CHAPTER 5

Lecture 5 (March 6)

1. Normal spaces

5.1. **From metric spaces to normal spaces:** We have already remarked that metric spaces (or topological spaces whose topology is induced by a metric) may enjoy special topological properties that distinguishes them from arbitrary topological spaces. We have seen two such properties: Hausdorfness and the first countability axiom. Here we look at another such topological properties (which is related to the Hausdorfness condition).

So, let us assume that $(X, \mathcal{T})$ is a topological space whose topology is induced by a metric d. How can we use d to find out properties of $\mathcal{T}$ ("topological properties")? We used the fact that $d(x, y) \neq 0$ for $x \neq y$, combined with the triangle inequality for d to conclude that X is Hausdorff. Let's now use again the distance, but in a smarter way: to measure the distance from a point to a given set $A \subset X$:

$$f : X \longrightarrow \mathbb{R}, f(x) = d(x, A).$$

We know from Proposition 4.6 that f is continuous. If A is closed, we also know that $f^{-1}(0) = \{A\}$ (see Corollary 3.19). It is worth to emphasize: given an arbitrary topological space $(X, \mathcal{T})$, then the level set $f^{-1}(0)$ of any continuous function $f : X \longrightarrow \mathbb{R}$ is closed. What we are pointing out here is that *if the topology is induced by a metric, then any closed set $A \subset X$ arises as the level set $f^{-1}(0)$ of some continuous map $f : X \longrightarrow \mathbb{R}$.*

A consequence of this is that in a metric space X, for any disjoint closed sets $A, B \subset X$, one can find a continuous map

$$f : X \longrightarrow \mathbb{R}, \text{ such that } f|_A = 0, f|_B = 1.$$

Indeed, one can take

$$f(x) = \frac{f^A(x)}{f^A(x) + f^B(x)},$$

where f^A is a function on X whose level set is A (e.g. $f^A(x) = d(x, A)$), and similarly f^B. Note that the condition $A \cap B = \emptyset$ together with the fact that A and B are closed insure that the denominator in the function above is never zero (use Corollary 3.19 again). In conclusion, *if the topology is induced by a metric, then any two disjoint closed sets A and B can be separated by a continuous function, i.e. there exists $f : X \longrightarrow \mathbb{R}$ such that $f|_A = 0, f|_B = 1$.*

Of course, these kind of "special properties" that metric spaces have might deserve a name, and may be formulated as axioms, but they would not be very usefull since proving the existence of a continuous function with certain properties is not easy in general. Also, an axiom that would require the existence of a continuous function, although is of topological nature, would be a bit far from the topology itself (open sets would be involved in a rather implicit way). In particular, regarding the last property discussed -the separation of disjoint closed sets by continuous functions- , it is worth pointing out a more topological version of it.

Definition 5.2. *We say that two subsets A and B of a topological space $(X, \mathcal{T})$ can be separated topologically if there are open sets U and V such that*

$$A \subset U, B \subset V, \text{ and } U \cap V = \emptyset.$$

With this, the Hausdorff axiom says that any two distinct points can be separated topologically. Also, it is easy to see that if A and B can be separated by a continuous function f, i.e. if there exists $f : X \longrightarrow \mathbb{R}$ such that $f|_A = 0$, $f|_B = 1$, then A and B can be separated topologically. Indeed, we can take

$$U = f^{-1}((-\infty, \frac{1}{2})), V = f^{-1}((\frac{1}{2}, \infty)).$$

In particular, in a topological space X, *if the topology is induced by a metric, then any two disjoint closed sets A and B can be separated topologically.* Now, this is a property that does deserve a name.

Definition 5.3. *We say that a topological space $(X, \mathcal{T})$ is normal if it is Hausdorff and each two disjoint closed subsets A and B can be separated topologically.*

Exercise 5.1. *Extract from the discussion above a shorter proof of the fact that any metric space is normal.*

Exercise 5.2. *With respect to which of the topologies $\mathcal{T}_{discr}$, $\mathcal{T}_{triv}$, $\mathcal{T}_{eucl}$, $\mathcal{T}_l$ and $\mathcal{T}_K$ is $\mathbb{R}$ normal? (see Example 2.13 in Lecture 2).*

Exercise 5.3. *Let X and Y be two topological spaces. If their product $X \times Y$, show that X and Y are normal. Is the converse true?*

What is amazing about the normality axiom is that it does imply the (apparently stronger) property that we avoided to state as an axiom (because it looked like something difficult to check): separation by continuous functions.

Theorem 5.4. *(Urysohn Lemma) If $(X, \mathcal{T})$ is a normal topological space then, for any two disjoint closed subsets A and B, there exsist a continuous function*

$$f : X \longrightarrow [0, 1]$$

such that $f|_A = 0$ and $f|_B = 1$.

Before we discuss the proof, it is worth to emphasize that this is potentially an useful step in the metrizability problem ("when is a topology induced by a metric?"). Indeed, any metrizability result would have to produce *a continuous function* (the metric itself) out of topological properties. And this guess is correct: we will show that the previous theorem implies the following

Theorem 5.5. *(Urysohn metrization theorem) Any topological space which is second countable and normal is metrizable.*

In other words, for second countable topological spaces, the normality condition is equivalent to the metrizability condition! It is also worth pointing out that, since not all metric spaces are second countable (see Exercise 3.9), this theorem is not the most satisfactory one (and, indeed, there exist much stronger versions of it which solve the metrizability problem completely). However, since spaces which arise from geometry are second countable, the previous theorem is all we want for this course.

Another important property of normal spaces is the existence of (finite) "partitions of unity". To explain this, we need the notion of support.

DEFINITION 5.6. *Given a topological space* $(X, \mathcal{T})$ *and a function* $f : X \longrightarrow \mathbb{R}$*, define the support of* f *as*

$$supp(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

EXERCISE 5.4. *Show that* $U = X - supp(f)$ *is the largets open set with the property that* $f = 0$ *outside* U*.*

REMARK 5.7. It is important that the support has been defined as <u>the closure</u> of $\{x : f(x) \neq 0\}$. By doing that, given $U \subset X$ open, saying that $supp(f) \subset U$ can be interpreted as saying that "f is interesting only inside U" (and it is trivial outside U). One reason that for these is pointed out in the following exercise.

EXERCISE 5.5. *Let* $(X, \mathcal{T})$ *be a topological space,* $U \subset X$ *open and* $g : X \longrightarrow \mathbb{R}$ *continuous. Define*

$$\tilde{g} : X \longrightarrow \mathbb{R},\ \tilde{g}(x) = \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}.$$

Show that $\tilde{g}$ *may fail to be continuous, but that it is continuous if we assume that*

$$\overline{\{x \in X : g(x) \neq 0\}} \subset U.$$

(the closure is the closure in X*).*

DEFINITION 5.8. *Let* $(X, \mathcal{T})$ *be a topological spaces, and let*

$$\mathcal{U} = \{U_1, \ldots, U_n\}$$

be a finite open covering of X *(i.e. each* U_i *is open and* $X = \cup_i U_i$*. A partition of unity subordinated to* $\mathcal{U}$ *is a family of continuous functions*

$$\eta_i : X \longrightarrow [0, 1] \quad (1 \leq i \leq n)$$

such that

- $supp(\eta_i) \subset U_i$.
- $\sum_{i=1}^{n} = 1$.

REMARK 5.9. The main use of "partitions of unity" is due to the following fact: given a topological space $(X, \mathcal{T})$, it often happens that we have a very good "local understanding" of X, i.e., for each $x \in X$, there exists a small enough open neighborhood U of x, such that U is really well understood (e.g., for manifolds, one can allways find U homeomorphic to an euclidean space). As a particular case of this instance, let us assume that X has a finite open cover $\mathcal{U}$ such that each member U_i of this covering is well understood. From this "local knowleadge", one would like to derive "global properties" (i.e. properties of X itself). One way of doing this is through the use of continuous functions. To achieve this, one would like to be able to brake any continuous function defined globally, i.e. any $f : X \longrightarrow \mathbb{R}$, into pieces $f_i : X \longrightarrow \mathbb{R}$ with the property that "f_i is interesting only on U_i" (and it is trivial outside "U_i"). I.e., one would like to write

$$f = \sum_i f_i,$$

where each f_i is supported in U_i. A simple remark is that it is enough to solve this problem for constant 1 function. I.e., it is enough to have a partition of unity subordinated to $\mathcal{U}$. Indeed, if $1 = \sum \eta_i$ is such a partition, then multiplying by f gives $f = \sum f_i$ where $f_i = \eta_i f$ is supported in U_i.

THEOREM 5.10. *(finite partitions of unity)*
If $(X, \mathcal{T})$ is a normal space, then any finite open cover $\mathcal{U} = \{U_i\}$ admits a partition of unity subordinated to $\mathcal{U}$.

Finally, normal spaces have the following important property.

THEOREM 5.11. *(The Tietze extension theorem) Let $(X, \mathcal{T})$ be a normal space $A \subset X$ closed, and let $I \subset \mathbb{R}$ be a closed or an open interval.*
Then for any $f : A \longrightarrow I$ continuous, there exists $\tilde{f} : X \longrightarrow I$ continuous such that $\tilde{f}|_A = f$.

EXERCISE 5.6. *Show that the Tietze extension theorem implies the Urysohn lemma.*

Before we turn to the proof of these results, we need to estabilish the following property of normal spaces.

LEMMA 5.12. *If $(X, \mathcal{T})$ us a normal space, then for each closed set A and each open set D such that $A \subset D$, there exists an open set V such that*

$$A \subset V \subset \overline{V} \subset D.$$

PROOF. Consider the closed set $B = X - D$. The condition $A \subset D$ is equivalent to $A \cap B = \emptyset$. Hence we find opens U and V such that

$$A \subset V, B \subset U, U \cap V = \emptyset.$$

The second condition is $X - D \subset U$, which is equivalent to

$$X - U \subset D.$$

The condition $U \cap V = \emptyset$ is equivalent to $V \subset X - V$, which implies

$$\overline{V} \subset \overline{X - U} = X - U,$$

where we have used that $X - U$ is closed. The last two inclusions imply

$$\overline{V} \subset X - U \subset D,$$

hence V does the job. □

EXERCISE 5.7. *Prove the converse of the previous lemma.*

5.13. **Proof of the Urysohn Lemma:**

Claim 1: There is a family of open sets

$$\{U_q : q \in \mathbb{Q}\}$$

such that

(1) for $q < 0$: $U_q = \emptyset$.
(2) for $q = 0$: $A \subset U_0$.
(3) for $0 < q < 1$: no condition yet.
(4) for $q = 1$: $U_1 = X - B$.
(5) for $q > 1$: $U_q = X$.

and such that

$$q, q' \in \mathbb{Q}, q < q' \Longrightarrow \overline{U}_q \subset U_{q'}. \tag{1.1}$$

PROOF. The conditions (1), (4) and (5) force the definition of U_q for $q < 0$ and for $q \geq 1$. For $q = 0$, we choose U_0 to be any open set such that

$$A \subset U_0 \subset \overline{U}_0 \subset U_1.$$

This is possible since $A \cap B = \emptyset$ means that $A \subset X - B = U_1$ (open) hence we can apply Lemma 5.12.

Hence we are left with the construction of U_q for $q \in \mathbb{Q} \cap (0, 1)$. Writing

$$\mathbb{Q} \cap [0, 1] = \{q_0, q_1, q_2, \dots\},$$

with $q_0 = 0$, $q_1 = 1$, we will define U_{q_n} by induction on n such that (1.1) holds for all $q = q_i$, $q' = q_j$ with $0 \leq i, j \leq n$. Assume that U_q is constructed for $q \in \{q_0, \dots, q_n\}$ and we construct it for $q = q_{n+1}$. Looking at all intervals of type (q_i, q_j) with $0 \leq i, j \leq n$, there is a smallest one containing q_{n+1}. Call it (q_a, q_b). Since $q_a < q_b$, by the induction hypothesis we have

$$\overline{U}_a \subset U_b$$

hence, by Lemma 5.12, we find an open U such that

$$\overline{U}_a \subset U \subset \overline{U} \subset U_b.$$

Define $U_{q_{n+1}} = U$. We have to check that (1.1) holds for $q, q' \in \{q_0, \dots, q_{n+1}\}$. Fix q, q'. If $q \neq q_{n+1}$ and $q' \neq q_{n+1}$, $\overline{U}_q \subset U_{q'}$ holds by the induction hypothesis. Hencewe may assume that $q = q_{n+1}$ or $q' = q_{n+1}$. We treat the case $q = q_{n+1}$, the other one being similar. Write $q' = q_j$ with $j \in \{0, 1, \dots, n\}$. The assumption is that $q_{n+1} < q_j$ and we want to show that

$$\overline{U}_{q_{n+1}} \subset U_j.$$

But, since $q_{n+1} < q_j$ and (q_a, q_b) is the smallest interval of this type containing q_{n+1}, we must have $q_j \geq a$. But then

$$\overline{U}_{q_{n+1}} = \overline{U} \subset U_{j_0} \subset U_j.$$

□

Claim 2: Put

$$f : X \longrightarrow [0, 1],\ f(x) = inf\{q \in \mathbb{Q} : x \in U_q\}.$$

Then

(1) $f(x) > q \Longrightarrow x \notin \overline{U}_q$.
(2) $f(x) < q \Longrightarrow x \in U_q$.

(in particular, $f(x) = q$ for $x \in \partial U_q$).

PROOF. For (1), we prove its negation, i.e. that $x \in \overline{U}_q$ implies $f(x) \leq q$. Hence assume that $x \in \overline{U}_q$. From (1.1 we deduce that $x \in U_{q'}$ for all $q' > q$. Hence $f(x) \leq q'$ for all $q' > q$. This implies $f(x) \leq q$.

For (2), we assume that $f(x) < q$. By the definition of $f(x)$ (as an infimum), there exists $q' < q$ such that $x \in U_{q'}$. But $q' < q$ implies $U_{q'} \subset U_q$, hence $x \in U_q$. □

Claim 3: $f|_A = 0$, $f|_B = 1$, and f is continuous.

PROOF. The first two conditions are immediate from the definition of f and properties (1), (2), (4), (5) of the first claim. We now prove that f is continuous. We have to prove that for any open interval (a, b) in $\mathbb{R}$, and any $x \in f^{-1}((a, b))$, there exists an open U containing x such that $f(U) \subset (a, b)$. Fix (a, b) and x such that $f(x) \in (a, b)$ and look for U satisfying the desired condition. Choosing $p, q \in \mathbb{Q}$ such that

$$a < p < f(x) < q < b,$$

$U := U_q - \overline{U}_p$ will do the job. Indeed:

1. using the previous claim, $f(x) > p$ implies $x \notin \overline{U}_p$, while $f(x) < q$ implies $x \in U_q$. Hence $x \in U$.
2. for $y \in U$ arbitrary, we have
   - $y \in U_q \subset \overline{U}_q$ which, by the previous claim, implies $f(y) \leq q < b$.
   - $y \notin \overline{U}_p$, hence $x \notin U_p$ which, by the previous claim, implies $f(x) \geq p > a$.

   Hence $f(y) \in (a, b)$ for all $y \in U$.

□

5.14. **Proof of the Urysohn metrization theorem:** *to come*

Claim 1: There exists a countable family $(f_n)_{n \geq 0}$ of continuous functions

$$f_n : X \longrightarrow [0, 1]$$

such that

$$\forall\, U \in \mathcal{T},\ \forall\, x \in U, \exists\, N \in \mathbb{Z} : f_N(x) = 1, f_N = 0 \text{ outside } U. \tag{1.2}$$

Proof. Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable basis for the topology $\mathcal{T}$. Consider

$$I = \{(n, m) : \overline{B}_n \subset B_m\} \subset \mathbb{N} \times \mathbb{N}.$$

This is countable (subset of countable is countable), hence we can enumerate it as

$$I = \{(n_0, m_0), (n_1, m_1), \ldots\}.$$

For each i, we choose a continuous function $f_i : X \longrightarrow [0, 1]$ such that

$$f_i|_{\overline{B}_{n_i}} = 1,\ f_i|_{X - B_{m_i}} = 0$$

(which exists by the Urysohn lemma). Then $(f_i)_{i \geq 0}$ has the desired properties: for $U \in \mathcal{T}$ and $x \in U$, we can choose m such that $x \in B_m \subset U$. By Lemma 1.1, we find V-open containing x such that $x \in V \subset \overline{V} \subset B_m$. Since $\mathcal{B}$ is a basis, we find n such that $x \in B_n \subset V$. Then $\overline{B}_n \subset \overline{V} \subset B_m$, hence $(n, m) \in I$. Writing $(n, m) = (n_N, m_N)$ with $N \in \mathbb{N}$, since $x \in B_n$ we have $f_N(x) = 1$, and since $B_m \subset U$, we have $f_N = 0$ outside B_m hence also outside U. □

Claim 2: The function

$$d : X \times X \longrightarrow \mathbb{R},$$

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

is a metric on X.

Proof. First of all, we have to show that, for $x, y \in X$, the infinite sum above (a serie of real numbers) makes sense (is convergent). For this remark that

$$|f_n(x) - f_n(y)| \leq 2$$

for all n, so our serie is dominated by the convergent serie

$$\sum_{n=0}^{\infty} \frac{2}{2^n}.$$

Hence it is convergent, and d is well-defined. For the triangle inequality for d, write

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_n(z)| + |f_n(z) - f_n(y)|,$$

divide by 2^n, and take the sum over n to conclude

$$d(x,y) \leq d(x,z) + d(z,y).$$

Finally, if $x \neq y$, then $d(x,y) \neq 0$. Indeed, since $x \neq y$, we find $U \in \mathcal{T}$ containing x and not containing y. Choosing N insured by (1.2), we have $f_N(x) = 1$ and $f_N(y) = 0$, hence

$$d(x,y) \geq \frac{|f_N(x) - f_N(y)|}{2^N} = \frac{1}{2^N} > 0.$$

$\square$

Claim 3: $\mathcal{T} \subset \mathcal{T}_d$.

PROOF. Let $U \in \mathcal{T}$. We have to show that $U \in \mathcal{T}_d$, i.e. that:

$$\forall\ x \in U,\ \exists\ \epsilon > 0 : B_d(x,\epsilon) \subset U.$$

Let $x \in U$ and choose N as in (1.2). Then

$$\epsilon = \frac{1}{2^N}$$

does the job. Indeed, if $y \in B_d(x,\epsilon)$, then

$$\frac{|1 - f_N(y)|}{2^N} = \frac{|f_N(x) - f_N(y)|}{2^N} \leq d(x,y) < \frac{1}{2^N},$$

hence $f_N(y) \neq 0$ nad this can only happen if $y \in U$. $\square$

Claim 4: $\mathcal{T}_d \subset \mathcal{T}$.

PROOF. We have to show that each ball $B(x,\epsilon)$ is in $\mathcal{T}$. Fix $x \in X$ and $\epsilon > 0$. To see that $B(x,\epsilon) \in \mathcal{T}$ we have to show that for each $y \in B(x,\epsilon)$, there exists $U \in \mathcal{T}$ such that $y \in U \subset B(x,\epsilon)$. Fix y and look for U. Since $\epsilon - d(x,y) > 0$ and

$$\lim_{N \to \infty} \frac{1}{2^{N-1}} = 0,$$

we find N such that

$$\frac{1}{2^{N-1}} < \epsilon - d(x,y).$$

We now choose $\epsilon_0, \epsilon_1, \ldots, \epsilon_N$ such that

$$\epsilon_0 + \frac{\epsilon_1}{2} + \ldots + \frac{\epsilon_N}{2^N} < \epsilon - \frac{1}{2^{N-1}}. \tag{1.3}$$

We claim that

$$U = \cap i = 0^N \{z \in X : |f_i(z) - f_i(y)| < \epsilon_i\}$$

satisfy the desired conditions. First of all, since f_i is continuous, each of the sets in this intersection is open; since finite intersection of opens is open, we deduce that $U \in \mathcal{T}$. That $x \in U$ is obvious. We still have to show that $U \subset B(x,\epsilon)$. Let $z \in U$ and we prove that $d(x,z) < \epsilon$. First of all, write

$$d(x,z) \leq d(x,y) + d(z,y).$$

Next, write

$$d(z,y) = \sum_{i=0}^{\infty} \frac{|f_i(z) - f_i(y)|}{2^i} \leq$$

$$\sum_{i=0}^{N} \frac{|f_i(z) - f_i(y)|}{2^i} + \sum_{i=N+1}^{\infty} \frac{2}{2^i} = \sum_{i=0}^{N} \frac{|f_i(z) - f_i(y)|}{2^i} + \frac{1}{2^{N-1}}.$$

Since $z \in U$, this is smaller then

$$\sum_{i=0}^{N} \frac{\epsilon_i}{2^i} + \frac{1}{2^{N-1}}.$$

Combining with the triangle inequality above and the choice of ϵ_i (see (1.3) above), we deduce $d(x, z) < \epsilon$. □

5.15. **Constructions of finite partitions of unity:**

We now discuss the proof of Theorem 5.10. The main ingredient in the proof is the follwing lemma which is probably more improtant then the result we want to prove itself. Indeed, there are various other versions of "finite partitions of unnity" (e.g., in differential geometry, one uses partitions which are "smooth"). Although they are not direct consequences of our result, they all have a similar proof, and they are all based on the following property of normal spaces.

LEMMA 5.16. *For any finite open covering $\mathcal{U} = \{U_i : 1 \leq i \leq n\}$ of a normal space X, there exists a covering $\mathcal{V} = \{V_i : 1 \leq i \leq n\}$ such that*

$$\overline{V}_i \subset U_i.$$

PROOF. Let

$$A = X - (U_2 \cup \ldots \cup U_n), D = U_1.$$

Then A is closed, U is open, and $A \subset U$. By lemma 5.12, we find V_1 open such that

$$A \subset V_1 \subset \overline{V}_1 \subset D.$$

This means that

$$\{V_1, U_2, \ldots, U_n\}$$

is a new open cover of X with $\overline{V}_1 \subset U_1$. In other words, we have mannaged to "refine U_1". Applying the same argument to this new cover (to refine U_2), we find a new open cover

$$\{V_1, V_2, U_3, \ldots, U_n\}$$

with $\overline{V}_1 \subset U_1$, $\overline{V}_2 \subset U_2$. Continuing this argument, we obtain the desired oper cover $\mathcal{V}$. □

We now prove Theorem 5.10. Let $\mathcal{U} = \{U_i\}$ be the given open cover. By the previous lemma, we find an open cover $\mathcal{V} = \{V_i\}$ with $\overline{V}_i \subset U_i$. Applying again the previous lemma, we find a cover $\mathcal{W} = \{W_i\}$ with $\overline{W}_i \subset V_i$. Now, for each i, we use the Urysohn lemma to the pair of disjoint closed sets $(\overline{W}_i, X - V_i)$ to produce a continuous function

$$f_i : X \longrightarrow [0, 1]$$

with $f_i = 1$ on $\overline{W}_i$ and $f_i = 0$ outside V_i. Note that

$$f = f_1 + \ldots + f_n$$

is nowhere zero. Indeed, if $f(x) = 0$, we must have $f_i(x) = 0$ for all i, hence, for all i, $x \notin W_i$. But this contradicts the fact that $\mathcal{W}$ is a cover of X. We can now conclude that each of the functions

$$\eta_i = \frac{f_i}{f_1 + \ldots + f_n} : X \longrightarrow [0, 1]$$

is well-defined and continuous. Clearly, $\sum \eta_i = 1$. Finally, for each i,

$$\{x : \eta_i(x) \neq 0\} \subset \{x : f_i(x) \neq 0\} \subset V_i,$$

hence

$$\mathrm{supp}(\eta_i) \subset \overline{V}_i \subset U_i.$$