Inleiding Topologie 2011

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CHAPTER 1

Introduction: some standard spaces

1. Keywords for this course

In this course we study *topological spaces*. One may remember that in group theory one studies groups- and a group is a set G together with some extra-structure (the group operation) which allows us to multiply the elements of G. Similarly, a topological space is a set X together with some "extra-structure" which allows us to make sense of "two points getting close to each other" or, even better, it allows us to make sense of statements like: a sequence $(x_n)_{n\geq 1}$ of elements of X converges to $x \in X$. Of course, if X is endowed with a "metric" (i.e. a way to measure, or to give sense to, "the distance between two points of X"), then such statements have a clear intuitive meaning and can easily be made precise. However, the correct extra-structure that is needed is a bit more subtle- it is the notion of *topology on* X which will be explained in the next chapter.

The interesting functions in topology are the *continuous functions*. One may remember that, in group theory, the interesting functions between two groups G_1 and G_2 are not all arbitrary functions $f : G_1 \longrightarrow G_2$, but just those which "respect the group structure" (group homomorphisms). Similarly, in topology, the interesting maps between two topological spaces X and Y are those functions $f : X \longrightarrow Y$ which are continuous. Continuous means that "it respects the topological structures"- and this will be made precise later. But roughly speaking, f being continuous means that it maps convergent sequences to convergent sequences: if $(x_n)_{n\geq 1}$ is a sequence in X converging to $x \in X$, then the sequence $(f(x_n))_{n\geq 1}$ of elements of Y converges to $f(x) \in Y$.

The correct notion of isomorphism in topology is that of *homeomorphism*. In particular, we do not really distinguish between spaces which are *homeomorphic*. Thinking again back at group theory, there we do not really distinguish between groups which are isomorphic- and there the notion of isomorphism was: a bijection which preserves the group structure. Similarly, in topology, a homeomorphism between two topological spaces X and Y is a bijection $f : X \to Y$ so that f and f^{-1} are both continuous (note the apparent difference with group theory: there, a group isomorphism was a bijection $f : G_1 \to G_2$ such that f is a group homomorphism. The reason that, in group theory, we do not require that f^{-1} is itself a group homomorphism, is simple: it follows from the rest!).

Some of the *main questions* in topology are:

1. how to decide whether two spaces are homeomorphic (= the same topologically) or not?
2. how to decide whether a space is metrizable (i.e. the topology comes from a metric)?
3. when can a space be embedded ("pictured") in the plane, in the space, or in a higher $\mathbb{R}^n$?

These questions played the role of a driving force in Topology. Most of what we do in this course is motivated by these questions; in particular, we will see several results that give answers to them. There are several ways to tackle these questions. The first one - and this will keep us busy for a while- is that of finding special properties of topological spaces, called *topological properties* (such as Hausdorffness, connectedness, compactness, etc). For instance, a space which is compact (or connected, or etc) can never be homeomorphic to one which is not. Another way is that of associating *topological invariants* to topological spaces, so that, if two spaces have distinct topological invariant, they cannot be homeomorphic. The topological invariants could be numbers (such as "the number of distinct connected components", or "the number of wholes", or "the Euler characteristic"), but they can also be more complicated algebraic objects such as groups. The study of such topological invariants is another field on its own (and is part of the course "Topologie en Meetkunde"); what we will do here is to indicate from time to time the existence of such invariants.

In this course we will also devote quite some time to *topological constructions*- i.e. methods that allow us to construct new topological spaces out of ones that we already know (such as taking the product of two topological spaces, the cone of a space, quotients).

Finally, I would like to mention that these lecture notes are based on the book "Topology" by James Munkres. But please be aware that the lecture notes should be self contained (however, you can have a look at the book if you want to find out more). The reason for writing lecture notes is that the book itself requires a larger number of lectures in order to achieve some of the main theorems of topology. In particular, in this lecture notes we present more direct approaches/proofs to such theorems. Sometimes, the price to pay is that the theorem we prove are not in full generality. Our principle is that: choose the version of the theorem that is most interesting for examples (as opposed to "most general") and then find the shortest proof.

2. Spaces

In this chapter we present several examples of "topological spaces" before introducing the formal definition of "topological space" (but trying to point out the need for one). Hence please be aware: some of the statements made in this chapter are rather loose (un-precise)- and I try to make that clear by using quotes; the spaces that we mention here are rather explicit and intuitive, and when saying "space" (as opposed to "set"), we have in mind the underlying set (of elements, also called points) as well as the fact that we can talk (at least intuitively) about its points "getting closer to each other " (or, even better, about convergence of sequences of points in the set). For those who insist of being precise, let us mention that, in this chapter, all our spaces are metric spaces (so that convergence has a precise meaning); even better, although in some examples this is not entirely obvious, all the examples from this chapter are just subspaces of some Euclidean space $\mathbb{R}^n$. Recall here:

Definition 1.1. *Let X be a set. A metric on X is a function*

$$d : X \times X \to \mathbb{R}$$

which associates to a pair (x, y) of points x and y of X, a real number $d(x, y)$, called the distance between x and y, such that the following conditions hold:

(M1) $d(x, y) \geq 0$ *for all* $x, y \in X$.
(M2) $d(x, y) = 0$ *if and only if* $x = y$.
(M3) $d(x, y) = d(y, x)$.
(M4) *(triangle inequality)* $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$.

A metric space is a pair (X, d) consisting of a set X together with a metric d.

Metric spaces are particular cases of "topological spaces"- since they allow us to talk about convergence and continuity. More precisely, given a metric space (X, d), and a sequence $(x_n)_{n\geq 1}$ of points of X, we say that $(x_n)_{n\geq 1}$ converges to $x \in X$ (in (X, d), or with respect to d) if $\lim_{n\to\infty} d(x_n, x) = 0$. When there is no danger of confusion (i.e. most of the times), we will just say that X is a metric space without specifying d. Given two metric spaces X and Y, a function $f : X \to Y$ is called continuous if for any convergent sequence $(x_n)_{n\geq 1}$ in X, converging to some $x \in X$, the sequence $(f(x_n))_{n\geq 1}$ converges (in Y) to $f(x)$. A continuous map f is called a homeomorphism if it is bijective and its inverse f^{-1} is continuous as well. Two spaces are called homeomorphic if there exists a homeomorphism between them.

The most intuitive examples of spaces are the real line $\mathbb{R}$, the plane $\mathbb{R}^2$, the space $\mathbb{R}^3$ or, more generally, the Euclidean space

$$\mathbb{R}^k = \{(x^1, \ldots, x^k) : x^1, \ldots, x^n \in \mathbb{R}\}$$

defined for any integer $k \geq 1$. For them we use the Euclidean metric and the notion of convergence and continuity with respect to this metric:

$$d(x, y) = \sqrt{(x^1 - y^1)^2 + \ldots + (x^k - y^k)^2}$$

(for $x = (x^1, \ldots, x^k)$, $y = (y^1, \ldots, y^k) \in \mathbb{R}^k$). Another interesting metric on $\mathbb{R}^k$ is the square metric ρ, defined by:

$$\rho(x, y) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

The next exercise exercise shows that, although the notion of metric allows us to talk about convergence, metrics do not encode convergence faithfully (two very different looking metrics can induce the same convergent sequences, hence the same "space"). The key of understanding the "topological content" of metrics (i.e. the one that allows us to talk about convergent sequences) is the notion of open subsets- to which we will come back later.

EXERCISE 1.1. *Show that a sequence of points of* $\mathbb{R}^n$ *is convergent with respect to the Euclidean metric if and only if it is convergent with respect to the square metric.*

Inside these Euclidean spaces sit other interesting topological spaces such as intervals, circles, spheres, etc. In general, any subset

$$X \subset \mathbb{R}^k$$

can naturally be viewed as a "space" (and as metric spaces with the Euclidean metric).

EXERCISE 1.2. *Show that, for any two numbers* $a < b$

1. *the interval* $[a, b]$ *is homeomorphic to* $[0, 1]$.
2. *the interval* $[a, b)$ *is homeomorphic to* $[0, 1)$ *and also to* $[0, \infty)$.
3. *the interval* (a, b) *is homeomorphic to* $(0, 1)$ *and also to* $(0, \infty)$ *and to* $\mathbb{R}$.

EXERCISE 1.3. *Explain why the three subset of the plane drawn in Figure 1 are homeomorphic.*

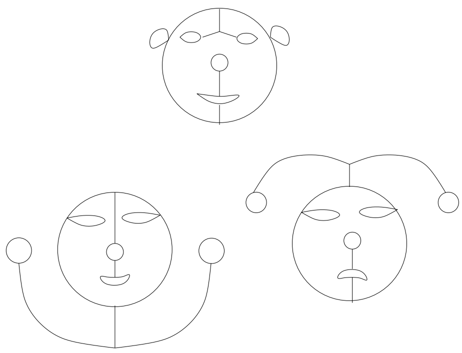


FIGURE 1.

EXERCISE 1.4. *Which of the subset of the plane drawn in Figure 2 do you think are homeomorphic? (be aware that, at this point, we do not have the tools to prove which two are not!).*

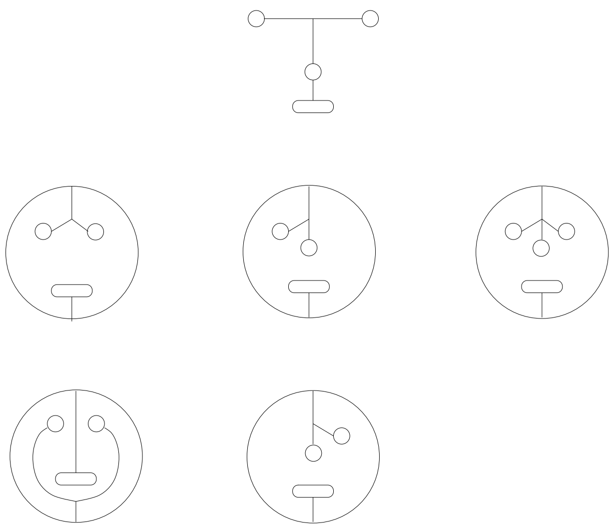


FIGURE 2.

3. The circle

In $\mathbb{R}^2$ one has the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

the open disk

$$\overset{\circ}{D}{}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

the closed disk

$$D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Next, we mentione the standard parametrization of the unit circle: any point on the circle can be written as

$$e^{it} := (cos(t), sin(t))$$

for some $t \in \mathbb{R}$. This gives rise to a function

$$e : \mathbb{R} \to S^1, \quad e(t) = e^{it}$$

which is continuous (explain why!). A nice picture of this function is obtained by first spiraling $\mathbb{R}$ above the circle and then projecting it down, as in Figure 3.

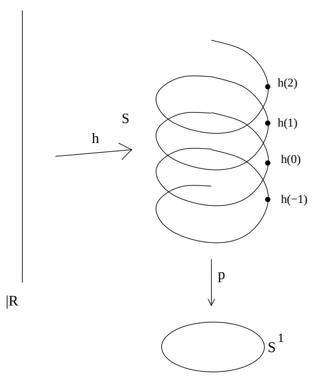


FIGURE 3.

EXERCISE 1.5. *Make Figure 3 more precise. More precisely, find an explicit subspace $S \subset \mathbb{R}^3$ which looks like the spiral, and a homeomorphism h between $\mathbb{R}$ and S, so that the map e above is obtained by first applying h and then applying the projection p ($p(x, y, z) = (x, y)$).*
(Hint: $\{(x, y, z) \in \mathbb{R}^3 : x = cos(z), y = sin(z)\}$).

Note also that, if one restricts to $t \in [0, 2\pi)$, we obtain a continuous bijection

$$f : [0, 2\pi) \to S^1.$$

However, $[0, 2\pi)$ and S^1 behave quite differently as topological spaces, or, more precisely, they are not homeomorphic. Note that this does not only mean that f is not a homeomorphism; it means that neither f nor any other bijection between $[0, 2\pi)$ and S^1 is a homeomorphism. It will be only later, after some study of topological properties (e.g. compactness), that we will be able to prove this statement. At this point however, one can solve the following

EXERCISE 1.6. *Show that the map $f : [0, 2\pi) \to S^1$ is not a homeomorphism.*

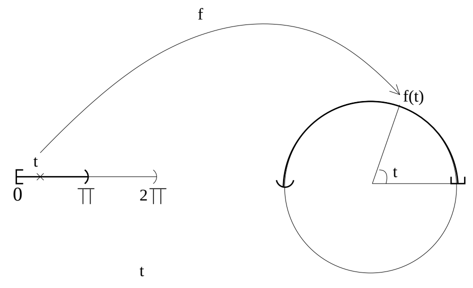


FIGURE 4.

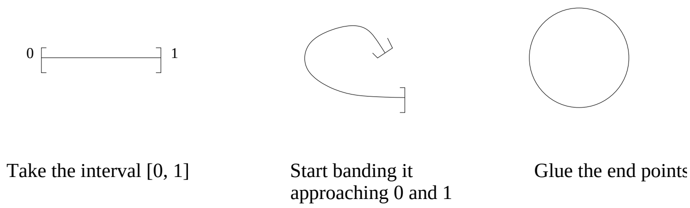


FIGURE 5.

Next, there is yet another way one can look at the unit circle: as obtained from the unit interval $[0, 1]$ by "banding it" and "gluing" its end points, as pictured in Figure 5. This "gluing process" will be made more precise later and will give yet another general method for constructing interesting topological spaces.

In general, by a (topological) circle we mean any space which is homeomorphic to S^1. In general, they may be placed in the space in a rather non-trivial way. Some examples of circles are:

- circles S^1_r with a radius $r > 0$ different from 1, or other circles placed somewhere else in the plane.
- pictures obtained by twisting a circle in the space, such as in Figure 6.
- even pictures which, in the space, are obtained by braking apart a circle, knotting it, and then gluing it back (see Figure 7).

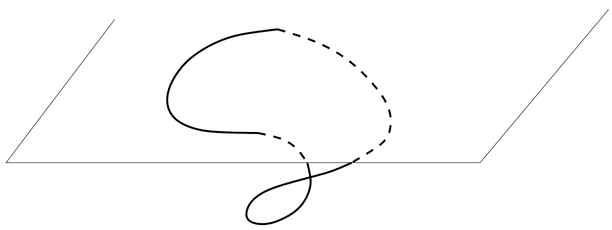


FIGURE 6.

EXERCISE 1.7. *Explain on pictures that all the spaces enumerated above are homeomorphic to* S^1*. If you find it strange, try to explain to yourself what makes it look strange (is it really the circles, or is it more about the ambient spaces in which you realize the circles?).*

Similarly, by a (topological) disk we we mean any space which is homeomorphic to D^2. For instance, the unit square

$$[0, 1] \times [0, 1] = \{(x, y) \in \mathbb{R}^2 : x, y \in [0, 1]\}$$

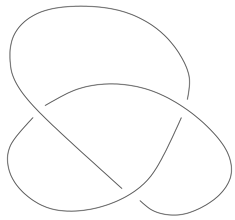
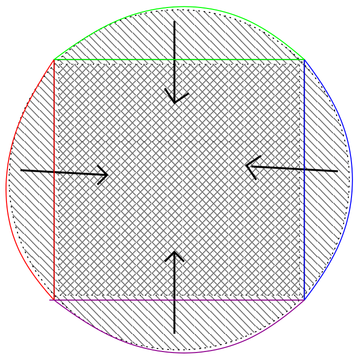


FIGURE 7.

is an important example of topological disk. More precisely, one has the following:

EXERCISE 1.8. *Show that the unit disk D^2 is homeomorphic to the unit square, by a homeomorphism which restricts to a homeomorphism between the unit circle S^1 and the boundary of the unit square.*



The unit disk and the square are homeomorphic

FIGURE 8.

4. The sphere and its higher dimensional versions

For each n, we have the <u>n-sphere</u>

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_0)^2 + \ldots + (x_n)^2 = 1\} \subset \mathbb{R}^{n+1},$$

the <u>open $(n+1)$-disk</u>

$$\overset{\circ}{D}{}^{n+1} = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_0)^2 + \ldots + (x_n)^2 < 1\} \subset \mathbb{R}^{n+1},$$

and similarly the <u>closed $(n+1)$-disk</u> D^{n+1}.

The points

$$p_N = (0, \ldots, 0, 1), p_S = (0, \ldots, 0, -1) \in S^n$$

are usually called the north and the south pole, respectively, and

$$S^n_+ = \{(x_0, \ldots, x_n) : x_n \geq 0, S^n_- = \{(x_0 \ldots, x_n) : x_n \leq 0\}$$

are called the north and the south hemisphere, respectively. See figure 9.

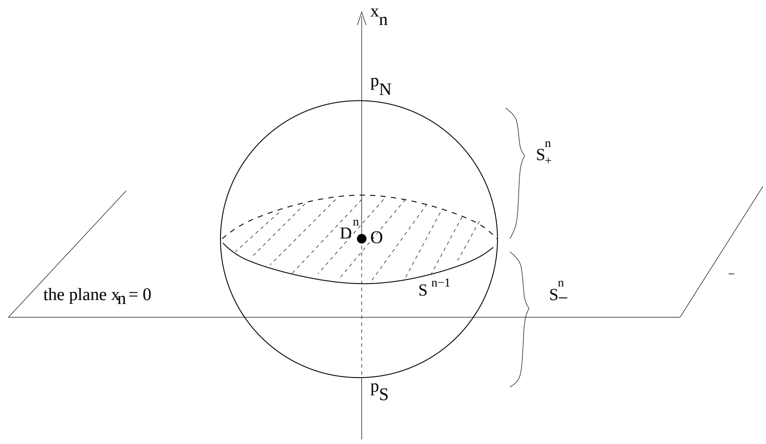


FIGURE 9.

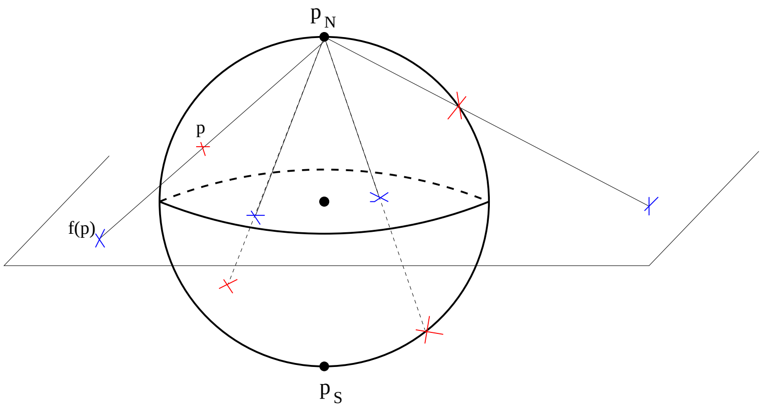
EXERCISE 1.9. *Explain on a picture that $S^n_+ \cap S^n_-$ is homeomorphic to S^{n-1}, and S^n_+ and S^n_- are both homeomorphic to D^n.*

As with the circle, we will call a sphere (or an n-sphere) any space which is homeomorphic to S^2 (or S^n). Also, we call a disk (or n-disk, or open n-disk) any space which is homeomorphic to D^2 (or D^n, or $\overset{\circ}{D}{}^n$). For instance, the previous exercise shows that the two hemispheres S^n_+ and S^n_- are n-disks, and S^n is the union of two n-disks whose intersection is an $n-1$-sphere. This also indicates that S^n can actually be obtained by gluing two copies of D^n along their boundaries. And, still as for circles, there are many subspaces of Euclidean spaces which are spheres (or disks, or etc), but look quite different from the actual unit sphere (or unit disk, or etc). An important example has already been seen in Exercise 1.8 .

Another interpretation of S^n is as adding to $\mathbb{R}^n$ "a point at infinity". (to be made precise later on). This can be explained using the stereographic projection

$$f : S^n - \{p_N\} \to \mathbb{R}^n$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 10).

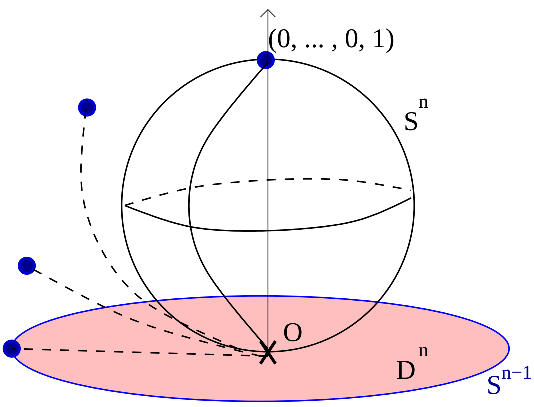


The stereographic projection (sending the red points to the blue ones)

FIGURE 10.

EXERCISE 1.10. *Explain on the picture that the stereographic projection is a homeomorphism between* $S^n - \{p_N\}$ *and* $\mathbb{R}^n$*, and that it cannot be extended to a continuous function defined on the entire* S^n*. Then try to give a meaning to: "*S^n *can be obtained from* $\mathbb{R}^n$ *by adding a point at infinity to". Also, find the explicit formula for* f*.*

Here is anther construction of the n-sphere. Take a copy of D^n, grab its boundary $S^{n-1} \subset D^n$ and glue it together (so that it becomes a point). You then get S^n (see Figure 11).



glue the boundary of D n to a point to obtain S n

FIGURE 11.

EXERCISE 1.11. *Find explicitly the function*

$$f : D^2 \to S^2$$

from Figure 11, check that $f^{-1}(p_N)$ *is precisely* $S^1 \subset D^2$*, then generalize to arbitrary dimensions.*

Another interesting way of obtaining the sphere S^2 is by taking the unit disk D^2, dividing its boundary circle S^1 into two equal sides and gluing the two half circles as indicated in the Figure 12.

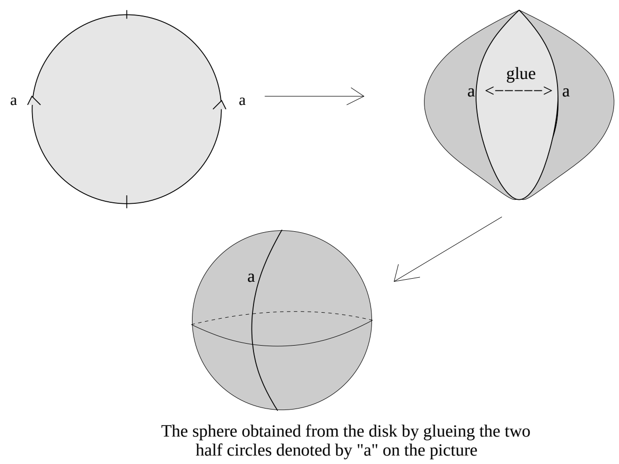


FIGURE 12.

A related construction of the sphere, which is quite important, is the following: take the disk D^2 and divide now it boundary circle into four equal sides, or take the unit square and its sides, and label them as in Figure 13. Glue now the two arcs denoted by a and the two arcs denoted by b.

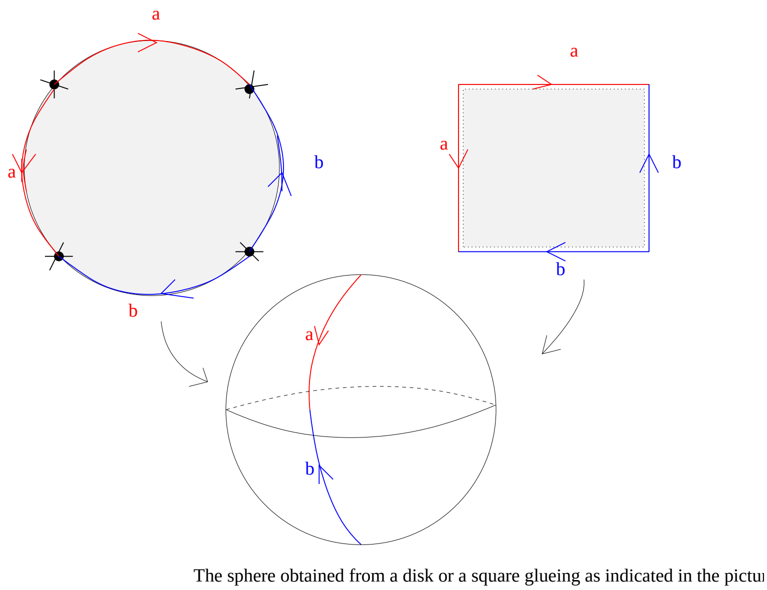


FIGURE 13.

5. The Moebius band

"The Moebius band" is a standard name for subspaces of $\mathbb{R}^3$ which are obtained from the unit square $[0,1]\times[0,1]$ by "gluing" two opposite sides after twisting the square one time, as shown in Figure 14. As in the discussion about the unit circle (obtained from the unit interval by "gluing"

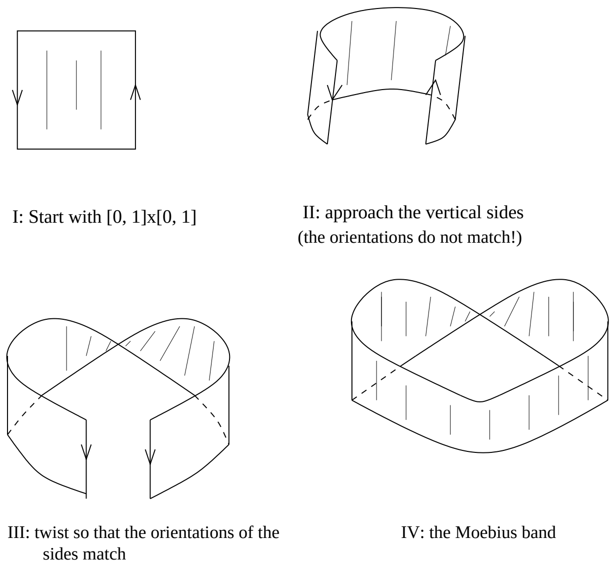


FIGURE 14.

its end points), this "gluing process" should be understood intuitively, and the precise meaning in topology will be explained later. The following exercise provides a possible parametrization of the Moebius band (inside $\mathbb{R}^3$).

EXERCISE 1.12. *Consider*

$$f : [0,1] \times [0,1] \to \mathbb{R}^3,$$
$$f(t,s) = ((2 + (2s-1)sin(\pi t))cos(2\pi t), (2 + (2s-1)sin(\pi t))sin(2\pi t), (2s-1)cos(\pi t)).$$

You may want to check that $f(t,s) = f(t',s')$ holds only in the following cases:

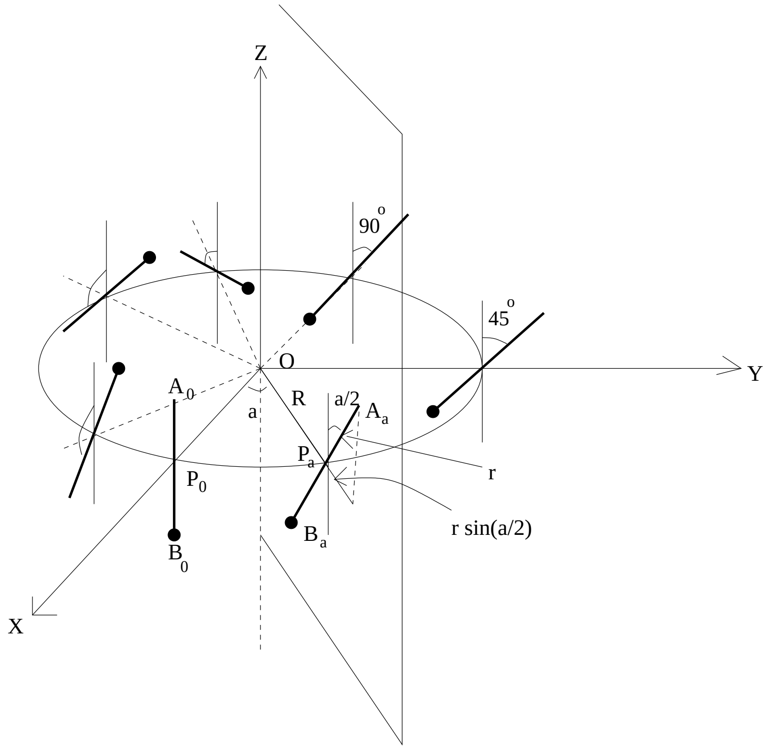
1. $(t,s) = (t',s')$.
2. $t = 0$, $t' = 1$ *and* $s' = 1 - s$.
3. $t = 1$, $t' = 0$ *and* $s' = 1 - s$.

(but this also follows from the discussion below). Based on this, explain why the image of f can be considered as the result of gluing the opposite sides of a square with the reverse orientation.

To understand where these formulas come from, and to describe explicit models of the Moebius band in $\mathbb{R}^3$, we can imagine the Moebius band as obtained by starting with a segment in $\mathbb{R}^3$ and rotating it around its middle point, while its middle point is being rotated on a circle. See Figure 15.

The rotations take place at the same time (uniformly), and while the segment rotates by $180°$, the middle point makes a full rotation ($360°$). To write down explicit formulas, assume that

- the circle is situated in the XOY plane, is centered at the origin, and has radius R.
- the length of the segment is $2r$ and the starting position A_0B_0 of the segment is perpendicular on XOY with middle point $P_0 = (R,0,0)$.
- at any moment, the segment stays in the plane through the origin and its middle point, which is perpendicular on the XOY plane.



Explicit realization of the Moebius band in |R3 : M R, r

FIGURE 15.

We denote by $M_{R,r}$ the resulting subspace of $\mathbb{R}^3$ (note that we need to impose the condition $R > r$). To parametrize $M_{R,r}$, we parametrize the movement by the angle a which determines the middle point on the circle:

$$P_a = (Rcos(a), Rsin(a), 0).$$

At this point, the precise position of the segment, denoted A_aB_a, is determined by the angle that it makes with the perpendicular on the plane XOY through P_a; call it b. This angle depends on a. Due to the assumptions (namely that while a goes from 0 to 2π, b only goes from 0 to π, and that the rotations are uniform), we have $b = a/2$ (see 15). We deduce

$$A_a = \{(R + rsin(a/2))cos(a), (R + rsin(a/2))sin(a), rcos(a/2)),$$

and a similar formula for B_a (obtained by replacing r by $-r$). Then, the Moebius band $M_{R,r}$ is:

$$M_{R,r} = \{(R+tsin(a/2))cos(a), (R+tsin(a/2))sin(a), tcos(a/2)) : a \in [0, 2\pi], t \in [-r, r]\} \quad (5.1)$$

Note that, although this depends on R and r, different choices of R and r produce homeomorphic spaces. To fix one example, one usually takes $R = 2$ and $r = 1$.

EXERCISE 1.13. *Do the following:*

1. *Make a model of the Moebius band and cut it through the middle circle. You get a new connected object. Do you think it is a new Moebius band? Then cut it again through the middle circle and see what you get.*
2. *Prove (without using a paper model) that if you cut the Moebius band through the middle circle, you obtain a (space homeomorphic to a) cylinder. What happens if you cut it again?*

6. The torus

"The torus" is a standard name for subspaces of $\mathbb{R}^3$ which look like a doughnut.

The simplest construction of the torus is by a gluing process: one starts with the unit square and then one glues each pair of opposite sides, as shown in Figure 16.

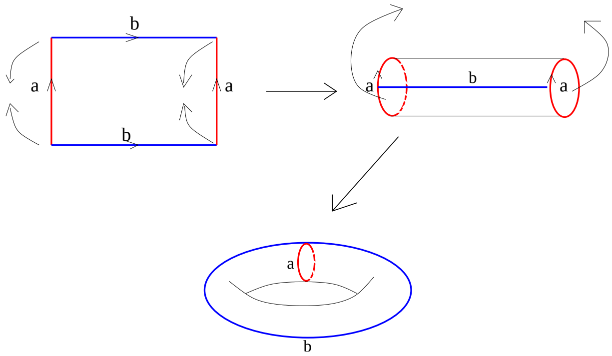


FIGURE 16.

EXERCISE 1.14. *Show that the surface of a cup with a handle is homeomorphic to the torus (what about a cup with no handle?).*

As in the case of circles, spheres, disks, etc, by a torus we mean any space which is homeomorphic to the doughnut. Let's find explicit models (in $\mathbb{R}^3$) for the torus. To achieve that, we will build it by placing our hand in the origin in the space, and use it to rotate a rope which at the other end has attached a non-flexible circle. The surface that the rotating circle describes is clearly a torus (see Figure 17).

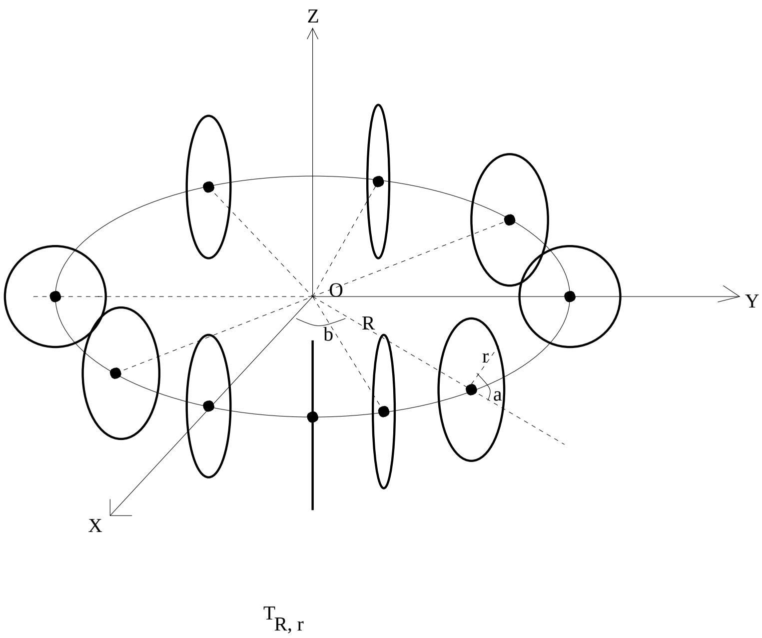


FIGURE 17.

To describe the resulting space explicitly, we assume that the rope rotates inside the XOY plane (i.e. the circle rotates around the OZ axis). Also, we assume that the initial position of the circle is in the XOZ plane, with center of coordinates $(R, 0, 0)$, and let r be the radius of the circle ($R > r$ because the length of the rope is $R - r$). We denote by $T^2_{R,r}$ the resulting subspace of $\mathbb{R}^3$. A point on $T^2_{R,r}$ is uniquely determined by the angles a and b indicated on the picture (Figure 17), and we find the parametric description:

$$T^2_{R,r} = \{(R + rcos(a))cos(b), (R + rcos(a))sin(b), rsin(a)) : a, b \in [0, 2\pi]\} \subset \mathbb{R}^3. \tag{6.1}$$

EXERCISE 1.15. *Show that*

$$T^2_{R,r} = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}. \tag{6.2}$$

Although $T^2_{R,r}$ depends on R and r, different choices of R and r produce homeomorphic spaces.

There is yet another interpretation of the torus, as the Cartesian product of two circles:

$$S^1 \times S^1 = \{(z, z') : z, z' \in S^1\} \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4.$$

Note that, priory, this product is in the 4-dimensional space. The torus can be viewed as a homeomorphic copy inside $\mathbb{R}^3$.

EXERCISE 1.16. *Show that*

$$f : S^1 \times S^1 \to T^2_{R,r},\ \ f(e^{ia}, e^{ib}) = (R + rcos(a))cos(b), (R + rcos(a))sin(b), rsin(a))$$

is a bijection. Explain the map in the picture, and convince yourself that it is a homeomorphism.

Proving directly that f is a homeomorphism is not really pleasant, but the simplest way of proving that it is actually a homeomorphism will require the notion of compactness (and we will come back to this at the appropriate time).

Related to the torus is the double torus, pictured in Figure 18. Similarly, for each $g \geq 1$ integer, one can talk about the torus with g-holes.

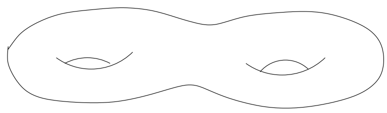


FIGURE 18.

EXERCISE 1.17. *The surface of a cup with two handles is homeomorphic to the double torus.*

Note that the double torus can be realized from two disjoint copies of the torus, by removing a small ball from each one of them, and then gluing them along the resulting circles (Figure 19).

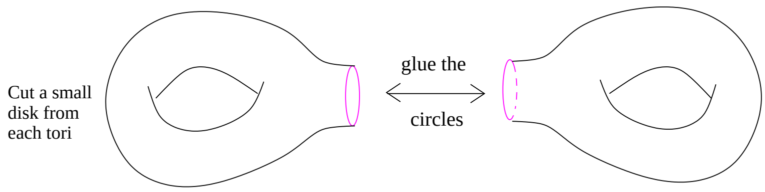


FIGURE 19.

EXERCISE 1.18. *How should one glue the sides of a pentagon so that the result is a cut torus? (Hint: see Figure 20 and try to understand it. Try to make a paper model).*

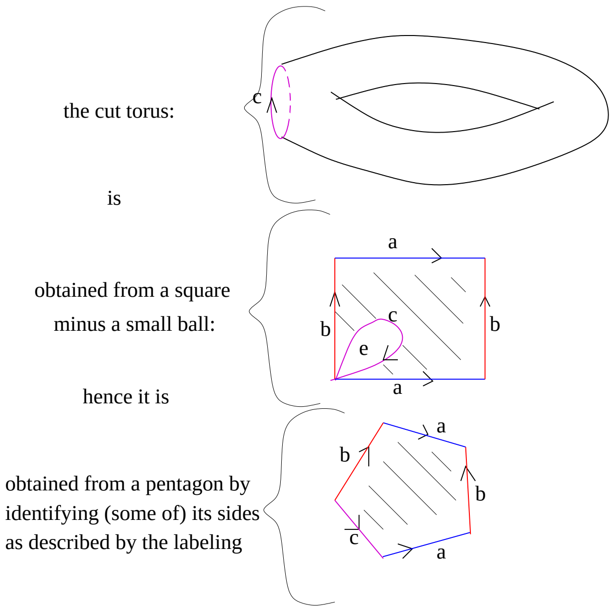


FIGURE 20.

EXERCISE 1.19. *Show that the double torus can be obtained from an octagon by gluing some of its sides. (Hint: see Figure 21 and try to understand it. Try to make a paper model).*

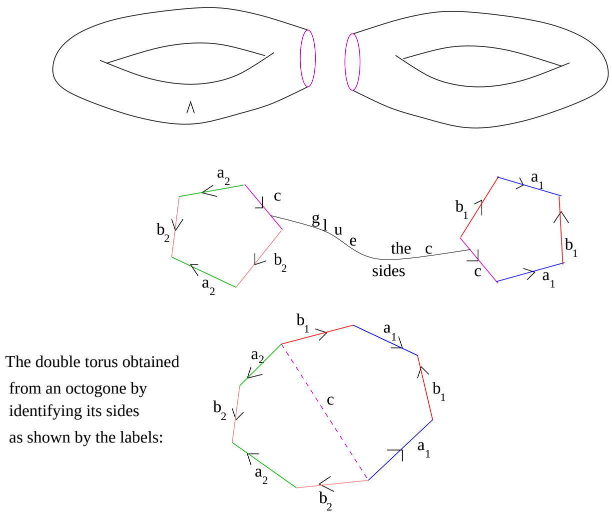


FIGURE 21.

7. The Klein bottle

We have seen that, starting from an unit square and gluing some of its sides, we can produce the sphere (Figure 13), the Moebius band (Figure 14) or the torus (Figure 16). What if we try to glue the sides differently? The next in this list of example would be the space obtained by gluing the opposite sides of the square but reversing the orientation for one of them, as indicated in Figure 22. The resulting space is called the Klein bottle, denoted here by K. Trying to repeat what we have done in the previous examples, we have trouble when "twisting the cylinder". Is that really a problem? It is now worth having a look back at what we have already seen:

- starting with an interval and gluing its end points, although the interval sits on the real line, we did not require the gluing to be performed without leaving the real line (it would not have been possible). Instead, we used one extra-dimension to have more freedom and we obtained the circle.
- similarly, when we constructed the Moebius band or the torus, although we started in the plane with a square, we did not require the gluing to take place inside the plane (it wouldn't even have been possible). Instead, we used an extra-dimension to have more freedom for "twisting", and the result was sitting in the space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

Something very similar happens in the case of the Klein bottle: what seems to be a problem only indicates the fact that the gluing cannot be performed in $\mathbb{R}^3$; instead, it indicates that K cannot be pictured in $\mathbb{R}^3$, or, more precisely, that K cannot be embedded in $\mathbb{R}^3$. Instead, K can be embedded in $\mathbb{R}^4$. The following exercise is an indication of that.

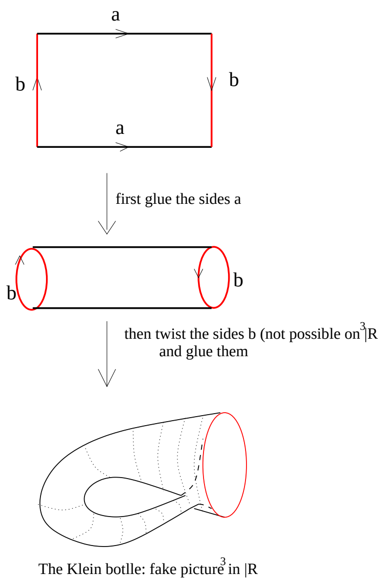


FIGURE 22.

EXERCISE 1.20. *Consider the map*

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^4,$$

$$\tilde{f}(t,s) = ((2 - cos(2\pi s)cos(2\pi t), (2 - cos(2\pi s)sin(2\pi t), sin(2\pi s)cos(\pi t), sin(2\pi s)sin(\pi t)).$$

Explain why the image of this map in $\mathbb{R}^4$ can be interpreted as the space obtained from the square by gluing its opposite sides as indicated in the picture (hence can serve as a model for the Klein bottle).

EXERCISE 1.21. *Explain how can one obtain a Klein bottle by starting with two Moebius bands and gluing them along their boundaries.*
(Hint: look at Figure 23).

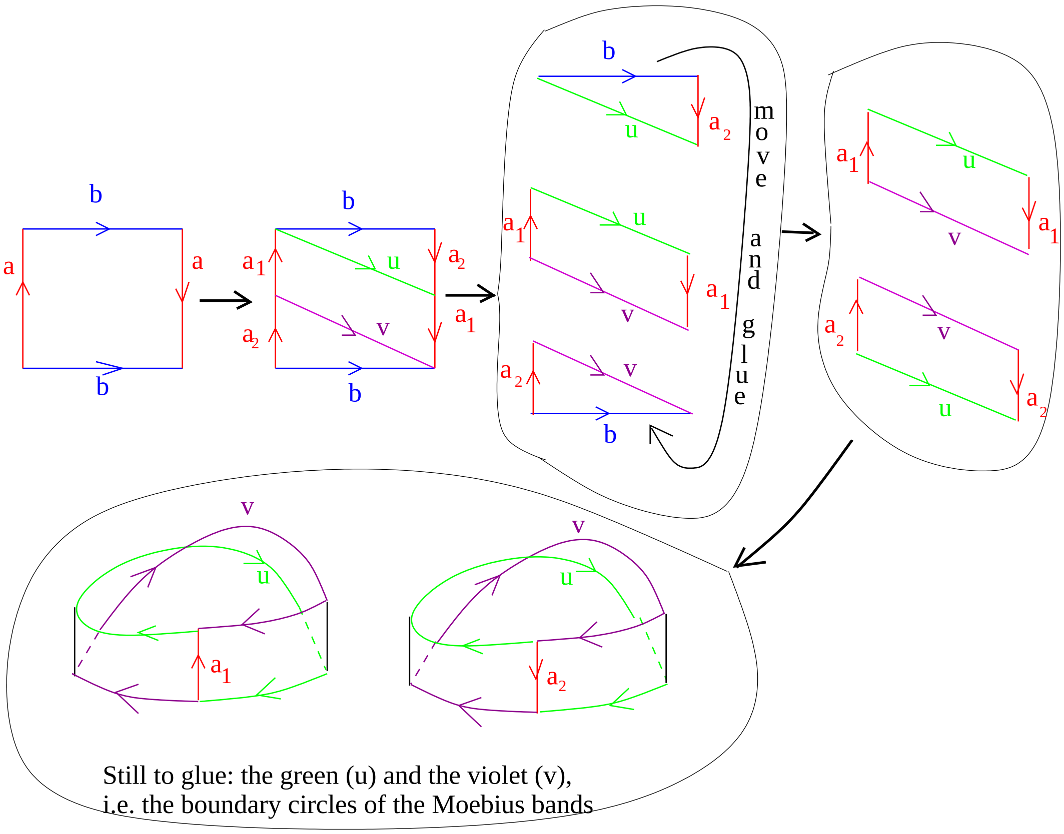


FIGURE 23.

8. The projective plane $\mathbb{P}^2$

In the same spirit as that of the Klein bottle, let's now try to glue the sides of the square as indicated on the left hand side of Figure 24.

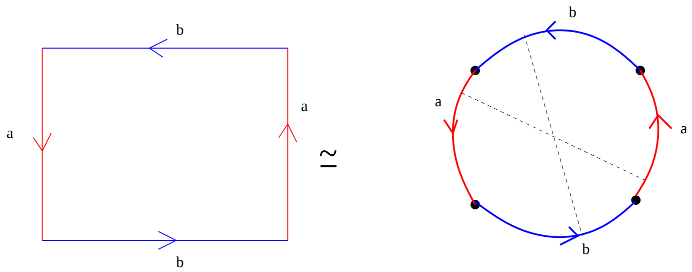


FIGURE 24.

Still as in the case of the Klein bottle, it is difficult to picture the result because it cannot be embedded in $\mathbb{R}^3$ (but it can be embedded in $\mathbb{R}^4$!). However, the result is very interesting: it can be interpreted as the set of all lines in $\mathbb{R}^3$ passing through the origin- denoted $\mathbb{P}^2$. Let's us adopt here the standard definition of the projective space:

$$\mathbb{P}^2 := \{l : l \subset \mathbb{R}^3 \text{ is a line through the origin}\},$$

(i.e. $l \subset \mathbb{R}^3$ is a one-dimensional vector subspace). Note that this is a "space" in the sense that there is a clear intuitive meaning for "two lines getting close to each other". We will explain how $\mathbb{P}^2$ can be interpreted as the result of the gluing that appears in Figure 24.

Step 1: First of all, there is a simple map:

$$f : S^2 \to \mathbb{P}^2$$

which associates to a point on the sphere, the line through it and the origin. Since every line intersects the sphere exactly in two (antipodal) points, this map is surjective and has the special property:

$$f(z) = f(z') \iff z = z' \text{ or } z = -z' \ (z \text{ and } z' \text{ are antipodal}).$$

In other words, $\mathbb{P}^2$ can be seen as the result of gluing the antipodal points of the sphere.

Step 2: In this gluing process, the lower hemisphere is glued over the upper one. We see that, the result of this gluing can also be seen as follows: start with the upper hemisphere S^2_+ and then glue the antipodal points which are on its boundary circle.

Step 3: Next, the upper hemisphere is homeomorphic to the horizontal unit disk (by the projection on the horizontal plane). Hence we could just start with the unit disk D^2 and glue the opposite points on its boundary circle.

Step 4: Finally, recall that the unit ball is homeomorphic to the square (by a homeomorphism that sends the unit circle to the contour of the square). We conclude that our space can be obtained by the gluing indicated in the initial picture (Figure 24).

Note that, since $\mathbb{P}^2$ can be interpreted as the result of gluing the antipodal points of S^2, the following exercise indicates why $\mathbb{P}^2$ can be seen inside $\mathbb{R}^4$:

EXERCISE 1.22. *Show that*

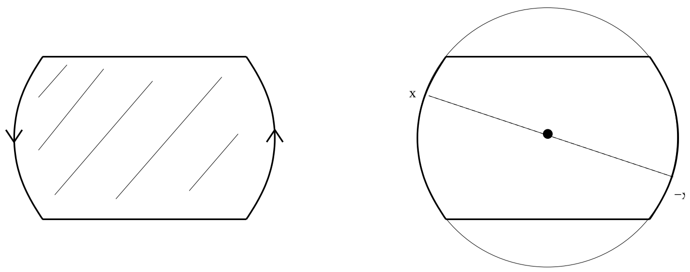
$$\tilde{f} : S^2 \to \mathbb{R}^4 \ , \tilde{f}(x,y,z) = (x^2 - y^2, xy, xz, yz)$$

has the property that, for $p, p' \in S^2$*,* $f(p) = f(p')$ *holds if and only if* p *and* p' *are either equal or antipodal.*

Note also that (a model of) the Moebius band can be seen as sitting inside (a model of) the projective plane $M \hookrightarrow \mathbb{P}^2$. To see this, recall that $\mathbb{P}^2$ can be seen as obtained from D^2 by gluing the antipodal points on its boundary. Consider inside D^2 the "band"

$$B = \{(x,y) \in D^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\}. \subset D^2.$$

The gluing process that produces $\mathbb{P}^2$ affects B in the following way: it glues the "opposite curved sides" of B as in the picture (Figure 25), and gives us the Moebius band.



The Moebius band inside the projective plane

FIGURE 25.

Paying attention to what happens to $D^2 - B$ in the gluing process, you can now try the following.

EXERCISE 1.23. *Indicate how* $\mathbb{P}^2$ *can be obtained by starting from a Moebius band and a disk, and glue them along the boundary circle.*

9. Gluing (or quotients)

We have already seen some examples of spaces obtained by gluing some of their points. When the gluing becomes less intuitive or more complicated, we start asking ourselves:

1. What gluing really means?
2. What is the result of such a gluing?

Here we address these questions. In examples such the circle, torus or Moebius band, the answer was clear intuitively:

1. Gluing had the intuitive meaning- done effectively by using paper models.
2. the result was a new object, or rather a shape (it depends on how much we twist and pull the piece of paper).

Emphasize again that there was no preferred torus or Moebius band, but rather models for it (each two models being homeomorphic). Moving to the Klein bottle, things started to become less intuitive, since the result of the gluing cannot be pictured in $\mathbb{R}^3$ (and things become probably even worse in the case of the projective plane). But, as we explained above, if we use an extra-dimension, the Klein bottle exist in $\mathbb{R}^4$- and Exercise 1.20 produces a subset of $\mathbb{R}^4$ which is an explicit *model* for it.

And things become much worse if we now start performing more complicated gluing of more complicated objects (one can even get "spaces" which cannot be "embedded" in any of the spaces $\mathbb{R}^n$!). It is then useful to a have a more conceptual (but abstract) understanding of what "gluing" and "the result of a gluing" means.

Start with a set X and assume that we want to glue some of its points. Which points we want to glue form the initial "gluing data", which can be regarded as a subset

$$R \subset X \times X$$

consisting of all pairs (x, y) with the property that we want x and y to be glued. This subset must have some special properties (e.g., if we want to glue x to y, y to z, then we also have to glue x to z). This brings us to the notions of equivalence relation that we now recall.

DEFINITION 1.2. *An equivalence relation on a set X is a subset $R \subset X \times X$ satisfying the following:*

1. *If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.*
2. *If $(x, y) \in R$ then also $(y, x) \in R$.*
3. *$(x, x) \in R$ for all $x \in X$.*

Hence, as a gluing data we start with any equivalence relation on X.

EXERCISE 1.24. *Describe explicitely the equivalence relation R on the square $X = [0, 1] \times [0, 1]$ which describe the gluing that we performed to construct the Moebius band. Similarly for the equivalence relation on the disk that described the gluing from Figure 11.*

What should the result of the gluing be? First of all, it is going to be a new set Y. Secondly, any point of X should give a point in Y, i.e. there should be a function $\pi : X \to Y$ which should be surjective (the gluing should not introduce new points). Finally, π should really reflect the gluing, in the sense that $\pi(x) = \pi(y)$ should only happen when $(x, y) \in R$. Here is the formal definition.

DEFINITION 1.3. *Given an equivalence relation R on a set X, a quotient of X modulo R is a pair (Y, π) consisting of a set Y and a surjection $\pi : X \to Y$ (called the quotient map) with the property that*

$$\pi(x) = \pi(y) \Longleftrightarrow (x, y) \in R.$$

Hence, a quotient of X modulo R can be viewed as a model for the set obtained from X by gluing its points according to R.

EXERCISE 1.25. *Describe the equivalence relation R on $[0,1] \times [0,1]$ that encodes the gluing that we performed when constructing the torus. Then prove that the explicit model $T^2_{R,r}$ of the torus given by formula (6.2) is a quotient of $[0,1] \times [0,1]$ modulo R, in the sense of the previous definition (of course, you also have to describe the map π).*

One can wonder "how many" abstract quotients can one build? Well, only one- up to isomorphism. More precisely:

EXERCISE 1.26. *Show that if (Y_1, π_1) and (Y_2, π_2) are two quotients of X modulo R, then there exists and is unique a bijection $f : Y_1 \to Y_2$ such that $f \circ \pi_1 = \pi_2$.*

One can also wonder: can one always build quotients? The answer is yes but, in full generality (for arbitrary X and R), one has to construct the model abstractly. Namely, for each $x \in X$ we define the R-equivalence class of x as

$$R(x) := \{y \in X : (x,y) \in R\}$$

(a subset of X) and define

$$X/R = \{R(x) : x \in X\}$$

(a new set whose elements are subsets of X). There is a simple function

$$\pi_R : X \to X/R, \quad \pi_R(x) = R(x),$$

called the canonical projection. This is the abstract quotient of X modulo R.

EXERCISE 1.27. *Show that:*

1. *For any equivalence relation R on a set X, $(X/R, \pi_R)$ is a quotient of X modulo R.*
2. *For any surjective map, $f : X \to Y$, there is a unique relation R on X such that (Y, f) is a quotient of X modulo R.*

REMARK 1.4. This discussion has been set-theoretical, so let's now go back to the case that X is a subset of some $\mathbb{R}^n$, and R is some equivalence relation on X. It is clear that, in such a "topological setting", we do not look for arbitrary models (quotients), but only for those which are in agreement with our intuition. In other words we are looking for "topological quotients" ("topological models"). What that really means will be made precise later on (since it requires the precise notion of topology). As a first attempt one could look for quotients (Y, π) of X modulo R with the property that

(1) Y is itself is a subspace of some $\mathbb{R}^k$.
(2) π is continuous.

These requirements pose two problems:

- Insisting that Y is a subspace of some $\mathbb{R}^k$ is too strong- see e.g. the exercise below. (Instead, Y will be just "topological space").
- The list of requirements is not complete. This can already be seen when $R = \{(x,x) : x \in X\}$ (i.e. when no gluing is required). Clearly, in this case, a "topological model" is X itself, and any other model should be homeomorphic to X. However, the requirements above only say that $\pi : X \to Y$ is a continuous bijection which, as we have already seen, does not imply that π is a homeomorphism.

As we already said, the precise list of requirements will be made precise later on. (There is a good news however: if X is "compact", then any quotient (Y, π) of X modulo R which satisfies (1) and (2) above, is a good topological model!).

Exercise 1.28. *Let $X = S^1$, and we want to glue any two points $e^{ia}, e^{ib} \in S^1$ with the property that $b = a + 2\pi\sqrt{2}$. Show that there is no model (Y, π) with Y-a subset of some space $\mathbb{R}^n$ and $\pi : X \to Y$ continuous.*

Exercise 1.29. *Let*

$$\mathbb{P}^n := \{l : l \subset \mathbb{R}^{n+1} \text{ is a line through the origin}\},$$

(i.e. $l \subset \mathbb{R}^{n+1}$ is a one-dimensional vector subspace). Explain how $\mathbb{P}^n$ (with the appropriate quotient maps) can be seen as:

1. *a quotient of $\mathbb{R}^{n+1}$ modulo an equivalence relation that you have to specify (see second part of Exercise 1.27).*
2. *obtained from S^n by gluing every pairs of its antipodal points.*
3. *obtained from D^n by gluing every pair of antipodal points situated on the boundary sphere S^{n-1}.*

10. Metric aspects versus topological ones

Of course, most of what we discussed so far can be done withing the world of metric spaces- a notion that did allow us to talk about convergence, continuity, homeomorphisms. There are however several reasons to allow for more flexibility and leave this world.

One was already indicated in Exercise 1.1 which shows that metric spaces do not encode convergence faithfully. Very different looking metrics on $\mathbb{R}^n$ (e.g. the Euclidean d and the square one ρ- in that exercise) can induce the same notion of convergence, i.e. they induce the same "topology" on $\mathbb{R}^n$- the one that we sense with our intuition. Of course, that exercise just tells us that $\mathrm{Id} : (\mathbb{R}^n, d) \to \mathbb{R}^n, \rho)$ is a homeomorphism, or that d and ρ induce the same "topology" on $\mathbb{R}^n$ (we will make this precise a few line below).

The key of understanding the "topological content" of metrics (i.e. the one that allows us to talk about convergent sequences) is the notion of opens with respect to a metric. This is the first step toward the abstract notion of topological space.

DEFINITION 1.5. *Let (X, d) be a metric space. For $x \in X$, $\epsilon > 0$ one defines the* <u>*open ball*</u> *with center x and radius ϵ (with respect to d):*

$$B_d(x, \epsilon) = \{y \in X : d(y, x) < \epsilon\}.$$

A a set $U \subset X$ is called <u>*open with respect to d*</u> *if*

$$\forall \;\; x \in U \;\; \exists \;\; \epsilon > 0 \;\; \text{such that} \;\; B(x, \epsilon) \subset U. \tag{10.1}$$

<u>*The topology induced by d*</u>*, denoted $\mathcal{T}_d$, is the collection of all such opens $U \subset X$.*

With this we have:

EXERCISE 1.30. *Let d and d' be two metrics on the set X. Show that convergence in X with respect to d coincides with convergence in X with respect to d' if and only if $\mathcal{T}_d = \mathcal{T}_{d'}$.*

It is not a surprise that the notion of convergence and continuity can be rephrased using opens only .

EXERCISE 1.31. *Let (X, d) be a metric space, $(x_n)_{n\geq 1}$ a sequence in X, $x \in X$. Then $(x_n)_{n\geq 1}$ converges to x (in (X, d)) if and only if: for any open $U \in \mathcal{T}_d$ containing x, there exists an integer n_U such that*

$$x_n \in U \quad \forall \quad n \geq n_U.$$

EXERCISE 1.32. *Let (X, d) and (Y, d') be two metric spaces, and $f : X \to Y$ a function. Then f is continuous if and only if*

$$f^{-1}(U) \in \mathcal{T}_d \quad \forall \quad U \in \mathcal{T}_{d'}.$$

The main conclusion is that the topological content of a metric space (X, d) is retained by the family $\mathcal{T}_d$ of opens in the metric space. This is the first example of a topology. I would like to emphasize here that our previous discussion does NOT mean that we should not use metrics and that we should not talk about metric spaces. Not at all! When metrics are around, we should take advantage of them and use them! However, one should be aware that some of the simple operations that we make with metric spaces (e.g. gluing) may take us out of the world of metric spaces. But, even when staying withing the world of metric spaces, it is extremely useful to be aware of what depends on the metric itself and what just on the topology that the metric induces. We give two examples here.

Compactness The first example is the notion of compactness. You have probably seen this notion for subspaces of $\mathbb{R}^n$: a subset $\mathbb{R}^n$ is called compact if it is bounded and closed in $\mathbb{R}^n$ (see Dictaat Inleiding Analyse, Stelling 4.20, page 78). With this definiton it is very easy to work with compactness (... of subspaces of $\mathbb{R}^n$). E.g., the torus, the Moebius band, etc, they are all compact. However, one should be carefull here: what we can say is that all the models of the torus, etc that we built are compact. What about the other ones? In other words, is compactness a topological condition? I.e., if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are homeomorphic, is it true that the compactness of A implies the compactness of B? With the previous definition of compactness, the answer may be no, as both the condition "bounded" and "closed" make reference to the way that A sits inside $\mathbb{R}^n$, and even to the Euclidean metric on $\mathbb{R}^n$ (for boundedness). See also Exercise 1.34 below. However, as we shall see, the answer is: yes, compactness if a topological property (and this is extremely useful).

Completeness Another notion that is extremely important when we talk about metric spaces is that of completeness. Recall (see Dictaat Inleiding Analyse, page 74):

DEFINITION 1.6. *Given a metric space (X, d) and a sequence $(x_n)_{n\geq 1}$ in X, we say that $(x_n)_{n\geq 1}$ is a Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

i.e., for each $\epsilon > 0$, there exists an integer n_ϵ such that

$$d(x_n, x_m) < \epsilon$$

for all $n, m \geq n_\epsilon$. One says that (X, d) is complete if any Cauchy sequence is is convergent. We say that $A \subset X$ is complete if A, together with the restriction of d to A, is complete.

For instance, $\mathbb{R}^n$ with the Euclidean metric is complete, as is any closed subspace of $\mathbb{R}^n$. Now, is completeness a topological property? This time, the answer is no, as the following exercise shows. But, again, this does not mean that we should ignore completeness in this course (and we will not). We should be aware that it is not a topological property, but use it whenever possible!

EXERCISE 1.33. On $\mathbb{R}$ we consider the metric $d'(x, y) = |e^x - e^y|$. Show that

1. d' induces the same topology on $\mathbb{R}$ as the Euclidean metric d.
2. although $(\mathbb{R}, d)$ is complete, $(\mathbb{R}, d')$ is not.

(Hint: $log(\frac{1}{n})$).

EXERCISE 1.34. For a metric space (X, d) we define $\hat{d} : X \times X \to \mathbb{R}$ by

$$\hat{d}(x, y) = \min\{d(x, y), 1\}.$$

Show that:

1. $\hat{d}$ is a metric inducing the same topology on X as d.
2. (X, d) is complete if and only if $(X, \hat{d})$ is.