}$
This is to say that two bundles $\xi_1 = (E_1, p_1, B_1)$, $\xi_2 = (E_2, p_2, B_2)$ are isomorphic, when E_1 and E_2 are homeomorphic, B_1 and B_2 are homeomorphic, and the following diagram is commutative:
$$\begin{array}{ccc} E_2 & \overset{u,u_2}{\longleftrightarrow} & E_2 \\ \downarrow^{p_1} & & \downarrow^{p_2} \\ B & \overset{f,f_2}{\longleftrightarrow} & B_2 \end{array}$$

Definition 1.5. The restriction $\xi|A$ of a bundle $\xi = (E, p, B)$ to a subspace $A \subset B$ is the bundle $\xi|A = (p^{-1}(A), p_A, A)$ where p_A is the map $p_A : p^{-1}(A) \to A : p_A(a) = p(a)$.
We call a bundle ξ with fibre F **Locally Trivial** if for any $b \in B$, there is a neighborhood U of b such that $\xi|U$ is isomorphic to a trivial bundle $(U \times F, p, U)$.

Example 1.6. Take as base space the circle, $B = S^1$, and as total space E the Moebius-band $M \simeq \{((2+(2s-1)\sin(\pi t))\cos(2\pi t), (2+(2s-1)\sin(\pi t))\sin(2\pi t), (2s-1)cos(\pi t)) \in \mathbb{R}^3 | t \in [0,1], s \in [0,1]\}$. Then take the projection $p : M \to S^1 : (s,t) \to (\cos(2\pi t), \sin(2\pi t))$, then it follows that $\xi = (E, p, B) = (M, p, S^1)$ is a bundle with fibre $[0,1]$. This bundle is not trivial, but it is locally trivial, since it looks locally just like $([0,0.5] \times [0,1], (s,t) \to t, [0,1])$, in fact it is precisely two times this trivial bundle, glued together in a non-trivial way.

Example 1.7. A vector bundle is a bundle with an additional vector space structure on each fibre such that a local triviality condition is satisfied. This will be made precise in the following definition. First, let F denote the field of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$.

Definition 1.8. A k-dimensional vector bundle ξ over F is a bundle (E, p, B) together with a structure of a k-dimensional vector space over F on each fibre $p^{-1}(b)$ such that the following local triviality condition is satisfied: For each $b \in B$, there is an open neighborhood $U(b)$ of b and an isomorphism $h : U(b) \times F^k \to p^{-1}(U(b))$ such that the restriction $b \times F^k \to p^{-1}(b)$ is a vector space isomorphism for each $b \in U$.

The trivial example of a vector bundle is the k-dimensional product bundle over a space B, that is, the bundle $(B \times F^k, p, B)$, where the vector space structure of F^k defines the vector space structure on $b \times F^k = p^{-1}(b)$. The local triviality condition is satisfied by letting $U = B$ and h be the identity map.

1.2. **The Pull-Back Bundle $f^*(\xi)$.** Given a bundle $\xi = (E, p, B)$ and a function $f : B_1 \to B$ there is an pull-back bundle of ξ along f with base space B_1, called $f^*(\xi)$. This construction will later on be used to discuss the notion of classification of bundles over a space B_1, using (homopopy classes of) functions $f : B_1 \to B$ with B a classifying space. Here we will define this pull-back bundle, so it can be used later on.

Definition 1.9. The pull-back bundle $f^*(\xi)$ of $\xi = (E, p, B)$ along a function $f : B_1 \to B$ is the triple $f^*(\xi) = (E_1, p_1, B_1)$ where E_1 is defined by $E_1 = \{(b_1, x) \in B_1 \times E | f(b_1) = p(x)\}$ and p_1 is defined by $p_1 : E_1 \to B_1$ by $(b_1, x) \to b_1$.

Proposition 1.10. *The induced bundle $f^*(\xi) = (E_1, p_1, B_1)$ of ξ under a function f is a bundle, and if ξ is a bundle with Fibre F, $f^*(\xi) = (E_1, p_1, B_1)$ is also a bundle with Fibre F.*

Proof. To see that $f^*(\xi) = (E_1, p_1, B_1)$ is a bundle it's only needed that p_1 is surjective, continuous and open, and since for any b_1, the relation $p_1(b_1, x) = b_1$ holds, it is surjective, and since p is open and continuous, p_1 is also open and continuous.
The fibre of $f^*(\xi)$ over b_1, for a given $b_1 \in B_1$, is given by $F = \{x \in E | f(b_1) = p(x)\}$. If ξ has fibre F, this fibre is $p^{-1}(p(y)) = \{x \in E | p(y) = p(x)\}$ for any $y \in E$. Choose $y \in E$ such that $f(b_1) = p(y)$, then the fibres are homeomorphic, as requested. This means that if ξ is a bundle with fibre F, $f^*(\xi) = (E_1, p_1, B_1)$ is also a bundle with fibre F. □

1.3. **Principal G-Bundles.** We will now continue with the definition of principal G-bundles: a principal G-bundle is a bundle $\xi = (X, p, B)$ together with a group G such that X is a principal G-space. This means that the group G acts on the total space X, and as we will prove later, that G can be seen as the fibre of the principal G-bundle. The group G needed is a topological group, what means that there is a topology defined on G which is compatible with the group operator. This will all be made precise in the following definitions.

Definition 1.11. A **Topological Group** G is a set G together with a group structure and topology on G such that the function $(s,t) \to st^{-1}$ as a map $G \times G \to G$ is continuous.

Example 1.12. $(\mathbb{R}, +)$ is the topological group of the real numbers with standard addition, where the topology is the Euclidian topology. Since it has $t^{-1} = -t$, $G \times G \to G : (s,t) \to st^{-1} = s - t$ is clearly continuous.

Example 1.13. S^1 or more completely $(S^1, \cdot)$ is the topological group consisting of the elements of $S^1 \subset \mathbb{C}$, what means that an element can be written $e^{2\pi i\omega}$ with $\omega \in [0,1]$ and $e^{2\pi i0} = e^{2\pi i1}$, where the group structure is given by standard multiplication and it has the Euclidean topology (from $\mathbb{C}$). $G \times G \to G : (e^{2\pi i\omega_1}, e^{2\pi i\omega_2}) \to e^{2\pi i\omega_1}(e^{2\pi i\omega_2})^{-1} = e^{2\pi i(\omega_1 - \omega_2)}$ is clearly continuous.

Example 1.14. $U(n)$ is the topological group of n by n unitary($u^H = u^{-1}$ where u^H is the conjugate transpose of u) complex matrices with matrix multiplication as group structure, and it gets its topology from the inclusion $U(n) \to \mathbb{C}^{n \cdot n}$, where $\mathbb{C}^{n \cdot n}$ has the Euclidian topology. The inverse of $u \in U(n)$ is u^* and $G \times G \to G : (s,t) \to u_1 u_2^{-1} = u_1 u_2^*$ is continuous.
$U(1)$ consists of the complex numbers with norm 1 and standard multiplication, and has the topology of $\mathbb{C}$. The elements of $U(1)$ are the elements on the unit circle, and since S^1 has the same topology and group structure, $U(1)$ and S^1 are isomorphic.

Example 1.15. $SU(n)$ is the topological group of n by n complex matrices satisfying $A^t A = I$ and $\det(A) = 1$. Its multiplication is standard matrix multiplication and it gets its topology from the inclusion $SU(n) \to \mathbb{C}^{n \cdot n}$.

Definition 1.16. Let G be a topological group. A G**-space** is a topological space X together with a continuous right side multiplication $X \times G \to X$ defined from G on X, satisfying the axioms $xe = x$ and $(xg)h = x(gh)$.
A **free** G-space is a G-space with the following property: $xg = x$, $x \in X$ and $g \in G$, implies that $g = 1$.
A **continuous translation-function** is a continuous function $\tau : X^* \to G$ with the property that $x\tau(x, x') = x'$, where X^* is given by $X^* = \{(x, xg) \in X \times X | x \in X, g \in G\}$.
X is called a **principal** G**-space** if X is a free G-space with continuous translation-function $\tau : X^* \to G$. In this case we denote $B = X/G$ and we say that $\xi = (X, p, B)$ is a **principal** G**-bundle**. Note that this is a fibre bundle with fibre G(see proposition 1.22).
We call a principal G-bundle **trivial** if the total space E is homeomorphic to $B \times G$, where G is the fibre.
We call a principal G-bundle **locally trivial** if, for each point $e \in B$, there is a neighborhood of e in E that is homeomorphic to $B_e \times G$ where B_e is a neighborhood of e in B.

Proposition 1.17. *If τ is a continuous translation-function of a principal G-bundle $\xi = (X, p, B)$, the following facts hold:*

(1) $\tau(x, x) = 1$
(2) $\tau(x, x')\tau(x', x'') = \tau(x, x'')$
(3) $\tau(x, x')^{-1} = \tau(x', x)$

Proof. Let τ be a continuous translation-function of a principal G-bundle $\xi = (X, p, B)$.

(1) We know that $x\tau(x, x') = x'$. Now take x=x'. Then $x\tau(x, x) = x$. Since x∈X, $\tau(x, x) \in G$ and X is free, it follows that $\tau(x, x) = 1$
(2) From $x\tau(x, x') = x'$ and $x'\tau(x', x'') = x''$ it follows that $x\tau(x, x')x'\tau(x', x'') = x'x''$ thus that $x\tau(x, x')\tau(x', x'') = x''$. Since this is true for all x, x' and x" it follows that $\tau(x, x')\tau(x', x'') = \tau(x, x'')$
(3) From $x\tau(x, x') = x'$ and $x'\tau(x', x) = x$ it follows that $x\tau(x, x')x'\tau(x', x) = x'x$, thus $\tau(x, x')\tau(x', x) = 1$. □

A few examples of principal G-bundles will now follow.

Example 1.18. The bundle $(B \times G, p, B)$ is a principal G-bundle when defined in the following way: Take the product G-space $B \times G$, where the action from G on $B \times G$ is given by the relation $(b, t)g = (b, tg)$, and let τ be defined by $\tau((b, t), (b, t')) = t^{-1}t'$. (Note that b in (b,t') is the same b as in (b,t): if $(b, t)(b', t') \in (B \times G)^*$ then $(b', t') = (b, t)g = (b, tg)$ thus b=b'.)
Since G is a group, $(b, t)g = (b, t)$ implies $tg = t$ and that implies that $g = 1$ thus $B \times G$ is a free G-space. For this to be a principal G-bundle we have to show that $(b, t)\tau((b, t), (b, t')) = (b, t')$. Now, $(b, t)\tau((b, t), (b, t')) = (b, t)t^{-1}t' = (b, tt^{-1}t') = (b, t')$ as it should be.

Example 1.19. Let G be the group $\{-1, 1\}$ with multiplication. Let X, the space that will be a principal G-space, be the n-dimensional sphere $X = S^n$. Then $X^* = \{(x, x) \in X \times X\} \cap \{(x, -x) \in X \times X\}$ where $-x$ is defined as $-x = x \cdot -1$. Since $x \cdot \pm 1 = \pm x$ and X does not contain 0, X is free.
Since $\tau(x, \pm x) = \pm 1$, it follows that $x\tau(x, \pm x) = \pm x$ as it should be.
Define the base space $B = S^n/G$. This space is given by two opposite points on the circle, and when one draws a line through these points, the space is given by the lines in $\mathbb{R}^n$, and is $B = S^n/G \simeq P^n$, the projective space. Thus $(S^n, p, S^n/G \simeq P^n)$ as defined above is a principal G-bundle.

Example 1.20. Take as base space the circle S^1, and let G be the circle group S^1. The torus is obtained by putting a 'smal' circle S^1 on each point of the 'bigger' base space S^1, and gluing these circles together in the trivial way. The obtained bundle is clearly a principal S^1-bundle, where the action of S^1 on the torus is just multiplication of S^1, that is, multiplication by g_2 with a point b_1g_1 on the fibre of $b_1 \in S^1$ gives the point $b_1(g_1g_2)$ on the fibre of b_1. The construction is illustrated by figure 1. There, the arrows indicate the direction in which the sides are glued together.

Example 1.21. The Klein bottle is just a strangely twisted (in the fourth dimension) torus, but the above construction cannot be carried over to obtain the Klein bottle. The problem is that the action of G on the Klein bottle cannot be defined continuously. Figure 2 illustrates the problem. In this figure, a is glued to a' such

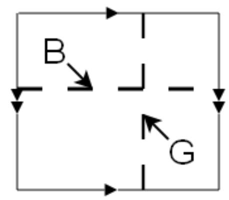

FIGURE 1. The Torus as S^1-bundle over $B = S^1$.

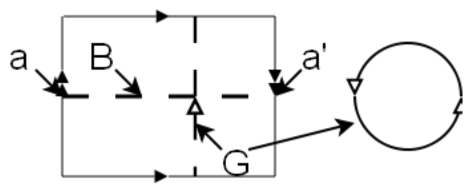

FIGURE 2. The Klein Bottle is not an S^1-bundle over $B = S^1$, since the action of G cannot be defined well and continuously, because there is a twist at $a = a'$.

that B is just the circle S^1, and on each point in B there is a circle $G = S^1$ attached. If we define the action of G on the Klein bottle going 'upwards', as the picture indicates, then the action is not well-defined in $a = a'$. If we define the action in $a = a'$, this causes the action to be discontinuous. Therefore, the Klein bottle is not a principal S^1-bundle over S^1 in any trivial way, although it is a bundle over S^1 with fibre S^1.

Proposition 1.22. *Let $\xi = (X, p, B)$ be a principal G-bundle. Then ξ is a bundle with fibre G.*

Proof. Lets find an homeomorphism between $p^{-1}(b)$ and G, for each b. Take an $x \in p^{-1}(b)$ and define $u(s) = xs$. This is a bijective map $u : G \to p^{-1}(b)$(the bijectivity is obtained using the group axioms). The inverse is $x' \to \tau(x, x')$, and since $x\tau(x, xs) = xs$ implies $\tau(x, xs) = s$, which is continuous, it follows that u is a homeomorphism. $\square$

Example 1.23. Let $Vect_n(B)$ denote the set of all n-dimensional vector bundles over a base space B with $F = \mathbb{R}$, and let $GL_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices. Let $Princ_{GL_n(\mathbb{R})}(B)$ denote the set of all principal $GL_n(\mathbb{R})$-bundles over B. It can be shown that there is a $1-1$ correspondence between $Vect_n(B)$ and $Princ_{GL_n(\mathbb{R})}(B)$. This means that every n-dimensional real (that is, $F = \mathbb{R}$) vector bundle can be associated to a principal $GL_n(\mathbb{R})$-bundle. Using this the problem of computing $Vect_k(B)$, can be reduced to the calculation of the set of homotopy classes of maps from B to $B_{GL_n(\mathbb{R})}$, that is, the set $[B, B_{GL_n(\mathbb{R})}]$, where $B_{GL_n(\mathbb{R})}$ denotes the base space of $GL_n(\mathbb{R})$. In this way $Vect_k(B)$ can be 'classified' by the set $[B, B_{GL_n(\mathbb{R})}]$. Classifying spaces will be defined in section 3 below.

1.4. Numerable Principal G-Bundles over $B \times [0, 1]$.

Definition 1.24. An open covering $\{U_i\}_{i\in S}$ of a topological space B is called numerable if there is a locally finite partition of unity $\{u_i\}_{i\in S}$ such that $\overline{u_i^{-1}((0,1])} \subseteq$

U_i for each $i \in S$.
A principal G-bundle ξ over B is called numerable if there is a numerable open covering $\{U_i\}_{i\in S}$ of B such that $\xi|U_i$ is trivial for each $i \in S$.

Proposition 1.25. *Let $f : B' \to B$ be a map and let ξ be a numerable bundle over B, then $f^*(\xi)$ is a numerable bundle over B'.*

Proof. Let ξ be numerable over B and take $\{U_i\}_{i\in S}$ such that $\xi|U_i$ is trivial for each $i \in S$. We have to show that $f^*(\xi)|f^{-1}(U_i)$ is trivial for each $i \in S$. It is given that $\xi|U_i \simeq (U_i \times G, p, U_i)$. Since $f^*(\xi)|f^{-1}(U_i) \simeq f^*(\xi|U_i) \simeq f^*((U_i \times G, p, U_i))$ is trivial, we are done. □

Proposition 1.26. *Let $\xi = (X, p, B)$ be a numerable principal G-bundle, and let $f_0, f_1 : B_1 \to B$ be two homotopic maps. Then $f_0^*(\xi) \simeq f_1^*(\xi)$.*

Proof. $f_0, f_1 : B_1 \to B$ are two homotopic maps, what means that there is a homotopy $f : B_1 \times I \to B$, $I = [0, 1]$, with $f_0 = f(-, 0)$ and $f_1 = f(-, 1)$. Define, for $i = 0, 1$, the maps $\epsilon_i : B_1 \to B_1 \times I$ by $\epsilon_i(b) = (b, i)$. We now clearly have $f_i = f\epsilon_i$ for $i = 0, 1$, and therefore we have that $f_i^*(\xi)$ and $\epsilon_i^* f^*(\xi)$ are isomorphic. To prove that $f_0^*(\xi)$ is isomorphic to $f_1^*(\xi)$, it is now enough to show that $\epsilon_0^*(\eta)$ is isomorphic to $\epsilon_1^*(\eta)$, where η is a numberable principal G-bundle with some base space $B_1 \times I$ ($f^*(\xi)$ is a bundle of this form).
Let $r : B_1 \times I \to B_1 \times I$ be the map $r(b, t) = (b, 1)$. Then $r\epsilon_0 = \epsilon_1$, and the bundles $\epsilon_1^*(\eta)$ and $\epsilon_0^* r^*(\eta)$ are isomorphic.
If we now show that $\epsilon_0^* r^*(\eta)$ is isomorphic to $\epsilon_0^*(\eta)$, we are done. Since the pullbacks $\epsilon_0^* r^*(\eta)$ and $\epsilon_0^*(\eta)$ only use points in $B_1 \times 0 \subset B_1 \times I$, it is enough to show that $r^*(\eta)|B \times 0$ is isomorphic to $\eta|B \times 0$. Using the numerability of η, one can construct a morphism $(g, r) : \eta \to \eta$ and this morphism gives rise to a homeomorphism[1]. Therefore, $r^*(\eta)|B \times 0$ is isomorphic to $\eta|B \times 0$.
This completes the proof, since now $\epsilon_1^* f^*(\xi)$ is isomorphic to $\epsilon_0^* f^*(\xi)$, what means that $f_0^*(\xi)$ is isomorphic to $f_1^*(\xi)$. □

[1]See Dale Husemoller, *Fibre bundles*, New york: Springer-Verlag, 1994. Third edition (first edition 1966), p. 51-52.

2. The Hopf Bundle and Quantum Entanglement

The Hopf bundle, sometimes called the Hopf fibration or Hopf map, is named after Heinz Hopf who studied it an 1931 article. It has a widely variety of physical applications. I will give a short introduction into the Hopf bundle and I will give an application in quantum theory. The simplest Hopf-bundle is a bundle $h_1 = (S^3, p, S^2)$ where the fibre is S^1. This means that every point on S^2 corresponds to a circle on S^3.

Definition 2.1. The **first Hopf bundle** h_1 is the bundle $h_1 = (S^3, p, S^2)$ with $p : S^3 \to S^2 : (a, b, c, d) \to (2(ac+bd), 2(ad-bc), a^2+b^2-c^2-d^2)$. We also consider the action of S^1 on S^3, for $s \in S^1$ where ϕ is the angle of the point s in S^1, by

$$\begin{aligned}(x_1, x_2, x_3, x_4)s &= (x_1, x_2, x_3, x_4)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & \sin(\phi) & \cos(\phi) \end{pmatrix} \\ &= (x_1, x_2, x_3\cos(\phi) + x_4\sin(\phi), -x_3\sin(\phi) + x_4\cos(\phi))\end{aligned}$$

Proposition 2.2. *$h_1 = (S^3, p, S^2)$ is a principal S^1-bundle.*

Proof. First note that $(2(ac+bd))^2 + (2(ad-bc))^2 + (a^2+b^2-c^2-d^2)^2 = 4(a^2 + b^2)(c^2+d^2) + (a^2+b^2-c^2-d^2)^2 = (a^2+b^2+c^2+d^2)^2 = 1$ so p maps to S^2 as it should do.
To show that the fibre is S^1, take a point $x = (x_1, x_2, x_3)$ on S^2, then $p^{-1}(x) = \{(a, b, c, d) \in S^3 | 2(ac+bd) = x_1, 2(bc-ad) = x_2, a^2+b^2-c^2-d^2 = x_3\}$. Some calculations can show that this defines a circle.
When the fibre S^1 is identified with the circle group, $G = S^1$, it acts on S^3 by rotation. Therefore S^3 is a free S^1-space with continuous translation-function $\tau : S^{3*} \to S^1$ that sends (x, x') to the unique $e^{2\pi i\omega} \in S^1$ that rotates x to x'.

□

Remark 2.3. There are three other bundles with a sphere S^i as base space, S^j as total space and S^k as fibre.The **second Hopf bundle** is the bundle $h_2 = (S^7, p, S^4)$ with fibre S^3. The **third Hopf bundle** is the bundle $h_3 = (S^{15}, p, S^8)$ with fibre S^7. And of course there is also the **trivial Hopf bundle** $h_0 = (S^1, p, S^1)$ with fibre $S^0 = \{1\}$ and multiplication $(a, b)1 = (a, b)$ for $(a, b) \in S^1 \subset \mathbb{R}^2$, $1 \in S^0$. The projection of the trivial Hopf bundle can easily made explicit, namely the identity map $p : S^1 \to S^1 : (a, b) \to (a, b)$.

In the next subsection, I will give an application of Hopf bundles, and in that application an explicit projection formula for the second Hopf bundle will be given. Here I will give a way to understand the first Hopf bundle, by looking to the base space as $\mathbb{C}P^1 \simeq SU(2)/U(1)$, and I will generalize this bundle to a bundle $(SU(n), p, \mathbb{C}P^{n-1} \simeq SU(n)/U(n-1))$. First I will introduce a way of constructing H-bundles using a normal subgroup $H \subset G$ and the quotient G/H.

Proposition 2.4. *Let G be a topological group and let H be a topological subgroup $H \subset G$. Then $\xi = (G, p, G/H)$ is a principal H-bundle, where p is the projection $p : G \to G/H : g \to g \mod H$.*

Proof. G is a free H-space by letting H act on G the same way as the elements of the subgroup $H \subset G$ act on elements of the group G. The continuous translation-function is just the function $\tau : G^* \to H : (g, g') \to g'^{-1}g$, that is, $(g, gh) \to h^-1g^-1g = g$. Therefore, $\xi = (G, p, G/H)$ is a principal H-bundle.

$\square$

To define the bundle $(SU(2), p, \mathbb{C}P^1)$ and to show that it is a principal $U(1)$-bundle, let us define an action of $U(1)$ on $SU(2)$ in the following way:

$$x(g) = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{1,1}g & x_{1,2} \\ x_{2,1}g & x_{2,2} \end{pmatrix}$$

for $x = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in SU(2)$ and $(g) \in U(1)$. Since $|(g)| = 1$, this is a map $SU(2) \times U(1) \to SU(2)$. We can now define a bundle by $h'_1 = (SU(2), p, SU(2)/U(1))$ which has fibre $U(1)$. The following proposition will prove that this is the first Hopf bundle.

Proposition 2.5. *The bundle $h'_1 = (SU(2), p, SU(2)/U(1))$ is a principal $U(1)$-bundle with base space $\mathbb{C}P^1$, and is isomorphic to the first Hopf bundle.*

Proof. The projection p is defined by $p : SU(2) \to SU(2)/U(1) : g \to G \mod U(1)$, and since $U(1)$ is a subgroup of $SU(2)$, this proposition also gives us that $h'_1 = (SU(2), p, SU(2)/U(1))$ is a principal $U(1)$-bundle.
To show that it is isomorphic to the first Hopf bundle, I will show that $SU(2) \simeq S^3$, $U(1) \simeq S^1$ and $\mathbb{C}P^1 \simeq S^2$.
$U(1)$ is isomorphic to S^1 as shown in example 1.14.
$SU(2)$ is homeomorphic with S^3, and this can be shown in the following way: an element of $SU(2)$ is of the form $\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$ and $\alpha^2 + \beta^2 = 1$. This formula defines the sphere $S^3 \subset \mathbb{C}^2$, and since $SU(2)$ gets its topology from injection Euclidean complex space, S^3 and $SU(2)$ are homeomorphic.
S^2 and $\mathbb{C}P^1$ are homeomorphic, since $\mathbb{C}P^1$ is given by the complex lines in $\mathbb{C}^2$, it can be seen as the Riemann sphere, and is just the sphere S^2 up to homomorphism.
To show that $\mathbb{C}P^1 \simeq SU(2)/U(1)$, $\mathbb{C}P^1$ is given by the complex lines in $\mathbb{C}^2$, and $SU(2)/U(1)$ is given by the 2 by 2 matrices with determinant 1 where the elements of the form $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ are glued to a point. Since we can see $SU(2)$ as S^3, $SU(2)/U(1)$ is S^3 with the circles $U(1)$ glued to a point, and this is the same as $\mathbb{C}^2$ with the complex lines glued to a point, that is, $\mathbb{C}P^1 \simeq SU(2)/U(1)$.
This completes the proof. $\square$

This first Hopf bundle can be generalized to the bundle $(SU(n), p, \mathbb{C}P^{n-1} \simeq SU(n)/U(n-1))$. This is a bundle with spheres as total space, base space and fibre only when $n \in \{1, 2\}$ (when $n = 1$, it is the trivial Hopf bundle $(SU(1) \simeq S^1, p, SU(1)/U(0) \simeq S^1/S^0)$). But for each $n \in \mathbb{N}$, it is a principal $U(n-1)$-bundle, and for $n = 2$, it is the first Hopf bundle.

In general, define the action of $U(n-1)$ on $SU(n)$ by

$$xg = x \left(\begin{array}{ccc|c} g_{1,1} & \cdots & g_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{n-1,1} & \cdots & g_{n-1,n-1} & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right)$$

for $x \in SU(n)$ and $g \in U(n-1)$, given by matrix multiplication.

Proposition 2.6. *For each $n \in \mathbb{N}$, $(SU(n), p, \mathbb{C}P^{n-1} \simeq SU(n)/U(n-1))$ is a principal $U(n-1)$-bundle.*

Proof. Since for each $n \in \mathbb{N}$, $U(n-1)$ is a subgroup of $SU(n)$, proposition 2.4 gives us that $(SU(n), p, SU(n)/U(n-1))$ is a principal $U(n-1)$-bundle. $\mathbb{C}P^{n-1}$ is given by the complex lines in $\mathbb{C}^n$, and $SU(n)$ can be injected into $\mathbb{C}^n$ in an analogue way as $SU(2)$ is injected in $\mathbb{C}^2$ in proposition 2.5. $SU(n)/U(n-1)$ is then given in the same way as an element of $\mathbb{C}P^{n-1}$ is given, and they have the same topology. Therefore $\mathbb{C}P^{n-1} \simeq SU(n)/U(n-1)$. This completes the proof. □

2.1. **Quantum Entanglement.** We will now discuss an application of the Hopf bundle h_2 in standard non-relative quantum mechanics[2]. In this application an explicit projection p will be given. For this some explanation of quantum mechanical notation is needed, and this explanation will be given first.

We will discuss a qubit state, that is, the quantum state of a two-level system. An example of a two-level system is a system with only one particle, where this particle has two possible outcomes upon measuring, for example, spin up and spin down (what means that it is a spin-1/2 particle). Every other qubit state can be seen as analogous to the one-particle spin-1/2 system[3].
The state of such a qubit system is written as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha, \beta \in \mathbb{C}$. Upon measurement there are two possible outcomes, $|1\rangle$ (for example, spin up) with chance $\frac{|\alpha|^2}{|\alpha|^2+|\beta|^2}$ and $|0\rangle$ (for example, spin down) with chance $\frac{|\beta|^2}{|\alpha|^2+|\beta|^2}$. In calculations, $|0\rangle$ can be seen as the vector $(1,0)^T$ and $|1\rangle$ can be seen as the vector $(0,1)^T$, such that $|0\rangle$ and $|1\rangle$ span the space of possible states. This means that for this given basis, the qubit state is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = (\alpha, \beta)^T$.
$\langle\psi|$ is just the complex conjugate and transposed $|\psi\rangle$, such that $\langle\psi||\psi\rangle$ gives the normal inner product. That means that $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$.

Since the chances $\frac{|\alpha|^2}{|\alpha|^2+|\beta|^2}$ and $\frac{|\beta|^2}{|\alpha|^2+|\beta|^2}$ do not change when the state $|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$ is multiplied with a constant $c \in \mathbb{C}$ (and for some other reasons), the space of the different possible states of the qubit is given by the complex lines in $\mathbb{C}^2$, that is, $\mathbb{C}P^1$.

We will now first translate the theory of the Hopf bundle into the language of quantum mechanics using h_1 and a qubit system as example. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

[2]The applications in this subsection are taken mostly from Dariusz Chruscinski, *Geometric Aspects of Quantum Mechanics and Quantum Entanglement*, Journal of Physics: Conference Series 30, 2006.

[3]This spin-1/2 system is well-described in David J. Griffiths, *Inroduction to Quantum Mechanics*, Pearson Education, inc., New Jersey, Second edition, 2005 (first edition 1995), p. 171-177.

be a normalized state vector of a qubit. The normalization requirement means that $|\alpha|^2 + |\beta|^2 = 1$ where $\alpha, \beta \in \mathbb{C}$. This formula defines a sphere S^3. Now we will define the bundle h_1 on the qubit, that is, the bundle $(S^3, p, \mathbb{C}P^1 \simeq S^2)$, in quantum-mechanical terms. This gives in mathematical terms how the possible $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ can be projected upon the possible states of the qubit.

Definition 2.7. Let h_1^{QM} be the bundle defined by

$$h_1^{QM} = \{S^3, p^{QM}, S^2\}$$

with

$$p^{QM} : S^3 \to S^2 : \psi \to (\langle\psi|\sigma_1|\psi\rangle, \langle\psi|\sigma_2|\psi\rangle, \langle\psi|\sigma_3|\psi\rangle)$$

where σ_i are the pauli matrices

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Proposition 2.8. $h_1^{QM} = \{S^3, p^{QM}, S^2\}$ *is a bundle with fibre* S^1 *and is isomorphic to the first Hopf bundle.*

Proof. For the first assertion we have to show that p^{QM} is correctly defined. Let us calculate the projection $p(\phi)$:

$$\begin{aligned} \langle\psi|\sigma_1|\psi\rangle &= |\alpha|^2 \langle 0| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |0\rangle + \alpha\beta^* \langle 1| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |0\rangle \\ &+ \alpha^*\beta\langle 0| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |1\rangle + |\beta|^2 \langle 1| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |1\rangle \\ &= 0 + \alpha^*\beta + \alpha\beta^* + 0 = \alpha^*\beta + \alpha\beta^* \end{aligned}$$

and in the same way $\langle\psi|\sigma_2|\psi\rangle = \alpha\beta^* i - \alpha^*\beta i$ and $\langle\psi|\sigma_3|\psi\rangle = |\alpha|^2 - |\beta|^2$. Further calculations give, where $\alpha = \alpha_1 + \alpha_2 i$ and $\beta = \beta_1 + \beta_2 i$, $\alpha_n, \beta_n \in \mathbb{R}$, that

$$\begin{aligned} \langle\psi|\sigma_1|\psi\rangle &= \alpha^*\beta + \alpha\beta^* = (\alpha_1 - \alpha_2 i)(\beta_1 + \beta_2 i) + (\alpha_1 + \alpha_2 i)(\beta_1 - \beta_2 i) \\ &= 2(\alpha_1\beta_1 + \alpha_2\beta_2) \\ \langle\psi|\sigma_2|\psi\rangle &= \alpha\beta^* i - \alpha^*\beta i = (\alpha_1 + \alpha_2 i)(\beta_1 - \beta_2 i)i - (\alpha_1 - \alpha_2 i)(\beta_1 + \beta_2 i)i \\ &= 2(\alpha_1\beta_2 - \alpha_2\beta_1) \\ \langle\psi|\sigma_3|\psi\rangle &= |\alpha|^2 - |\beta|^2 = \alpha_1\alpha_1 + \alpha_2\alpha_2 - \beta_1\beta_1 - \beta_2\beta_2 \end{aligned}$$

such that

$$\begin{aligned} & |\langle\psi|\sigma_1|\psi\rangle|^2 + |\langle\psi|\sigma_2|\psi\rangle|^2 + |\langle\psi|\sigma_3|\psi\rangle|^2 \\ =\ & |2(\alpha_1\beta_1 + \alpha_2\beta_2)|^2 + |2(\alpha_1\beta_2 - \alpha_2\beta_1)|^2 + |\alpha_1\alpha_1 + \alpha_2\alpha_2 - \beta_1\beta_1 - \beta_2\beta_2|^2 \\ =\ & \left| |\alpha|^2 + |\beta|^2) \right|^2 = 1 \end{aligned}$$

When we take $a = \alpha_1$, $b = \alpha_2$, $c = \beta_1$ and $d = \beta_2$, the projection of this bundle is the same as the projection of the first Hopf-bundle. Therefore the bundle h_1^{QM} is isomorphic to the first Hopf bundle and is a bundle with fibre S_1.

□

Remark 2.9. The fact that $p^{QM} : S^3 \to S^2 : \psi \to (\langle\psi|\sigma_1|\psi\rangle, \langle\psi|\sigma_2|\psi\rangle, \langle\psi|\sigma_3|\psi\rangle)$ is the projection of the first Hopf bundle, can be physically understood by noticing that $\langle\psi|\sigma_1|\psi\rangle$ can be seen as the chance of measuring 'σ_1' upon measurement when the state is ψ, $\langle\psi|\sigma_2|\psi\rangle$ the chance for measuring 'σ_2', and $\langle\psi|\sigma_3|\psi\rangle$ for measuring 'σ_3', where 'σ_1', 'σ_2' and 'σ_3' are the only possibilities. Therefore, these chances must count op to one, and since the case is symmetric (there is no a priori difference between σ_1, σ_2 and σ_3), these possible measurements must be equally reached by possible states, that is, elements of S^3. Therefore, p^{QM} constitutes a bundle with fibre S^1.

2.1.1. *Composite 2-Qubit System.* I will give a second example where a Hopf bundle can help understanding a quantum-mechanical situation. This example also gives an explicit projection formula for the second Hopf bundle. Here I will work with a composite 2-qubit system, that is, a two-particle system where both the particles are two-level systems, as in the normal qubit system. For example, both are spin-1/2 particles, what means that they can have spin up or spin down upon measurement.
Such a 2-qubit state can be given by $|\psi\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle$. This means that upon measurement, there are four possible outcomes $|00\rangle$, $|10\rangle$, $|01\rangle$ and $|11\rangle$ (both spin down, one up second down, one down second up, both spin down) with chances $\frac{|\alpha|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2}$, $\frac{|\beta|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2}$, $\frac{|\gamma|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2}$ and $\frac{|\delta|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2}$.
The state is normalized if $\|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2 + \|\delta\|^2 = 1$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. This formula defines S^7.
The space of all different possible states is a tensor product of the spaces of different possible states of each of the two particles, what means that it is $CP^1 \otimes CP^1$.
If $|\psi_1\rangle$ is a state of particle one and $|\psi_2\rangle$ is a state of particle 2, then the joined system state is $|\psi_1\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. But in general, a composite 2-qubit state cannot be written in this way. If a composite 2-qubit state $|\psi\rangle$ can be written as $|\psi_1\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ where $|\psi_1\rangle$ is a state for the first particle and $|\psi_2\rangle$ is a state for the second particle, then it is called a separable state.

Here I will give a criteria for a 2-qubit state to be separable using the bundle h_2. The next proposition gives a good starting criteria to know when a state is separable.

Proposition 2.10. *A 2-qubit state $|\psi\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle$ is separable iff $\alpha\delta = \gamma\beta$.*

Proof. A 2-qubit $|\psi\rangle$ is separable if it can be written $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ with $|\psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\psi_2\rangle = c|0\rangle + d|1\rangle$ where $a, b, c, d \in \mathbb{C}$. This means that a state is separable iff $|\psi\rangle = ac|00\rangle + bc|10\rangle + ad|01\rangle + bd|11\rangle$ holds.
If this holds, then $\alpha\delta = acbd = bcad = \beta\gamma$.
If, for the proof in the other direction, assume that $\alpha\delta = \beta\gamma$. Then, if $\delta = 0$, it follows that $\beta = 0$ or $\gamma = 0$. Assume that $\beta = 0$, then define $b = 0$, $a = 1$, $c = \alpha$ and $d = \gamma$, then $|\psi\rangle = ac|00\rangle + bc|10\rangle + ad|01\rangle + bd|11\rangle$ holds. It is analogue for $\gamma = 0$.
If $\alpha = 0$, $\beta = 0$ or $\gamma = 0$, it can be shown in the same way that $|\psi\rangle = ac|00\rangle + bc|10\rangle + ad|01\rangle + bd|11\rangle$ holds. Therefore we can assume that none of them are zero.

Let $d \in \mathbb{C}$, define $a = \gamma/d$, $b = \delta/d$ and $c = \alpha/a = d\alpha/\gamma$. Now we can prove that $\beta = bc$ holds, in the following way: $bc = (\delta/d)(d\alpha/\gamma) = \alpha\delta/\gamma = \beta$. Therefore, we are done. Note that we can choose $d \in \mathbb{C}$ randomly, because we have a choice to make what basis we use for the states of the separated particles. $\square$

Let me introduce the operator $E : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, defined by

$$E|\psi\rangle = (\sigma_2 \otimes \sigma_2)(|\psi\rangle)^*$$

Using E, let $x_0 \dots x_4$ be defined by

$$\begin{aligned}
x_0 &= \langle\psi|\sigma_3 \otimes \mathbf{1}|\psi\rangle \\
x_1 &= \langle\psi|\sigma_1 \otimes \mathbf{1}|\psi\rangle \\
x_2 &= \langle\psi|\sigma_2 \otimes \mathbf{1}|\psi\rangle \\
x_3 &= Re(\langle\psi|E|\psi\rangle) \\
x_4 &= Im(\langle\psi|E|\psi\rangle)
\end{aligned}$$

The next theorem says that the above map $|\psi\rangle \to (x_0, x_1, x_2, x_3, x_4)$ defines a Hopf bundle (S_7, p, S_4). This means that the projection of the second Hopf bundle can be given by $p : S^7 \to S^4 : |\psi\rangle \to (x_0, x_1, x_2, x_3, x_4)$.

Proposition 2.11. *The bundle (S_7, p, S_4) with $p : S^7 \to S^4 : |\psi\rangle \to (x_0, x_1, x_2, x_3, x_4)$ is correctly defined and is the second Hopf bundle.*

Proof. For p to be correctly defined, $x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ must hold. This follows trivially from some calculations using $|\psi\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle$ and $\|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2 + \|\delta\|^2 = 1$, for example

$$\begin{aligned}
x_0 &= \langle\psi|\sigma_3 \otimes \mathbf{1}|\psi\rangle \\
&= (\alpha^*\langle 00| + \beta^*\langle 10| + \gamma^*\langle 01| + \delta^*\langle 11|) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle) \\
&= (\alpha^*\langle 00| + \beta^*\langle 10| + \gamma^*\langle 01| + \delta^*\langle 11|) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (\alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle) \\
&= |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2
\end{aligned}$$

Further calculations show that

$$\begin{aligned}
x_1 &= 2(\alpha_1\gamma_1 + \alpha_2\gamma_2) + 2(\beta_1\delta_1 + \beta_2\delta_2) \\
x_2 &= 2(\alpha_1\gamma_2 - \alpha_2\gamma_1) + 2(\beta_1\delta_2 - \beta_2\delta_1) \\
x_3 &= 2RE(-\alpha\delta + \beta\gamma) = 2(\alpha_2\delta_2 - \alpha_1\delta_1) + 2(\beta_1\gamma_1 - \beta_2\gamma_2) \\
x_4 &= 2Im(\alpha\delta - \beta\gamma) = 2(\alpha_1\delta_2 + \alpha_2\delta_1) - 2(\beta_1\gamma_2 + \beta_2\gamma_1)
\end{aligned}$$

what means that the projection can be written

$$\begin{array}{rcl} p: S^7 & \to & S^4: \\ (\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2) & \to & (\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2-\gamma_1^2-\gamma_2^2-\delta_1^2-\delta_2^2, \\ & & 2(\alpha_1\gamma_1+\alpha_2\gamma_2)+2(\beta_1\delta_1+\beta_2\delta_2), \\ & & 2(\alpha_1\gamma_2-\alpha_2\gamma_1)+2(\beta_1\delta_2-\beta_2\delta_1), \\ & & 2(\alpha_2\delta_2-\alpha_1\delta_1)+2(\beta_1\gamma_1-\beta_2\gamma_2), \\ & & 2(\alpha_1\delta_2+\alpha_2\delta_1)-2(\beta_1\gamma_2+\beta_2\gamma_1)) \end{array}$$

$x_0,\dots,x_4$ are all defined in a continuous and open way, and therefore p is continuous and open.
The projection is surjective and a point on S^4 is reached by a sphere S^3 on S^7. □

The special thing about this bundle, is that we can tell whether the 2-qubit state is separable, by looking at the bundle. Since $x_3 = 2RE(-\alpha\delta+\beta\gamma)$ and $x_4 = 2Im(\alpha\delta-\beta\gamma)$(as one can check with some calculations, see proof of next proposition), they are both 0 if and only if the state is separable. This is what the final proposition of this chapter gives us.

Proposition 2.12. *A state of a 2-qubit system is separable iff its image under the Hopf map belongs to the 2-dimensional sphere defined by the intersection of S^4 with the plane $x_3 = x_4 = 0$.*

Proof. The only thing that is left to be proven is that $x_3 = 2RE(-\alpha\delta+\beta\gamma)$ and $x_4 = 2Im(\alpha\delta-\beta\gamma)$ hold. Now,

$$\begin{array}{rcl} \langle\psi|E|\psi\rangle & = & \langle\psi|(\sigma_2\otimes\sigma_2)(|\psi\rangle)^* = \\ & = & \langle\psi| \begin{pmatrix} 0 & 0 & 0 & -i\cdot -i \\ 0 & 0 & -i\cdot i & 0 \\ 0 & -i\cdot i & 0 & 0 \\ i\cdot i & 0 & 0 & 0 \end{pmatrix} (|\psi\rangle)^* \\ & = & -\alpha^*\delta^*+\beta^*\gamma^*+\beta^*\gamma^*-\alpha^*\delta^* \\ & = & -2\alpha^*\delta^*+2\beta^*\gamma^* \\ x_3 = RE(\langle\psi|E|\psi\rangle) & = & RE(-2\alpha^*\delta^*+2\beta^*\gamma^*) = 2RE(-\alpha\delta+\beta\gamma) \\ x_4 = IM(\langle\psi|E|\psi\rangle) & = & IM(-2\alpha^*\delta^*+2\beta^*\gamma^*) = 2IM(\alpha\delta-\beta\gamma) \end{array}$$

Now, $x_3 = 0$ and $x_4 = 0$ clearly hold if and only if $\alpha\delta-\beta\gamma = 0$. This completes the proof. □

3. Classifying Spaces

3.1. $Princ_G(B)$ and K_G.

Definition 3.1. Let $Princ_G(B)$ be the set of isomorphism classes of numerable principal G-bundles over B. Denote the isomorphism class of ξ by $\{\xi\}$.
For a homotopy class $[f] : X \to Y$ define a function $K_G([f]) : Princ_G(Y) \to Princ_G(X)$ by the relation $K_G([f])\{\xi\} = \{f^*(\xi)\}$.
Let H denote the category of all spaces and homotopy classes of maps.

Remark 3.2. The function $K_G([f]) : Princ_G(Y) \to Princ_G(X)$ is well defined using homotopy classes of maps $[f] : X \to Y$ to send isomorphism classes of numerable principal G-bundles over Y to isomorphism classes of numerable principal G-bundles over X, since when f_1 and f_2 are homotopic, their pull-back bundles $f_1^*(\xi)$ and $f_2^*(\xi)$ are isomorphic (see proposition 1.26).

3.2. Universal Principal G-Bundles.

We will now define when a principal G-bundle is universal. We will call a principal G-bundle (E, p, B) universal if for any space X there is a bijection between the homotopy classes of functions from X to B and isomorphism classes of numerable principal G-bundles over X, or, formally defined:

Definition 3.3. For each space X and for each principal G-bundle $\omega = (E_0, p_0, B_0)$, let $\phi_\omega(X) : [X, B_0] \to Princ_G(X)$ be defined by the relation $\phi_\omega(X)[u] = \{u^*(\omega)\}$.
A principal G-bundle $\omega = (E_0, p_0, B_0)$ is **universal** provided that ω is numerable and $\phi_\omega : [-, B_0] \to Princ_G(-)$ is an isomorphism.
We say that $\omega = (E_0, p_0, B_0)$ is **universal for G-bundles on finite CW-complexes** provided that ω is numerable and $\phi_\omega : [X, B_0] \to Princ_G(X)$ is a bijection for all finite CW-complexes X.

$\phi_\omega : [-, B_0] \to Princ_G(-)$ is an isomorphism if and only if each function $\phi_\omega : [X, B_0] \to Princ_G(X)$ is a bijection for all X. This gives rise to the following criterium of universality.

Proposition 3.4. *Let $\omega = (E_0, p_0, B_0)$ be a numerable principal G-bundle. Then ω is universal if and only if the following are true.*

(1) *For each numerable principal G-bundle ξ over X there exists a map $f : X \to B_0$ such that ξ and $f^*(\omega)$ are isomorphic over X.*

(2) *If $f, g : X \to B_0$ are two maps such that $f^*(\omega)$ and $g^*(\omega)$ are isomorphic over X, then f and g are homotopic.*

Proof. We will prove that (1) says that $\phi_\omega(X)$ is surjective and that (2) says it's injective.
Now, (1) says that for every numerable principal G-bundle over X, that is, for every element of $Princ_G(X)$, there is a map $f : X \to B_0$ such that ξ and $f^*(\omega)$are isomorphic over X, what means that for every element $\xi \in Princ_G(X)$, there is an map f such that $\phi_\omega(X)[f] = \{f^*(\omega)\} \simeq \{\xi\}$. This means that (1) says that $\phi_\omega(X)$ is surjective.
(2) says that for any element $\xi \in Princ_G(X)$, if $\phi_\omega(X)[f] = \{f^*(\omega)\} = \{\xi\}$ and $\phi_\omega(X)[g] = \{g^*(\omega)\} = \{\xi\}$, that then follows that f and g are homotopic, or in

other words, $[f] = [g]$. This means that $\phi_\omega(X)$ is injective.

$\square$

Later we will prove that this criterium of universality is equivalent to the criterium that the Total space E_0 is contractible.

3.3. Definition of a Classifying Space.

Definition 3.5. A classifying space for a topological group G is a Base space B_0 of a universal principal G-bundle $\omega = (E_0, p_0, B_0)$.

B_0 is called a classifying space because it classifies principal G-bundles in the following sense. For any space X, there is a $1-1$ correspondence between the homotopy classes $[f]$ of maps from X to B_0, and the isomorphism-classes $\{\xi\}$ of principal G-bundles over X, using the pull-back bundle $\phi_\omega(X)[f] = \{f^*(\omega)\}$. This means that the bundle $\xi = (P, p, X)$ is reconstructed as the pullback of E_0 over B_0 via $\phi_\omega(X)$.

I will first give two examples of bundles that do **not** make the base space into a classifying space for the given group, since they are not universal. These examples will help clarify what is special about classifying spaces.

Example 3.6. Let $\mathbb{R}$ be the group of real numbers with addition as group operator, and let $\omega = (\mathbb{R}\oplus\mathbb{R}, p : \mathbb{R}\oplus\mathbb{R} \to \mathbf{1}\oplus\mathbf{1} : (i, r) \to i), \mathbf{1}\oplus\mathbf{1})$ be the principal $\mathbb{R}$-bundle, where $\mathbb{R}\oplus\mathbb{R} \simeq \{(i, r) \in \{1, 2\} \times \mathbb{R}\}$ is the disjoint union of $\mathbb{R}$ and $\mathbb{R}$, $\mathbf{1}\oplus\mathbf{1} \simeq \{1, 2\}$ is the disjoint union of two points, and the projection p sends a point $(1, r)$ on the first line, $r \in \mathbb{R}$, to the first point, and sends a point $(2, r)$ on the second line, $r \in \mathbb{R}$, to the second point. This clearly is a principal $\mathbb{R}$-bundle, but it is not universal. To show this, let $X = \mathbb{R}$ be the topological space consisting of the real line, then there are two functions $f_0 : X \to \mathbf{1}\oplus\mathbf{1} : x \to 1$ (sending x to the first point) and $f_1 : X \to \mathbf{1}\oplus\mathbf{1} : x \to 2$ (sending x to the second point). It is clear that $f_0^*(\omega) \simeq f_1^*(\omega) \simeq (\mathbb{R}\times\mathbb{R}, p : \mathbb{R}\times\mathbb{R} \to \mathbb{R} : (r_1, r_2) \to r_1, \mathbb{R})$, but f_1 and f_2 are not homotopic maps($f_1(x)$ cannot be pulled from the first point to the second point continuously). There is not 1-1 correspondence between the homotopy classes $[f]$ of maps from X to B_0, and the isomorphism-classes $\{\xi\}$ of principal G-bundles over X, using the pull-back bundle $\phi_\omega(X)[f] = \{f^*(\omega)\}$.

Example 3.7. The principal $\mathbb{R}$-bundle that has the cylinder as total space, $\omega = (S^1 \times \mathbb{R}, p : S^1 \times \mathbb{R} \to S^1 : (\phi, r) \to \phi), S^1)$, is also not a universal bundle. Let $X = S^1$. There are again two maps $f_0 : X \to S^1 : \phi \to \phi$ and $f_1 : X \to S^1 : \phi \to -\phi$ which constitute isomorphic pull-back bundles $f_0^*(\omega) \simeq (S^1 \times \mathbb{R}, p : S^1 \times \mathbb{R} \to S^1 : (\phi, r) \to \phi), S^1) \simeq f_1^*(\omega)$, but the maps f_1 and f_2 are not homotopic. f_1 runs counterclockwise and f_2 runs clockwise, and these maps cannot be homotopic since the hole in the circle cannot be passed continuously.

Proposition 3.8. *Let G be a topological group, and let $\xi_1 = (E_1, p_1, B_1)$ and $\xi_2 = (E_2, p_2, B_2)$ be two universal numerable principal G-bundles. Then B_1 and B_2 are homotopic equivalent as spaces, and E_1 and E_2 are homotopic equivalent as spaces.*

Proof. Let $\xi_1 = (E_1, p_1, B_1)$ and $\xi_2 = (E_2, p_2, B_2)$ be two universal numerable principal G-bundles. Then there is a unique homotopy class $\{f\}$ of functions $f : B_1 \to B_2$ with $f^*(\xi_2) \simeq \xi_1$, and there is a unique homotopy class $\{g\}$ of

functions $g : B_2 \to B_1$ with $g^*(\xi_1) \simeq \xi_2$.
Since $f^*(g^*(\xi_1)) \simeq \xi_1$, it follows that, when changing f, g up to homotopy, $g \circ f = id_{B_1}$ and analoque that $f \circ g = id_{B_2}$. Therefore, B_1 and B_2 are homotopic equivalent as spaces.

An element $e_1 \in E_1$ can be written $e_1 = p(e_1)\tau(p(e_1), e_1) = b_1g_1$ where $b_1 \in B_1$ and $g_1 \in G$. Then, define $i : E_1 \to E_2 : b_1g_1 \to f(b_1)g_1$, and define $j : E_2 \to E_1 : b_2g_2 \to g(b_2)g_2$. Then $i \circ j(b_1g_1) = i(f(b_1)g_1) = gf(b_1)g_1 = b_1g_1$, and therefore $i \circ j = id_{E_2}$. In the same way, $j \circ i = id_{E_1}$. Therefore, E_1 and E_2 are homotopic equivalent as spaces. □

It is not trivial to see that there exists a classifying space for a group like S^1. At the end of this section a classifying space of S^1 and some other examples of classifying spaces of Lie groups will be given. Before I can do this, another criterium of universality will be given and proved. General constructions of classifying spaces for any group will be considered in the next two sections, from which shall follow that for any topological group, there exists a classifying space.

3.4. **Contractibility and Universality.** The next proposition gives a criterium for a numerable principal G-bundle to be universal. It says that a bundle is universal if the total space is contractible. This criterium is much easier to use in explicit examples then the criterium given before, such that we can easily see when a bundle is universal.

Proposition 3.9. *Let $\eta = (E, p, B)$ be a numerable principal G-bundle. If E is contractible, η is universal and B is a classifying space.*

Remark 3.10. The inverse, that when ξ is universal and B is a classifying space, E is contractible, is also true. This can be proven in the following way. Proposition 3.8 has as a corollary that if there is a universal principal numerable G-bundle with a contractible total space E, then all universal principal numerable G-bundles have a contractible total space. In the next section a construction of a universal principal numerable G-bundle will be given for any group G, and it can be shown that this construction constructs a contractible total space. Therefore, for any topological group G, all universal principal numerable G-bundles have a contractible total space, what means that the inverse of proposition 3.9 is also true.

For proving proposition 3.9, the following lemma is necessary:

Lemma 3.11. *A numerable principal G-bundle over B with contractible fibre has the property: If a section over $A \subset B$ can be extended to a neighborhood $V = \tau^{-1}(0, 1]$ of $A \subset \tau^{-1}(1)$, where $\tau : B \to [0, 1]$ is continuous, then it can be extended over B. This possibility to be extended to a neighborhood $V = \tau^{-1}(0, 1]$ of $A \subset \tau^{-1}(1)$ is called the section extension property or SEP.*

Dold gave a proof of this lemma in an 1962 article[4]. The proof is very technical and extensive and there is no need to give it here.

Proof. (of proposition 3.9)I will give the proof Dold gave in the 1962 article[5]. The proof will first show that for any X $\phi_\eta : [X, B] \to Princ_G(X)$ is surjective and then

[4]Albrecht Dold, Partitions of unity in the Theory of Fibrations, 2.7, p. 229
[5]Same, 7.5, p. 249

that it is injective. He does this in a way you could expect; he first proves that if η is a universal principal G-bundle, then for an arbitrary principal G-bundle ζ, there is a bundle map $\zeta \to \eta$, that is, ζ can be construed as the pullback of η over this map. Then he shows that if $f_0, f_1 : X \to B$ induce equivalent bundles, then there is a homotopy between f_0 and f_1.
He writes E_η for the total space of a principal G-bundle η, and in the same way B_η and p_η as the base space and the projection.

Assume E_η is contractible. Let ζ be an arbitrary principal G-bundle, and define a new bundle (ζ, η) over $X = B_\zeta$ as follows. The fibre $(\zeta, \eta)_x$ of (ζ, η) over $x \in X$ consists of all admissible maps (bundle maps) of ζ_x into η; clearly $(\zeta, \eta)_x \approx E_\eta$ (these maps are determined by the image of one point). The local product structure in ζ gives a local product structure in (ζ, η). In fact, (ζ, η) is the associated bundle of ζ with fibre E_η on which G operates by $(g, e) \to eg^{-1}$, $e \in E_\eta$, $g \in G$. In particular, (ζ, η) is numerable if ζ is numerable.
A section s in (ζ, η) over $V \subset X$ associates, in a continuous fashion, with every $v \in V$ a map $s_v : \zeta \to \eta$, i.e., a section s over B is the same as a bundle map $\zeta|V \to \eta$.
If E_η is contractible and ζ is numerable then $p_{(\zeta,\eta)} : E_{(\zeta,\eta)} \to X$ has the section extension property [see lemma 3.11]. In particular, (ζ, η) admits a section, i.e., ζ admits a bundle map $\zeta \to \eta$. This shows $\phi_\eta(X)$ is surjective.
If $f_0, f_1 : X \to B$ induce equivalent bundles $\zeta_i = f_i^*\zeta$, $i = 0, 1$, let $s_i : \zeta \to \eta$ be the induced bundle maps, and $h : \zeta_0 \to \zeta_1$ an equivalence. Define $\zeta = \zeta_0 \times [0, 1]$ (this is a bundle over $X \times [0, 1]$; cf. Steenrod 11.1), and a partial map of ζ,

$$\begin{aligned} s \quad &: \zeta|X \times ([0, \frac{1}{2} \cup \frac{1}{2}, 1]) \to \eta, \\ s \quad &(z, t) = \begin{cases} s_0(z) & \text{for } t < \frac{1}{2} \\ s_1 h(z) & \text{for } t > \frac{1}{2}, z \in E_{\zeta_0} \end{cases} \end{aligned}$$

View s as a section of (ζ, η) over $X \times ([0, \frac{1}{2} \cup \frac{1}{2}, 1])$. Since this set is a halo around $X \times (\{0\} \cup \{1\})$(cf. 2.1 [in Dold; A halo around $A \subset B$ is a subset V of B such that there exists a continuous function $\tau : B \to [0, 1]$ with $A \subset \tau^{-1}(1)$, $V \subset \tau^{-1}(0)$]; take $\tau(x, t) = |2t - 1|$), there exists a global section S in (ζ, η) which agrees with s over $X \times (\{0\} \cup \{1\})$, i.e, there exists a bundle map $S : \zeta \to \eta$ which over $X \times (\{0\} \cup \{1\})$ agrees with s. On the base, we then have $B_s : X \times [0, 1] = B_\zeta \to B_\eta$, a homotopy between f_0 and f_1. Therefore $\phi_\eta(X)$ is injective.

□

3.5. **A few examples: Classifying Spaces of Lie Groups.** Some examples of classifying spaces of Lie groups will now follow. A Lie group is a differentiable manifold that obeys the group properties and that satisfies the condition that the group operations are differentiable. Our first example of a Lie Group is $\mathbb{R}$ under addition.

Example 3.12. Take $G = \{\mathbb{R}, +\}$ the Lie-group of the real line under addition. Define $E_G = \mathbb{R}$ and $B_G = 1$. Take $\omega = (E_G, p, B_G) = (\mathbb{R}, p, \{1\})$. Since $\mathbb{R}$ is contractible, this defines a classifying space. Let X be a space, then the only function $f : X \to B_G$ is the trivial one, and the only isomorphism-class of principal G-bundles over X is the trivial $\{\xi\} = (X \times \mathbb{R}, p, X)$.

The next two examples give classifying spaces of the Lie Groups Z and Z_2.

Example 3.13. Take $G = Z$ and take as the total space the real line $\mathbb{R}$. Z acts free on $\mathbb{R}$ by standard multiplication. Therefore $BG = \mathbb{R}/Z = S^1$ is a classifying space of the Lie group Z.

Example 3.14. Take $G = Z_2$ and take as total space the space $E_G = S^\infty$. Since S^∞ is still a contractible space, the bundle (S^∞, p, BG) is universal where the base space BG is $E_G = S^\infty/Z_2$, which is the same as the infinite dimensional real projective space RP^∞. RP^∞ is a classifying space of the Lie group Z_2.

The final example will give a classifying space of the circle group, seen as $U(1)$.

Example 3.15. Take $G = U(1)$ and define $E_G = S^\infty = \lim_{n\to\infty} S^n$, the infinite dimensional sphere in Euclidean complex space (it therefore has an Euclidean topology). We can let $U(1)$ act freely on S^∞ by rotation. Since S^∞ is a contractible total space (generally S^n is contractible in $n-1$ dimensions, what means that S^∞ is contractible in all dimensions), the bundle $(S^\infty, p, S^\infty/U(1))$ is universal and gives rise to a classifying space of $U(1) : S^\infty/U(1)$ is given by the complex lines in $\mathbb{C}^\infty$ and is $\mathbb{C}P^\infty$, what means that the space $\mathbb{C}P^\infty$ is the classifying space of the circle group $U(1)$.

4. Milnor's Construction

It is not trivial from the definition of a classifying space, that there is a classifying space for every topological group G. This is nonetheless true. In this section and in the next I will give two constructions of classifying spaces for an arbitrary topological group G.
The Milnor's construction constructs a topological space E_G out of an arbitrary group G. This topological space E_G is a G-space and $B_G \equiv E_G/G$ is a classifying space, where the bundle is given by $\omega_G = (E_G, p, B_G)$. E_G will now be defined.

4.1. **The Construction.**

Definition 4.1. The total space E_G in Milnor's construction is the space

$$E_G = (G * G * \cdots * G * \dots)/\sim \ \text{(an infinite join)}$$

defined as follows: an element of E_G is written $\langle x, t\rangle \in E_G$ and stands for

$$\langle x, t\rangle = (t_0x_0, t_1x_1, \dots, t_nx_n, \dots)$$

where $x_i \in G$ and $t_i \in [0, 1]$ such that there are only a finite number of $t_i \neq 0$ and such that $\sum_{0\leq i} t_i = 1$. The equivalence $\sim$ is defined by $\langle x, t\rangle \sim \langle x', t'\rangle$ provided that $t_i = t'_i$ for all i, and if $t_i = t'_i > 0$, $x_i = x'_i$. In other words, two elements are the same if they have the same t, and for each i where t_i is not 0, they have the same x_i. That means that when t_i is zero, it does not matter what x_i is.
Define a right action of G on E_G by $\langle x, t\rangle g = \langle xg, t\rangle$, what means that

$$(t_0x_0, t_1x_1, \dots, t_nx_n, \dots)g = (t_0x_0g, t_1x_1g, \dots, t_nx_ng, \dots)$$

We must now define a topology on E_G. For this, define two families of functions: let

$$t_i : E_g \to [0, 1] : (t_0x_0, t_1x_1, \dots, t_nx_n, \dots) \to t_i$$

and

$$x_i : {t_i}^{-1}(0, 1] \to G : (t_0x_0, t_1x_1, \dots, t_nx_n, \dots) \to x_i$$

Note that x_i is not defined for i when $t_i = 0$, because x_i does not matter there.
Now, let E_G be the space with the smallest topology such that each function $t_i : E_g \to [0, 1]$ and $x_i : {t_i}^{-1}(0, 1] \to G$ is continuous, where ${t_i}^{-1}(0, 1]$ has the subspace topology.
Let

$$B_G \equiv E_G/G$$

and let the bundle be defined by

$$\omega_G = (E_G, p, B_G).$$

In the next proposition says that $\omega_G = (E_G, p, B_G)$ is a numerable principal G-bundle.

Proposition 4.2. *$\omega_G = (E_G, p, B_G)$, as obtained in Milnor's construction, is a numerable principal G-bundle.*

Proof. There are several things to show here. First I will show that the topology on E_G is well defined and that G acts on it freely. After that I will give a continuous translation-function τ, what means that $\omega_G = (E_G, p, B_G)$ is a principal G-bundle. Finally I will show that it is numerable.
Since the set E_G is a set, it has several topologies, one of them the discrete topology, that is, all subsets of E_G as topology. In this topology, t_i and x_i are continuous, so there is a topology in which these functions are continues. The topology on E_G is well defined as being the smallest topology having this property.
Since $(t_0x_0, \dots, t_kx_k, \dots) \in E_G$ implies that there is an i such that $t_i \neq 0$, it follows that, for $g \in G$, if $(t_0x_0, \dots, t_kx_k, \dots)g = (t_0x_0, \dots, t_kx_k, \dots)$, then $x_ig = x_i$, where $x_i \in G$, and this is only possible if $g = 1$. So G acts freely on E_G.
Define τ, for an i (for example the smallest i) such that $t_i \neq 0$, as

$$\tau : \{(\langle x,t\rangle, \langle x,t\rangle g) \in E_G \times E_G | \langle x,t\rangle \in E_G, g \in G\} \to G$$
$$\tau(\langle x,t\rangle, \langle x',t\rangle) = x_i(\langle x,t\rangle)^{-1}x_i(\langle x',t\rangle)$$

Then

$$\begin{aligned}\langle x,t\rangle\tau(\langle x,t\rangle, \langle x',t\rangle) &= \langle x,t\rangle x_i(\langle x,t\rangle)^{-1}x_i(\langle x',t\rangle) \\ &= \langle t_0x_0x_i^{-1}x_ig, \dots, t_nx_nx_i^{-1}x_ig, \dots\rangle \\ &= \langle x,t\rangle g = \langle x',t\rangle\end{aligned}$$

Consequently, $\omega_G = (E_G, p, B_G)$ is a principal G-bundle.

Finally, to show that ω_G is numerable, define the map $u_i : B_G \to [0,1]$ with the property that $u_ip = t_i$ on E_G. Since $t_i(ag) = t_i(a)$ for any $g \in G$, u_i is unique and well-defined. Now define

$$w_i : B_G \to [0,1]$$
$$w_i(b) = \max(0, u_i(b) - \sum_{j<i} u_j(b))$$

For $b \in B$, let m be the smallest i such that $u_i(b) \neq 0$, and let n be the largest(since there only a finite number of $t_i \neq 0$, this is possible). Then we have $\sum_{m\leq i\leq n} u_i(b) = 1$. Then $u_m(b) = w_m(b)$ and B_G is covered by the open sets $w_i^{-1}(0,1]$. For $i > n$ and for all b' with $\sum_{0\leq i\leq n} u_i(b') > \frac{1}{2}$, we have $w_i(b') = 0$. Since $u_i(b) = 0$ for $n < 1$, it follows that the set consisting of all these b', $N_n(b)$, is an open neighborhood of b such that $N_n(b) \cap w_i^{-1}(0,1]$ is empty for $i > 1$. The open covering $\{w_i(0,1]\}$ is locally finite. For it to be a partition of unity, replace w_i with $v_i = w_i/\sum_j w_j$. Hence $\omega_G = (E_G, p, B_G)$ is numerable. □

What is left to be shown is that this bundle $\omega_G = (E_G, p, B_G)$ is universal. Before I do this, I will give some examples to clarify the Milnor's construction in some specific cases of a group G. For this examples, I will now define the subspaces $E_G(n) \subset E_G$ and $B_G(n) \subset B_G$.

Definition 4.3. For every n, the subspaces $E_G(n) \subset E_G$ and $B_G(n) \subset B_G$ are the subspaces such that $(t_0x_0, t_1x_1, \dots) \in E_G(n)$ provided that $t_i = 0$ for $i > n$, with $p(E_G(n)) = B_G(n)$.

Example 4.4. Take $G = Z_2$. Then the elements of $E_G(1)$ are of the form $(t_0\hat{1}, t_1\hat{1})$, $(t_0\hat{0}, t_1\hat{1})$, $(t_0\hat{1}, t_1\hat{0})$ and $(t_0\hat{0}, t_1\hat{0})$, where $t_0, t_1 \in [0,1]$ and $t_0 + t_1 = 1$ and $\hat{1}$, $\hat{0}$ are the elements of Z_2. These four forms of elements constitute four line pieces, and these four line pieces are glued together by the relations $(0\hat{1}, 1\hat{1}) = (0\hat{0}, 1\hat{1})$, $(0\hat{1}, 1\hat{0}) = (0\hat{0}, 1\hat{0})$, $(1\hat{1}, 0\hat{1}) = (1\hat{1}, 0\hat{0})$ and $(1\hat{0}, 0\hat{1}) = (1\hat{0}, 0\hat{0})$. Therefore, $E_G(1)$ is just S^1 up to homeomorphism and the action of Z_2 on $E_G(1) = S^1$ is given by the identity and the antipodal map. The space $B_G(1)$ is given by $S^1/Z_2 \simeq RP^1 \simeq \mathbb{R}$, the real line. S^1 is not contractible in one dimension, so this bundle (S^1, p, RP^1) is not universal.
In the same way, the space $E_G(n)$ is just the n-sphere S^n up to homeomorphism, and the action of Z_2 on $E_G(n) = S^n$ is given by the identity and the antipodal map. The space $B_G(n)$ is given by the real lines in R^n: $B_G(n) = RP^n$. S^n is contractible in $n-1$ dimensions.
The Total space E_G is S^∞ and the Base space B_G is $B_G = RP^\infty$ and the corresponding bundle $(S^\infty, p, RP^\infty \simeq S^\infty/Z_2)$ is contractible in all dimensions and thus universal.

Example 4.5. Take $G = S^1$. Then the elements of $E_G(1)$ are of the form (t_0x_0, t_1x_1) where $x_0, x_1 \in S^1$ and $t_0, t_1 \in [0,1]$ such that $t_0 + t_1 = 1$, and this is glued together by, for any $x_0, x_1, x_0', x_1' \in S^1$, $(0x_0, 1x_1) = (0x_0', 1x_1)$ and $(1x_0, 0x_1) = (1x_0, 0x_1')$. The space $E_G(1)$ is just S^3 up to homeomorphism, and the action from G on S^3 is given by the standard multiplication with $e^{i\delta} \in S^1$. The base space is $B_G(1) \simeq S^3/S^1 \simeq \mathbb{C}P^1$, the projective space of (complex) lines in $\mathbb{C}^2$, which is homeomorphic to S^2.
The space $E_G(n)$ is just S^{2n+1} up to homeomorphism, and the action is the standard multiplication. The space $B_G(n)$ is $\mathbb{C}P^n$, the n-dimensional projective space. S^{2n+1} is contractible in $2n$ dimensions.
The space E_G is S^∞ and B_G is $\mathbb{C}P^\infty$, and the corresponding bundle is universal.

4.2. **Universality.** We will now prove that the bundle as obtained in Milnor's construction is universal for any group G.

Proposition 4.6. *$\omega_G = (E_G, p, B_G)$, as obtained in Milnor's construction, is universal.*

Proof. As proof I will give two lemmas, this proposition will follow trivially from the lemmas. The first lemma gives that the first criterium of proposition 3.4 is met, and the second lemma gives that the second criterium of proposition 3.4 is met. □

Lemma 4.7. *Let $\omega_G = (E_G, p, B_G)$ be the principal G-bundle that comes from Milnor's construction. Then, for each numerable principal G-bundle ξ over a space B, there exists a map $f : B \to B_G$ such that ξ and $f^*(\omega_G)$ are isomorphic principal G-bundles.*

Proof. It is enough to show that there is a G-morphism $(g, f) : \xi \to \omega_G$, for then ξ and $f^*(\omega_G)$ are isomorphic[6]. I will first show that there is a countable partition of unity $\{u_n\}_{0 \le n}$ on B such that $\xi | u_n^{-1}(0,1]$ is trivial for $n \ge 0$. After that I will define g and with that f, such that (f, g) will prove the lemma.
Let $\{v_i\}_{i \in T}$ be a partition of unity on B, such that ξ is trivial over each $v_i^{-1}(0,1]$.

[6]This is shown in Dale Husemoller, *Fibre bundles*, New york: Springer-Verlag, 1994, Third edition (first edition 1966), chapter 4, 4.2 (p. 44).

This is possible since ξ is numerable. For each $b \in B$, we denote the finite set of $i \in T$ with $v_i(b) > 0$ by S(b), and for each finite subset S of T we denote the open set of $b \in B$ such that $v_i(b) > v_j(b)$ for each $i \in S$, $j \in T - S$, by $W(S)$. Let $u_s : B \to [0,1]$ be the continuous map defined by

$$u_s(b) = \max[0, \min_{i \in S, j \in T-S} (v_i(b) - v_j(b))]$$

Then $W(s) = u_s^{-1}(0,1]$.
Now let $\#S$ denote the number of elements in S. I claim that if $\#S = \#S'$, that then $W(S) \cap W(S')$ is empty. If $S = S'$ this is trivial, if not, then $S - S'$ and $S' - S$ are both non-empty. Let $i \in S - S'$ and let $j \in S' - S$. Let $B \in W(S) \cap W(S')$, then $v_i(b) > v_j(b)$ because $b \in W(S)$ and $v_j(b) > v_i(b)$ because $b \in W(S')$, which is a contradiction.
Now denote W_m and $w_m(b)$ by

$$W_m = \bigcup_{\#S=m} W(S) \text{ and } w_m(b) = \sum_{\#S=m} u_S(b)$$

Then we have $w_m^{-1}(0,1] = W_m$. Let $u_m(b) = w_m(b) / \sum_{0 \leq n} w_n(b)$, then the family $\{u_n\}$ is the desired partition of unity since $u_n^{-1}(0,1] = W_n$. Since $\xi | v_i^{-1}(0,1]$ is trivial, $\xi | u_S^{-1}(0,1]$ is trivial, what means that $\xi | W_n$ is trivial because $\xi | W(S)$ is trivial and W_n is a disjoint union.

Now lets find a G-morphism $(g,f) : \xi \to \omega_G$ to prove the lemma. Let, as we just proved that exists, $\{u_n\}_{0 \leq n}$ be a countable partition of unity on B such that $\xi | u_n^{-1}(0,1]$ is trivial for $n \leq 0$. Let $h_n : u_n^{-1}(0,1] \times G \to E(\xi | u_n^{-1}(0,1]) \subset E(\xi)$ be an isomorphism defining the locally trivial character of ξ(it has this character since $\{u_n\}$ exists as it does). Now define $g : E(\xi) \to E_G$ by the relation

$$g(z) = (u_0 p(z)(q_o h_0^{-1}(z)), \dots, u_n p(z)(q_n h_n^{-1}(z)), \dots)$$

where $q_n : U_n \times G \to G$ is the projection on the second factor. If for a $z \in E(\xi)$, the map h_n^{-1} is undefined, we have $u_n(p_z) = 0$ what means that the map g is well defined. Since, for each $s \in G$, $h_n(zs) = h_n(z)s$ and $p(zs) = p(z)$,the relation $g(zs) = g(z)s$ holds. The map induces $f : B \to B_G$, using the projection maps p, and $(g,f) : \xi \to \omega_G$ is a bundle morphism. □

Lemma 4.8. *Let $\omega_G = (E_G, p, B_G)$ be a universal principal G-bundle that comes from Milnor's construction. Let $f_0, f_1 : X \to B_G$ be two maps such that $f_0^*(\omega_G)$ and $f_1^*(\omega_G)$ are isomorphic. Then f_0 and f_1 are homotopic.*

Proof. First we will consider several maps, after which we can define a G-morphism $(k,f) : \xi \times [0,1] \to \omega_G$ with $f | X \times 0 = f_0$ and $f | X \times 1 = f_1$, what will prove the lemma.
Let E_G^{od} denote the subspace of all $(x,t) \in E_G$ with $t_{2i+1} = 0$ for all $i \geq 0$, and let E_G^{ev} denote the subspace of all $(x,t) \in E_G$ with $t_{2i} = 0$ for all $i \geq 0$. Let B_G^{od} be the subspace $p(E_G^{od})$ and let B_G^{ev} be the subspace $p(E_G^{ev})$. Define the linear functions α_n by

$$\begin{aligned} \alpha_n : & \quad [1 - (1/2)^n, 1 - (1/2)^{n+1}] \to [0,1] \\ \alpha_n(t) &= 2^{n+1} t - 2^{n+1} + 2 \end{aligned}$$

Then $\alpha_n(1-(1/2)^n) = 0$ and $\alpha_n(1-(1/2)^{n+1}) = 1$. We define the homotopy $h_s^{od} : E_G \to E_G$ by $h_s^{od}(x,t) = (x',t')$ such that $h_s^{od}(x,t)y = h_s^{od}(xy,t)$ with, for $s \in I_n = [1-(1/2)^n, 1-(1/2)^n+1]$,

$$x'_i = \begin{cases} x_i & 0 \leq i \leq n \\ x_n + j & i = n+2j-1 \text{ for } 0 < j < \infty \\ x_n + j & i = n+2j \text{ for } 0 < j < \infty \end{cases}$$

$$t'_i = \begin{cases} t_i & 0 \leq i \leq n \\ \alpha_n(s)t_{n-j} & i = n+2j-1 \text{ for } 0 < j < \infty \\ (1-\alpha_n(s))t_{n-j} & i = n+2j \text{ for } 0 < j < \infty \end{cases}$$

and for s=1 the relation $(x',t') = (x,t)$. This is well defined for $s = 1-2^n \in I_n \cap I_{n-1}$ as it should be. The homotopy is continuous since it is continuous on the locally finite open covering of $v_i^{-1}(0,1]$, where $\{v_i\}$ is the partition of unity. The maps $h_s^{od} : E_G \to E_G$ induce $g_s^{od} : B_G \to B_G$ such that $(h_s^{od}, g_s^{od}) : \omega_G \to \omega_G$ is a homotopy of bundle morphisms. Note that $h_0^{od}(E_G) = E_G^{od}$ and $g_0^{od} = B_G^{od}$. By a similar construction we get $(h_s^{ev}, g_s^{ev}) : \omega_g \to \omega_G$ where $h_0^{ev} = E_G^{ev}$ and $g_0^{ev} = B_G^{ev}$.
Now I claim that the bundles $\omega_G, (g_0^{od})^*\omega_G$ and $(g_0^{ev})^*\omega_G$ are all isomorphic. This follows from $(h_1^{od}, g_1^{od}) = (h_1^{ev}, g_1^{ev}) = 1$ and the fact that when there is a homotopy f_t from a space B' to a space B with ξ a numerable principal G-bundle over B, the principal G-bundles $f_0^*(\xi)$ and $f_1^*(\xi)$ are isomorphic.

For the final part of the proof, let ξ be any numerable principal G-bundle isomorphic to $f_0^*(\omega_G)$ and $f_1^*(\omega_G)$. Then f_0 is homotopic to $g_0^{od}f_0$ and f_1 is homotopic to $g_0^{ev}f_1$. Consequently, when changing f_0 and f_1 up to homotopy, we can assume that $f_0(X) \subset B_G^{od}$ and $f_1(X) \subset B_G^{ev}$.
Now we define the G-morphism $(k,f) : \xi \times [0,1] \to \omega_G$ with $f|X \times 0 = f_0$ and $f|X \times 1 = f_1$. For k we have

$$k(z,0) = (t_0(z)x_0(z), 0, t_2(z)x_2(z), 0, \dots)$$
$$k(z,1) = (0, t_0(z)x_0(z), 0, t_3(z)x_3(z), \dots)$$

and we prolong this to $k : E(\xi) \times I \to E_G$ by the function

$$k(z,s) = ((1-s)t_0(z)x_0(z), st_0(z)x_0(z), (1-s)t_2(z)x_2(z), st_3(z)x_3(z), \dots)$$

Since, for $y \in G$, $t_i(zy) = t_i(z)$ and $x_i(zy) = x_i(z)y$, we have $k(zy,s) = k(z,s)y$. Then the map $k(z,s)$ induces a map $f : X \times I \to B_G$ such that $f(b,0) = f_0(b)$ and $f(b,1) = f_1(b)$. This proves the lemma. □

The conclusion is that, for any G, the bundle $\omega_G = (E_G, p, B_G)$ which arises from Milnor's construction is universal, which means that B_G is a classifying space.

5. Simplicial Techniques

In this section I will provide constructions of classifying spaces using simplicial techniques. First, I will give an appropriate criterium of universality, then simplicial objects will be defined, after which universal principal G-bundles can be constructed.

The following proposition is a characterization of the classifying spaces, given by Steenrod[7].

Proposition 5.1. *Let G be a topological group, and let $\xi = (E, p, Z)$ be a principal G-bundle. Then ξ is a universal bundle for G-bundles on finite CW-complexes if and only if $\pi_i(E) = 0$ for all i.*

Remark 5.2. If E is itself a finite CW-complex, $\pi_i(E) = 0$ is equivalent with E being contractible, what means that $\xi = (E, p, Z)$ is universal not only for G-bundles on finite CW-complexes, but for all G-bundles (proposition 3.9).

5.1. **Simplicial Objects.**

Definition 5.3. Let Ord be the category whose objects are finite sets ordered by a transitive $\leq$ relation, and whose morphisms are the order-preserving set maps. The objects of Ord are called $0, 1, 2, \ldots$, where $0 = \{0\}$, $1 = \{0, 1\}$, $2 = \{0, 1, 2\}$,$\ldots$ and are ordered by inclusion. The order-preserving property of the morphisms means that for a morphism f, if $x \leq y$, then $f(x) \leq f(y)$.

Two kinds of morphisms, face maps and degeneracy maps, will now be defined. All morphisms in Ord are compositions of these maps (see proposition 5.5). Therefore when considering the morphisms in Ord, we only have to consider the face and degeneracy maps.

Definition 5.4. The injections $\delta_i : n \to n+1$, called **face maps**, are defined, for $j \in n = \{0, \ldots, n\}$ and $i \in n+1$, by

$$\begin{aligned} \delta_i(j) &= j \quad \text{if} \quad j < i \\ \delta_i(j) &= j+1 \quad \text{if} \quad j \geq i \end{aligned}$$

That means that δ_i maps n to $n+1$ injectively, preserving order and omitting $i \in n+1$ from its range.
The surjections $\epsilon_i : n \to n-1$, called **degeneracy maps**, are defined, for $j \in n = \{0, \ldots, n\}$ and $i \in n-1$, by

$$\begin{aligned} \epsilon_i(j) &= j \quad \text{if} \quad j \leq i \\ \epsilon_i(j) &= j-1 \quad \text{if} \quad j > i \end{aligned}$$

That means that ϵ maps n to $n-1$ surjectively, preserving order and mapping i and $i+1$ to i.

Note that there are $n+2$ face maps δ_i from n to $n+1$, and n degeneracy maps ϵ_i from n to $n-1$.

[7] Steenrod, N., "The topology of fibre bundles", Princeton University Press, Princeton, 1951, p. 102.

Proposition 5.5. *All morphisms in Ord can be written as a composition of face and degeneracy maps.*

Proof. Let $f : k \to l$ be an order-preserving map, where $k, l \in Ord$.
First assume $l > k$. Then there is a trivial injection $I : k \to l$, $I(j) = j$, defined by $I = \delta_l \circ \delta_{l-1} \circ \cdots \circ \delta_{k+2} \circ \delta_{k+1}$(this sends $(0, 1, \dots, k) \to (0, 1, \dots, k, -) \to (0, 1, \dots, k, -, -) \to \cdots \to (0, 1, \dots, k, -, \dots)$ and therefore injects k trivial into l using face maps). If there is a map $f' : l \to l$ that is a composition of face and degeneracy maps such that $f' \circ I = f$, the problem is solved for $l > k$. Therefore, the case $l > k$ is reduced to the case where $k = l$.

Now, let $k = l$. If $f(0) = 0$, let $f_0 : l \to l : j \to j$ be the identity map. If $f(0) = i_0 > 0$, let for all $a \in \{1, \dots i_0\}$, x_a be the biggest i_0 elements (ordered $x_1 > x_2 > \cdots > x_a$) for which $f(x_i) = f(x_i + 1)$ and $x_a = x_{a'}$ only when $a = a'$. Let $f_0 : l \to l$ be defined by first letting a degeneracy map send $x_1, x_1 + 1$ to the same spot, then letting a face map ommit the current place of 0 from its range, and continu doing this for $x_2, x_3, \dots$ and $1, 2, \dots$, until 0 is in place. Then $f_0(0) = f(0)$ and $f_0(j_1) = f_0(j_2)$ only when $f(j_1) = f(j_2)$.
Now assume $f_0, \dots, f_i$ is defined such that $f_i \circ \cdots \circ f_0(b) = f(b)$ for $b \in \{0, \dots, i\}$ and $f_i \circ \cdots \circ f_0(b)(j_1) = f_i \circ \cdots \circ f_0(b)(j_2)$ only when $f(j_1) = f(j_2)$. We will now define $f_{i+1} : l \to l$. If $f_i \circ \cdots \circ f_0(i+1) = f(i+1)$, let f_{i+1} be the identity map. If $f(i+1) - f_i \circ \cdots \circ f_0(i+1) = -1$, let $f_{i+1} = \delta_l \epsilon_{f(i+1)}$. If $f(i+1) - f_i \circ \cdots \circ f_0(i+1) > 0$, let $x_{i_0+1}, \dots, x_{i_1-1}$ be the next few biggest elements for which $f(x_{i+1}) = f(x_{i+1}+1)$ and let f_{i+1} be defined by first letting a degeneracy map send $x_{i+1}, x_{i+1} + 1$ to the same spot, then letting a face map omit the current place of $i + 1$ from its range, and continu doing these two things for $x_i + 2, \dots$ and $i + 1, \dots$, until $i + 1$ is in place. Then $f_{i+1} \circ \cdots \circ f_0(j_1) = f(j_1)$ for $j_1 \in \{0, 1, \dots, i + 1\}$ and $f_{i+1} \circ \cdots \circ f_0(j_1) = f_{i+1} \circ \cdots \circ f_0(j_2)$ only when $f(j_1) = f(j_2)$. Then, by induction, $f_l \circ \cdots \circ f_0 = f$, what means that f can be written as composed of face and degeneracy maps in the case that $k = l$.

Now assume $k > l$. Then we can use degeneracy maps to define a surjection $J : k \to l$, that has the property that $J(j_1) = J(j_2)$ only when $f(j_1) = f(j_2)$. Then again, the problem is reduced to the case where $k = l$, since finding an composition in face and degeneracy maps of $f' : l \to l$ where $f' \circ J = f$, is now enough. Therefore, all morphisms in Ord can be written as compositions of face and degeneracy maps. □

Now, let C be any category. A simplicial object in C is a functor A from Ord to C.

Definition 5.6. Let C be any category. A simplicial object A in C is a contravariant functor from Ord to C, that is, a mapping that associates to each object $i \in Ord$ an object $A(i)$ in C and associates to each morphism $f : i \to j$, for $i, j \in Ord$ a morphism $A(f) : A(i) \to A(j)$ such that $A(id_X) = id_A(X)$ and $A(g \circ f) = A(f) \circ A(g)$ for $f : i \to j$ and $g : j \to k$, for $i, j, k \in Ord$(contravariant functors reverse the direction of composition).
Explicitly this means that for every $n \in Ord$ and for each face map $\delta : n \to n + 1$, there is a map $A(\delta_i) : A(n + 1) \to A(n)$ and for each map $\epsilon : n \to n - 1$, there is a map $A(\epsilon_i) : A(n - 1) \to A(n)$.

If B is a covariant functor from Ord to C, what means that $B(g \circ f) = B(g) \circ B(f)$ for $f : i \to j$ and $g : j \to k$, then B is called a co-simplicial object in C. What follows explicitly is that, where $B(n)$ is associated with an object $n \in Ord$, there is a map $B(\delta_i) : B(n) \to B(n+1)$ for each face map $\delta_i : n \to n+1$ and a map $B(\epsilon_i) : B(n) \to B(n-1)$ for each degeneracy map $\epsilon_i : n \to n-1$.

5.2. **Construction of BG for a Discrete Group G.** Here I wil give a simplicial construction of a base space BG for any discrete group G.

This construction starts with choosing a point to be the 0-skeleton of BG. Next, we add an oriented loop for each element $x \in G$. We then attach 2-simplexes (triangles) to kill each relation of the form $xy = z$ with $x, y, z \in G$. This means we have a 2-simplex corresponding to each pair (x, y).
In the same way, we attach n-simplexes (the n-dimensional analogue of a triangle) for each n corresponding to the n-tuple $(x_1, x_2, \dots, x_n)$ where $x_1x_2 \dots x_n = y$. Having done this, it is not hard to see that $\pi_i(E) = 0$ for all i.
I will now define a simplicial set NG based on this construction, to describe it in simplicial terms. Later, BG will be the geometric realization of NG.

Definition 5.7. Let NG be the simplicial set from Ord to $Sets$, with

$$\begin{aligned} NG(0) &= \text{a point} \\ NG(1) &= G \\ NG(2) &= G \times G \\ NG(p) &= G^p = G \times \cdots \times G \text{ (p factors)} \end{aligned}$$

and

$$\begin{aligned} NG(\delta_i) : G^p \to G^{p-1} & \qquad i = 0, \dots, p \\ NG(\delta_0)(x_1, \dots, x_p) &= (x_2, \dots, x_p) \\ NG(\delta_p)(x_1, \dots, x_p) &= (x_1, \dots, x_{p-1}) \\ NG(\delta_i)(x_1, \dots, x_p) &= (x_1, \dots, x_i x_{i+1}, \dots, x_p) \end{aligned}$$

and

$$\begin{aligned} NG(\epsilon_i) : G^{p-1} \to G^p & \qquad i = 0, \dots, p \\ NG(\epsilon_0)(x_1, \dots, x_{p-1}) &= (1, x_1, \dots, x_p) \\ NG(\epsilon_i)(x_1, \dots, x_{p-1}) &= (x_1, \dots, x_i, 1, x_{i+1}, \dots, x_{p-1}) \end{aligned}$$

We will now define the geometric realization of a simplicial space. Since a simplicial set can be made into a topological space by giving it the discrete topology, a geometric realization of a simplicial set is also possible. We will use the next definition to make the geometric realization BG of the simplicial set NG.

Definition 5.8. Let X be a simplicial space from Ord to $TopologicalSpaces$. Define a co-simplicial space Δ from Ord to $TopologicalSpaces$, by

$$\Delta(i) = \Delta^i,$$

where Δ^i us the Euclidian n-simplex, and

$$\begin{aligned} \Delta(\delta_i) : \Delta^{n-1} \to \Delta^n, & \qquad i = 0, \ldots, p \\ \Delta(\delta_i)(t_0, \ldots, t_{n-1}) &= (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \Delta(\epsilon_i) : \Delta^n \to \Delta^{n-1}, & \qquad i = 0, \ldots, p \\ \Delta(\epsilon_i)(t_0, \ldots, t_n) &= (t_0, \ldots, t_{i-1} + t_i, \ldots, t_n) \end{aligned}$$

both in barycentric coordinates, that means, coordinates that are defined by the vertices of the simplex.
Define $|X|$, the **geometric realization** of X, by

$$(\coprod_n (X(n) \times \Delta^n))/\sim$$

which is the quotient space of the topological sum (disjoint union) of the spaces $X(n) \times \Delta^n$ where the equivalence is defined by

$$(x, \Delta(f)(t)) \sim (X(f)(x), t)$$

for $x \in X(m)$, $t \in \Delta^n$ and $f : n \to m$ any morphism in Ord.

Now take the simplicial NG as defined above. This is a simplicial space with the discrete topology, that is, with the collection of all subsets of NG as topology. That is what the next definition says.

Definition 5.9. Define

$$BG = |NG| = \text{the geometric realization of } NG$$

That means that

$$BG = (\coprod_{p=0}^{\infty} (G^p \times \Delta^p))/\sim$$

where $\sim$ means equivalence via face and degeneracy maps. This means that

$$\begin{aligned} BG &= \text{point} \cup (\text{a 1-simplex for each } x \in G) \\ &\cup \ (\text{a 2-simplex for each pair } (x, y) \in G \times G) \cup \ldots \end{aligned}$$

with the 2-simplex (x, y) glued to the 1-skeleton by $(x, y) = xy$ and in the same way with the n-simplex glued to the $(n-1)$-skeleton by $(x_1, \ldots, x_n) = x_1 \cdot \cdots \cdot x_n$, for each n.

5.3. **Construction of BG for a Topological Group G.** The construction of BG for a discrete group G as given above carries almost completely over to the case where G is a topological group. The only difference is that now we topologize G with its given topology instead of the discrete topology. This means that $G^p \times \Delta^p$ in $BG = (\coprod_{p=0}^{\infty} (G^p \times \Delta^p))/\sim$ has the product topology obtained from the topology of G and the standard topology of Δ^i. This construction gives us, with $\sim$ gluing via the maps δ_i and ϵ_i, that

$$BG = |NG| = \big((\star \times \Delta^0) \coprod (G \times \Delta^1) \coprod (G \times G \times \Delta^2) \coprod \ldots\big)/\sim$$

5.4. **Construction of PG and $EG = |PG|$.** To prove that $BG = |NG|$ is a classifying space for G-bundles, we will construct a simplicial space PG such that $(|PG|, \pi, |NG|)$ is a universal principal G-bundle.

Definition 5.10. Define the simplicial space PG from Ord to $TopologicalSpaces$ by

$$PG(n) = G^{n+1}$$

where the degeneracy and face-maps are given by

$$\begin{array}{rcl} PG(\delta_i) : G^{p+1} \to G^p, & & i = 0, \dots, p \\ PG(\delta_i)(x_0, \dots, x_p) & = & (x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p) \end{array}$$

and

$$\begin{array}{rcl} PG(\epsilon_i) : G^{p+1} \to G^{p+2}, & & i = 0, \dots, p \\ PG(\epsilon_i)(x_0, \dots, x_p) & = & (x_0, \dots, x_i, x_i, \dots, x_p) \end{array}$$

Now there is a map

$$\pi : \qquad PG \to NG$$

sending

$$G \xrightarrow{\pi} \star, G \times G \xrightarrow{\pi} G, G^3 \xrightarrow{\pi} G^2, \dots$$

given by

$$\pi(x_0, \dots, x_p) \quad = \quad (x_0x_1^{-1}, x_1x_2^{-1}, \dots, x_{p-1}x_p^{-1})$$

The next proposition says that (PG, π, NG) is a principal G-bundle. This is an important step in proving that the bundle of geometric realizations of PG and NG is also a principal G-bundle. Proposition 5.12 will give us that $(|PG|, \pi, |NG|)$ is a universal principal G-bundle, and hence that $|NG|$ is a classifying space, when G is a Lie group.

Proposition 5.11. *(PG, π, NG) is a principal G-bundle.*

Proof. First we will show that π is correctly defined.
For π to be a morphism of simplicial spaces, π has to commute with δ_i and with ϵ_i, so we must check that

$$\pi PG(\delta_i) = NG(\delta_i)\pi \text{ for all maps } \delta_i$$
$$\pi PG(\epsilon_i) = NG(\epsilon_i)\pi \text{ for all maps } \epsilon_i$$

Therefore, for $i \neq 0$, p, we compute

$$\begin{array}{rcl} \pi PG(\delta_i)(x_0, \dots, x_p) & = & \pi(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_p) \\ & = & (x_0x_1^{-1}, \dots, x_{i-1}x_{i+1}^{-1}, \dots, x_{p-1}x_p^{-1}) \\ & = & (x_0x_1^{-1}, \dots, x_{i-1}x_i^{-1}x_ix_{i+1}^{-1}, \dots, x_{p-1}x_p^{-1}) \\ & = & NG(\delta_i)(x_0x_1^{-1}, \dots, x_{p-1}x_p^{-1}) = NG(\delta_i)\pi(x_0, \dots, x_p) \end{array}$$

and for $i = 0$, p, we compute

$$\begin{aligned}\pi PG(\delta_0)(x_0, \ldots, x_p) &= \pi(x_1, \ldots, x_p) \\ &= (x_1 x_2^{-1}, \ldots, x_{p-1} x_p^{-1}) \\ &= NG(\delta_0)(x_0 x_1^{-1}, \ldots, x_{p-1} x_p^{-1}) = NG(\delta_0)\pi(x_0, \ldots, x_p)\end{aligned}$$

Computations for ϵ_i are analogue to this and are left to the reader.
G acts free on the right on PG, for $(x_0, \ldots, x_p) \in PG(p)$, $x \in G$, by

$$(x_0, \ldots, x_p) \cdot x = (x_0 x, \ldots, x_p x)$$

Since $x_i \in G, x_i x = x_i$ means that $x = 1$ what means that $(x_0, \ldots, x_p) \cdot x = (x_0, \ldots, x_p)$ implies $x = 1$, PG is a free G-space.
Define the continues translation function $\tau : PG^* = \{(x, xg) \in PG \times PG | x \in PG, g \in G\} \to G$ by

$$\tau((x_0, \ldots, x_p), (x_0', \ldots, x_p')) \equiv x_0^{-1} x_0'$$

It then follows that $(x_0, \ldots, x_p)\tau((x_0, \ldots, x_p), (x_0', \ldots, x_p')) = (x_0, \ldots, x_p)x_0^{-1}x_0 g = x_0, \ldots, x_p)g$.
Therefore, (PG, π, NG) is a principal G-bundle.

□

What is left to be shown is that $(|PG|, \pi, |NG|)$ is a principal G-bundle and that $|PG|$ is contractible. It can be shown that $(|PG|, \pi, |NG|) = (EG, \pi, BG)$ is a universal principal G-bundle, when G is a topological group with reasonable topology, for example, when G is a Lie group(This is shown by Segal[8]). I will not prove this here.

Proposition 5.12. *For any Lie group G, $(|PG|, \pi, |NG|) = (EG, \pi, BG)$ is a universal principal G-bundle and hence BG is a classifying space.*

[8]Segal, G., *Classifying spaces and spectral sequences.* Inst. des Hautes Etudes Scientifiques, Publ. Math., 34, 105-112.

6. Quantum Geometry

In my eyes, where mathematics becomes more and more abstract, it also becomes more and more beautiful. Durdevich wrote in an introduction to Quantum Geometry that

> Quantum geometry is a new branch in mathematics. It introduces a completely new concept of a space, by unifying methods of classical geometry with non-commutative $\mathbb{C}^*$-algebras and functional analysis, and incoprorating(sic) into geometry various ideas from quantum physics.

I think this means that there are ways to generalize the concepts in this thesis, into more abstract and with that more beautiful forms of geometry.
In this final section we will take some steps towards generalizing our concepts of G-bundles and classifying spaces in quantum geometry, following some introduction papers of Micho Durdevich. We will continue in a somewhat less formal way then the previous chapters, for two reasons. Firstly, my main source, Durdevich, writes in an informal way, and secondly, this informal way is best to get a good picture and short introduction of what Quantum geometry and Quantum classifying spaces are. This means that not everything that is stated will be proved. Although somewhat less formal, I will try to stay consistent and complete in what I say.

First Quantum spaces will be explained, then Quantum Groups and Quantum bundles will be explained and the section will end with the notion of a Quantum Classifying space.

6.1. **Quantum Spaces.** Quantum geometry deals with quantum spaces and has the classical concept of a space as a special case. The classical concept of a space is a collection of points equipped with an additional structure. Quantum spaces are not interpretable in this way; in general, Quantum spaces have no points at all, but rather something like intervals or indistinguishable points.
An interesting potential application of Quantum geometry in physics is to provide a mathematical coherent description of the physical space-time, where quantum geometry might help to unify gravity (general relativity) and quantum theory. This is because in Quantum theory, the space has no real points either, only intervals with a certain(very small, probably $\sqrt{\frac{\gamma h}{c^3}}$) length.
Quantum spaces are described by certain non-commutative complex *-algebras, that is, certain complex sets (of functions) with algebraic operations on them and a *-operation (for example, standard complex conjugation), which do not need to be commutative. When the algebras are commutative, we are back in the classical geometry. I will explain what I just said in two conceptual steps: first I will translate geometry into commutative algebra language, and then do non-commutative generalizations.

6.1.1. *Geometry in Commutative Algebra Language.* A definition of an algebra and of a character will now be given. Later on, characters will play the role in an algebra that points play in a space.

Definition 6.1. An **algebra** is a vector space V over a field F with a multiplication. The multiplication must be distributive and, for every $f \in F$ and $x, y \in V$, it must

satisfy $f(xy) = (fx)y = x(fy)$.
A **Banach algebra** is an algebra V over a field F such that V is a Banach space (that is, a complete vector space with a norm $\|\cdot\|$).
A $\mathbb{C}^*$**-algebra** is a Banach algebra with an operator $*$ which satisfies the following axioms:

$$\begin{aligned} (x^*)^* &= x \\ x^*y^* &= (yx)^* \\ x^* + y^* &= (x+y)^* \\ (cx)^* &= \overline{c}x^* \\ \|aa^*\| &= \|a\|^2 \end{aligned}$$

where $x, y \in V$ and $c \in \mathbb{C}$, with $\overline{c}$ standard complex conjugation.
A **character** of a $\mathbb{C}^*$-algebra is a $*$-homomorphism $k : V \to \mathbb{C}$, that is, a morphism from the algebra into the complex numbers, which satisfies the following axioms:

$$\begin{aligned} k(x+y) &= k(x) + k(y) \\ k(cx) &= ck(x) \text{ for } c \in \mathbb{C} \\ k(xy) &= k(x)k(y) \\ k(x^*) &= k(x)^* \\ k &\neq 0 \end{aligned}$$

where $x, y \in V$.

We will now show how some geometrical concepts are translated into the language of algebra. First, let us define the unitary commutative $\mathbb{C}^*$-algebra $A = C(X)$ for a topological space X. This can be seen as a translation of the concept of a compact topological space into the language of algebra.

Definition 6.2. Let X be any compact topological space. Define the **unitary commutative** $\mathbb{C}^*$**-algebra** $A = C(X)$ as the $\mathbb{C}^*$-algebra of continuous complex-valued functions on X. The algebraic operations in A are the standard multiplication an addition of functions. The *-operation is the standard complex conjugation. The norm is the maximum norm $\|f\| = \max_{x \in X} |f(x)|$. It is commutative and it is unitary, that is, it has identity elements, namely the $\mathbf{1}$ map $(x \to 1)$ for multiplication and the $\mathbf{0}$ map $(x \to 0)$ for addition.

The classical theorem of Gelfand an Naimark characterizes the algebras $A = C(X)$, as unitary commutative C^*-algebras:

Proposition 6.3. ***Gelf-Naimark Theorem.*** *For every unitary commutative C^*-algebra A there exists an up to homeomorphisms unique compact topological space such that $A \simeq C(X)$.*

Let us now give a translation of the **concept of a point** into the language of algebra. For every element $x \in X$, define $k_x : A \to \mathbb{C}$ by $k(f) = f(x)$. Since f is continuous and complex valued, it is easy to check that k_x satisfies the axioms of a character and therefore k_x is a character. This means that for every point $x \in X$, there is a character k_x.
Conversely, that for every character there is a point, let us consider an arbitrary

character $k : A \to \mathbb{C}$. Since $k(f+g) = k(f) + k(g)$, k cannot be a constant (it is not 0) and must depend on the function. Since $k(cf) = ck(f)$ must also hold, k is linear in $\mathbb{C}$. Since $k(fg) = k(f)k(g)$, it can only be that k depends on the function in a very direct way. The character must be of the form $k(f) = f(x)$ for an $x \in X$. Therefore $k = k_x$.
Therefore, there is a bijection between points of X and characters of A. We can now talk about characters in the commutative unitary $\mathbb{C}^*$-algebra A in the same way as we can talk about points of a compact topological space X.

We will now give a translation of the **concept of a continuous map**. A unitary *-homomorphism will play this role in algebra. First, we will define a *-homomorphism:

Definition 6.4. Let A and B be two *-algebras. A ***-homomorphism** is an algebraic homomorphism $\Phi : A \to B$ which satisfies $\Phi(a^*) = \Phi(a)^*$.

Consider two compact topological spaces X and Y, and denote by A and B the $\mathbb{C}^*$-algebras of continuous functions over X and Y respectively.
For any continuous map $F : X \to Y$, define the map $\Phi_F : B \to A$ by composition $\Phi_F(g) = g \circ F$. Then Φ_F sends a map from Y to $\mathbb{C}$ to a map from X to $\mathbb{C}$ using the map $F : X \to Y$. Since it satisfies $\Phi_F(g^*) = g^* \circ F = (g \circ F)^* = \Phi(g)^*$ it is a $*$-homomorphism.
Conversely, consider an arbitrary unitary $*$-homomorphism $\Phi : B \to A$. This sends a map from Y to $\mathbb{C}$ to a map from X to $\mathbb{C}$ in a way satisfying the axioms of a *-homomorphism. It can be shown that Φ is always of the form $\Phi = \Phi_F$ for a uniquely determined continuous map $F : X \to Y$, and this can be understood by seeing that this is the only way for Φ to use the map from X to $\mathbb{C}$ to get a map from Y to $\mathbb{C}$ in a way it satisfies $\Phi(a^*) = \Phi(a)^*$.
This means there is a natural bijection between continuous maps from X to Y, and unitary $*$-homomorphisms from B to A. Properties of the map F are reflected as properties of Φ_F and vice versa. For example, F is surjective iff Φ_F is injective and F is injective iff Φ_F is surjective.
If C is the $\mathbb{C}^*$-algebra of continuous functions on the direct product $X \times Y$, then the natural identification $C \leftrightarrow A \otimes B$ holds, where $\otimes$ is the $\mathbb{C}^*$-algebraic tensor product.

The **concept of a compact topological group**[9] can be translated in the following way. Let X be a compact topological group. Then X is a compact topological space with a group structure, such that the product map $X \times X \to X$ is continuous. The topological space is represented by a commutative unitary $\mathbb{C}^*$-algebra A and the product map is represented by the $*$-homomorphism $\Phi : A \to A \times A$, as described above. The first group axiom, associativity of the product, is equivalent to the co-associativity property $(\phi \otimes id)\phi = (id \otimes \phi)\phi$, and it can be shown that the remaining two group axioms (existence of the identity element and of the inverse) are equivalent to the single assumption that the elements of the form $a\phi(b)$ and $\phi(b)a$, where $a, b \in A$, span two everywhere dense linear subspaces of $A \otimes A$.

[9]Originally characterized in the paper by S.L. Woronowicz, *Compact Quantum Groups*, Symetries quantiques (Les Houches, 1995), 845–884, North-Holland, Amsterdam, 1998.

The following table is a mini dictionary between geometry and algebra, as given by Durdevich. I discussed the most important translations above, the other ones I will not use here.[10]:

Compact topological space X	Unitary commutative $\mathbb{C}^*$-algebras
X =compact topological space	$A = \mathbb{C}(X) = \{$complex continuous functions on $X\}$
Points $x \in X$	Characters $k = k_x$ of A
Continuous maps between compacts X any $Y \leftrightarrow B$	unitary $*$-homomorphisms from B to A
The direct product $X \times Y$	The $\mathbb{C}^*$-tensor product $A \otimes B$
Symmetries of X	Automorphisms of A
Group structure on X	Coproduct map $\Phi : A \to A \otimes A$
Probability measures on a metrizable compact X	Positive normalized linear functionals on A
Locally-compact/noncompact topological spaces	Non-unitary commutative $\mathbb{C}^*$-algebras
Measure theory	Commutative von Neuman algebras
X =Compact smooth manifold	$A = \mathbb{C}^\infty(X) = \{$complex smooth functions on $X\}$
Vector bundles over X	Finite projective modules over A
Vector fields on X	Hermitian derivations on A
Differential forms on X	Graded-differential algebra $\Omega(X) \subset \mathbb{C}(\Xi, A)$

6.1.2. *Non-Commutative Generalizations.* The second main conceptual step consists in non-commutative generalizations. This means that the reformulated classical geometry (reformulated in the language of algebra) will be generalized by relaxing the assumption of commutativity of the algebra $A = \mathbb{C}(X)$, by allowing A to be the appropriately chosen non-commutative $\mathbb{C}^*$-algebra. Having done this we arrive at quantum spaces, the main objects of study in quantum geometry.
The elements of these non-commutative $\mathbb{C}^*$-algebras are intuitively interpreted as continuous functions over quantum spaces. In contrast to classical geometry, is the existence of quantum spaces implicit, as they generally appear in the formalism only via the corresponding $\mathbb{C}^*$-algebras.

6.2. **Quantum Groups.** A compact quantum group will now be defined. As can be expected from the translations above, the group structure is described by a *-homomorphism $\phi : A \to A \otimes A$.

Definition 6.5. A compact quantum group G is a quantum space G together with a group structure which is described by a $*$-homomorphism $\phi : A \to A \otimes A$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A \otimes A \\ \downarrow^{\phi} & & \downarrow^{id\otimes\phi} \\ A \otimes A & \xrightarrow{\phi\otimes id} & A \otimes A \otimes A \end{array}$$

[10]blah

is commutative, and such that

$$A \otimes A = \overline{\{\sum a\phi(b) | a, b \in A\}}$$

$$A \otimes A = \overline{\{\sum \phi(b)a | a, b \in A\}}$$

Remark 6.6. The two criteria $\phi : A \to A \otimes A$ must fulfill are the criteria stated above in the discussion of the translation of the concept of a compact topological group to the language of algebra and are in the case of a commutative $\mathbb{C}^*$-algebra (that is, when the quantum space G is commutative, when we are back at the classical case) equivalent to the standard group axioms.

6.3. **Quantum Principal G-Bundles.** Quantum principal G-bundles can be defined using quantum groups, and the main geometrical idea is the same as in classical theory, namely that of a fibered space on which the structure group acts freely on the right, so that the fibres are the orbits of this action.

Let the quantum principal G-bundle and the structure group be represented by $\mathbb{C}^*$-algebras B and A respectively. The right action of G on P, the total quantum space, is represented by a $*$-homomorphism $F : B \to B \otimes A$, such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & B \otimes A \\ \downarrow^{\phi} & & \downarrow^{id \otimes \phi} \\ B \otimes A & \xrightarrow{B \otimes id} & B \otimes A \otimes A \end{array}$$

is commutative. The classical condition that the structure group is acting freely on the bundle is expressed by the density condition

$$B \otimes A = \overline{\{\sum bF(q) | b, q \in B\}}$$

6.4. **Quantum Classifying Spaces.** The construction of classifying spaces can be incorporated in the quantum context. Let G be a compact quantum group, that is, an abstract structure which is determined by a noncommutative $\mathbb{C}^*$-algebra whose elements represent continuous complex-valued functions on the compact quantum group. For any such compact quantum group G, it is possible to construct a quantum space B_G and a quantum principal G-bundle E_G over E_B, such that it is universal in the following sense: every quantum principal bundle P over a quantum space M can be obtained as a pull back of the bundle E_G, via a classifying map from M to B_G. The classification exists in the fact that there is a bijection between homotopy classes of quantum principal G-bundles P over M and homotopy classes of classifying maps from M to B_G.

In the classical case, we had isomorphism classes of classical bundles P over M, not homotopy classes of them, as we have in the quantum case. The explanation is that in classical geometry homotopic bundles over the same space, and with the same group structure, are always isomorphic, what is not the case in quantum geometry. There are homotopic but not isomorphic quantum spaces and bundles.

References

1. Armstrong, MA. *Groups and Symmetry.* Springer-Verlag New York inc, 1988.
2. Abad, Camilo Arias. *Representations up to homotopy and cohomology of classifying spaces.* Ph.D Thesis, University Utrecht. 2008.
3. Chruscinski, Dariusz. *Geometric Aspects of Quantum Mechanics and Quantum Entanglement.* Journal of Physics: Conference Series 30, 2006.
4. Durdevich, Micho. *Introduction to Quantum Principal Bundles.*
5. Durdevich, Micho. *A brief introduction to Quantum Geometry.*
6. Griffiths, David J.. *Inroduction to Quantum Mechanics.* Pearson Education, inc., New Jersey. Second edition, 2005 (first edition 1995).
7. Hopf, Heinz. *Uber die Abbildungen der dreidimensionalen Sphäre auf die Kugeläche*. Math. Ann. 104, 1931, 637665.
8. Husemoller, Dale. *Fibre bundles.* New york: Springer-Verlag, 1994. Third edition (first edition 1966).
9. Lyons, David W. *An elementary Introduction to the Hopf Fibration.* Lebanon Valley College, 2003.
10. Moslehian, Mohammad Sal and Weisstein, Eric W. "Banach Algebra." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/BanachAlgebra.html
11. Lees, J. Alexander *Notes on bundle theory*, lecture note series No. 42. August 1974.
12. Mostow, M and Perchik, J. *Notes on Gelfand-Fuks cohomology and Characteristic Classes.* New Mexico State University, December 1973.
13. Renze, John and Weisstein, Eric W. "Algebra." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Algebra.html
14. Sakurai, J. J.. *Modern Quantum Mechanics.* Addison-Wesley Publishing Company. Revised edition, 1994.
15. Segal, Graeme. *Classifying spaces and spectral sequences.* Inst. des Hautes Etudes Scientifiques, Publ. Math., 34. 105-112.
16. Weisstein, Eric W. "$\mathbb{C}^*$-Algebra." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/C-Star-Algebra.html
17. Woronowicz, S.L.. *Compact Quantum Groups.* Symetries quantiques (Les Houches, 1995). p. 845-884. North-Holland, Amsterdam, 1998.
18. http://knowledgerush.com/kr/encyclopedia/Algebra_over_a_field/