Solutions to exercises for the course Topologie en Meetkunde 1

For a metric space (X, d), one defines $\mathcal{T}_d$ as the collection of all sets $U \subset X$ which are open with respect to d. This defines a topology, called the topology induced by the metric.

Exercise 1. *Show that $\mathcal{T}_d$ is a topology.*

Proof. It is obvious that X and ϕ belong to $\mathcal{T}_d$. Next, assume that $U, V \in \mathcal{T}_d$. We want to show that $U \cap V \in \mathcal{T}_d$. If $U \cap V = \phi$ we are done. Assume then that $x \in U \cap V$. By definition there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$B(x, \epsilon_1) \subset U$$

and

$$B(x, \epsilon_2) \subset V.$$

Define $\epsilon = min\{\epsilon_1, \epsilon_2\}$, then

$$B(x, \epsilon) \subset U \cap V.$$

This proves that $U \cap V \in \mathcal{T}_d$.
Now suppose that there is a family $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ with $V_\alpha \in \mathcal{T}_d$, we need to prove that $U = \bigcup_{\alpha \in \mathcal{A}} V_\alpha \in \mathcal{T}_d$. Again, if U is empty we are done. Now, take $x \in U$. Then $x \in V_\alpha$ for some α and by definition there exists an $\epsilon > 0$ such that

$$B(x, \epsilon) \subset V_\alpha \subset U,$$

this shows that $U \in \mathcal{T}_d$.

□

Exercise 2. *Show that in a Hausdorff topological space, if a sequence has a limit, then the limit is unique.*

Proof. Let X be a Hausdorff topological space and suppose that $x_n \to x$ and $x_n \to y$. We need to prove that $x = y$. Suppose that $x \neq y$, then there are neighborhoods U_x and U_y of x and y respectively with

$$U_x \cap U_y = \phi$$

Since $x_n \to x$ there exists $N_x >> 0$ such that for all $k > N_x$, $x_k \in U_x$. Similarly, there exists $N_y >> 0$ such that for all $k > N_y$, $x_k \in U_y$. Define $N = max\{N_x, N_y\}$, and one has that $x_{N+1} \in U_x \cap U_y$, which is a contradiction. We conclude that $x = y$. □

Exercise 3. *Prove that the euclidean and the square metric induce the same topology in $\mathbb{R}^n$.*

Proof. In view of Lemma 2.11 from the notes it is enough to show that given $\epsilon > 0$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$ there exist δ_1, δ_2 such that:

$$B_d(x, \delta_1) \subset B_\rho(x, \epsilon)$$

and

$$B_\rho(x, \delta_2) \subset B_d(x, \epsilon)$$

Define

$$\delta_1 = \epsilon$$

and

$$\delta_2 = \epsilon/\sqrt{n}$$

Now suppose that $y = (y_1, \cdots y_n) \in B_d(x, \delta_1)$ then:

$$\rho(x, y) = max\{|x_1 - y_1|, \cdots |x_n - y_n|\} \leq \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = d(x, y) < \epsilon$$

Thus, $y \in B_\rho(x, \epsilon)$.
On the other hand, assume that $y = (y_1, \cdots y_n) \in B_\rho(x, \delta_2)$, then:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} < \sqrt{n\delta_2^2} = \epsilon$$

Thus, $y \in B_d(x, \epsilon)$ □

Exercise 4. *Find an example of a space that satisfies the first countability axiom but does not satisfy the second one.*

Proof. Take the space $\mathbb{R}$ with the discrete topology. For any point $x \in \mathbb{R}$, $\{\{x\}\}$ is a basis of neighborhoods for x. However, this space has no countable basis: suppose $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a countable basis. Then for each $x \in \mathbb{R}$ there is $\alpha \in \mathcal{A}$ such that $x \in U_\alpha \subset \{x\}$ which implies that $U_\alpha = \{x\}$. So the basis cannot be countable. □

Exercise 5. *Consider two topological spaces (X, T_X), (Y, T_Y), $A \subset X$ endowed with the induced topology T_A, $B \subset Y$ endowed with the induced topology T_B. Show that*

- *If $\mathbb{B}$ is a basis for the topology T_X, then $\mathbb{B}|_A := \{U \cap A : U \in \mathbb{B}\}$ is a basis for the induced topology on A.*
- *The restriction of the product topology $T_X \times T_Y$ to $A \times B$ coincides with the product topology $T_A \times T_B$.*

Proof. For the first part we need to show that given an open set U of A and a point $x \in U$ there exist $W \in \mathbb{B}|_A$ with:

$$x \in W \subset U$$

Since U is open in A, there exists U' open in X such that $U' \bigcap A = U$. Since $\mathbb{B}$ is a basis for the topology of X, there exists $W' \in \mathbb{B}$ such that

$$x \in W' \subset U'.$$

Then one can choose $W = W' \cap A$. This shows that $\mathbb{B}|_A$ is a basis for T_A.

Let us now look at the second part: Let $\mathbb{B}'$ be a basis for the topology of Y. It is enough to show that

$$\mathbb{B}|_A \times \mathbb{B}'|_B$$

is a basis for $T_A \times T_B$ and for $(T_X \times T_Y)_{A \times B}$. Let us first see that $\mathbb{B}|_A \times \mathbb{B}'|_B$ is a basis for $T_A \times T_B$. From Part (a) we know that $\mathbb{B}|_A$ and $\mathbb{B}'|_B$ are basis for T_A and T_B respectively. Then we know from Lemma 3.2 in the notes that $\mathbb{B}|_A \times \mathbb{B}'|_B$ is a basis for $T_A \times T_B$.
Now,

$$\mathbb{B}|_A \times \mathbb{B}'|_B = (\mathbb{B} \times \mathbb{B}')|_{A \otimes B}$$

so, by the first part of this exercise (applied to the space $X \times Y$) it follows that $\mathbb{B}|_A \times \mathbb{B}'|_B$ is a basis for $(T_X \times T_Y)_{A \times B}$. □

Exercise 6. *(ex. 6, pp. 101) Let A and B denote subsets of a topological space X. Prove that*

(a) *If $A \subset B$ then $\overline{A} \subset \overline{B}$.*
(b) *$\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

Proof. We will present to solutions to this exercise, the first one is more "formal" and the second one may be more "natural".
First solution:
Part(a) It is clear that

$$A \subset B \subset \overline{B}$$

now,

$$\overline{A} = \bigcap_{D \in \mathbb{D}} D$$

where $\mathbb{D}$ is the set of all closed subsets of X that contain A. Then since $\overline{B} \in \mathbb{D}$, it follows that $\overline{A} \subset \overline{B}$.
Part(b) Since

$$A \subset A \cup B$$

and

$$B \subset A \cup B$$

then from part (a) it follows that $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, hence

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B}$$

On the other hand:

$$A \cup B \subset \overline{A} \cup \overline{B}$$

so, again by part (a)

$$\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$$

Second solution: Here we use the characterization of points in the closure of a subset A given in Lemma 3.16:

$$x \in \overline{A} \text{ iff } U_x \cap A \neq \phi \text{ for all } U_x \text{ neighborhood of x}$$

Part(a) Suppose that $x \in \overline{A}$ and let U_x be a neighborhood of x. Then

$$U_x \cap A \subset U_x \cap B$$

and since $U_x \cap A$ is nonempty, so is $U_x \cap B$. We conclude that $x \in \overline{B}$.
Part(b) We argue by contradiction: Take $x \in \overline{A \cup B}$ and assume that

$$x \notin \overline{A}$$

and

$$x \notin \overline{B}$$

Then, there exist neighborhoods U, V of x such that

$$U \cap A = \phi$$

and

$$U \cap B = \phi$$

Define $W := U \cap V$, then W is neighborhood of x and

$$W \cap (A \cup B) = (W \cap A) \cup (W \cap V) = \phi$$

this is a contradiction because $x \in \overline{A \cup B}$. We conclude that x belongs to $\overline{A} \cup \overline{B}$.

$\square$