CHAPTER 2

Lecture 2 (February 13)

Although we changed the order a bit, the content of this lecture is in the book.

1. Abstract definitions

DEFINITION 2.1. *A topology on a set X is a collection $\mathcal{T}$ of subsets of X, called the open sets of the topology, satisfying the following axioms:*

(T1) *$\emptyset$ and X are open sets.*
(T2) *The intersection of any two open sets is an open set.*
(T3) *The union of any collection of open sets is an open set.*

A topological space is a pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.

Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, we say that $\mathcal{T}'$ is finer then $\mathcal{T}$, or that $\mathcal{T}$ is smaller then $\mathcal{T}'$, if $\mathcal{T} \subset \mathcal{T}'$. We say that $\mathcal{T}$ and $\mathcal{T}'$ are comparable if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

Given a topological space $(X, \mathcal{T})$, and a point $x \in X$, a neighborhood of x is any subset $V \subset X$ with the property that there exists an open U such that

$$x \in U \subset V.$$

We denote by $\mathcal{V}(x)$ the set of all neighborhoods of x, and we denote by $\mathcal{T}(x)$ the set of all open neighborhoods of x. We say that a sequence $(x_n)_{n\geq 1}$ of elements of X converges to x (with respect to $\mathcal{T}$, or in $(X, \mathcal{T})$), and write

$$x_n \xrightarrow{\mathcal{T}} x, \quad or \quad \lim_{n\to\infty} x_n = x \ in \ (X, \mathcal{T}),$$

if

$$\forall\ V \in \mathcal{V}(x),\ \exists\ n_V \in \mathbb{N} : x_n \in V\ \forall\ n \geq n_V.$$

2.2. **Remark on (T2):** Axiom (T2) is equivalent to:

(T2'): *The intersection of any finite number of open sets is an open set.*

Note also that it is not reasonable to require a stronger version of this axiom in which intersections of arbitrary collection of open sets is open. Already on the real line $\mathbb{R}$, the intersection of open intervals may be a point. For instance,

$$\bigcap_{n\geq 1} (-\frac{1}{n}, \frac{1}{n}) = \{0\}.$$

2.3. **Remark on (T3):** Axiom (T3) is equivalent to the following axiom:

(T3') *Criteria for openess: given $U \subset X$, if for any point $x \in U$ there exists an open set V such that $x \in V \subset U$, then U is open.*

PROOF. *(T3') $\Longrightarrow$ (T3):* Let $\{U_i : i \in I\}$ be an arbitrary collection of open sets (indexed by a set I), and put $U = \cup_i U_i$. Then, for any $x \in U$, we find and i such that $x \in U_i$, hence $V = U_i$ is an open set with $x \in V \subset U$. From (T3') we deduce that U is open.

(T3) $\Longrightarrow$ (T3'): Let $U \subset X$ satisfying the condition from (T3'). Then, for any $x \in U$, we find an open set V_x such that $x \in V_x \subset U$. Hence

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} V_x \subset U,$$

hence $U = \cup_{x \in U} V_x$ will be an union of open sets, hence open. □

Recall from the last lecture the following important examples of topologies:

- the trivial topology on X, denoted $\mathcal{T}_{\text{triv}}$: the only open sets are $\emptyset$ and X.
- the discrete topology on X, denoted $\mathcal{T}_{\text{dis}}$: all the subsets of X are open.
- the co-final topology on X, denoted $\mathcal{T}_{\text{cf}}$: the open sets are the empty set and complements of finite subsets of X.
- the topology associated to a metric space (X, d), denoted $\mathcal{T}_d$: the open sets are those subsets of X with the property that

$$\forall\ x \in U\ \exists\ \epsilon > 0\ :\ B(x, \epsilon) \subset U. \tag{1.1}$$

The first two examples describe the extreme cases (the smallest and the lowest topologies on the set X). The last example served (and will continue to do so) as motivation and inspiration. Note that not all topologies are induced by a metric. Also, two different metrics may induce the same topology (see the exercises given last time).

DEFINITION 2.4. *Given a topological space $(X, \mathcal{T})$, we say that $\mathcal{T}$ is a metric topology, or that the topological space $(X, \mathcal{T})$ is metrizable, if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.*

Understanding which topological spaces are metrizable is a non-trivial problem, and will be discussed later. Roughly speaking, what we have to do is to understand which are the special properties that metric topologies have. The simplest such property is the Hausdorff condition.

DEFINITION 2.5. *We say that a topological space $(X, \mathcal{T})$ is Hausdorff, or separated, if for any two distinct points $x, y \in X$, there exists two opens U, V, such that*

$$x \in U, y \in V, U \cap V = \emptyset.$$

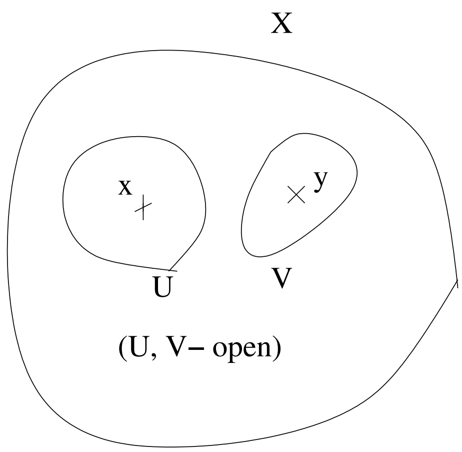

FIGURE 1.

EXAMPLE 2.6. Metric topologies have this property. Indeed, if $\mathcal{T} = \mathcal{T}_d$ for some metric d, since $x \neq y$, the distance $r = d(x, y)$ is non-zero. Take then $U = B(x, r/2)$, $V = B(y, r/2)$ and use the triangle inequality for d to prove that $U \cap V = \emptyset$.

Looking at the extreme examples: $\mathcal{T}_{\text{triv}}$ is not Haudorff (unless X is empty or consists of one point only), while $\mathcal{T}_{\text{dis}}$ is Hausdorff.

In the light of the Haudorfness property, the cofinal topology $\mathcal{T}_{cf}$ becomes more interesting:

EXERCISE 2.1. *Given a set X, show that any Hausdorff topology on X contains $\mathcal{T}_{cf}$. When is $\mathcal{T}_{cf}$ Hausdorff? When does it exist a smallest Hausdorff topology on X (i.e. a Hausdorff topology which is contained in all other Hausdorff topologies on X)?*

EXERCISE 2.2. *Show that in a Hausdorff topological space, if a sequence has a limit, then the limit is unique.*

2. Basis for a topology

Looking again at metric topologies, we see that the balls play a special role: (1.1) shows that in order to know the topology induced by the metric, one only needs to know the balls. This property that the balls have in a metric topology is important for general topologies and motivates the following definition.

DEFINITION 2.7. *Let $(X, \mathcal{T})$ be a topological space. A basis for $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X which satisfies the following $\mathcal{B}$-criteron for openess: a subset $U \subset X$ is open if and only if*

$$\forall\ x \in U\ \exists\ B \in \mathcal{B}\ :\ B \subset U. \tag{2.1}$$

Given a set X and $\mathcal{B} \subset \mathcal{P}(X)$, we say that $\mathcal{B}$ is a basis for a topology on X if there exists a topology $\mathcal{T}$ on X such that $\mathcal{B}$ is a basis for $\mathcal{T}$.

EXAMPLE 2.8. Let us repeat: given a metric space (X, d),

$$\mathcal{B}_d := \{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$$

is a basis for the topology $\mathcal{T}_d$. For more examples, see 2.13 below.

The point of view addopted in the last part of the definition, namely to start from $\mathcal{B}$ and to look for $\mathcal{T}$, is useful for constructing topologies (see also Remark 2.12) below). Note that, given a collection $\mathcal{B}$, if there exists a topology $\mathcal{T}$ as above, then it is unique. More precisely, introducing

$$\mathcal{T}(\mathcal{B}) = \{U \subset X : U \text{ satisfies } (2.1)\},$$

the "$\mathcal{B}$-criterion for openess" shows that $\mathcal{T}$ must coincide with $\mathcal{T}(\mathcal{B})$.

Hence $\mathcal{B}$ is a basis for a topology on X if and only if $\mathcal{T}(\mathcal{B})$ is a topology on X. The following describe an useful criterion for checking when a collection $\mathcal{B}$ is a basis for a topology.

PROPOSITION 2.9. *Let X be a set, let $\mathcal{B} \subset \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for a topology on X if and only if it satisfies the following two conditions:*

(B1) *for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.*
(B2) *for each $B_1, B_2 \in \mathcal{B}$, and each $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.*

Moreover, in this case $\mathcal{T}(\mathcal{B})$ is the unique topology on X that has $\mathcal{B}$ as basis.

PROOF. We have to show that conditions (T1), (T2) and (T3) (for $\mathcal{T}(\mathcal{B})$) are equivalent to conditions (B1) and (B2) (for $\mathcal{B}$). First of all, (T3) is satisfied without any assumption on B: this follows from (T3') and the fact that $\mathcal{B} \subset \mathcal{T}(\mathcal{B})$. Next, due to the definition of $\mathcal{T}(\mathcal{B})$, (B1) is equivalent to $X \in \mathcal{T}(\mathcal{B})$. Since clearly $\in \mathcal{T}(\mathcal{B})$, (B1) will be equivalent to (T1). Hence we it suffices to prove that (T2) (for $\mathcal{T}(\mathcal{B})$) is equivalent to (B2) (for $\mathcal{B}$). If (T2) holds then, since $\mathcal{B} \subset \mathcal{T}(\mathcal{B})$, $B_1 \cap B_2 \in \mathcal{T}(\mathcal{B})$ for any $B_1, B_2 \in \mathcal{B}$. From the definition of $\mathcal{T}(\mathcal{B})$, this means that for any $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$. Hence condition (B2). Conversely, assume that (B2) holds, and let $U, V \in \mathcal{T}(\mathcal{B})$. To prove that $U \cap V \in \mathcal{T}(\mathcal{B})$ we have to find $B \in \mathcal{B}$ such that $x \in B \subset U \cap V$. Since $x \in U \in \mathcal{T}(\mathcal{B})$, we find $B_1 \in \mathcal{B}$ such that $x \in B_1 \subset U$. Similarly, we find $B_2 \in \mathcal{B}$ such that $x \in B_2 \subset V$. By (B2) we find $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$. We deduce that $x \in B \subset U \cap V$, proving (T2). □

REMARK 2.10. As we have already remarked, (B1) says that $\mathcal{B} \subset \mathcal{T}(\mathcal{B})$. What actually happens is that $\mathcal{T}(\mathcal{B})$ is the smallest topology containing $\mathcal{B}$:

$$\mathcal{T} - \text{topology on } X, \mathcal{B} \subset \mathcal{T} \Longrightarrow \mathcal{T}(\mathcal{B}) \subset \mathcal{T}.$$

To see this, one first remarks that any $U \in \mathcal{T}(\mathcal{B})$ can be written as an union of sets from $\mathcal{B}$: for each $x \in U$ we find $B_x \in \mathcal{B}$ with $x \in B_x \subset U$ and then, as at the end of remark 2.3, $U = \cup_{x \in U} B_x$. On the other hand, each topology $\mathcal{T}$ containing $\mathcal{B}$ must contain also arbitrary unions of sets of $\mathcal{B}$, hence $\mathcal{T}(\mathcal{B}) \subset \mathcal{T}$.

Next, since many topologies will be defined with the help of a basis, it is useful to know how to compare topologies by only looking at basis elements.

LEMMA 2.11. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for two topologies on X. Then the topology $\mathcal{T}_2$ associated to $\mathcal{B}_2$ is finer then the topology $\mathcal{T}_1$ associated to $\mathcal{B}_1$ if and only if: for each $B_1 \in \mathcal{B}_1$ and each $x \in B_1$, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_1 \subset B_2$.*

PROOF. By the discussion above, $\mathcal{T}_1 \subset \mathcal{T}_2$ is equivalent to $\mathcal{B}_1 \subset \mathcal{T}(\mathcal{B}_2)$. □

EXERCISE 2.3. *Prove again Exercise 1.2: the euclidean and the square metrics induce the same topology.*

2.12. **Important conclusion:** As already mentioned, basis are very useful for constructing topologies. To summarize, here is a general recipe for constructing topologies: start with a colection $\mathcal{B}$ of subsets of X; usually, you collect in $\mathcal{B}$ all the sets that you want to be open in the topology you are looking for. Next, prove that $\mathcal{B}$ satisfies the conditions (B1) and (B2) above. You then consider $\mathcal{T}(\mathcal{B})$.

If you have two basis $\mathcal{B}_1$ and $\mathcal{B}_2$, to check that $\mathcal{B}_1$ and $\mathcal{B}_2$ induce the same topologies, one uses the previous lemma.

Probably one of the best illustration of the use of bases is the construction of product topologies. One starts with two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and one would like to define a "product topology" on the cartesian product

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

What we would like is that sets of type $U \times V$ with $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ are all open. However, the collection of all such sets is not a topology. Instead, it is a basis, so we just consider the associated topology. We will come back to this example in the next lecture.

2.13. **Example: various topologies on $\mathbb{R}$:** Here are several interesting topologies on $\mathbb{R}$ that can be defined with the help of a basis.

1. the discrete topology on $\mathbb{R}$, denoted $\mathcal{T}_{\text{discr}}$: start with the basis consisting of all ssubsets of $\mathbb{R}$ consisting of exactly one point. $\mathbb{R}$ together with this topology is denoted $\mathbb{R}_{\text{discr}}$.
2. the euclidean topology on $\mathbb{R}$, denoted $\mathcal{T}_{\text{eucl}}$: start with the basis consisting of all open intervals in $\mathbb{R}$. Alternatively, one could start with all the open intervals of type (p, q) with p, q-rational (check that they induce the same topology using the previous lemma!). $\mathbb{R}$ together with this topology is denoted $\mathbb{R}_{\text{eucl}}$.
3. the lower limit topology on $\mathbb{R}$, denoted $\mathcal{T}_{\text{l}}$: take as basis all the intervals of type $[r, s)$ with $r, s \in \mathbb{R}$. $\mathbb{R}$ together with this topology is denoted $\mathbb{R}_{\text{l}}$.

4. the K-topology on $\mathbb{R}$, denoted $\mathcal{T}_{\mathrm{K}}$: take as basis all the open intervals (r, s) together with the sets of the form $(r, s) - K$, where
$$K = \{\frac{1}{n} : n \in \mathbb{N}, n \neq 0\}.$$
$\mathbb{R}$ together with this topology is denoted $\mathbb{R}_{\mathrm{K}}$.

EXERCISE 2.4. *Show that $\mathcal{T}_l$ and $\mathcal{T}_K$ are both finer then $\mathcal{T}_{eucl}$, but they are not comparable.*

EXERCISE 2.5. *From the book: Exercise 3 and Exercise 4 from paragraph 13 (page 83).*