CHAPTER 1

Lecture 1 (February 6)

This lecture is introductory, rather informal, and most of it is not in the text book. The aim is to recall a few things that you already know, give you a first feeling of the word "topology", and motivate the abstract definition.

1. What will be done in this course? (informal description)

1.1. **Topology.** In topology one studies topological spaces. Some key words in topology are:

- limits
- convergent sequences
- continuous functions
- open sets, closed sets
- compact sets

But you should really have in mind that topologies are what is needed in order to talk about convergence, continuity, or, more informally, about "a point moving close to another point". As examples to have in mind: the real line $\mathbb{R}$, subsets of the real line (such as $[0,1]$ or $(0,1)$, or $[0,1)$, or $(0,1)\cup(1,2)$), the plane $\mathbb{R}^2$, the higher dimensional euclidean space $\mathbb{R}^p$, the unit circle, the unit sphere, the torus.

The topics that we will cover in the "Topology" part can be roughly divided into three types of topics:

(1) Methods to construct topologies (such as: using bases, or using metrics, or product topologies, etc). These methods will allow us to construct new topological spaces from old ones.
(2) General topological aspects (such as: convergence, continuity, interior, closure, etc). These are basic notions that make sense in any topological space.
(3) Particular classes of topological spaces (such as Hausdorff, or connected, or compact, etc). The point is that most of the topological spaces one encounters in examples have particular properties. In turn, these particular properties have interesting consequences about the topological space. We will look at several important "particular properties".

1.2. ***Algebraic Topology.*** In the second part we will have a first introduction to algebraic topology. What is "Algebraic Topology"? Roughly speaking, it provides a bridge between "Topology" and "Algebra":

$$\text{Topology} \stackrel{\text{Algebraic Topology}}{\longrightarrow} \text{Algebra}.$$

In algebraic topology, one associates algebraic structures (numbers, groups, etc) to topological spaces. Some keywords are: homotopies, homotopy groups.

2. Recall: the real line $\mathbb{R}$

Here we recalled some of the properties of $\mathbb{R}$ that are well known from the previous years, and we fixed some notations. In particular:

- For $x \in \mathbb{R}$ and $\epsilon > 0$ we denote
$$B(x, \epsilon) = \{y \in \mathbb{R} : d(y, x) < \epsilon$$
called the ball of center x and radius ϵ. Here, $d(x, y) = |x - y|$ is the distance between x and y. Of course, $B(x, \epsilon)$ is just the open interval centered at x, $(x - \epsilon, x + \epsilon)$.
- a set $U \subset \mathbb{R}$ is called open if for each $x \in U$ there exists $\epsilon > 0$ such that
$$B(x, \epsilon) \subset U.$$
- A sequence $(x_n)_{n \geq 1}$ of real numbers converges to x if for each $\epsilon > 0$ there exists an integer n_ϵ (depending on ϵ) such that $|x_n - y| < \epsilon$ for all $n \geq n_\epsilon$. Think that x_n approaches x when n increases.

  Equivalently: for each open $U \subset \mathbb{R}$, there exists an integer n_U (depending on U) such that $x_n \in U$ for all $n \geq n_U$.
- A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if for each $x \in \mathbb{R}$ and each $\epsilon > 0$ there exists $\delta > 0$ such that
$$|f(x) - f(y)| < \epsilon \text{ for all } y \in \mathbb{R} \text{ with } |y - x| < \delta.$$
(think: $f(y)$ approaches $f(x)$ if y approaches x). Write this using balls.

Exercise 1.1. *Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is open for all opens $U \subset \mathbb{R}$.*

3. What do we need to do topology? First attempt: metric spaces

Read the story on page 75 of the book (ending at the beginning of page 76).

One first attempt to answer the question in the title: distances (so that, like on the real line, one can talk about the distance between two points, hence also make sense of sentences like "two points are close to each other").

Definition 1.3 ((beginning of paragraph 20 in the book)). *A metric on a set X is a function*
$$d : X \times X \longrightarrow \mathbb{R}$$
which associates to a pair (x, y) of points x and y of X, a real number $d(x, y)$, called the distance between x and y, such that the following conditions hold:

(M1) $d(x, y) \geq 0$ *for all* $x, y \in X$.
(M2) $d(x, y) = 0$ *if and only if* $x = y$.
(M3) $d(x, y) = d(y, x)$.
(M4) *(triangle inequality)* $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$.

A metric space is a pair (X, d) consisting of a set X together with a metric d.

Exactely like for the real l ine, one defines the following concepts.

Definition 1.4. *Given a metric space (X, d):*

- *for $x \in X$, $\epsilon > 0$ one defines the open ball of center x and radius ϵ (with respect to d):*
$$B_d(x, \epsilon) = \{y \in X : d(y, x) < \epsilon\}.$$
- *a set $U \subset X$ is called open (with respect to d) if for each $x \in U$ there exists $\epsilon > 0$ such that*
$$B(x, \epsilon) \subset U.$$

- *given $x \in X$ and a sequence $(x_n)_{n \geq 1}$ of elements of X, we say that x_n converges to x (with respect to d), and write*
$$\lim_{n \to \infty} x_n = x, \quad \text{or} \quad x_n \xrightarrow{d} x$$
if for each $\epsilon > 0$ there exists an integer n_ϵ such that
$$x_n \in B_d(x, \epsilon), \quad \text{for all } n \geq n_\epsilon.$$
(equivalently: for each open $U \subset X$ containing x, there exists an integer n_U such that
$$x_n \in U, \quad \text{for all } n \geq n_U.$$

EXAMPLE 1.5. Example 1, page 120 in the book.

EXAMPLE 1.6. The real line that we have already discussed.

EXAMPLE 1.7. Take $X = \mathbb{R}^2$, the plane. The euclidean metric, denoted d, is given by the equation
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$
for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. The balls with respect to d are usual balls in the plane/ The square metric, denoted ρ, is given by the equation
$$\rho(x, y) = max\{|x_1 - y_1|, |x_2 - y_2|\}.$$
The name "square metric" comes from the fact that the balls with respect to ρ are squares in the plane (draw the picture!).

EXERCISE 1.2. (i) *Show that d and ρ are metrics.*
(ii) *Show that a set $U \subset \mathbb{R}^2$ is open with respect to d if and only if it is open with respect to ρ.*
(iii) *Generalize this to $\mathbb{R}^p$, with $p \geq 1$ integer.*

Note the following important consequence of (ii): since convergence can be formulated by using open sets with respect to the metric (and not the metric itself), it follows that convergence with respect to d and ρ is the same. (and the same for continuity). In other words, although d and ρ are different, we have a strong indication that they "are the same topologically" (they define the same topology).

4. What do we need to do topology? Topological spaces: the abstract definition

Our discussion about $\mathbb{R}$, and then about metric spaces, and the repeated appearance of open sets, indicates that, in order to do topology, all that we need to do is to prescribe the open sets. Hence the abstract definition:

DEFINITION 1.8. *See the definition in page 76 of the book, and the definition on page 77 in the book.*

EXAMPLE 1.9. For a metric space (X, d), one defines $\mathcal{T}_d$ as the collection of all sets $U \subset X$ which are open with respect to d. This defines a topology, called the topology induced by the metric.

EXERCISE 1.3. *Show that $\mathcal{T}_d$ is a topology.*

EXAMPLE 1.10. See example 2 and example 3 on page 77.

EXERCISE 1.4. ** Define the notion of convergence of sequences in a topological space. Work this out in the examples 2 and 3 from the book (page 77).*