CHAPTER 8

Lecture 8 (April 3)

Finnish what was left from the previous lecture.

1. Locally compact spaces

In this section we discuss spaces which are compact only locally. The aim is to explain how to pass from such spaces to compact ones, and to show that, if they are second countable, then they are metrizable (in particular, they are normal). Although the notion of local compactness exists in general, we will restrict here to locally compact spaces which are Hausdorff.

DEFINITION 8.1. *A locally compact Hausdorff space is a Hausdorff topological space $(X, \mathcal{T})$ with the following property:*
(LC): each point admits a basis of neighborhoods consisting of compact sets.

EXAMPLE 8.2.

(1) $\mathbb{R}^n$ is locally compact. Indeed, for any $x \in \mathbb{R}^n$, and any $\epsilon > 0$, the closed ball
$$B[x, \epsilon] = \{y \in \mathbb{R}^n : d(x, y) \leq \epsilon\},$$
is compact (closed and bounded!), and varying ϵ, we obtain a basis of compact neighborhoods of x.

(2) any open subset $U \subset \mathbb{R}^n$ is locally compact. Indeed, if $x \in U$, then there exists ϵ_0 such that
$$B(x, \epsilon_0) \subset U,$$
and then
$$\{B[x, \epsilon] : \epsilon < \epsilon_0\}$$
is a basis of compact neighborhoods of x in U. More generally, the same type of argument shows that an open subset of a locally compact Hausdorff space has the same property.

(3) any closed subset $A \subset \mathbb{R}^n$ is locally compact. Indeed, for any $a \in A$,
$$\{B[a, \epsilon] \cap A : \epsilon > 0\}$$
is a basis of neighborhoods of a (in A), and each $B[a, \epsilon] \cap A$ is closed in $B[a, \epsilon]$ (because A is closed), hence compact (use again that closed inside compact is compact). More generally, the same type of argument shows that a closed subset of a locally compact Hausdorff space has the same property.

1. although the interval $(0, 1]$ is neither open nor closed (in $\mathbb{R}$), it is locally compact: a basis of neighborhoods of points $t \neq 1$ can be described as in (2), while around $t = 1$ we use a basis as in (3).
2. $\mathbb{Q} \subset \mathbb{R}$ is not locally compact. To see this, assume that the origin has a compact neighborhood N in $\mathbb{Q}$. Hence $N \subset \mathbb{Q}$ and there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \mathbb{Q} \subset N$. Since N is compact, it must be closed in $\mathbb{R}$. Passing to closures in the previous inclusion, we obtain $[-\epsilon, \epsilon] \subset N$ which is impossible because $N \subset \mathbb{Q}$.

3. in general, any compact Hausdorff space $(X, \mathcal{T})$ is locally compact Hausdorff. To see this, we claim that for any $x \in X$,
$$\mathcal{B}_x := \{\overline{V} : V \in \mathcal{T}, x \in V\}$$
is a basis of neighborhoods of x (they are compact as closed subsets of the compact space X). To see this, we have to check that for any open neighborhood U of x, there exists $V \in \mathcal{B}_x$ such that $\overline{V} \subset U$. But this follows from Lemma 5.12 which can be applied because compact Hausdorff spaces are normal.

Remark 8.3. There are various versions of condition (LC); more precisely, under the assumption that $(X, \mathcal{T})$ is Hausdorff, the following are equivalent:

(1) the condition (LC).
(2) for each $x \in X$, the collection of *all* compact neighborhoods of x is a basis of neighborhoods of x.
(3) for each $x \in X$ and any open neighborhood U of x, there exists a compact neighborhood N of x such that $N \subset U$.
(4) for each $x \in X$ and any open neighborhood U of x, there exists an open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.
(5) any point $x \in X$ has a compact neighborhood.

The condition (LC) as stated in the previous definition, or its reformulation (2), are probably the most natural formulations of local compactness- they fit into the general idea of what it means for a property (compactness in our case) to hold locally. The reformulation (4) is probably the most useful one to use (if we already know that the space is locally compact). When it comes to checking that a Hausdorff space is locally compact however, it is condition (5) that is clearly the easiest to check (however, the proof of the fact that (5) implies the others is postponed untill Exercise 8.1).

A few words about the proof that these conditions are equivalent. That (2) implies (1), that (4) implies (3), and that (5) is implied by the others- these are all clear. The equivalence between (2) and (3) is also clear: (3) is just (2) after we spell out the definition of a basis of neighborhoods. That (1) implies (2) is easy: the collection from (2) contains any basis as in (1), hence it must be itself a basis (of neighborhoods). Next, (3) implies (4). To see this, since N is a neighborhood of x, we find an open V such that $V \subset N$. This implies that $\overline{V} \subset \overline{N}$. But, since N is compact and X is Hausdorff, N must be closed (see Theorem 6.6), hence we get $\overline{V} \subset N \subset U$. Also, since $\overline{V}$ is closed in X, it is also closed in N hence it is compact (see Remark 6.5). This proves (4). Finally, the proof of the fact that (5) implies (4) is hinted in the Exercise 8.1 below.

1.1. One-point compactification. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Roughly speaking, the one-point compactification is a compact Hausdorff space obtained from X by adding "the infinity", and which contains X as a subspace. To make this precise, we will use a new element which is not a point of X, and which we denote by the symbol ∞ for convenience. We consider
$$X^+ = X \cup \{\infty\}.$$
Note that $X \subset X^+$ and any subset of X can be viewed as a subset of X^+. We consider the family of subsets of X^+:
$$\mathcal{T}^+ = \mathcal{T} \cup \mathcal{T}(\infty),$$
where
$$\mathcal{T}(\infty) = \{X^+ - K : K \subset X, K - \text{compact}\}.$$

THEOREM 8.4. *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Then*

(i) *$(X^+, \mathcal{T}^+)$ is a compact Hausdorff space.*
(ii) *$(X, \mathcal{T})$ is a subspace of $(X^+, \mathcal{T}^+)$ (i.e. the restriction of the topology $\mathcal{T}^+$ to X coincides with $\mathcal{T}$).*

PROOF. We first have to show that $\mathcal{T}^+$ is a topology on X^+. Clearly $X^+, \emptyset \in \mathcal{T}^+$. Next, we show that the intersection of two sets U and V from $\mathcal{T}^+$ is in $\mathcal{T}^+$. Since $\mathcal{T}^+ = \mathcal{T} \cup \mathcal{T}(\infty)$, we have three cases. If $U, V \in \mathcal{T}$, we know that $U \cap V \in \mathcal{T}$. If U and V are both of type $X - K$ with K-compact, then so is their intersection, because union of two compacts is compact (show this!). Finally, if U is of type $U = X^+ - K$ with $K \subset X$ compact, while V is open in X, then $U \cap V = U - K = U \cap (X - K)$ is open in X because U and $X - K$ are (we use again that compact is closed).

Next, we show that arbitrary union of sets from $\mathcal{T}^+$ is in $\mathcal{T}^+$. Note that we already know that the same statement holds for $\mathcal{T}$, and also for $\mathcal{T}(\infty)$ since intersections of compacts is compact (why?). We deduce that we only have to show that, if $U \in \mathcal{T}$ and $V \in \mathcal{T}(\infty)$, then $U \cup V \in \mathcal{T}^+$. Writing $V = X^+ - K$ with $K \subset X$ compact, we have

$$U \cap V = X^+ - K',$$

where $K' = K \cap (X - U)$. Since $K - K' = K \cap U$, $K - K'$ is open in K, i.e. K' is closed in the compact K, hence K' is compact (Remark 6.5 again!). This shows that $U \cap V \in \mathcal{T}^+$. This ends the proof of the fact that $\mathcal{T}^+$ is a topology.

Next, we show that X^+ is compact. Hence assume that we have an open cover $\mathcal{U}$ of X. Since no set in $\mathcal{T}$ contains ∞, $\mathcal{U}$ must contain a set of type $U = X - K$ with $K \subset X$ compact. Let $\mathcal{U}' = \{V \cap X : V \in \mathcal{U}, V \neq U\}$. Then $\mathcal{U}'$ is an open covering of K in X^+ hence, from the compactness of K, we find a finite subcollection $\mathcal{V} \subset \mathcal{U}'$ which covers K. But then

$$\mathcal{V} \cup \{X^+ - V\}$$

is a subcollection of $\mathcal{U}$ which covers X^+. This proves that X^+ is compact.

Next, we show that X^+ is Hausdorff. So, let $x, y \in X^+$ distinct, and we are looking for $U, V \in \mathcal{T}^+$ such that $U \cap V = \emptyset$, $x \in U, y \in V$. If $x, y \in X$ we are done because X is Hausdorff and $\mathcal{T} \subset \mathcal{T}^+$. Hence we may assume that $x \in X$, $y = \infty$. We have to find an open $U \subset X$ containing x and a compact $K \subset X$ such that

$$U \cap (X^+ - K) = \emptyset.$$

For that, we choose any compact neighborhood K of x and we consider $U \subset X$ open such that $x \in U \subset K$. This propves that X^+ is compact.

Next, we have to show that the restriction of $\mathcal{T}^+$ to X coincides with $\mathcal{T}$. From the definition of $\mathcal{T}^+$ we have

$$\mathcal{T}^+|_X = \mathcal{T} \cup \{U \cap X : U \in \mathcal{T}(\infty)\}.$$

But for $U = X - K \in \mathcal{T}(\infty)$ we have $U \cap X = X - K \in \mathcal{T}$ (again, we use that compacts in Hausdorff are closed); hence $\mathcal{T}^+|_X = \mathcal{T}$. $\square$

EXERCISE 8.1. *(1/2-point exercise) Let $(X, \mathcal{T})$ be a topological space.*

(1) *What can one say about X^+ if we only assume that X is Hausdorff?*
(2) *In the previous proof, where did we use that X is locally compact? Actually, what did we use there?*
(3) *Show now that condition (5) in Remark 8.3 implies (4).*

(Hint for (3): Answer the last question in (2), and then try to use Lemma 5.12).

Note: if X is an arbitrary topological space (Hausdorff or not), one says that X is locally compact if any point in X has a compact neighborhood. Hence (3) shows that a space is locally compact Hausdorff if and only if it is both locally compact and Hausdorff.

Let us also point out that the properties from the previous theorem determine X^+ uniquely up to homeomorphism, as explained in the next proposition. This will be useful in examples, when one wants to produce the one-point compactification more explicitely.

PROPOSITION 8.5. *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Then any compact Hausdorff space $(\tilde{X}, \tilde{T})$ with the property that there exists an embedding*

$$i : X \longrightarrow \tilde{X}$$

such that $\tilde{X} - X$ consists of one point only, is homeomorphic to $(X^+, \mathcal{T}^+)$.

PROOF. Let $\tilde{x} \in \tilde{X}$ be the point with the property that

$$\tilde{X} = i(X) \cup \{\tilde{x}\}.$$

We define

$$f : \tilde{X} \longrightarrow X^+$$

as follows. For $y \in \tilde{X}$, we have two cases

1. $y \in i(X)$, and then we write $y = i(x)$ with $x \in X$ and we put $f(y) = x$.
2. $y = \tilde{x}$ and then we put $f(y) = \infty$.

We check that f is continuous, i.e. $f^{-1}(U) \in \tilde{T}$ for any $U \in \mathcal{T}^+$. We have two cases. If $U \in \mathcal{T}$, then $f^{-1}(U) \subset i(X)$ and

$$f^{-1}(U) = \{y = i(x) : f(y) \in U\} = \{i(x) : x \in U\} = i(U)$$

which is open in $i(X)$ since i is an embedding. However, $i(X)$ is open in $\tilde{X}$ (its complement consists of one point only, hence it is closed because $\tilde{X}$ is Hausdorff). We deduce that $f^{-1}(U)$ is open in $\tilde{X}$. The second case is when $U = X^+ - K$ with $K \subset X$ compact. Then $f^{-1}(U)$ consist of $\tilde{x}$ and $f^{-1}(X - K) = i(X) - i(K)$, i.e.

$$f^{-1}(U) = X^+ - i(K).$$

Since $i(K)$ is compact (the image of a compact is compact), it is closed in X^+, hence $f^{-1}(U)$ is open. This proves that f is continuous. Since f is clearly bijective and we know that $\tilde{X}$ is compact and X^+ is Hausdorff, we deduce (see Thorem 7.5) that f is a homeomorhism. $\square$

EXAMPLE 8.6.

1. If $X = (0, 1]$, then X^+ is (homeomorphic to) $[0, 1]$. Indeed, $\tilde{X} = [0, 1]$ is compact, and the inclusion $i : (0, 1] \longrightarrow [0, 1]$ satisfies the properties from the previous proposition.
2. If $X = (0, 1)$, then X^+ is (homeomorphic to) the circle S^1. Indeed, $i : (0, 1) \longrightarrow S^1$ as in Figure 1 (e.g. $i(t) = (cos(2\pi t), sin(2\pi t))$ has the properties from the previous proposition.
3. If $X = [-1, 0) \cup (1, 2) \subset \mathbb{R}$, then X^+ is homeomorphic with the subset of $\mathbb{R}^2$ drawn in Figure 2.

EXERCISE 8.2. *What is the one-point compactification of $X = (0, 1) \cup (2, 3)$?*

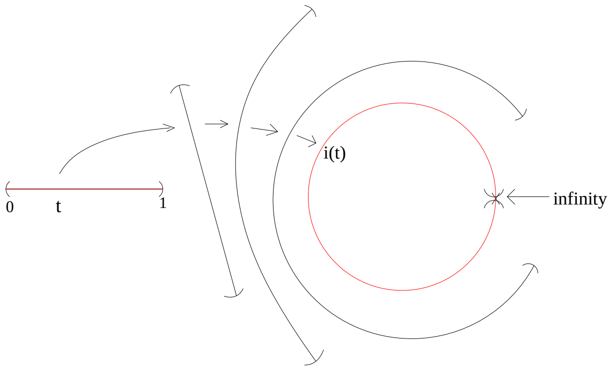

FIGURE 1.

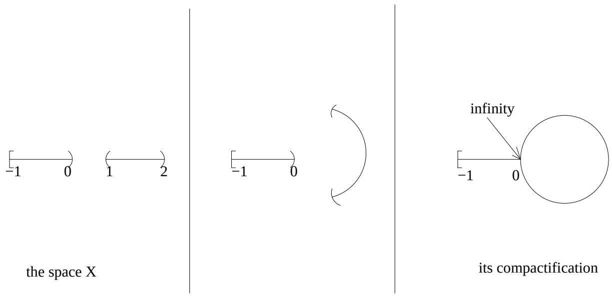

FIGURE 2.

1.2. Exhaustions and metrizability. [1] We now want to prove that locally compact Hausdorff spaces which are second countable are metrizable. The argument itself reveals a notion that is interesting on its own, that we now introduce.

DEFINITION 8.7. *Let $(X, \mathcal{T})$ be a topological space. An exhaustion of X is a family $\{K_n : n \in \mathbb{Z}_+\}$ of compact subsets of X such that $X = \cup_n K_n$ and*

$$K_n \subset \overset{\circ}{K}_{n+1}$$

for all n.

PROPOSITION 8.8. *If $(X, \mathcal{T})$ is a locally compact Hausdorff space which is second countable, then it admits an exhaustion.*

PROOF. We first show that X admits a basis with the property that the closure of each of the sets in the basis is compact. Let $\mathcal{U} = \{U_n : n \in \mathbb{Z}_+\}$ be an arbitrary countable basis for the topology $\mathcal{T}$. Let I be the set of positive integers n with the property that there exists an open V such that $\overline{V}$ is compact ant $U_n \subset V$. For each $n \in I$, we choose V_n such that $\overline{V}_n$ is compact and $U_n \subset V_n$. We show that $\mathcal{V} = \{V_n : n \in I\}$ is a basis. Let $x \in X$ and let U be an open neighborhodd of x. Since X is locally compact Hausdorff, we find an open V containing x such that $\overline{V} \subset U$ and $\overline{V}$ is compact. Since $\mathcal{B}$ is a basis, we find n such that $U_n \subset V$ and $x \in U_n$. This implies that $n \in I$ and $x \in V_n$ (since $U_n \subset V_n$). This shows that $\mathcal{V}$ is a basis; it is countable bacause $I' \subset \mathbb{Z}_+$.

Hence, we may assume that we have a basis

$$\mathcal{V} = \{V_n : n \in \mathbb{Z}_+\}$$

[1] this part will not be presented in the class

consisting of opens with the property that $\overline{V}_n$ is compact for each n. We define the exhaustion $\{K_n\}$ inductively, as follows. We put $K_1 = \overline{V}_1$. Since $\mathcal{V}$ covers the compact K_1, we find i_1 such that

$$K_1 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_1}.$$

Denoting by D_1 the right hand side of the inclusion above, we put

$$K_2 = \overline{D}_1 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_1}.$$

This is compact because it is a finite union of compacts. Since $D_1 \subset K_2$ and D_1 is open, we must have $D_1 \subset \overset{\circ}{K}_2$; since $K_1 \subset D_1$, we have

$$K_1 \subset \overset{\circ}{K}_2 .$$

Next, we choose $i_2 > i_1$ such that

$$K_2 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_2},$$

we denote by D_2 the right hand side of this inclusion, and we put

$$K_3 = \overline{D}_2 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_2}.$$

As before, K_3 is compact, its interior contains D_2, hence also K_2. Continuing this process, we construct the family K_n, which clearly covers X. $\square$

THEOREM 8.9. *Any locally compact Hausdorff topological space $(X, \mathcal{T})$ which is second countable is metrizable.*

PROOF. Let $\mathcal{B}$ be a countable basis for the topology $\mathcal{T}$. Using the previous proposition, we know we find an exhaustion $\{K_n : n \in \mathbb{Z}_+\}$ of X. Consider

$$\mathcal{B}(\infty) = \{X^+ - K_n : n \in \mathbb{Z}_+\}.$$

We will show that $\mathcal{B}^+ = \mathcal{B} \cup \mathcal{B}(\infty)$ is a basis for the topology $\mathcal{T}^+$. This will prove that X^+ is second countable. Since it is compact, it follows that X^+ is metrizable. Since X is a subspace of X^+, it follows that X is metrizable.

We still have to show that $\mathcal{B}^+$ is a basis for $\mathcal{T}^+$. We have to show that for any $U \in \mathcal{T}^+$ and any $x \in U$ there exists $B \in \mathcal{B}^+$ such that $x \in B \subset U$. If $U \in \mathcal{T}$, we use the fact that $\mathcal{B}$ is a basis for $\mathcal{T}$ and we can choose $B \in \mathcal{B}$ as desired. Assume now that $U = X^+ - K$ with $K \subset X$ compact. If $x \in X$, since $x \notin K$ we find $B \in \mathcal{B}$ containing x such that $B \cap K = \emptyset$, i.e. $x \in B \subset U$. Hence we are left with the case $x = \infty$, and we are looking for $B = X - K_n$ such that $B \subset U$, i.e. $K \subset K_n$. But such an K_n exists: the interiors of the K_m's cover X, hence also K; since K is compact, a finite number of such interiors cover K. From the definition of an exhaustion (beeing increasing) we deduce that there exists and n such that $K \subset K_n$. $\square$