CHAPTER 11

Attaching cells

1. Cells

DEFINITION 11.1. *Let X be a Hausdorff topological space. An* <u>*open n-cell*</u> *in X is a subspace $e \subset X$ together with a homeomorphism*

$$\mathring{h}_e : \mathring{D}^n \longrightarrow e \subset X.$$

It is called an <u>*n-cell*</u> *if $\mathring{h}_e$ extends to a continuous map*

$$h_e : D^n \longrightarrow X.$$

We call h_e <u>*the defining map*</u> *of the n-cell, and we also say that e is an n-cell with defining map h_e, or that* <u>*e is the image of*</u> *the n-cell. The* <u>*cell boundary*</u> *of e is defined by:*

$$\partial_{cell}(e) := \overline{e} - e,$$

where $\overline{e}$ is the closure of e in X. The characteristic map χ_e of e is defined as the restriction

$$\chi_e := h_e|_{S^{n-1}} : S^{n-1} \longrightarrow X.$$

REMARK 11.2. Since $\mathring{D}^n$ is dense in D^n, given $\mathring{h}_e$, the extension h_e will be unique (so the real condition is the existence). Also, we could have defined the notion of n-cell in X just as a continuous map $h_e : D^n \longrightarrow X$ which, when restricted to $\mathring{D}^n$, becomes an embedding.

Here are some simple examples (more will come later).

EXAMPLE 11.3. Note that, for $n = 0$, a 0-cell in X is the same thing as a point of X. An interesting 1-cell is $e = S^1 - \{(1, 0)\}$ which is a 1-cell in S^1 with defining map

$$h_e(t) = (cos(\pi t), sin(\pi t)).$$

EXAMPLE 11.4. In $\mathbb{R}$, $e = (1, \infty)$ together with

$$\mathring{h}_e : (-1, 1) \longrightarrow \mathbb{R}, t \mapsto \frac{2}{t+1}$$

is an open 1-cell which is not an 1-cell. Actually, the subsets of $\mathbb{R}$ which are images of open 1-cells are precisely the open intervals (prove this!). Among all the open intervals, the only ones that can be realized as the image of an 1-cell are bounded ones (see the next proposition).

Here are the main properties of n-cells.

PROPOSITION 11.5. *If $e \subset X$ is an n-cell with defining map $h_e : D^n \longrightarrow X$, then*

$$h_e(D^n) = \overline{e}, \ h_e(S^{n-1}) = \partial_{cell}(e).$$

Moreover, as a map from D^n into $\overline{e}$, h_e is a topological quotient map.

Proof. Since h_e is continuous, we have

$$h_e(\overline{B}) \subset \overline{h_e(B)}$$

for all $B \subset D^n$ (prove this!). Choosing $B = \overset{\circ}{D}{}^n$, we obtain $h_e(D^n) \subset \overline{e}$.

On the other hand,

$$e = h_e(\overset{\circ}{D}{}^n) \subset h_e(D^n),$$

and $h_e(D^n)$ is compact (as the image of a compact by a continuous map), hence closed in X (since X is Hausdorff). This implies $\overline{e} \subset h_e(D^n)$. Since the opposite inclusion has been proven, we get $h_e(D^n) = \overline{e}$. We now prove

$$h_e(S^{n-1}) = \overline{e} - e.$$

We first show the inverse inclusion: for $y \in \overline{e} - e$, by the first part, $y = h_e(x)$ for some $x \in D^n$ and, since $e = h_e(\overset{\circ}{D}{}^n)$, x cannot be in $\overset{\circ}{D}{}^n$; hence $x \in S^{n-1}$. We now prove the direct inclusion. So let $y = h_e(x)$ with $x \in S^{n-1}$, and we want to prove that $y \notin e$. Assume the contrary, i.e. $y = h_e(x')$ with $x' \in \overset{\circ}{D}{}^n$. Since x and x' are distinct, we find $U, V \subset D^n$ opens (in D^n) such that

$$x \in U, x' \in V, U \cap V = \emptyset.$$

Since $\overset{\circ}{h}_e$ is a homeomorphism, $h_e(V)$ is open in e, hence also in $\overline{e}$. Since $h_e : D^n \longrightarrow \overline{e}$ is continuous, $h_e^{-1}(h_e(V))$ will be open in D^n; but it contains x, hence we find $\epsilon > 0$ such that

$$D^n \cap B(x, \epsilon) \subset h_e^{-1}(h_e(V))$$

(where the ball is with respect to the usual Euclidean metric). Since $x \in U$ and U is open in D^n, we may choose ϵ so small so that

$$D^n \cap B(x, \epsilon) \subset U.$$

Pick up an element $z \in \overset{\circ}{D}{}^n \cap B(x, \epsilon)$. By the inclusion above, we find $z' \in V$ such that $h_e(z) = h_e(z')$. But then both z and z' are in $\overset{\circ}{D}{}^n$ hence we must have $z = z'$. But $z \in U$ and $z' \in V$-and that contradicts the condition $U \cap V = \emptyset$.

Finally, h_e is a topological quotient map as a continuous surjection from a compact space to a Hausdorff space. □

2. Attaching one n-cell

Definition 11.6. *Let X be a Hausdorff space and $A \subset X$ closed.*

We say that X is obtained from A by attaching an n-cell if there exists an n-cell $e \subset X$, with some defining map h_e, such that

$$X = A \cup e, A \cap e = \emptyset.$$

Example 11.7. It is clear that the n-ball D^n is obtained from $\partial D^n = S^{n-1}$ by attaching and n-cell (the defining map being the identity map). From example 11.3, we see that S^1 can be obtained from a point (which is a 0-cell!) by attaching a 1-cell. In general, S^n can be obtained from a point by attaching an n-cell (see also example 11.10 below).

To treat examples treated in the previous lectures such as the torus, the Moebius band etc, the following lemma is very useful.

Lemma 11.8. *Let $X = D^n$ or $X = [0,1]^n$, and let $Y = X/\sim$ be a Hausdorff quotient obtained by identifying only (certain) points on the boundary $\partial(X)$. Also, let $B = \partial X/\sim$. Then Y is obtained from B by attaching an n-cell.*

PROOF. Since D^n and $X = [0,1]^n$ are homeomorphic by a homeomorphism which preserves their boundary (see e.g. the end of Example 5.5), we may assume that $X = D^n$. We then choose as defining map for the n-cell the quotient map

$$h : D^n \longrightarrow Y,$$

hence the n-cell will be $e := h(\overset{\circ}{D}{}^n)$. Note that $B = \pi(S^{n-1})$. Clearly, $Y = B \cup e$. Next, since no element on the boundary of D^n is equivalent (identified) with an interior element, we have $B \cap e = \emptyset$. Next, since no two interior points of D^n are equivalent, the restriction of h to $\overset{\circ}{D}{}^n$ is injective, hence

$$h|_{\overset{\circ}{D}{}^n} \longrightarrow e$$

is a continuous bijection. We still have to prove that this map is a homeomorphism, and for this it is enough to show that it sends closed sets to closed sets. So, let F be closed in $\overset{\circ}{D}{}^n$; write $F = \overset{\circ}{D}{}^n \cap K$ with $K \subset D^n$ closed. Then K is compact (as a closed inside a compact), hence $h(K)$ will be compact. Since Y was assumed to be Hausdorff, we deduce that $h(K)$ is closed in Y. So, to show that $h(F)$ is closed in e, it suffices to show that

$$h(\overset{\circ}{D}{}^n \cap K) = e \cup h(K).$$

The direct inclusion is clear. For the converse, note that if y belongs to the right hand side we have $y = h(x)$ and $y = h(x')$ with $x \in \overset{\circ}{D}{}^n$ and $x' \in K$, but since interior points are not being identified with any other points, we must have $x = x'$, hence y belongs to the left hand side. $\square$

EXAMPLE 11.9. Consider the torus as a quotient of $X = [0,1] \times [0,1]$ (see Example 5.3) and apply the previous lemma. The resulting space B is shown in the picture (Figure 1) and it consists of two circles touching each other. This space can be drawn (is homeomorphic to) the space consisting of two circles in the plane touching in one point only (a bouquet of two circles). In conclusion, T^2 can be obtained from a bouquet of two spheres by attaching a 2-cell.

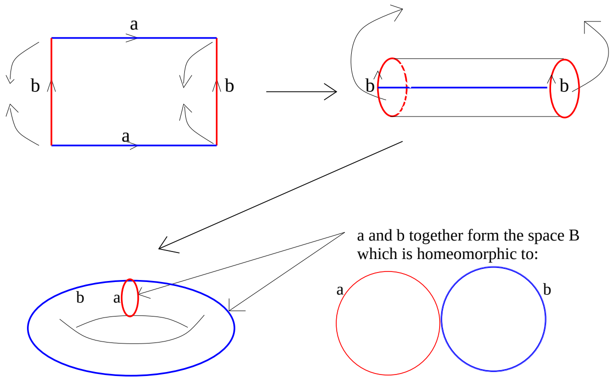


FIGURE 1.

Similarly, we see that the Moebius band is obtained by adjoining a 2-cell to the space B consisting of the circle together with a segment joining the north and the south pole (Figure 2).

EXERCISE 11.1. What about the Klein bottle?

EXAMPLE 11.10. Consider the sphere S^n. We already know (Example 5.5) that $S^n = D^n/\partial D^n$, i.e. S^n can be obtained from D^n by identifying all the points of $\partial D^n = S^{n-1}$ to a point. Hence we can apply the previous lemma. The resulting space B clearly consists of one point only, hence S^n can be obtained from one point by adjoining an n-cell.

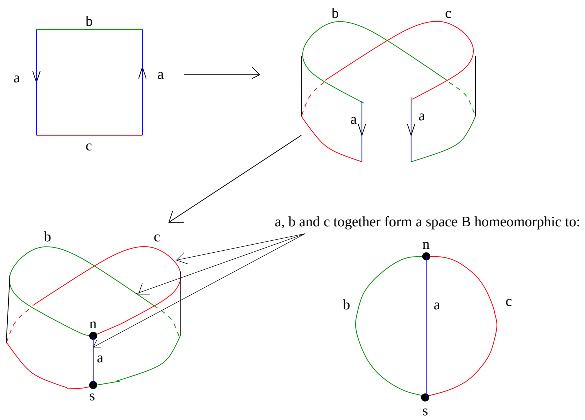


FIGURE 2.

EXAMPLE 11.11. Consider the projective space $\mathbb{P}^n$. To apply the previous lemma, we recall that $\mathbb{P}^n$ "is equal to" the space $D^n/\sim$ obtained by identifying the antipodal points on $\partial D^n = S^{n-1}$ (Example 5.6). Hence, the space B resulting from the lemma will be $S^{n-1}/\sim$- and that "is equal to" the projective space $\mathbb{P}^{n-1}$. In conclusion, $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining an n-cell.

EXERCISE 11.2. In the previous example, "is equal to" really means that "is homeomorphic to". Via the sequence of "is equal to" that is used in the example, it appears that $\mathbb{P}^{n-1}$ is a subspace of $\mathbb{P}^n$. Write out the "equalities" (i.e. homeomorphisms) that we used to conclude that the way we see $\mathbb{P}^{n-1}$ as a subspace of $\mathbb{P}^n$ is via the canonical inclusion

$$\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$$

which associates to a line l inside $\mathbb{R}^n$ the line inside $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ given by

$$l \times \{0\} = \{(x, 0) : x \in l\} \subset \mathbb{R}^{n+1}.$$

3. The characteristic map

Note that, using Proposition 11.5, we see that the characteristic map of e can be viewed as a map

$$\chi_e : S^{n-1} \longrightarrow A.$$

The role of the characteristic map is that "it describes the way that A and e interact inside X", or, equivalently, that it describes "the relationship between the inclusion $i : A \hookrightarrow X$ and the defining map $h : D^n \longrightarrow X$. This relationship is $h|_{S^{n-1}} = i \circ \chi_e$ which should be interpreted as a commutative diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\text{inclusion}"] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"] \\
A \arrow[r, "i"] & X
\end{tikzcd}$$

But the real reason that the characteristic map χ_e is important comes from the fact that X can be reconstructed from the subspace A and the characteristic map χ_e. The following gives already a strong indication of this fact.

PROPOSITION 11.12. *(the universal property) Assume that X is obtained from A by adjoining an n-cell e with defining map $h_e : D^n \longrightarrow X$. Let Y be another topological space. Then a map*

$$f : X \longrightarrow Y$$

is continuous if and only if

$$f|_A : A \longrightarrow Y, \;\; f \circ h_e : D^n \longrightarrow Y$$

are continuous. Moreover, the correspondence

$$f \mapsto (f|_A, f \circ h_e)$$

defines a 1-1 correspondence between

- *continuous maps $f : X \longrightarrow Y$*
- *pairs (f_A, f_e) with $f_A : A \longrightarrow Y$ and $f_e : D^n \longrightarrow Y$ continuous satisfying $f_A \circ \chi_e = f_e|_{S^{n-1}}$.*

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"'] \arrow[ddr, "f_e"] & \\
A \arrow[r, "i"] \arrow[drr, "f_A"'] & X \arrow[dr, "f"] & \\
& & Y
\end{tikzcd}$$

PROOF. If f is continuous, it is clear that $f|_A$ and $f \circ h_e$ are continuous. For the converse, assume that $f|_A$ and $f \circ h_e$ are continuous. To show that f is continuous, we will show that, for $B \subset Y$ closed, $f^{-1}(B)$ is closed in X. Note that

$$f^{-1}(B) \cap A = (f|_A)^{-1}(B), f^{-1}(B) \cap \overline{e} = h_e((f \circ h_e)^{-1}(B)).$$

From the first equality and the continuity of $f|_A$, we deduce that $f^{-1}(B) \cap A$ is closed in A, hence (since A is closed in X), also in X. On the other hand, $(f \circ h_e)^{-1}(B)$ is closed in D^n, hence compact. Then $h_e((f \circ h_e)^{-1}(B))$ is a compact inside the Hausdorff X, hence it is closed in X. Since $f^{-1}(B)$ is the union of two closed subspaces of X (namely $f^{-1}(B) \cap A$ and $f^{-1}(B) \cap \overline{e}$), it is itself closed in X. This concludes the proof of the equivalence.

For the second part we remark that, given f_A and f_e, the conditions

$$f_A = f|_A, f_e = f \circ h_e$$

determine f uniquely because $X = A \cup \mathrm{Im}(h_e)$:

$$f(x) = f_A(x) \text{ if } x \in A, f(x) = f_e(v) \text{ if } x = h_e(v) \in \overline{e}.$$

Moreover, under the assumption $f_A \circ \chi_e = f_e|_{S^{n-1}}$, the previous formulas define f un-ambiguously: if x is both in A and of type $h_e(v)$ with $v \in D^n$, then $f_A(x) = f_e(v)$. Indeed, since $h_e(v) \in A$, v must be in S^{n-1}, hence we can write $x = \chi_e(v)$ and we can use the assumption. The continuity of the maps involved follows from the first part. □

The fact that X only depends on A and χ_e is even stronger indicated by the following:

COROLLARY 11.13. *Let A be a Hausdorff space and let $\chi : S^{n-1} \longrightarrow A$ be a continuous map. For $i \in \{1, 2\}$, assume that X_i is a space which is obtained from A by adjoining an n-cell e_i, and let $\chi_i : S^{n-1} \longrightarrow A$ be the characteristic map of e_i. If $\chi_1 = \chi_2$, then X_1 and X_2 are homeomorphic.*

PROOF. Let $\chi = \chi_1 = \chi_2$, and let h_i be the maps defining the n-cell e_i. We apply the last part of the previous proposition to $X = X_1$, $Y = X_2$ and to the pair (i_2, h_2), where $i_2 : A \longrightarrow X_2$ is the inclusion. We find that there is one and only one continuous map $f_{1,2} : X_1 \longrightarrow X_2$ continuous such that

$$f_{1,2}i_1 = i_2,\ f_{2,1}h_1 = h_2.$$

These conditions mean that: for $a \in A$, $f_{1,2}(a) = a$, while for $y \in h_1(D^n)$, writing $y = h_1(x)$, $f_{1,2}(y) = h_2(x)$. Since $X_1 = A \cup h_{e_1}(D^n)$, these formulas do define $f_{1,2}$ uniquely, but what the universal property is telling us is that $f_{1,2}$ is well defined and it is continuous. Of course, this could have been checked directly (and that is precisely what we have done when proving the universal property), but the hole point of the "universal property philosophy" is that everything can be done by using this property alone (and not the fact that X_i is obtained from A by adjoining and n-cell).

Leaving aside the "philosophical arguments", and choosing one of the two (rather equivalent) ways of defining the map $f_{1,2}$, we end up with our continuous map $f_{1,2} : X_1 \longrightarrow X_2$. Similarly (by exchanging the role of X_1 and X_2), we obtain a map $f_{2,1} : X_1 \longrightarrow X_2$. Our maps fit into the following diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_1"] \arrow[ddr, "h_2"] \arrow[dddrr, "h_1"] & & \\
A \arrow[r, "i_1"] \arrow[drr, "i_2"] \arrow[ddrrr, "i_1"'] & X_1 \arrow[dr, "f_{1,2}"] & & \\
& & X_2 \arrow[dr, "f_{2,1}"] & \\
& & & X_1
\end{tikzcd}$$

We claim that $f_{1,2}$ and $f_{2,1}$ are inverse to each other, i.e. $f_{1,2}f_{2,1} = \mathrm{Id}_{X_1}$ and $f_{2,1}f_{1,2} = Id_{X_2}$. If we choose to use the explicit description of $f_{1,2}$ and $f_{2,1}$, this is immediate and concludes the proof. Alternatively (but equivalently!), these relations are, again, a consequence of the universal property. For instance, using the formulas that $f_{1,2}$ and $f_{2,1}$ satisfy (by their definition via the universal property), we deduce that $f := f_{1,2}f_{2,1} : X_1 \longrightarrow X_2$ satisfies:

$$f \circ i_1 = f_{2,1} \circ i_2 = i_1 = \mathrm{Id}_{X_1} \circ i_1,$$

$$f \circ h_{e_1} = f_{2,1} \circ h_{e_2} = h_{e_1} = \mathrm{Id}_{X_1} \circ h_{e_1}.$$

Using now the universal property for $X = X_1$, $Y = X_1$ (the uniqueness part!), we deduce that $f = \mathrm{Id}_{X_1}$. Similarly, $f_{2,1}f_{1,2} = \mathrm{Id}_{X_2}$. This proves that X_1 and X_2 are homeomorphic (... for any two spaces X_1 and X_2 which satisfy the universal property). □

4. Reconstruction

Finally, we show how to reconstruct X from the subspace A and the characteristic map χ_e. So, let's start with a Hausdorff topological space A and a continuous map

$$\chi : S^{n-1} \longrightarrow A.$$

We construct our space in two steps.

- We consider the disjoint union
$$A \coprod D^n.$$
As a set, it is the union of disjoint copies of A and D^n. Any subset of $A \coprod D^n$ is of type $U \coprod V$ with $U \subset A$ and $V \subset D^n$. On $A \coprod D^n$ we consider the topology consisting of those subsets $U \coprod V$ with U open in A and V open in D^n.
- We consider the quotient of $A \coprod D^n$ obtained by identifying each point $x \in S^{n-1}$ with $\chi(x) \in A$; we denote by $A \cup_\chi D^n$ the result. Hence
$$A \cup_\chi D^n = A \coprod D^n / \sim,$$
where $\sim$ is the smallest equivalence relation in $A \coprod D^n$ with the property that
$$x \sim \chi(x), \quad \forall\ x \in S^{n-1}.$$

Remark 11.14. Note that the construction of X is inspired by the universal property; actually, X is constructed by "brute force", forcing the universal property to hold. To explain this, we remind that maps out of X (i.e. defined on X) should correspond to certain pairs of maps, one out of A and one out of D^n. The first step (taking the disjoint union of A and D^n) takes care of this property. However, the inclusion of A into the disjoint union, and the inclusion of D^n into the disjoint union (a map out of A together with a map out of D^n!) do not satisfy the formula appearing in the universal property (for $x \in S^{n-1}$, x itself and $\chi(x) \in A$ are distinct elements in the disjoint union!). The second step in our construction forces the desired formula (by identifying x with $\chi(x)$).

Some more explanations now.

- First about A. In the first step, A is a subspace of $A \coprod D^n$. Passing to the quotient $A \cup_\chi D^n$, note that the equivalence class of an element a coming from A can be safely denoted by the same symbol a (two elements coming from A are equivalent only if they are equal!). In this way we view A as a *subset* of $A \cup_\chi D^n$.
- Now D^n. As before, D^n is a subspace of $A \coprod D^n$. Passing to the quotient, we obtain a map which associates to x the equivalence class of x, and which we denote by
$$h : D^n \longrightarrow A \cup_\chi D^n.$$
In other words, h is the restriction of the quotient map $\pi : A \coprod D^n \longrightarrow A \cup_\chi D^n$ to D^n. The map h will be the defining map for our n-cell.
- In particular, we define the n-cell
$$e := h(\overset{\circ}{D}{}^n) \subset A \cup_\chi D^n.$$

Exercise 11.3. Show that A is a subspace of $A \cup_\chi D^n$, i.e. the topology induced on A (the restricted topology) coincides with the original topology of A.

Exercise 11.4. Show that the restriction of h to $\overset{\circ}{D}{}^n$ defines a homeomorphism between $\overset{\circ}{D}{}^n$ and e.

Exercise 11.5. We now know that $A \cup_\chi D^n$ is obtained from A by adjoining the n-cell e. Check that the characteristic map of e coincides with χ.

Exercise 11.6. (1-point exercise) Haven't we forgotten something? Prove that $A \cup_\chi D^n$ is Hausdorff.

Putting together everything, we deduce:

THEOREM 11.15. *For any Hausdorff space A and any continuous map $\chi : S^{n-1} \longrightarrow A$, there is a space X which is unique up to homeomorphism, with the property that X is obtained from A by adjoining an n-cell with characteristic map equal to χ.*

5. Adjoining more cells; cell complexes

DEFINITION 11.16. *Let X be a Hausdorff space and let $A \subset X$ be closed. We say that X is obtained from A by adjoining n-cells if we are given n-cells $e_i \subset X$ (with i running in an index set I) such that*

$$X = A \cup (\cup_{i \in I} e_i)$$

with $A \cap e_i = \emptyset$ for all i and $e_i \cap e_j = \emptyset$ for all $i \neq j$, and such that the following condition is satisfied: $F \subset X$ is closed if and only if $F \cap A$ is closed, and $F \cap \overline{e_i}$-closed for all $i \in I$.

REMARK 11.17. It follows that

- e_i is open in X (for this one uses the last condition for $F = X - e_i$).
- since e_i's are open and $e_i \cap e_j = \emptyset$ for $i \neq j$, we deduce that $\overline{e}_i \cap e_j = \emptyset$ for all $i \neq j$ (indeed, $X - e_j$ will be a closed set containing e_i, hence also its closure).
- In particular, we deduce that $\partial(e_i) \subset A$ for all i.

Hence the characteristic maps χ_i of e_i will be continuous maps

$$\chi_i : S^{n-1} \longrightarrow A.$$

As in the case of one n-cell, the characteristic maps together with A determine X uniquely (the arguments we have presented before extend to an arbitrary number of cells without much trouble).

DEFINITION 11.18. *Let X be a Hausdorff space. A* <u>*cell decomposition*</u> *of X is a sequence of closed subspaces*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \dots$$

with:

- *$\cup_n X_n = X$.*
- *For each $n \geq 0$ integer, X_n is obtained from X_{n-1} by attaching n-cells (the n-cells, together with the defining maps, are part of the structure!).*
- *A subset $F \subset X$ is closed if and only if $F \cap X_n$ is closed for all $n \geq 0$.*

The space X together with a cell decomposition is called a <u>*cellular space*</u> *(note that the cells themselves, together with their defining maps, are viewed as part of the structure). The subspace X^n is called* <u>*n-skeleton*</u> *of X.*

Given a cell complex X which is compact, define the Euler number of the cell complex X as

$$\chi(X) = \#(0 - cells) - \#(1 - cells) + \#(2 - cells) - \dots.$$

REMARK 11.19. Hence, for each n, X_n is obtained from X_{n-1} by adjoining a family of n-cells:

$$\{e_i^n : i \in I_n\}$$

(I_n- a set indexing the n-cells). Note that

$$X_0 = \{e_i^0 : i \in I_0\},$$

is a closed set of points in X,

$$X = \cup_{n, i \in I_n} e_i^n,$$

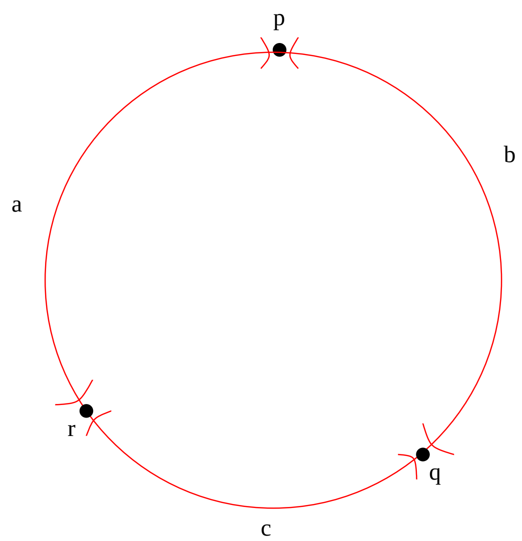
and each e_i^n is open in X_n, but not in X.

The following exercise explains the compactness condition we added when defining the Euler number of a cell complex.

EXERCISE 11.7. *(1/2 point exercise) Let X be a cell complex. Show that*

- *For any e, $\overline{e}$ and $\partial_{cell}(e)$ intersect only a finite number of cells.*
- *The total number of cells is finite if and only if X is compact.*

EXAMPLE 11.20. Given a topological space X, it may admit many different cell decompositions. For instance, for $X = S^1$, we know that S^1 can be obtained from a point by adjoining a 1-cell, i.e. it has a cell decomposition with one 0-cell and one 1-cell. However, it can also has a cell decomposition with two 0-cells and two 1-cells, or one with three 0-cells and three 1-cells, etc (see Figure 3).



A cell decomposition of the circle with three 0−cells (p, q and r)
and three 1−cells (the arches a, b and c)

FIGURE 3.

We would also like to mention here that, although the definition of $\chi(X)$ uses a cell-decomposition of X, it does not depend on the cell-decomposition one uses, but only X as a topological space X. This is a non-trivial result of algebraic topology which will not be proven in this course.

EXAMPLE 11.21. Here is a cell-decomposition of $\mathbb{R}$ (figure 4) the zero cells are the integers, while the one cells are

$$e_n = (n, n+1),$$

one for each integer n, with defining map

$$h_n : D^1 = [-1, 1] \longrightarrow \mathbb{R}, t \mapsto \frac{t + 2n + 1}{2}.$$

EXERCISE 11.8. *Describe a cell-decomposition of the plane $\mathbb{R}^2$.*

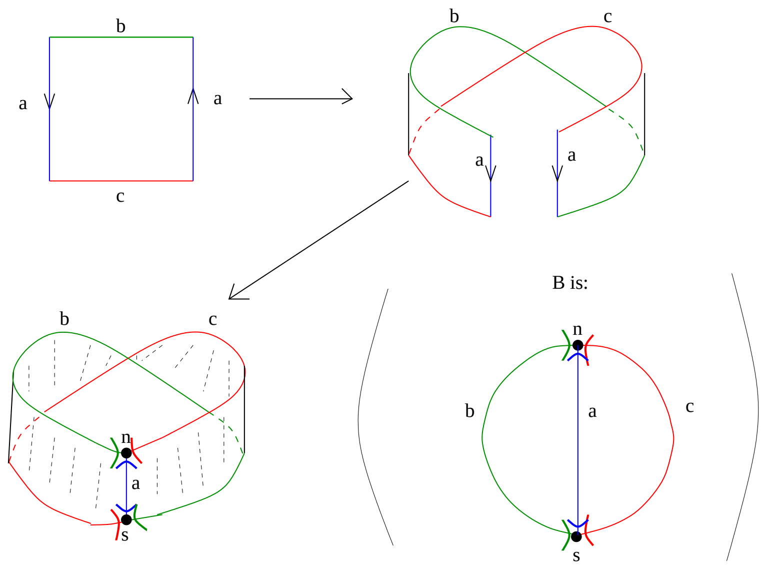
EXAMPLE 11.22. Consider the space B which, after adjoining a 2-cell, gives the Moebius band M (see Example 11.9). This is obtained the south and the north pole by adjoining three 1-cells, as shown in Figure 5: the 0-cells are denoted n and s there, while the 1-cells are a, b

Cell decomposition of the real line

FIGURE 4.

and c. Hence the Moebius band has a cell decomposition with two 0-cells, three 1-cells and one 2-cell (see the picture):

$$M = c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_2.$$



Moebius band: two 0−cells (n and s), three 1−cells (a, b and c), one 2−cell.

FIGURE 5.

EXERCISE 11.9. *What about the Klein bottle?*

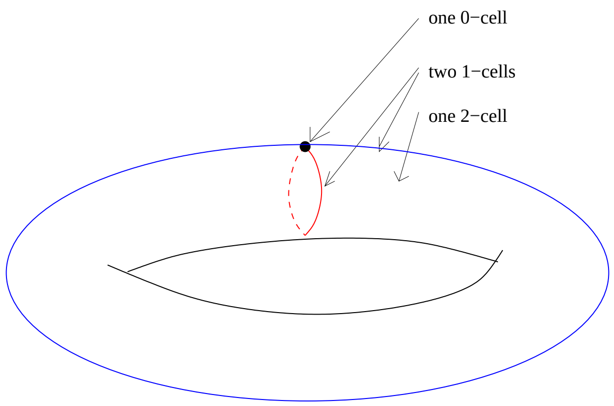
EXAMPLE 11.23. Consider X to be a bouquet of two circles (two unit circles touching each other in one point). It is obtained from a point (the common point) by attaching two 1-cells (the connected components of the space that remains after removing the point). On the other hand, we know that by (suitably) attaching a 2-cell to X one ends up with the torus T^2. Hence T^2 has a cell decomposition with one 0-cell (the point), two 1-cells, and a 2-cell (see Figure 6). Symbolically, we write

$$T^2 = c_0 \cup c_1 \cup c_1 \cup c_2.$$

EXAMPLE 11.24. From example 11.10 we know that

$$S^n = c_0 \cup c_n$$

(i.e. the n-sphere has a cell decomposition with one 0-cell and one n-cell). This can be pictured as shown in the first picture in Figure 7. However, there is yet another interesting cell decomposition



Cellular decomposition of the torus

FIGURE 6.

of S^n. We already know that the two (open) semi-spheres S^n_+ and S^n_- are n-cells, hence S^n can be obtained from S^{n-1} by adjoining two n-cells (see Figure 7).

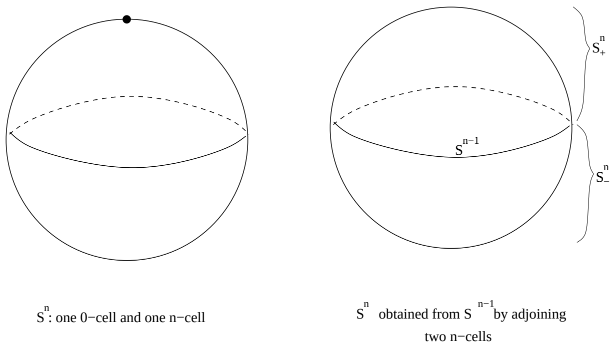


FIGURE 7.

One could now use the previous cell decomposition applied to S^{n-1} to deduce that

$$S^n = c_0 \cup c_{n-1} \cup c_n \cup c_n.$$

Alternatively, one could iterate our argument and deduce that S^n can be obtained from S^0 (two points!) by adjoining two 1-cells, and then two 2-cells, etc, and at the end two n-cells:

$$S^n = c_0 \cup c_0 \cup c_1 \cup c_1 \cup \ldots \cup c_n \cup c_n.$$

EXAMPLE 11.25. The way we expressed S^n as obtained from S^{n-1} by adjoining two n-cells (previous example) is interesting when looking at the projective space $\mathbb{P}^n$. Realizing $\mathbb{P}^n$ as $S^n/\mathbb{Z}_2$, note that the two cells will be identified in the quotient, and will define a single n-cell in $\mathbb{P}^n$. Since S^{n-1} goes in the quotient to a copy of $\mathbb{P}^{n-1}$, we deduce that $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining an n-cell. One can check that this coincides with the decomposition that we have discussed in Example 11.11- this is a matter of unraveling the definitions (exercise).

EXAMPLE 11.26. Since $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining and n-cell, continuing inductively, we find a cell decomposition of $\mathbb{P}^n$ of type

$$\mathbb{P}^n = c_0 \cup c_1 \cup \ldots \cup c_n.$$

EXERCISE 11.10. *For* $\mathbb{P}^2$*:*

(i) *describe a cell decomposition of type*

$$c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_2 \cup c_2.$$

(ii) *can you find one of type*

$$c_0 \cup c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_1 \cup c_2 \cup c_2 \cup c_2?$$

(Hint for (i): Moebius).

EXERCISE 11.11. *Find a cell decomposition of the annulus*

$$A(R, r) = \{(x, y) \in \mathbb{R}^2 : r^2 \leq x^2 + y^2 \leq R^2.$$

Then generalize this to arbitrary dimensions.