CHAPTER 12

Fundamental group

1. Homotopies

DEFINITION 12.1. *Let X and Y be topological spaces. We denote by* $\underline{\mathrm{Cont}(X,Y)}$ *the set of all continuous maps from X to Y. Given $f_0, f_1 \in \mathrm{Cont}(X,Y)$, a* homotopy *between f_0 and f_1 is a continuous map*

$$H : X \times [0,1] \longrightarrow Y$$

with the property that

$$H(x,0) = f_0(x),\ H(x,1) = f_1(x),$$

for all $x \in X$. We say that f_0 and f_1 are homotopic, *and we write*

$$f_0 \sim f_1,$$

if there exists a homotopy H between f_0 and f_1.

We say that a space X is contractible *if $Id_X : X \longrightarrow X$ is homotopic to a constant map.*

We say that a continuous function $f : X \longrightarrow Y$ is a homotopy equivalence *if there exists a continuous function $g : Y \longrightarrow X$ such that*

$$g \circ f \sim Id_X,\ \ f \circ g \sim Id_Y.$$

The map g will be called a homotopy inverse *of f. Given two topological spaces X and Y, we say that they are* homotopic equivalent *if there exists a homotopy equivalence $f : X \longrightarrow Y$.*

REMARK 12.2. A homotopy between f_0 and f_1 should be viewed as a "continuous deformation" of f_0 into f_1: the homotopy H defines for each $t \in [0,1]$ a continuous map $H_t \in \mathrm{Cont}(X,Y)$,

$$H_t : X \longrightarrow Y,\ x \mapsto H(t,x),$$

and the family $\{H_t\}$ goes from $H_0 = f_0$ to $H_1 = f_1$.

EXAMPLE 12.3. If $Y = \mathbb{R}^n$, any two continuous maps $f_0, f_1 : X \longrightarrow \mathbb{R}^n$ are homotopic. Indeed,

$$H(x,t) = (1-t)f_0(x) + tf_1(x)$$

is a homotopy between f_0 and f_1. In particular, $\mathbb{R}^n$ is contractible. More generally, by the same argument, any subspace $X \subset \mathbb{R}^n$ which is convex (i.e. $tx + (1-t)y \in X$ for all $x, y \in X$ and all $t \in [0-,1]$) is contractible.

EXERCISE 12.1. Show that X is contractible if and only if it is homotopic equivalent to the space consisting of one point only. Show that any contractible space is path-connected.

EXAMPLE 12.4. $\mathbb{R}^2 - \{0\}$ is homotopic equivalent to S^1. To see this, we consider the inclusion $i : S^1 \longrightarrow \mathbb{R}^2$ and the map

$$r : \mathbb{R}^2 - \{0\} \longrightarrow S^1, r(x) = \frac{x}{||x||}.$$

Clearly, $r \circ i = \mathrm{Id}_{S^1}$. We still have to check that

$$i \circ r : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}, x \mapsto \frac{x}{||x||}$$

is homotopic to the identity map. For this we consider

$$H(x,t) = (1-t)\frac{x}{||x||} + tx$$

(which takes values in $\mathbb{R}^2 - \{0\}$!).

Similarly, $\mathbb{R}^n - \{0\}$ and to $D^n - \{0\}$ are homotopic equivalent to S^{n-1} for any $n \geq 1$ integer, with the homotopy equivalences given by the inclusion of S^{n-1}. More generally, for any convex subspace $X \subset \mathbb{R}^n$ and any interior point $x_0 \in \overset{\circ}{X}$, one can prove that $X - \{x_0\}$ is homotopic equivalent to S^{n-1}.

EXERCISE 12.2. Assume that X is obtained from A by adjoining an n-cell. Prove that, for any point $x \in X - A$, the inclusion

$$i : A \longrightarrow X - \{x\}$$

is a homotopy equivalence.

The previous example belongs the following situation:

PROPOSITION 12.5. *Given topological spaces X, Y, Z,*

(i) *The homotopy relation $\sim$ is an equivalence relation on* $\mathrm{Cont}(X,Y)$.
(ii) *If $f_0, f_1 \in \mathrm{Cont}(Y,Z)$ are homotopic, and $g_0, g_1 \in \mathrm{Cont}(X,Y)$ are homotopic, then $f_0 g_0$ and $f_1 g_1$ are homotopic.*

PROOF. We first check (i). For any $f : X \longrightarrow Y$ we have $f \sim f$ via the homotopy $h(x,t) = f(x)$. If $f \sim g$ via the homotopy H, then $g \sim f$ via the homotopy H^- defined by $H^-(x,t) = H(x, 1-t)$. Assume now that $f \sim g$ via a homotopy H and $g \sim h$ via a homotopy H'. Then we define H'' by

$$H''(x,t) = \begin{cases} H(x,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ H'(x,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} .$$

Clearly, it satisfies $H''_0 = f$, $H''_1 = h$, while it is continuous because of the following general result.

LEMMA 12.6 ((The pasting lemma)). *Let X and Y be two topological spaces, and let $A_1, \ldots, A_n \subset X$ be a finite number of closed sets covering X. Assume that $f_i : A_i \longrightarrow Y$ are continuous functions satisfying*

$$f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j},$$

for all i, j. Then $f : X \longrightarrow Y$ defined by $f|_{A_i} = f_i$ is continuous.

PROOF. We check that $f^{-1}(F)$ is closed for any $F \subset Y$ closed. For this we first remark that

$$f^{-1}(F) = \cup_i f_i^{-1}(F).$$

Since f_i is continuous, $f_i^{-1}(F)$ is closed in A_i, hence also in X (since A_i is closed in X), hence the previous union is a finite union of closed subsets, hence closed. □

□

REMARK 12.7 ((The homotopy category)). Some of the constructions here can be nicely interpreted using the language of "category theory". Precisely, a category $\mathcal{C}$ consists of

- certain objects of $\mathcal{C}$, which we will denote by capital letters (e.g. X, A, etc).

- for any two objects X and Y, a set of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ which are called morphisms (or arrows) from X to Y, and which we will denote by small letters. Given an arrow $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we also write $f : X \longrightarrow Y$.
- for any three objects X, Y and Z, a map

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z),$$

which is called the composition of arrows, and which is denoted by $(a,b) \mapsto a \circ b$. The composition is required to satisfy:

$$(a \circ b) \circ c = a \circ (b \circ c),$$

(whenever the composition is defined).
- for any object X, there is a specified arrow $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, called the identity arrow of X. It is required that, for any $a \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$,

$$a \circ \mathrm{Id}_X = \mathrm{Id}_Y \circ a = a.$$

Each category $\mathcal{C}$ has its own notion of isomorphisms: we say that a morphism $a \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism if there exists $b \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that

$$b \circ a = \mathrm{Id}_X, \;\; a \circ b = \mathrm{Id}_Y.$$

Standard examples are:

(i) The category of sets has as objects the sets, the morphisms in this category between two sets X and Y are the usual functions $f : X \longrightarrow Y$, and the composition is the usual composition of functions. The isomorphisms of this category correspond to bijective functions.

(ii) The category of groups has as objects all groups, the morphisms in this category between two groups G and H are the usual group homomorphisms $f : G \longrightarrow H$, and the composition is the usual composition of maps. The isomorphisms in this category correspond to the usual group isomorphisms.

(iii) The category of topological spaces has as objects the topological spaces, the morphisms between two space X and Y are the continuous maps $f : X \longrightarrow Y$ and, again, the composition is the usual composition of maps. The isomorphisms of this category are the homeomorphisms.

Back to the previous proposition, for any two spaces X and Y we consider the set of homotopy classes of maps from X to Y:

$$[X,Y] := \mathrm{Cont}(X,Y)/\sim .$$

For $f : X \longrightarrow Y$ continuous, we denote by $[f] \in [X,Y]$ the equivalence class of f. The proposition ensures that there is a well-defined map

$$[Y,Z] \times [X,Y] \longrightarrow [X,Z], ([f],[g]) \mapsto [f] \circ [g] := [f \circ g]$$

What we have really done was to define a new category, called the homotopy category of spaces: it has as objects all topological spaces, while the set of morphisms from X to Y is $[X,Y]$ with the composition that we have just described. In this language, two spaces are homotopic equivalent if and only if they are isomorphic in the homotopy category,

2. Path homotopies

Recall that a path in a topological space X is a continuous map

$$\gamma : [0,1] \longrightarrow X.$$

The point $x = \gamma(0)$ is called the initial (or start) point of γ, while the point $y = \gamma(1)$ is called the final (or end) point of γ. We also say that γ is a path (in X) from x to y. We denote by

$$P(X, x, y)$$

the set of all paths in X from x to y.

DEFINITION 12.8. *Given a space X, $x, y \in X$, we say that two paths*

$$\gamma, \gamma' \in P(X, x, y)$$

are path homotopic, and we write

$$\gamma \sim_p \gamma'$$

if there exists a homotopy H between γ and γ' such that each $H_t(-) = H(-, t) : [0, 1] \longrightarrow X$ is a path from x to y.

We say that a path $\gamma \in P(X, x, x)$ is null-homotopic if it is path homotopic to the constant path $c_x \in P(X, x, x)$ ($c_x(t) = t$ for all t).

REMARK 12.9. A path homotopy H should be viewed a "continuous deformation" of γ into γ' through paths from x to y. Explicitly,

$$H : [0, 1] \times [0, 1] \longrightarrow X$$

must satisfy

$$H(s, 0) = \gamma(s), H(s, 1) = \gamma'(s),$$

$$H(0, t) = x, H(1, t) = y.$$

for all $s, t \in [0, 1]$.

PROPOSITION 12.10. *Let X be a space, $x, y \in X$. The path homotopy relation $\sim_p$ is an equivalence relation on $P(X, x, y)$.*

PROOF. One just remarks that all the homotopies in the proof of Proposition 12.5 are path-homotopies. □

How can one "compose" paths?

DEFINITION 12.11. *Let X be a topological space, $x, y, z \in X$. For $\gamma \in P(X, x, y)$ and $\gamma' \in P(X, y, z)$, we define the concatenation of γ and γ' as the new path $\gamma * \gamma' \in P(x, z)$ defined by*

$$(\gamma * \gamma')(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma'(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} .$$

Note that, by the pasting lemma (Lemma 12.6), $\gamma * \gamma'$ is continuous.

Note that the concatenation of paths does not behave "like a composition"- for instance, it is not associative. Things become much nicer is we pass to homotopy classes of paths.

DEFINITION 12.12. *Given a space X and $x, y \in X$, we define the set of homotopy classes of paths from x to y as*

$$\Pi(X, x, y) := P(X, x, y)/\sim_p .$$

For $\gamma \in P(X, x, y)$, the induced equivalence class is denote by $[\gamma] \in \Pi(X, x, y)$.

PROPOSITION 12.13. *Let X be a topological space. Then, for any $x, y, z \in X$, the map*

$$\Pi(X, x, y) \times \Pi(X, y, z) \longrightarrow \Pi(X, x, z),$$

$$([\gamma], [\gamma']) \mapsto [\gamma] * [\gamma'] := [\gamma * \gamma']$$

is well defined. Moreover

1. *For $x \in X$, denote by $c_x \in P(X, x, x)$ the constant path, and $1_x = [c_x] \in \Pi(X, x, x)$. Then for any $a \in \Pi(X, x, y)$,*
$$1_x * a = a * 1_y = a.$$
2. *Given $a = [\gamma] \in \Pi(X, x, y)$, define $a^{-1} \in \Pi(X, y, x)$ as follows: write $a = [\gamma]$ with $\gamma \in P(X, x, y)$ and put $a^{-1} = [\gamma^-]$ where $\gamma^- \in P(X, y, x)$ is given by $\gamma^-(t) = \gamma(1-t)$. Then a^{-1} is well-defined and*
$$a * a^{-1} = 1_x, \;\; a^{-1} * a = 1_y.$$
3. *The new operation $*$ is associative:*
$$(a * b) * c = a * (b * c),$$
for all $a \in \Pi(X, x, y)$, $b \in \Pi(X, y, z)$, $c \in \Pi(X, z, u)$ (with $x, y, z, u \in X$).

Proof. We will use the following construction: for $a, b \in \mathbb{R}$, we consider the affine function $l_{a,b}$ which sends a to 0 and b to 1:
$$l_{a,b} : [a, b] \longrightarrow [0, 1], \; l_{a,b}(s) = \frac{l - a}{b - a}.$$
With the help of this, if we have a path $\gamma : [0, 1] \longrightarrow X$, and we want to reparametrize it to obtain a path defined on $[a, b]$ which has the same image as γ, we will consider
$$\gamma \circ l_{a,b} : [a, b] \longrightarrow X, \gamma_{a,b}(s) = \gamma(\frac{l - a}{b - a}).$$
For the first part of the theorem, write $a = [\gamma]$, and we have to show that $c_x * \gamma \sim_p \gamma$, $\gamma * c_y \sim_p \gamma$. We prove the first one (the second one being similar). To produce paths h_t with $h_0 = \gamma$, $h_1 = c_x * \gamma$, look first at $c_x * \gamma$:
$$(c_x * \gamma)(s) = \begin{cases} x & \text{if } 0 \le s \le \frac{1}{2} \\ \gamma(2s - 1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}.$$
We will construct H_t by considering the points $p_t \in [0, 1]$ and defining $H_t|_{[0,p_t]}$ to be constant equal to x, and $H_t|_{[p_t,1]}$ to be γ reparametrized (i.e. γ_{1,p_t} described above). We need to choose p_t so that $p_0 = 0$ (so that $H_0 = \gamma$) and $p_1 = \frac{1}{2}$ (so that $H_1 = c_x * \gamma$). The simples choice is the linear one: $p_t = \frac{t}{2}$, which produces the homotopy
$$H(s, t) = H_t(s) = \begin{cases} x & \text{if } 0 \le s \le \frac{t}{2} \\ \gamma(\frac{2s-t}{2-t}) & \text{if } \frac{t}{2} \le s \le 1 \end{cases}.$$
Next, for $\gamma \in P(X, x, y)$, we have to show that $\gamma * \gamma^- \sim_p c_x$ and $\gamma^- * \gamma \sim_p c_y$. We prove the first one (the second one being similar). The intuition is the following: $(\gamma * \gamma^-)(s)$ covers $\gamma([0, 1])$ when s covers the first half of $[0, 1]$, and then comes back covering the same path on the second half of the interval. To construct the homotopy H_t, we consider the path which, in the first half of the interval covers $\gamma([0, t])$ and then comes back. With this intuition, we define
$$H(s, t) = \begin{cases} \gamma(2st) & \text{if } 0 \le s \le \frac{1}{2} \\ \gamma(2(1 - s)t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$
which realizes the desired path homotopy.

For the last part of the theorem, assume that $a = [\gamma]$, $b = [\gamma']$, $c = [\gamma'']$, and consider the paths $H_0 = (\gamma * \gamma') * \gamma''$, $H_1 = \gamma * (\gamma' * \gamma'')$. Hence
$$H_0(s) = \begin{cases} \gamma(4s) & \text{if } 0 \le s \le \frac{1}{4} \\ \gamma'(4s - 1) & \text{if } \frac{1}{4} \le s \le \frac{1}{2} \\ \gamma''(2s - 1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases},$$

$$H_1(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma'(4s-2) & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4} \\ \gamma''(4s-3) & \text{if } \frac{3}{4} \leq s \leq 1 \end{cases} .$$

Remark that both H_0 and H_1 are of the following type. One divides the interval $[0,1]$ into three intervals, by using two numbers p and q $(0 < p < q < 1)$. On the first interval, i.e. on $[0,p]$, one consider γ reparametrized by $l_{0,p}$, (which, when s goes from 0 to p, will cover the whole path γ). Next, on $[p,q]$, one considers γ' reparametrized by $l_{p,q}$ and, on $[q,1]$, γ'' reparametrized by $l_{q,1}$. Putting together these three pieces, we find a curve constructed out of the curves γ, γ' and γ'', and of the numbers p and q. With these, H_0 is obtained for

$$p = p_0 = \frac{1}{4}, q = q_0 = \frac{1}{2},$$

while H_1 is obtained for

$$p = p_1 = \frac{1}{2},\ q = q_1 = \frac{3}{4}.$$

We are looking for a homotopy consisting of paths H_t (between H_0 and H_1). For this we consider (see Figure 1)

$$p_t = \frac{t+1}{4},\ q_t = \frac{t+2}{4}$$

(the affine maps which have the described values $p_0, p_1,\ q_0, q_1$), and we consider the path H_t given by the procedure we have just described, applied to $p = p_t$, $q = q_t$. We find the desired homotopy

$$H(s,t) = H_t(s) = \begin{cases} \gamma(\frac{4s}{t+1}) & \text{if } 0 \leq s \leq \frac{t+1}{4} \\ \gamma'(4s-t-1) & \text{if } \frac{t+1}{4} \leq s \leq \frac{t+2}{4} \\ \gamma''(\frac{4s-t-2}{2-t}) & \text{if } \frac{t+2}{4} \leq s \leq 1 \end{cases} .$$

□

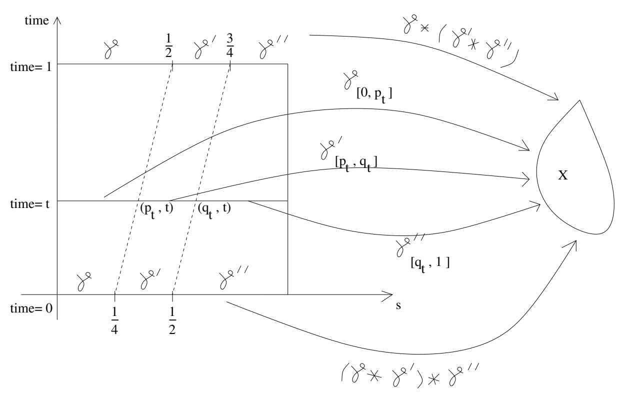

FIGURE 1.

EXERCISE 12.3. Let $\gamma : [0,1] \longrightarrow X$ be a path from x to y, and $0 = a_0 < a_1 < \ldots < a_p < a_{p+1} = 1$. For each $0 \leq i \leq p$ integer define

$$\gamma_i : [0,1] \longrightarrow X, \gamma_i(s) = \gamma((1-s)a_i + s a_{i+1}).$$

Show that

$$[\gamma] = [\gamma_p] * \ldots * [\gamma_1] * [\gamma_0].$$

REMARK 12.14 ((The homotopy groupoid)). The previous proposition too, should be viewed from the point of view of category theory. It says that, given the space X, there is an associated category $\Pi(X)$ with

- The objects of $\Pi(X)$ are the points of X.
- The set of morphisms from $x \in X$ to $y \in Y$ is $\Pi(X, x, y)$.
- The composition of morphisms comes from the concatenation of paths.

The last part of the proposition says that this category has one more property: any morphism is an isomorphism. Categories with this property are called groupoids. For this reason, $\Pi(X)$ is called the fundamental groupoid of X.

3. The fundamental group

DEFINITION 12.15. *The fundamental group of X with base point x, (X-a topological space, $x \in X$), is defined as*

$$\pi(X, x) := \Pi(X, x, x),$$

with the group structure is the one induced by the concatenation of paths.

We first discuss the functoriality of the fundamental group (what happens if we change the space?).

THEOREM 12.16. *If $f : X \longrightarrow Y$ is a continuous map and $x \in X$, then the map*

$$f_* : \pi(X, x) \longrightarrow \pi(Y, f(x)), [\gamma] \mapsto [f \circ \gamma]$$

is well defined and is a morphism of groups. If $g : Y \longrightarrow Z$ is another continuous map, then

$$g_* \circ f_* = (g \circ f)_*.$$

In particular, if $f : X \longrightarrow Y$ is a homeomorphism, then $\pi(X, x)$ is isomorphic to $\pi(Y, f(x))$, for all $x \in X$.

PROOF. The map $\gamma \mapsto f \circ \gamma$ (from paths in X to paths in Y) is compatible with path homotopies: if H is a path homotopy between γ and γ', then $f \circ f$ is a path homotopy between $f \circ \gamma$ and $f \circ \gamma'$. This implies that f_* is well defined. Next, using the definition of the concatenation of paths, we see that

$$f \circ (\gamma * \gamma') = (f \circ \gamma) * (f \circ \gamma')$$

or all $\gamma, \gamma' \in P(X, x, x)$, which implies that f_* is a group homomorphism. Given another function g, we have:

$$(g_* \circ f_*)([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma],$$

proving that $(g_* \circ f_*) = (g \circ f)_*$. Finally, if f is a homeomorphism, denoting by $g : Y \longrightarrow X$ its inverse and by $g_* : \pi(Y, y) \longrightarrow \pi(X, g(y))$ the induced map, where $y = f(x)$, we have $g_* \circ f_* = (g \circ f)_* = (Id_X)_* = Id$, and similarly $f_* \circ g_* = Id$, proving that f_* is an isomorphism. □

Next, what happens if we change the base point?

THEOREM 12.17. *Given a space X, and α a path in X from x to y, the map*

$$\widehat{\alpha} : \pi(X, x) \longrightarrow \pi(X, y),$$

$$\widehat{\alpha}([\gamma]) = [\alpha]^{-1} * [\gamma] * [\alpha]$$

is well defined and is an isomorphism of groups.

In particular, if X is path connected, then for any two points $x, y \in X$ the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.

PROOF. Let $a = [\alpha] \in \Pi(X, x, y)$. We have to show that

$$\widehat{a} : \pi(X, x) \longrightarrow \pi(X, y), \phi(u) = a^{-1} * \gamma * u$$

is a group isomorphism. First of all, note that if $b \in \Pi(Y, Z)$, then

$$\widehat{b} \circ \widehat{a} = \widehat{a * b}.$$

Indeed, for all $u \in \Pi(X, x, x)$,

$$\begin{aligned}(\widehat{b} \circ \widehat{a})(u) &= b^{-1} * (a^{-1} * u * a) * b \\ &= (b^{-1} * a^{-1}) * u * (a * b) \\ &= (a * b)^{-1} * u * (a * b) = \widehat{a * b}(u).\end{aligned}$$

In particular, this implies that $\widehat{a}$ is bijective with inverse $\widehat{b}$ with $b = a^{-1}$. Next, $\widehat{a}$ is a group homomorphism:

$$\begin{aligned}\widehat{a}(u) * \widehat{a}(v) &= (a^{-1} * u * a) * (a^{-1} * v * a) \\ &= a^{-1} * u * (a * a^{-1}) * v * a \\ &= a^{-1} * u * 1_y * v * a \\ &= a^{-1} * (u * v) * a = \widehat{a}(u * v). \qquad (3.1)\end{aligned}$$

When X is path connected, then for any two points $x, y \in X$ we can find a path γ from x to y, and then $\hat{\gamma}$ will provide an isomorphism between $\pi(X, x)$ and $\pi(X, y)$. □

Finally, we show that two spaces which are homotopic equivalent (Definition 12.1) have isomorphic fundamental groups.

THEOREM 12.18. *If $f : X \longrightarrow Y$ is a homotopy equivalence then, for any $x_0 \in X$, the induced map*

$$f_* : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0))$$

is an isomorphism of groups.

PROOF. Let $g : Y \longrightarrow X$ be a homotopy inverse of f (see Definition 12.1), $x_0 \in X$, and the maps

$$f_* : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0)), \ g_* : \pi(Y, f(x_0)) \longrightarrow \pi(X, g(f(x_0)))$$

the maps induced in the fundamental groups. It suffices to show that g_*f_* and f_*g_* are isomorphism. Indeed, the injectivity of g_*f_* implies that f_* is injective, while the surjectivity of f_*g_* implies that f_* is surjective (show this!).

Due to the symmetry, it suffices to show that $g_*f_* = (gf)_*$ is bijective. Let $h = gf$. We are in the following situation: we have a continuous function $h : X \longrightarrow X$ which is homotopic to the identity map, $x_0 \in X$, and we want to prove that

$$h_* : \pi(X, x_0) \longrightarrow \pi(X, h(x_0))$$

is an isomorphism. Since h is homotopic to Id_X, we find

$$H : X \times [0, 1] \longrightarrow X$$

such that $H(x, 0) = x$, $H(x, 1) = h(x)$ for all $x \in X$. Consider

$$\alpha : [0, 1] \longrightarrow X, \alpha(s) = H(x_0, s).$$

This is a path from x_0 to $h(x_0)$, hence the previous theorem (Theorem 12.17) provides us with a group isomorphism

$$\widehat{\alpha} : \pi(X, x_0) \longrightarrow \pi(X, h(x_0)), \ [\gamma] \mapsto [\alpha]^{-1} * [\gamma] * [\alpha].$$

We will show that $h_* = \widehat{\alpha}$. Given $[\gamma] \in \pi(X, x_0)$, we have to check that

$$[h \circ \gamma] = [\alpha]^{-1} * [\gamma] * [\alpha].$$

Using the properties of the concatenation (see Proposition 12.13), we have to check that

$$[\alpha] * [h \circ \gamma] = [\gamma] * [\alpha],$$

i.e. that the paths $\gamma * \alpha$ and $\alpha * (h \circ \gamma)$ are path homotopic. For this, we define

$$G : [0, 1] \times [0, 1] \longrightarrow X,$$

$$G(s, t) = \begin{cases} H(\alpha(2st), 2s(1-t)) & \text{if } 0 \leq s \leq \frac{1}{2} \\ H(\alpha(1 - 2(1-s)(1-t)), 1 - 2(1-s)t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

One checks directly that G is a path homotopy between $\gamma * \alpha$ and $\alpha * (h \circ \gamma)$ (check it!). □

EXERCISE 12.4. Explain (on a picture) the formula for the homotopy G given in the previous proof.

EXAMPLE 12.19.

- If X is a contractible space, the $\pi(X, x)$ is trivial (i.e. consisting only of the identity element 1_x) for all $x \in X$. In particular,
$$\pi(\mathbb{R}^n, x) = 0.$$
- Consider the inclusion $i : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$. Since this is a homotopy equivalence (Example 12.4), we deduce that we have an isomorphism
$$i_* : \pi_1(S^{n-1}, x) \xrightarrow{\sim} \pi_1(\mathbb{R}^n - \{0\}, x),$$
for all $x \in S^{n-1}$. We will see that the left hand side equals to $\mathbb{Z}$ if $n = 1$, and is trivial otherwise.
- Consider the Moebius band M (See Example 5.2). Inside it, one finds a copy of the circle S^1, and, as in the previous example, the inclusion $S^1 \longrightarrow M$ is a homotopy equivalence (provide the explicit formulas!). We deduce that the homotopy group of M is isomorphic to the homotopy group of S^1.

DEFINITION 12.20. *We say that a space X is* simply connected *if $\pi(X, x)$ is trivial for any $x \in X$. We say that X is* 1-connected *if it is path connected and simply connected.*

EXERCISE 12.5. Show that, for any two topological spaces X and Y, and $x \in X$, $y \in Y$, one has an isomorphism of groups:

$$\pi(X \times Y, (x, y)) \cong \pi(X, x) \times \pi(Y, y).$$