CHAPTER 9

Lecture 9 (April 10)

- first we finished the discussion on locally compact Hausdorff spaces.

1. Connected spaces

DEFINITION 9.1. *We say that a topological space $(X, \mathcal{T})$ is connected if X cannot be written as the union of two disjoint non-empty opens $U, V \subset X$.*

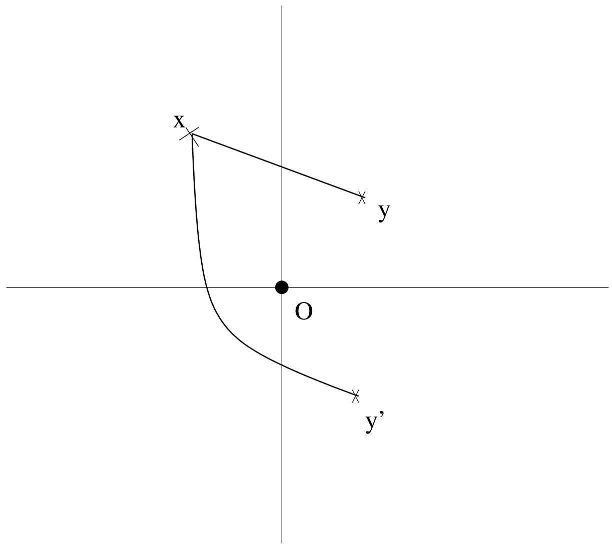
We say that a topological space $(X, \mathcal{T})$ is path connected if for any two points $x, y \in X$, there exists a path γ connecting x and y, i.e. a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x$, $\gamma(1) = y$.

Given $(X, \mathcal{T})$, we say that a subset $A \subset X$ is connected (or path connected) if A, together with the induced topology, is connected (path connected).

EXERCISE 9.1. *For a topological space $(X, \mathcal{T})$, show that the following are equivalent:*
(1) *X is connected.*
(2) *$\emptyset$ and X are the only subsets of X which are both open and closed.*
(3) *the only continuous functions $f : X \longrightarrow \{0,1\}$ are the constant ones (where $\{0,1\}$ is endowed with the discrete topology).*

EXAMPLE 9.2.
(1) $X = \{0,1\}$ with the discrete topology is not connected. Indeed, $U = \{0\}$, $V = \{1\}$ are be two disjoint non-empty opens (in X) whose union is X.
(2) $X = [0,1) \cup [2,3]$ is not connected. Indeed, $U = [0,1)$, $V = [2,3]$ are two disjoint non-empty opens (in X) whose union is X. More generally, if $X \subset \mathbb{R}$ is connected, then X must be an interval (the converse is true as well-see below). Indeed, if X is not interval, we find $r, s \in X$ and $t \in (r,s)$ such that $t \notin X$. But then
$$U = (-\infty, t) \cap X, V = (t, \infty) \cap X$$
are opens in X, nonempty (U contains r while V contains s), disjoint, and whose union is X (because $t \notin X$).
(3) Proving that a space is connected by checking directly the definition is not always straightforward. We will see however (see the theorem below) that if we can prove that a space is path connected, then connectedness follows. The use of this comes from the fact that, in contrast with connectedness, proving that a space is path connected can be done more directly. For instance, any interval $I \subset \mathbb{R}$ is path connected. Indeed, for $x, y \in I$, since I is an interval, $[x,y] \subset I$, hence the straight linear path from x to y is contained in I; in other words,
$$\gamma : [0,1] \longrightarrow X, \gamma(t) = (1-t)x + ty$$
is a path connecting x and y.
(4) By the same argument as above, any convex subset $X \subset \mathbb{R}^n$ is path connected (recall that X-convex means that for any $x, y \in X$, the whole segment $[x,y]$ is contained in X). An example which is not of this type, but is still path connected, is $X = \mathbb{R}^2 - \{0\}$: if

$x, y \in X$ and the segment $[x, y]$ doe not contain the origin, we use the linear path from x to y. But even if $[x, y]$ contains the origin, we can join them by a path going around the origin (see Figure 1).



$\mathbb{R}^2 - \{0\}$ is path connected

FIGURE 1.

Before proving that path connected spaces are connected we point out some simple properties of connected spaces.

PROPOSITION 9.3. (i) *If $f : X \longrightarrow Y$ is a continuous map and X is connected, then $f(X)$ is connected.*

(ii) *Given $(X, \mathcal{T})$, if for any two points $x, y \in X$, there exists $\Gamma \subset X$ connected such that $x, y \in \Gamma$, then X is connected.*

PROOF. For (i), replacing Y by $f(X)$, we may assume that f is surjective, and we want to prove that Y is connected. If it is not, we find $U, V \subset Y$ disjoint nonempty opens whose union is Y. But then $f^{-1}(U), f^{-1}(V) \subset X$ are disjoint (since U and V are), nonempty (since U and V are and f is surjective) opens (because f is continuous) whose union is X- and this contradicts the connectedness of X. For (ii) we reason again by contradiction, and we assume that X is not connected, i.e. $X = U \cup V$ for some disjoint nonempty opens U and V. Since they are non-empty, we find $x \in U$, $y \in V$. By hypothesis, we find Γ connected such that $x, y \in \Gamma$. But then

$$U' = U \cap \Gamma, \ V' = V \cap \Gamma$$

are disjoint non-empty opens in Γ whose union is Γ- and this contradicts the connectedness of Γ. $\square$

THEOREM 9.4. *Any path connected space is connected.*

PROOF. It is enough to prove that the closed interval $[0, 1]$ is connected. Indeed, using (i) of the previous proposition, it follows that the image $\gamma([0, 1])$ of any path from x to y is a connected set containing x and y, and then we can use (ii) of the proposition to conclude connectedness. To prove that $[0,1]$ is connected, we assume the contrary, hence

$$[0,1] = U \cup V$$

with U, V-non-empty disjoint opens in $[0,1]$. Since $U = [0,1] - V$, U must be closed in $[0,1]$, hence

$$R := \sup U \in U.$$

Since U is open this implies that $R = 1$ (if not, we find an interval $(R-\epsilon, R+\epsilon) \subset U$ and then $R + \frac{1}{2}\epsilon$ is an element in U strictly greater then its supremum-impossible). In conclusion, $1 \in U$. But exactly the same argument shows that $1 \in V$, and this contradicts the fact that $U \cap V = \emptyset$. □

The previous theorem and the remarks made in examples 2 and 3 in Example 9.2 imply the following:

Corollary 9.5. *The only connected subsets of* $\mathbb{R}$ *are the intervals.*

Using also (i) of Proposition 9.3 we deduce that the image of a connected space X by a continuous map $f : X \longrightarrow \mathbb{R}$ must be an interval. Using also the properties of compact sets (Proposition 7.4), we deduce the following:

Corollary 9.6. *If X is a compact connected space and $f : X \longrightarrow \mathbb{R}$ is a continuous function, then there exist $a, b \in \mathbb{R}$ such that*

$$f(X) = [a, b].$$

Example 9.7. There are a few more consequences that one can derive by playing with connectedness properties.

(1) $\mathbb{R}$ cannot be homeomorphic to S^1. If the two were homeomorphic, then so would be the spaces obtained by removing one point from each one of them. Indeed, if $f : \mathbb{R} \longrightarrow S^1$ is a homeomorphism, then the restriction of f to $\mathbb{R} - \{0\}$ defines a homeomorphism from $\mathbb{R} - \{0\}$ to $S^1 - \{p\}$, where $p = f(0)$ (the inverse is just the restriction of the inverse of f, hence continuous). But $\mathbb{R} - \{0\}$ is connected (it is clearly path connected) while $\mathbb{R} - \{0\}$ is not- hence they cannot be homeomorphic.
(2) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$. The argument is similar to the previous one (recall that $\mathbb{R}^2 - \{0\}$ is path connected, hence connected).
(3) The more general statement that $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic if $n \neq m$ is much more difficult to prove. One possible proof is a generalization of the argument given above (when $n = 1$, $m = 2$)- but that is based on "higher versions of connectedness", a notion which is at the core of algebraic topology.

Finally, one has the following partial converse of Theorem 9.4.

Theorem 9.8. *Let $(X, \mathcal{T})$ be a topological space with the property that any point of X has a path connected neighborhood. Then X is path connected if and only if it is connected.*

Proof. We still have to prove that, if X is connected, then it is also path connected. We define on X the following relation $\sim$: $x \sim y$ if and only if x and y can be joined by a (continuous) path. This is an equivalence relation. Indeed, for $x, y, z \in X$:

- $x \sim x$: consider the constant path.
- if $x \sim y$ then $y \sim x$: if γ is a path from x to y, then $\gamma^-(t) = \gamma(1-t)$ is a path from y to x.

- if $x \sim y$ and $y \sim z$, then $x \sim z$. Indeed, if γ_1 is a path from x to y, while γ_2 from y to z, then
$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } x \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$
is a path from x to z.

For $x \in X$, we denote by $C(x)$ the equivalence class of x:
$$C(x) = \{y \in X : y \sim x\}.$$
Note that each $C(x)$ is path connected. We claim that $C(x) = X$ for any $x \in X$. The fact that $\sim$ is an equivalence relation implies that
$$\{C(x) : x \in X\}$$
is a partition of X: if $C(x) \cap C(x') \neq \emptyset$, then $C(x) = C(x')$. What we want to prove is that this partition consists of one set only. Since X is connected, it is enough to prove that $C(x)$ is open for each $x \in X$. Fixing x, we want to prove that for any $y \in C(x)$, there is an open U such that $y \in U \subset C(x)$. To see this, we use the hypothesis and we choose any path connected neighborhood V of y. Since $y \sim x$ and $z \sim y$ fro any $z \in V$, we deduce that $z \sim x$ fro any $z \in V$, hence $V \subset C(x)$. Since V is a neighborhood of y, we find U open such that $y \in U \subset V$, and this U clearly has the desired properties. □

EXERCISE 9.2. *Show that S^1 and S^2 are not homeomorphic. Then show that S^1 and $S^1 \times [0, 1]$ are not homeomorphic.*

(hint: taking only one point out is not enough?)