Solutions 2 Meetkunde en Topologie

Exercise 1. *Give a short proof of the fact that a metric space (X, d) is normal.*

Proof. Let A and B be two closed disjoint subsets of X. We define

$$U = \{x \in X : d(x, A) < d(x, B)\}, V = \{x \in X : d(x, A) > d(x, B)\}.$$

Clearly, U and V are disjoint. Since the function $x \mapsto d(x, A) - d(x, B)$ is continuous (see Proposition 4.6 in the lecture notes), U and V are open. Next, we have $A \subset U$: if $x \in A$, then $d(x, A) = 0$. Also, since A and B are disjoint, $x \notin B$; since B is closed, we must have $d(x, B) > 0$ (see Corollary 3.19 in the lecture notes). This proves that $A \subset U$. Similarly, $B \subset V$. Hence U and V are two open sets which separate A and B topologically. □

Exercise 2. *Show that $U = X - supp(f)$ is the largest open set with the property that $f = 0$ when restricted to it.*

Proof. By definition

$$supp(f) = \overline{f^{-1}(\mathbb{R} - \{0\})}$$

It is obvious then that $f = 0$ when restricted to U. Now, assume that there is another open V such that $f = 0$ when restricted to V. I claim that $V \subset U$. Suppose the contrary. Then there exists $z \in supp(f) \bigcap V$. Since $z \in supp(f)$ there is a sequence $\{x_n\}$ of points in $f^{-1}(\mathbb{R} - \{0\})$ that converges to z. In particular, one can choose N big enough so that $x_N \in V$. But then $f(x_N) = 0$. This is a contradiction because $x_N \in f^{-1}(\mathbb{R} - \{0\})$. We conclude that $V \subset U$. □

Exercise 3. *Show that*

$$L = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$$

is a compact subset of $\mathbb{R}$.

Proof. Let $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ be an open cover of L. Then, there is some $\alpha \in \mathfrak{A}$ such that $0 \in U_\alpha$. Since U_α is open, there exists some $N >> 0$ such that $[0, \frac{1}{N}) \subset U_\alpha$. For all $k > N$ one has $\frac{1}{k} \in U_\alpha$. The remaining $x_1, \cdots, x_N$ can obviously be covered by finitely many open sets from the collection $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$. We conclude that one can find a finite subcover. □

Exercise 4. *Consider $X = \mathbb{R}^n$ with the euclidian metric. Prove that a subset A of X is bounded if and only if it is totally bounded.*

Proof. First assume that A is totally bounded. Then there exist $x_1, \cdots, x_N$ points of A such that

$$A \subset B(x_1, 1) \cup \cdots \cup B(x_N, 1)$$

Define

$$R = max\{||x_i|| + 1 : i \leq N\}$$

With this, we have:

$$A \subset B(0, R)$$

Indeed, if $x \in A$, there exists i such that $x \in B(x_i, 1)$, and then

$$||x|| \leq ||x - x_i|| + ||x_i|| \leq 1 + R.$$

Hence A is bounded. On the other hand, assume that A is bounded. Then $A \subset I^n$ for some interval I of the form $I = [-k, k]$ for $k \in \mathbb{N}$. It is enough to prove that I^n is totally bounded. Let $\epsilon > 0$ be given and choose $\delta = \frac{1}{m} < \frac{\epsilon}{\sqrt{n}}$ for some $m \in \mathbb{N}$ big enough. Consider the $(2mk+1)^n$ points of the form $x_{l_1,\cdots,l_n} = (\frac{l_1}{m}, \cdots, \frac{l_n}{m})$ with $l_i \in \{-km, -(km)+1, \cdots, km\}$. It is clear that given $\mathbf{y} = (y_1, \cdots, y_n) \in I^n$ one can find $l_1(\mathbf{y}), \cdots, l_n(\mathbf{y})$ such that

$$|\frac{l_i(\mathbf{y})}{m} - y_i| \leq \delta$$

Then one computes:

$$d(\mathbf{y}, (\frac{l_1(\mathbf{y})}{m}, \cdots, \frac{l_n(\mathbf{y})}{m})) \leq \sqrt{\sum_{i\leq n} |\frac{l_i(\mathbf{y})}{m} - y_i|^2} \leq \sqrt{n\delta^2} < \epsilon$$

We conclude that $\mathbf{y} \in B((\frac{l_1(\mathbf{y})}{m}, \cdots, \frac{l_n(\mathbf{y})}{m}), \epsilon)$. So we can cover I^n with finitely many balls of radius ϵ. □

Exercise 5. *Let (X, d) be a metric space. Show that*

1. *Any $A \subset X$ that is complete must be closed.*
2. *If X is complete then any closed subset of X is complete.*

Proof. Let us see the first part: Suppose that A is not closed, we need to show that A is not complete. Take $y \in \overline{A} - A$. There exist a sequence $\{x_n\}$ of elements of A that converges to y. Since the sequence $\{x_n\}$ is convergent (in X), it is a cauchy sequence. However, it cannot converge to any element of A since it already converges to y and in metric spaces limits are unique. We conclude that A is not complete. Now assume that X is complete and A is closed. We need to prove that A is complete. Take $\{x_n\}$ a cauchy sequence in A. In particular, $\{x_n\}$ is also a cauchy sequence in X and since X is complete, it converges to a limit $y \in X$. Since A is closed it follows that $y \in A$ and the sequence $\{x_n\}$ has a limit in A. □

Exercise 6. *What is the one point compactification of $X = (0,1)\bigcup(2,3)$?*

Proof. The one point compactification of X is the figure eight, which we denote by $\tilde{X}$ (see Figure 1).

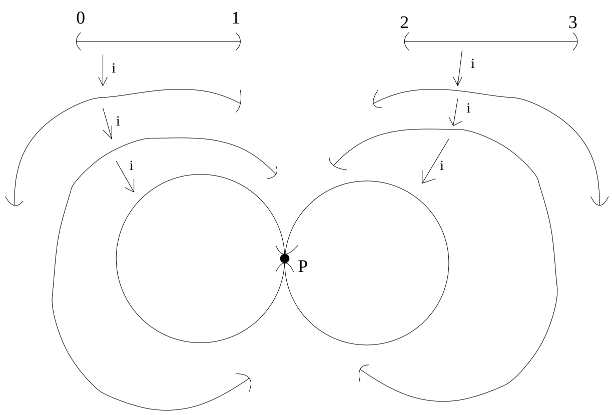

FIGURE 1. Figure eight

To see this, note that if X is homeomorphic to the space obtained from $\tilde{X}$ after removing the point P which is at the intersection of the two circles. The homeomorphism $i : X \to \tilde{X} - P$ is indicated in the picture. Since $\tilde{X}$ is closed and bounded, it is compact. Hence we can use Proposition 8.5 to conclude that the one point compactification of X is homeomorphic to $\tilde{X}$. □