1. Topological spaces

We start with the abstract definition of topological spaces.

DEFINITION 2.1. *A topology on a set X is a collection $\mathcal{T}$ of subsets of X, satisfying the following axioms:*

(T1) *$\emptyset$ and X belong to $\mathcal{T}$.*
(T2) *The intersection of any two sets from $\mathcal{T}$ is again in $\mathcal{T}$.*
(T3) *The union of any collection of sets of $\mathcal{T}$ is again in $\mathcal{T}$.*

A topological space is a pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.
A subset $U \subset X$ is called open in the topological space $(X, \mathcal{T})$ if it belongs to $\mathcal{T}$.
A subset $A \subset X$ is called closed in the topological space $(X, \mathcal{T})$ if $X - A$ is open.
Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, we say that $\mathcal{T}'$ is larger (or finer) than $\mathcal{T}$, or that $\mathcal{T}$ is smaller (or coarser) than $\mathcal{T}'$, if $\mathcal{T} \subset \mathcal{T}'$.

EXERCISE 2.1. Show that, in a topological space $(X, \mathcal{T})$, any finite intersection of open sets is open: for each $k \geq 1$ integer, $U_1, \ldots, U_k \in \mathcal{T}$, one must have $U_1 \cap \ldots \cap U_k \in \mathcal{T}$. Would it be reasonable to require that arbitrary intersections of opens sets is open? What can you say about intersections or union of closed subsets of $(X, \mathcal{T})$?

TERMINOLOGY/CONVENTIONS 2.2. *When referring to a topological space $(X, \mathcal{T})$, when no confusion may arise, we will simply say that "X is a topological space". Also, the opens in $(X, \mathcal{T})$ will simply be called "opens in X" (and similarly for "closed").*
In other words, we will not mention $\mathcal{T}$ all the time; its presence is implicit in the statement "X is a topological space", which allows us to talk about "opens in X".

EXAMPLE 2.3. *(Extreme topologies)* On any set X we can define the following:

1. The trivial topology on X, $\mathcal{T}_{\text{triv}}$: the topology whose open sets are only $\emptyset$ and X.
2. The discrete topology on X, $\mathcal{T}_{\text{dis}}$: the topology whose open sets are all subsets of X.
3. The co-finite topology on X, $\mathcal{T}_{\text{cf}}$: the topology whose open sets are the empty set and complements of finite subsets of X.
4. The co-countable topology on X, $\mathcal{T}_{\text{cc}}$: the topology whose open sets are the empty set and complements of subsets of X which are at most countable.

An important class of examples comes from metrics.

PROPOSITION 2.4. *For any metric space (X, d), the family $\mathcal{T}_d$ of opens in X with respect to d is a topology on X. Moreover, this is the smallest topology on X with the property that it contains all the balls*

$$B_d(x; r) = \{y \in X : d(x, y) < r\} \quad (x \in X, r > 0).$$

PROOF. Axiom (T1) is immediate. To prove (T2), let $U, V \in \mathcal{T}_{\text{d}}$ and we want to prove that $U \cap V \in \mathcal{T}_{\text{d}}$. We have to show that, for any point $x \in U \cap V$, there exists $r > 0$ such that $B_d(x, r) \subset U \cap V$. So, let $x \in U \cap V$. That means that $x \in U$ and $x \in V$. Since $U, V \in \mathcal{T}_{\text{d}}$, we find $r_1 > 0$ and $r_2 > 0$ such that

$$B_d(x, r_1) \subset U, B_d(x, r_2) \subset V.$$

Then $r = \min\{r_1, r_2\}$, has the desired property: $B_d(x, r) \subset U \cap V$.

To prove axiom (T3), let $\{U_i : i \in I\}$ be a family of elements $U_i \in \mathcal{T}_{\text{d}}$ (indexed by a set I) and we want to prove that $U := \cup_{i \in I} U_i \in \mathcal{T}_{\text{d}}$. We have to show that, for any point $x \in U$, there exists $r > 0$ such that $B_d(x, r) \subset U$. So, let $x \in U$. Then $x \in U_i$ for some $i \in I$; since $U_i \in \mathcal{T}_{\text{d}}$, we find $r > 0$ such that $B_d(x, r) \subset U_i$. Since $U_i \subset U$, r has the desired property $B_d(x, r) \subset U$.

Assume now that $\mathcal{T}$ is a topology on X which contains all the balls and we prove that $\mathcal{T}_d \subset \mathcal{T}$. Let $U \in \mathcal{T}_d$ and we prove $U \in \mathcal{T}$. From the definition of $\mathcal{T}_d$, for each $x \in U$ we find $r_x > 0$ with

$$\{x\} \subset B(x; r_x) \subset U.$$

Taking the union over all $x \in U$ we deduce that

$$U \subset \cup_{x \in U} B(x; r_x) \subset U.$$

Hence $U = \cup_x B(x, r_x)$ and then, since all the balls belong to $\mathcal{T}$, U belongs itself to $\mathcal{T}$. $\square$

DEFINITION 2.5. *A topological space $(X, \mathcal{T})$ is called* <u>*metrizable*</u> *if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.*

REMARK 2.6. And here is one of the important problems in topology:

which topological spaces are metrizable?

More exactly, one would like to find the special properties that a topology must have so that it is induced by a metric. Such properties will be discussed throughout the entire course.

The most basic metric is the Euclidean metric on $\mathbb{R}^n$ which was behind the entire discussion of Chapter 1. Also, the Euclidean metric can be (and was) used as a metric on any subset $A \subset \mathbb{R}^n$.

TERMINOLOGY/CONVENTIONS 2.7. *The topology on $\mathbb{R}^n$ induced by the Euclidean metric is called the* <u>*Euclidean topology*</u> *on $\mathbb{R}^n$. Whenever we talk about "the space $\mathbb{R}^n$" without specifying the topology, we allways mean the Euclidean topology. Similarly for subsets $A \subset \mathbb{R}^n$.*

For other examples of topologies on $\mathbb{R}$ you should look at Exercise 2.19. General methods to construct topologies will be discussed in the next chapter. Here we mention:

EXAMPLE 2.8. *(subspace topology)* In general, given a topological space $(X, \mathcal{T})$, any subset $A \subset X$ inherits a topology on its own. More precisely, one defines the restriction of $\mathcal{T}$ to A, or the topology induced by $\mathcal{T}$ on A (or simply the <u>induced topology</u> on A) as:

$$\mathcal{T}|_A := \{B \subset A : B = U \cap A \text{ for some } U \in \mathcal{T}\}.$$

EXERCISE 2.2. Show that $\mathcal{T}|_A$ is indeed a topology.

TERMINOLOGY/CONVENTIONS 2.9. *Given a topological space $(X, \mathcal{T})$, whenever we deal with a subset $A \subset X$ without specifying the topology on it, we allways consider A endowed with $\mathcal{T}|_A$.*

To remove ambiguities, you should look at Exercise 2.23.

DEFINITION 2.10. *Given a topological space $(X, \mathcal{T})$ and $A, B \subset X$, we say that B is* <u>*open in A*</u> *if $B \subset A$ and B is an open in the topological space $(A, \mathcal{T}|_A)$. Similarly, we say that B is* <u>*closed in A*</u> *if $B \subset A$ and B is closed in the topological space $(A, \mathcal{T}|_A)$.*

EXERCISE 2.3. The interval $[0, 1) \subset \mathbb{R}$:
(i) is neither open nor closed in $(-1, 2)$.
(ii) is open in $[0, \infty)$ but it is not closed in $[0, \infty)$.
(iii) is closed in $(-1, 1)$ but it is not open in $(-1, 1)$.
(iv) it is both open and closed in $(-\infty, -1) \cup [0, 1) \cup (2, \infty)$.

2. Continuous functions; homeomorphisms

DEFINITION 2.11. *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and a function $f : X \to Y$, we say that f is* <u>continuous</u> *(with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$) if:*

$$f^{-1}(U) \in \mathcal{T}_X \quad \forall \quad U \in \mathcal{T}_Y.$$

EXERCISE 2.4. Show that a map $f : X \to Y$ between two topological spaces (X with some topology $\mathcal{T}_X$, and Y with some topology $\mathcal{T}_Y$) is continuous if and only if $f^{-1}(A)$ is closed in X for any closed subspace A of Y.

EXAMPLE 2.12. Some extreme examples first:

1. If Y is given the trivial topology then, for any other topological space $(X, \mathcal{T}_X)$, any function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_{\text{triv}})$ is automatically continuous.
2. If X is given the discrete topology then, for any other topological space $(Y, \mathcal{T}_Y)$, any function $f : (X, \mathcal{T}_{\text{dis}}) \to (Y, \mathcal{T}_Y)$ is automatically continuous.
3. The composition of two continuous functions is continuous: If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ are continuous, then so is $g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$. Indeed, for any $U \in \mathcal{T}_Z$, $V := g^{-1}(U) \in \mathcal{T}_Z$, hence
$$(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U)) = g^{-1}(V)$$
must be in $\mathcal{T}_X$.
4. For any topological space $(X, \mathcal{T})$, the identity map $\text{Id}_X : (X, \mathcal{T}) \to (X, \mathcal{T})$ is continuous. More generally, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then the identity map $\text{Id}_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_2$ is smaller than $\mathcal{T}_1$.

EXAMPLE 2.13. Given $f : X \to Y$ a map between two metric spaces (X, d) and (Y, d'), Exercise 1.32 says that $f : X \to Y$ is continuous as a map between metric spaces (in the sense discussed in the previous chapter) if and only if $f : (X, \mathcal{T}_d) \to (Y, \mathcal{T}_{d'})$ is continuous as a map between topological spaces.

That is good news: all functions $f : \mathbb{R}^n \to \mathbb{R}^m$ between Euclidean spaces that we knew (e.g. from the Analysis course) to be continuous, are continuous in the sense of the previous definition as well. This applies in particular to all the elementary functions such as polynomial ones, exp, sin, cos, etc.

Even more, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ such that $f(A) \subset B$, then the restriction of f to A, viewed as a function from A to B, is automatically continuous (check that!). Finally, the usual operations of continuous functions are continuous:

EXERCISE 2.5. Let X be a topological space, $f_1, \ldots, f_n : X \to \mathbb{R}$ and consider

$$f := (f_1, \ldots, f_n) : X \to \mathbb{R}^n.$$

Show that f is continuous if and only if $f_1, \ldots, f_n$ are. Deduce that the sum and the product of two continuous functions $f, g : X \to \mathbb{R}$ are themselves continuous.

DEFINITION 2.14. *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a* <u>homeomorphism</u> *between them is a bijective function $f : X \to Y$ with the property that f and f^{-1} are continuous. We say that X and Y are* <u>homeomorphic</u> *if there exists a homeomorphism between them.*

REMARK 2.15. In the definition of the notion of homeomorphism (and as we have seen already in the previous chapter), it is not enough to require that $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous and bijective (it may happen that f^{-1} is not continuous!). For f^{-1} to be continuous, one would need that for each $U \subset X$ open, $f(U) \subset Y$ is open. Functions with this property (that send opens to opens) are called <u>open maps</u>.

For instance, the function

$$f : [0, 2\pi) \to S^1, f(t) = (cos(t), sin(t)),$$

that we also discussed in the previous chapter, is continuous and bijective, but it is not a homeomorphism. More precisely, it is not open: $[0, \pi)$ is open in X, while $f([0, 2\pi))$ is a half circle closed at one end and open at the other- hence not open (Figure 4 in the previous chapter).

REMARK 2.16. We would like to emphasize that the notion of "homeomorphism" is the correct notion "isomorphism in the topological world". A homeomorphism $f : X \to Y$ allows us to move from X to Y and backwards carrying along any topological argument (i.e. any argument which is based on the notion of opens) and without loosing any topological information. For this reason, in topology, homeomorphic spaces are not viewed as being different from each other.

Another important question in Topology is:

how do we decide if two spaces are homeomorphic or not?

Actually, all the topological properties that we will discuss in this course (the countability axioms, Hausdorffness, connectedness, compactness, etc) could be motivated by this problem. For instance, try to prove now that $(0, 2)$ and $(0, 1) \cup (1, 2)$ are not homeomorphic (if you managed, you have probably discovered the notion of connectedness). Try to prove that the open disk and the closed disk are not homeomorphic (if you managed, you have probably discovered the notion of compactness). Let us be slightly more precise about the meaning of "topological property".

TERMINOLOGY/CONVENTIONS 2.17. *We call topological property any property $\mathcal{P}$ of topological spaces (that a space may or may not satisfy) such that, if X and Y are homeomorphic, then X has the property $\mathcal{P}$ if and only if Y has it.*

For instance, the property of being metrizable (see Definition 2.5) is a topological property:

EXERCISE 2.6. Let X and Y be two homeomorphic topological spaces. Show that X is metrizable if and only if Y is.

DEFINITION 2.18. *A continuous function $f : X \to Y$ (between two topological spaces) is called an embedding if f is injective and, as a function from X to its image $f(X)$, it is a homeomorphism (where $f(X) \subset Y$ is endowed with the induced topology).*

EXAMPLE 2.19. There are injective continuous maps that are not embeddings. This is the case already with the function $f(\alpha) = (cos(\alpha), sin(\alpha))$ already discussed, viewed as a function $f : [0, 2\pi) \to \mathbb{R}^2$.

REMARK 2.20. Again, one of the important questions in Topology is:

understand when a space X can be embedded in another given space Y

When $Y = \mathbb{R}^2$, that means intuitively that Xcan be pictured topologically on a piece of paper. When $Y = \mathbb{R}^3$, it is about being able to make models of X in space. Of course, one of the most interesting versions of this question is whether X can be embedded in some $\mathbb{R}^N$ for some N. As we have seen, the torus and the Moebius band can be embedded in $\mathbb{R}^3$; one can *prove* that they cannot be embedded in $\mathbb{R}^2$; also, one can *prove* that the Klein bottle cannot be embedded in $\mathbb{R}^3$. However, all these proofs are far from trivial.

3. Neighborhoods and convergent sequences

DEFINITION 2.21. *Given a topological space $(X, \mathcal{T})$, $x \in X$, a neighborhood of x (in the topological space $(X, \mathcal{T})$) is any subset $V \subset X$ with the property that there exists $U \in \mathcal{T}$ such that*

$$x \in U \subset V.$$

When V is itself open, we call it an open neighborhood of x. We denote:

$$\mathcal{T}(x) := \{U \in \mathcal{T} : x \in U\}, \quad \mathcal{N}(x) = \{V \subset X : \ \exists\, U \in \mathcal{T}(x) \text{ such that } U \subset V\}.$$

EXAMPLE 2.22. In a metric space (X, d), from the definition of $\mathcal{T}_d$ we deduce:

$$\mathcal{N}(x) = \{V \subset X : \ \exists\, \epsilon > 0 \text{ such that } B(x, \epsilon) \subset V\}. \tag{3.1}$$

REMARK 2.23. What are neighborhoods good for? They are the " topological pieces" which are relevant when looking at properties which are "local", in the sense that they depend only on what happens "near points". For instance, we can talk about continuity at a point.

DEFINITION 2.24. *We say that a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at x if*

$$f^{-1}(V) \in \mathcal{N}_X(x) \quad \forall \ \ V \in \mathcal{N}_Y(f(x)). \tag{3.2}$$

PROPOSITION 2.25. *A function is continuous if and only if it is continuous at all points.*

PROOF. Assume first that f is continuous, $x \in X$. For $V \in \mathcal{N}(f(x))$, there exists $U \in \mathcal{T}(f(x))$ with $U \subset V$; then $f^{-1}(U)$ is open, contains x and is contained in $f^{-1}(V)$; hence $f^{-1}(V) \in \mathcal{N}(x)$. For the converse, assume that f is continuous at all points. Let $U \subset Y$ open; we prove that $f^{-1}(U)$ is open. For each $x \in f^{-1}(U)$, continuity at x implies that $f^{-1}(U)$ is a neighborhood of x, hence we find $U_x \subset f^{-1}(U)$, with U_x-open containing x. It follows that $f^{-1}(U)$ is the union of all U_x with $x \in f^{-1}(U)$, hence it must be open. ☐

Neighborhoods also allow us to talk about convergence.

DEFINITION 2.26. *Given a sequence $(x_n)_{n \geq 1}$ of elements of in a topological space $(X, \mathcal{T}_X)$, $x \in X$, we say that $(x_n)_{n \geq 1}$ converges to x in $(X, \mathcal{T})$), and we write $x_n \to X$ (or $\lim_{n \to \infty} x_n = x$) if for each $V \in \mathcal{N}_x$, there exists an integer n_V such that*

$$x_n \in V \quad \forall\ n \geq n_V. \tag{3.3}$$

EXAMPLE 2.27. Let X be a set. Then, in $(X, \mathcal{T}_{\text{triv}})$, any sequence $(x_n)_{n \geq 1}$ of points in X converges to any $x \in X$. In contrast, in $(X, \mathcal{T}_{\text{dis}})$, a sequence $(x_n)_{n \geq 1}$ converges to an $x \in X$ if and only if $(x_n)_{n \geq 1}$ is stationary equal to x, i.e. there exists n_0 such that $x_n = x$ for all $n \geq n_0$.

To clarify the relationship between convergence and continuity, we introduce:

DEFINITION 2.28. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $f : X \to Y$. We say that f is sequentially continuous if, for any sequence $(x_n)_{n \geq 1}$ in X, $x \in X$, we have:*

$$x_n \to x \text{ in } (X, \mathcal{T}_X) \Longrightarrow f(x_n) \to f(x) \text{ in } (Y, \mathcal{T}_Y).$$

THEOREM 2.29. *Any continuous function is sequentially continuous.*

PROOF. Assume that $x_n \to x$ (in $(X, \mathcal{T}_X)$). To show that $f(x_n) \to f(x)$ (in $(Y, \mathcal{T}_Y)$, let $V \in \mathcal{N}(f(x))$ arbitrary and we have to find n_V such that $f(x_n) \in V$ for all $n \geq n_V$. Since f is continuous, we must have $f^{-1}(V) \in \mathcal{N}(x)$; since $x_n \to x$, we find n_V such that $x_n \in f^{-1}(V)$ for all $n \geq n_V$. Clearly, this n_V has the desired properties. ☐

DEFINITION 2.30. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A basis of neighborhoods of x (in the topological space $(X, \mathcal{T})$) is a collection $\mathcal{B}_x$ of neighborhoods of x with the property that*

$$\forall\ V \in \mathcal{T}(x)\ \exists\ B \in \mathcal{B}_x : \ B \subset V.$$

EXAMPLE 2.31. If (X, d) is a metric space, $x \in X$, the family of all balls centered at x,

$$\mathcal{B}_d(x) := \{B(x; \epsilon) : \epsilon > 0\}, \tag{3.4}$$

is a basis of neighborhoods of x.

REMARK 2.32. What are bases of neighborhoods good for? They are collections of neighborhoods which are "rich enough" to encode the local topology around the point. I.e., instead of proving conditions for all $V \in \mathcal{N}(x)$, it is enough to do it only for the elements of a basis. For instance, in the the definition of convergence $x_n \to x$ (Definition 2.26), if we have a basis $\mathcal{B}_x$ of neighborhoods of x, it suffices to check the condition from the definition only for neighborhoods $V \in \mathcal{B}_x$ (why?). In the case of a metric space (X, d), we recover the more familiar description of convergence: using the basis (3.4) we find that $x_n \to x$ if and only if:

$$\forall\ \epsilon > 0,\ \exists\ n_\epsilon \in \mathbb{N} : d(x_n, x) < \epsilon\ \forall\ n \geq n_\epsilon.$$

A similar discussion applies to the notion of continuity at a point- Definition 2.24: if we have a basis $\mathcal{B}_{f(x)}$ of neighborhoods of $f(x)$, then it suffices to check (3.2) for all $V \in \mathcal{B}_{f(x)}$. As before, if f is a map between two metric spaces (X, d_X) and (Y, d_Y), we find the more familiar description of continuity: using the basis (3.4) (with x replaced by $f(x)$), and using (3.1), we find that f is continuous at x if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(y), f(x)) < \epsilon \quad \forall\ y \in X \text{ satisfying } d_X(y, x) < \delta.$$

DEFINITION 2.33. *We say that* $(X, \mathcal{T})$ *satisfies the first countability axiom, or that it is* <u>*1st-countable*</u>*, if for each point* $x \in X$ *there exists a countable basis of neighborhoods of* x.

EXERCISE 2.7. Show that the first-countability is a topological property.

EXAMPLE 2.34. Any metric space (X, d) is 1st countable: for $x \in X$,

$$\mathcal{B}'_d(x) := \{B(x; \frac{1}{n}) : n \in \mathbb{N}\}$$

is a countable basis of neighborhoods of x. Hence, in relation with the metrizability problem, we deduce: if a topological space is metrizable, then it must be 1st countable.

EXERCISE 2.8. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Show that if x admits a countable basis of neighborhoods, then one can also find a decreasing one, i.e. one of type

$$\mathcal{B}_x = \{B_1, B_2, B_3, \ldots\}, \text{with } \ldots \subset B_3 \subset B_2 \subset B_1.$$

(Hint: $B_n = V_1 \cap V_2 \cap \ldots \cap V_n$).

The role of the first countability axiom is a theoretical one: "it is the axiom under which the notion of sequence can be used in its full power". For instance, Theorem 2.29 can be improved:

THEOREM 2.35. *If* X *is 1st countable (in particular, if* X *is a metric space) then a map* $f : X \to Y$ *is continuous if and only if it is sequentially continuous.*

PROOF. We are left with the converse implication. Assume f-continuous. By Prop. 2.25, we find $x \in X$ such that f is not continuous at x. Hence we find $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$. Let $\{B_n : n \in \mathbb{N}\}$ be a countable basis of neighborhoods of x; by the previous exercise, we may assume it is decreasing. Since $f^{-1}(V) \notin \mathcal{N}(x)$, for each n we find $x_n \in B_n - f^{-1}(V)$. Since $x_n \in B_n$, it follows that $(x_n)_{n\geq 1}$ converges to x (see Remark 2.32). But note that $(f(x_n))$ cannot converge to $f(x)$ since $f(x_n) \notin V$ for all n. This contradicts the hypothesis. ☐

Another good illustration of the fact that, under the first-countability axiom, "convergent sequences contain all the information about the topology", is given in Exercise 2.43. Another illustration is the characterisation of Hausdorffness (Theorem 2.46 below).

4. Inside a topological space: closure, interior and boundary

DEFINITION 2.36. *Let $(X, \mathcal{T})$ be a topological space. Given $A \subset X$, define:*

- *the interior of A:*

$$\overset{\circ}{A} = \bigcup_{U-\text{open contained in } A} U.$$

(The union is over all the subsets U of A which are open in $(X, \mathcal{T})$). It is sometimes denoted by Int(A). Note that $\overset{\circ}{A}$ is open, is contained in A, and it is the largest set with these properties.

- *the closure of A:*

$$\overline{A} = \bigcap_{F-\text{closed containing } A} F.$$

(The intersection is over all the subsets A of X which contain A and are closed in $(X, \mathcal{T})$). It is sometimes denoted by Cl(A). Note that $\overline{A}$ is closed, contains A, and it is the smallest set with these properties.

- *the boundary of A:*

$$\partial(A) = \overline{A} - \overset{\circ}{A}.$$

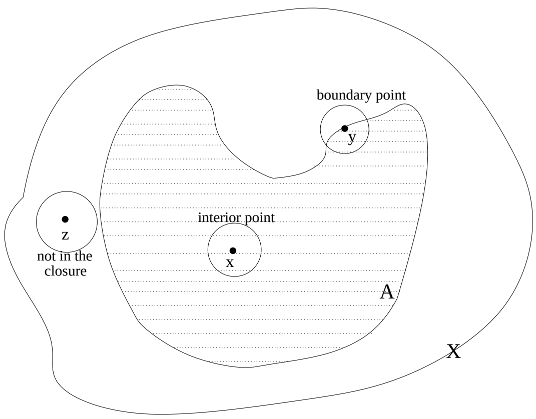


FIGURE 1.

LEMMA 2.37. *Let $(X, \mathcal{T})$ be a topological space, $x \in X$, and assume that $\mathcal{B}_x$ is a basis of neighborhoods around x (e.g. $\mathcal{B}_x = \mathcal{T}(x)$). Then:*

(i) *$x \in \overset{\circ}{A}$ if and only if there exists $U \in \mathcal{B}_x$ such that $U \subset A$.*
(ii) *$x \in \overline{A}$ if and only if, for all $U \in \mathcal{B}_x$, $U \cap A \neq \emptyset$.*
(iii) *If $(X, \mathcal{T})$ is metrizable (or just 1st countable) then $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_{n\geq 1}$ of elements of A such that $a_n \to x$.*

See Figure 1.

PROOF. (of the lemma) You should first convince yourself that (i) is easy; we prove here (ii) and (iii). To prove the equivalence in (ii), is sufficient to prove the equivalence of the negations, i.e.

$$[x \notin \overline{A}] \Longleftrightarrow [\exists\, U \in \mathcal{B}_x : U \cap A = \emptyset].$$

From the definition of $\overline{A}$, the left hand side is equivalent to:

$$\exists\ F - \text{closed} : A \subset F, x \notin F.$$

Since closed sets are those of type $F = X - U$ with U-open, this is equivalent to

$$\exists\ U - \text{open} : A \cap U = \emptyset, x \in U,$$

i.e.: there exists $U \in \mathcal{T}(x)$ such that $U \cap A = \emptyset$. On the other hand, any $U \in \mathcal{T}(x)$ contains at least one $B \in \mathcal{B}_x$, and the condition $U \cap A = \emptyset$ will not be destroyed if we replace U by B. This concludes the proof of (ii). For (iii), first assume that $x = \lim a_n$ for some sequence of elements of A. Then, for any $U \in \mathcal{T}(x)$, we find n_U such that $a_n \in U$ for all $n \geq n_U$, which shows that $U \cap A \neq \emptyset$. By (ii), $x \in \overline{A}$. For the converse, one uses that fact that $B(x, \frac{1}{n}) \cap A \neq \emptyset$ hence, for each n, we find $a_n \in A$ with $d(a_n, x) < \frac{1}{n}$. Clearly $a_n \to a$. □

EXAMPLE 2.38. Take the "open disk" in the plane

$$A = \overset{\circ}{D}{}^2 \ (= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}).$$

Then the interior of A is A itself (it is open!), the closure is the "closed disk"

$$A = D^2 (= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}),$$

while the boundary is the unit circle

$$\partial(A) = S^1 (= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}).$$

EXAMPLE 2.39. Take $A = [0, 1) \cup \{2\} \cup [3, 4)$ in $X = \mathbb{R}$. Using the lemma (and the basis given by open intervals) we find

$$\overset{\circ}{A} = (0, 1) \cup (3, 4),\ \overline{A} = [0, 1] \cup \{2\} \cup [3, 4],\ \partial(A) = \{0, 1, 2, 3, 4\}.$$

However, considering A inside $X' = [0, 4)$ (with the topology induced from $\mathbb{R}$),

$$\overset{\circ}{A} = [0, 1) \cup (3, 4),\ \overline{A} = [0, 1] \cup \{2\} \cup [3, 4), \partial(A) = \{1, 2, 3\}.$$

For the case of metric spaces, let us point out the following corollary. To state it, recall that given a metric space (X, d), $A \subset X$, and $x \in X$, one defines the distance between x and A as

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

COROLLARY 2.40. *If A is a subspace of a metric space (X, d), $x \in X$, then the following are equivalent:*

(1) $x \in \overline{A}$.
(2) *there exists a sequence (a_n) of elements of A such that $a_n \to x$.*
(3) $d(x, A) = 0$.

PROOF. The equivalence of (1) and (2) follows directly from (iii) of the lemma. Next, the condition (3) means that, for all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$. In other words, $A \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$. Using (iii) of the lemma (with $\mathcal{B}_x$ being the collection of all balls centered at x), we find that (3) is equivalent to (1). □

DEFINITION 2.41. *Given a topological space X, a subset $A \subset X$ is called dense in X if $\overline{A} = X$.*

EXAMPLE 2.42. $\overset{\circ}{D}{}^n$ is dense in D^n; $\mathbb{Q}$ is dense in $\mathbb{R}$.

5. Hausdorffness; 2nd countability; topological manifolds

One of the powers of the notion of topological space comes from its generality, which gives it a great flexibility when it comes to examples and general constructions. However, in many respects the definition is "too general". For instance, for proving interesting results one often has to impose extra-axioms. Sometimes these axioms are rather strong (e.g. compactness), but sometimes they are rather weak (in the sense that most of the interesting examples satisfy them anyway). The most important such (weak) axiom is "Hausdorffness". This axiom is also important for the metrizability problem, for which we have to understand the special topological properties that a topology must satisfy in order to be induced by a metric. And Hausdorffness is the most basic one.

Definition 2.43. *We say that a topological space $(X, \mathcal{T})$ is <u>Hausdorff</u> if for any $x, y \in X$ with $x \neq y$, there exist $V \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$ such that $V \cap W = \emptyset$.*

Example 2.44. Looking at the extreme topologies: $\mathcal{T}_{\text{triv}}$ is not Hausdorff (unless X is empty or consists of one point only), while $\mathcal{T}_{\text{dis}}$ is Hausdorff. In the light of the Hausdorffness property, the cofinal topology $\mathcal{T}_{cf}$ becomes more interesting (see Exercise 2.62).

Exercise 2.9. Show that Hausdorffness is a topological property.

As promised, one has:

Proposition 2.45. *Any metric space is Hausdorff.*

Proof. Given (X, d), $x, y \in X$ distinct, we must have $r := d(x, y) > 0$. We then choose $V = B(x; \frac{r}{2})$, $W = B(y; \frac{r}{2})$. We claim these are disjoint. If not, we find z in their intersection, i.e. $z \in Z$ such that $d(x, z)$ and $d(y, z)$ are both less than $\frac{r}{2}$. From the triangle inequality for d we obtain the following contradiction

$$r = d(x, y) < d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r.$$

□

However, one of the main reasons that Hausdorffness is often imposed comes from the fact that, under it, sequences behave "as expected".

Theorem 2.46. *Let $(X, \mathcal{T})$ be a topological space. If X is Hausdorff, then every sequence $(x_n)_{n\geq 1}$ has at most one limit in X. The converse holds if we assume that $(X, \mathcal{T})$ is 1st countable.*

Proof. Assume that X is Hausdorff. Assume that there exists a sequence $(x_n)_{n\geq 1}$ in X converging both to $x \in X$ and $y \in X$, with $x \neq y$; the aim is to reach a contradiction. Choose $V \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$ such that $V \cap W = \emptyset$. Then we find n_V and n_W such that $x_n \in V$ for all $n \geq n_V$, and similarly for W. Choosing $n > \max\{n_V, n_W\}$, this will contradict the fact that V and W are disjoint.

Let's now assume that X is 1st countable and each sequence in X has at most one limit, and we prove that X is Hausdorff. Assume it is not. We find then $x \neq y$ two elements of X such that $V \cap W \neq \emptyset$ for all $V \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$. Choose $\{V_n : n \geq 1\}$ and $\{W_n : n \geq 1\}$ bases of neighborhoods of x and y, which we may assume to be decreasing (cf. Exercise 2.8). For each n, we find an element $x_n \in V_n \cap W_n$. As in the previous proofs, this implies that x_n converges both to x and to y- which contradicts the hypothesis. □

Besides Hausdorffness, there is another important axiom that one often imposes on the spaces one deals with (especially on the spaces that arise in Geometry).

DEFINITION 2.47. *Let $(X, \mathcal{T})$ be a topological space. A basis of the topological space $(X, \mathcal{T})$ is a family $\mathcal{B}$ of opens of X with the property that any open $U \subset X$ can be written as a union of opens that belong to $\mathcal{B}$.*

We say that $(X, \mathcal{T})$ satisfies the second countability axiom, or that it is second-countable (also written 2nd countable) if it admits a countable basis.

EXERCISE 2.10. Given a topological space $(X, \mathcal{T})$ and a family $\mathcal{B}$ of opens of X, show that $\mathcal{B}$ is a basis of $(X, \mathcal{T})$ if and only if, for each $x \in X$,

$$\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\}$$

is a basis of neighborhoods of x. Deduce that any 2nd countable space is also 1st countable.

EXAMPLE 2.48. In a metric space (X, d), the collection of all balls

$$\mathcal{B}_d := \{B(x, r) : x \in X, r > 0\}$$

is a basis for the topology $\mathcal{T}_d$ (see the end of the proof of Proposition 2.4). Although metric spaces are allways 1st countable (cf. Example 2.34), not all are 2nd countable. However:

EXAMPLE 2.49. For $\mathbb{R}^n$, one can restrict to balls centered at points with rational coordinates:

$$\mathcal{B}_{\text{Eucl}}^{\mathbb{Q}} := \{B(x, \frac{1}{k}) : x \in \mathbb{Q}^n, k \in \mathbb{Q}_+\}.$$

This is a countable family since $\mathbb{Q}$ is countable and products of countable sets are countable.

EXERCISE 2.11. Show that $\mathcal{B}_{\text{Eucl}}^{\mathbb{Q}}$ is a basis of $\mathbb{R}^n$. Deduce that any $A \subset \mathbb{R}^n$ is 2nd countable.

Finally, we come at the notion of topological manifold.

DEFINITION 2.50. *An n-dimensional topological manifold is any Hausdorff, 2nd countable topological space X which has the following property: any point $x \in X$ admits an open neighborhood U which is homeomorphic to $\mathbb{R}^n$.*

REMARK 2.51. Of course, the most important condition is the one requiring X to be locally homeomorphic to $\mathbb{R}^n$. A pair (U, χ) consisting of an open $U \subset X$ and a homeomorphism

$$\chi : U \to \mathbb{R}^n, \ x \mapsto \chi(x) = (\chi_1(x), \ldots, \chi_n(x))$$

is called a (local) coordinate chart for X; U is called the domain of the chart; $\chi_1(x), \ldots, \chi_n(x)$ are called the coordinates of x in the chart (U, χ). Given another chart $\psi : V \to \mathbb{R}^n$,

$$c := \psi \circ \chi^{-1} : \chi(U \cap V) \to \psi(U \cap V)$$

(a homeomorphism between two opens in $\mathbb{R}^n$) is called the change of coordinates from χ to ψ (it satisfies $\psi_i(x) = c_i(\chi_1(x), \ldots, \chi_n(x))$ for all $x \in X$). By definition, topological manifolds can be covered by (domains of) coordinate charts; hence they can be thought of as obtained by "patching together" several copies of $\mathbb{R}^n$, glued according to the change of coordinates.

REMARK 2.52. One may wonder why the "2nd countability" condition is imposed. Well, there are many reasons. The simplest one: we do hope that a topological manifold can be embedded in some $\mathbb{R}^N$ for N large enough. However, as the previous exercise shows, this would imply that X must be 2nd countable anyway. Also, the 2nd countability condition implies that X can be covered by a countable family of coordinate charts (see Exercise 2.63).

EXAMPLE 2.53. Of course, $\mathbb{R}^n$ is itself a topological manifold. Using the stereographic projection (see the previous chapter), we see that the spheres S^n are topological n-manifolds. But note that, while the open disks are topological manifolds, the closed disks are not.

EXERCISE 2.12. Show that the torus is a 2-dimensional topological manifold. What about the Klein bottle? What about the Moebius band? Try to define "manifolds with boundary".

6. More on separation

The Hausdorffness is jut one of the possible "separation axioms" that one may impose (the most important one!). Such separation axioms are relevant to the metrizability problem, as they are automatically satisfied by metric spaces. Here is the precise definition.

DEFINITION 2.54. *We say that two subspaces A and B of a topological space $(X, \mathcal{T})$ can be separated topologically (or simply separated) if there are open sets U and V such that*

$$A \subset U, B \subset V, \text{ and } U \cap V = \emptyset.$$

We say that A and B can be separated by continuous functions if there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$, $f|_B = 1$ (and we say that f separates A and B).

EXAMPLE 2.55. If A and B can be separated by continuous functions, then they can be separated topologically as well. Indeed, if f separates A and B, then

$$U = f^{-1}((-\infty, \frac{1}{2})), V = f^{-1}((\frac{1}{2}, \infty))$$

are disjoint opens (as pre-images of opens by a continuous map) containing A and B.

The separation conditions are most natural when A and B are closed in X. For instance, inside $\mathbb{R}$, $[0,1)$ and $(1,2]$ cannot be separated by continuous functions, while $[0,1)$ and $[1,2]$ cannot be separated even topologically (see also Exercise 2.64).

In any metric space (X, d), any two disjoint closed subsets A and B can be separated: indeed,

$$U = \{x \in X : d(x, A) < d(x, B)\}, \ V = \{x \in X : d(x, A) > d(x, B)\}$$

are disjoint opens containing A and B. Here we use the continuity of the function $d_A : X \to \mathbb{R}$, $x \mapsto d(x, A)$ (see Exercise 2.32) and the similar function d_B. Actually, one can separate A and B even by continuous functions: take

$$f : X \to [0,1], \quad f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}.$$

There are several classes of separation conditions one may impose on a topological space X. At one extreme, when the separation is required for sets of one elements, we talk about the Hausdorffness condition. At the other extreme, one has "normality":

DEFINITION 2.56. *A topological space is called normal if it is Hausdorff and any two disjoint closed subsets can be separated topologically.*

From our previous discussion it follows that all metrizable spaces are normal. As we shall see, for normal spaces disjoint closed subsets can allways be separated by continuous functions (Urysohn lemma) and 2nd countable normal spaces are metrizable (Urysohn metrization theorem). That is why normal spaces are important. However, one should be aware that"normality" is a condition that is (so) important mainly inside the field of Topology; as soon as one moves to neighbouring fields (Geometry, Analysis, etc), although many of the topological spaces one meets there are normal, very little attention is paid to this condition (and you will almost never hear about "normal spaces" in other courses). For instance, in such fields, Urysohn lemma (so important for Topology), often follows by simple tricks (e.g., as many such spaces are already metrizable, it follows from the previous remark). For that reason we decided not to concentrate too much on normal spaces; also, although the proof of Urysohn's results could be presented right away, we have decided not to do them until they are absolutely needed. Note that, in contrast with "normality", the other topological conditions that we will study, such as Hausdorffness, connectedness, compactness, local compactness and even paracompactness, show up all over in mathematics whenever topological spaces are relevant, and they are indispensable.

7. More exercises

7.1. On topologies.

EXERCISE 2.13. How many distinct topologies can there be defined on a set with two elements? But with three?

EXERCISE 2.14. Consider the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (the set of strictly positive integers to which we add the infinity) and, for each $n \in \mathbb{N}$ put

$$U_n := \{k \in \overline{\mathbb{N}} : k \geq n\} = \{n, n+1, \ldots\} \cup \{\infty\}.$$

Show that the following is a topology on $\overline{\mathbb{N}}$:

$$\mathcal{T}_{\text{seq}} := \{\emptyset, U_1, U_2, U_3, \ldots\}.$$

EXERCISE 2.15. Prove that, for any set X, $\mathcal{T}_{cf}$ and $\mathcal{T}_{cc}$ are indeed topologies.

EXERCISE 2.16. On $\mathbb{R}$ consider the family $\mathcal{T}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of type $(-\infty, r)$ with $r \in \mathbb{R}$. Show that $\mathcal{T}$ is a topology on $\mathbb{R}$ and compare it with the Euclidean topology.

EXERCISE 2.17. On $\mathbb{R}$ consider the family $\mathcal{B}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of type $(-\infty, r]$ with $r \in \mathbb{R}$. Show that $\mathcal{B}$ is not a topology on $\mathbb{R}$ and find the smallest topology containing $\mathcal{B}$. Is it larger or smaller than the topology from the previous exercise? But than the Euclidean topology?

EXERCISE 2.18. Let $\mathcal{T}$ be a topology on $\mathbb{R}$. Show that $\mathcal{T}$ is the discrete topology if and only if $\{r\} \in \mathcal{T}$ for all $r \in \mathbb{R}$.

EXERCISE 2.19. Let $\mathcal{T}_l$ be the smallest topology on $\mathbb{R}$ which contains all the intervals of type $[a, b)$ with $a, b \in \mathbb{R}$. Similarly, define $\mathcal{T}_r$ using intervals of type $(a, b]$. Show that:

1. A subset $D \subset \mathbb{R}$ belongs to $\mathcal{T}_l$ if and only if the following condition holds: for any $x \in D$ there exists an interval $[a, b)$ such that
$$x \in [a, b) \subset D.$$
2. $\mathcal{T}_l$ and $\mathcal{T}_r$ are finer than $\mathcal{T}_{\text{eucl}}$, but $\mathcal{T}_l$ and $\mathcal{T}_r$ are not comparable.
3. $\mathcal{T}_{\text{dis}}$ is the only topology on $\mathbb{R}$ which contains both $\mathcal{T}_l$ and $\mathcal{T}_r$.

EXERCISE 2.20. Consider a set X, a set of indices I and, for each $i \in I$, a topology $\mathcal{T}_i$ on X. Show that $\mathcal{T} := \cap_i \mathcal{T}_i$ (i.e. the family consisting of subsets $U \subset X$ with the property that $U \in \mathcal{T}_i$ for all $i \in I$) is a topology on X.

EXERCISE 2.21. Given a set X and a family $\mathcal{S}$ of subsets of X, prove that there exists a topology $\mathcal{T}(\mathcal{S})$ on X which contains $\mathcal{S}$ and is the smallest with this property. (Hint: use the axioms to see what other subsets of X, besides the ones from $\mathcal{S}$, must $\mathcal{T}(\mathcal{S})$ contain.)

EXERCISE 2.22. On $\mathbb{R}^2$ we define the topology $\mathcal{T}_l \times \mathcal{T}_l$ as the smallest topology which contains all subsets of type

$$[a, b) \times [c, d)$$

with $a, b, c, d \in \mathbb{R}$. Define similarly $\mathcal{T}_r \times \mathcal{T}_r$, $\mathcal{T}_l \times \mathcal{T}_r$ and $\mathcal{T}_r \times \mathcal{T}_l$. Show that any two of these four topologies are not comparable.

EXERCISE 2.23. To remove ambiguities regarding the Convention 2.9 show that, inside any topological space $(X, \mathcal{T})$, for all $A \subset Y \subset X$ one has

$$(\mathcal{T}|_Y)|_A = \mathcal{T}|_A.$$

To remove ambiguities regarding Convention 2.7 and 2.9 show that, for $A \subset \mathbb{R}^n$, the Euclidean topology on A coincides with the restriction to A of the Euclidean topology of $\mathbb{R}^n$.

EXERCISE 2.24. Let (X, d) be a metric space, A, B subspaces of X such that

$$d(A, B) = 0.$$

(where $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$).

Is it true that A and B must have a common point (i.e. $A \cap B \neq \emptyset$)? What if we assume that both A and B are closed?

7.2. On induced topologies.

EXERCISE 2.25. Consider the real line $\mathbb{R}$ as a subset of the plane $\mathbb{R}^2$. Show that the induced topology on $\mathbb{R}$ coincides with the Euclidean topology on the real line.

EXERCISE 2.26. Find an example of a topological space X and $A \subset B \subset X$ such that A is closed in B, B is open in X, and A is neither open nor closed in X.

EXERCISE 2.27. Which of the following subsets of the plane are open?

1. $A = \{(x, y) : x \geq 0\}$.
2. $B = \{(x, y) : x = 0\}$.
3. $C = \{(x, y) : x > 0, y < 5\}$.
4. $D = \{(x, y) : xy < 1, x \geq 0\}$.
5. $E = \{(x, y) : 0 \leq x < 5\}$.

Note that all these sets are contained in A. Which ones are open in A?

EXERCISE 2.28. Let $(X, \mathcal{T})$ be a topological space and $B \subset A \subset X$.

1. If A is open in X, show that B is open in X if and only if it is open in A.
2. If A is closed in X, show that B is closed in X if and only if it is closed in A.

EXERCISE 2.29. Given a topological space $(X, \mathcal{T})$ and $A \subset X$, show that the induced topology $\mathcal{T}|_A$ is the smallest topology on A with the property that the inclusion map $i : A \to X$ is continuous.

7.3. On continuity.

EXERCISE 2.30. Consider

$$D := \{(x, y) : e^x > sin(y)cos(x)\},$$

$$A := \{(x, y) : x^7 - sin(y^7) \geq \frac{1}{x^2 + y^2 + 1}\}.$$

Show that D is open in $\mathbb{R}^2$, while A is closed in $\mathbb{R}^2$.

EXERCISE 2.31. Let $\mathbb{R}$ be endowed with the topology $\mathcal{T}_l$ from Exercise 2.19. Which one of the following functions $f : \mathbb{R} \to \mathbb{R}$ is continuous:

(i) $f(x) = x + 1$
(ii) $f(x) = -x$.
(iii) $f(x) = x^2$.

EXERCISE 2.32. If (X, d) is a metric space and A is a subspace of X, then the function

$$d_A : X \to \mathbb{R}, \quad d_A(x) = d(x, A)$$

is continuous.

Deduce that, for any closed subset A of a metric space X, there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(0)$.

EXERCISE 2.33. The space $M_n(\mathbb{R})$, of $n \times n$ matrices with real coefficients can be identified with $\mathbb{R}^{n^2}$ and in this way has a natural topology (coming from the Euclidean metric). Prove that:

1. the subspace $GL_n(\mathbb{R})$ of invertible matrices is open in $M_n(\mathbb{R})$.
2. the subspace $SL_n(\mathbb{R})$ consisting of invertible matrices of determinant equal to 1 is closed in $GL_n(\mathbb{R})$.
3. the subspace
$$O(n) := \{A \in GL_n(\mathbb{R}^n) : AA^* = \mathrm{Id}\}$$
is also closed (where A^* denotes the transpose of A and Id is the identity matrix).

EXERCISE 2.34. Let X and Y be two topological spaces, $f : X \to Y$. We say that f is continuous at a point $x \in X$ if, for any neighborhood V of $f(x)$ in Y there exists a neighborhood U of x in X such that $f(U) \subset V$. Show that f is continuous if and only if it is continuous at all points $x \in X$.

7.4. On homeomorphisms and embeddings.

EXERCISE 2.35. Show that the following three spaces are homeomorphic (giving the explicit homeomorphisms):
$$X = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$
$$Y = \{(x, y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$
$$Z = S^1 \times (0, 1] = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 < z \leq 1\} \subset \mathbb{R}^3.$$
Then compute the interiors and the boundaries of X, Y (in $\mathbb{R}^2$) and Z (in $\mathbb{R}^3$). How comes that, although X, Y and Z are homeomorphic, their interiors and boundaries are quite different?

EXERCISE 2.36. From the topologies that you found in Exercise 2.13, how many non homeomorphic ones are there?

EXERCISE 2.37. Show that $\mathbb{R}$ endowed with the Euclidean topology is not homeomorphic to $\mathbb{R}$ endowed with the topology from Exercise 2.16.

EXERCISE 2.38. Exhibit an embedding $f : M \to M$ of the Moebius band into itself which is not surjective. What is the boundary of $f(M)$ in M?

7.5. The "removing a point trick". The first part of the following exercise is extremely useful for some of the later exercises.

EXERCISE 2.39. *If X and Y are homeomorphic, prove that for any $x \in X$ there exists $y \in Y$ such that $X - \{x\}$ is homeomorphic to $Y - \{y\}$.*
Also explain that, "there exists $y \in Y$" cannot be replaced with "for any $y \in Y$".

7.6. On convergence.

EXERCISE 2.40. Let $X = (0, 1)$ endowed with the Euclidean topology. Is the sequence
$$x_n = \frac{1}{n}$$
convergent in the topological space X?

EXERCISE 2.41. Let $\mathbb{R}$ be endowed with the topology $\mathcal{T}_l$ from Exercise 2.19. Study the convergence of the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ where
$$x_n = \frac{1}{n},\ y_n = -\frac{1}{n}.$$

EXERCISE 2.42. What about convergence in $(X, \mathcal{T}_{\mathrm{cf}})$. But about $(X, \mathcal{T}_{\mathrm{cc}})$?

EXERCISE 2.43. We say that two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X have the same convergence of sequences if, for a sequence $(x_n)_{n\geq 1}$ of elements of X, and $x \in X$, one has:

$$x_n \xrightarrow{\mathcal{T}_1} x \Longleftrightarrow x_n \xrightarrow{\mathcal{T}_2} x.$$

Show that

- If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the first countability axiom and have the same convergence of sequences, then $\mathcal{T}_1 = \mathcal{T}_2$.
- This is no longer true if one gives up the first countability axiom.

(Hint for the second part: This is difficult. Try $\mathcal{T}_{\mathrm{discr}}$ and $\mathcal{T}_{\mathrm{cf}}$. If it does not work, try to change one of them).

7.7. On closure, interior, etc.

EXERCISE 2.44. Let $X = (-\infty, 1) \cup (1, 4) \cup [5, \infty)$. Find the closure, the interior and the boundary (in X) of $A = [0, 10 \cup (1, 2) \cup [3, 4) \cup (5, 6)$.

EXERCISE 2.45. Find the interior and the closure of $\mathbb{Q}$ in $\mathbb{R}$ in each of the cases: when $\mathbb{R}$ is endowed with

- the Euclidean topology.
- the discrete topology.
- the cofinite topology.
- the co-countable topology.

EXERCISE 2.46. Find the closure, the interior and the boundary of the following subsets of the plane:

1. $\{(x, y) : x \geq 0, y \neq 0\}$.
2. $\{(x, y) : x \in \mathbb{Q}, y > 0\}$.

EXERCISE 2.47. For each of the sets from Exercise 2.27, find the interior, closure and boundary in the plane. Then in A (i.e. as subspaces of A).

EXERCISE 2.48. Let $\mathcal{T}_l$ be the topology from Exercise 2.19. Find the closure and the interior in $(\mathbb{R}, \mathcal{T}_l)$ of each of the intervals

$$[0, 1), (0, 1], (0, 1), [0, 1].$$

EXERCISE 2.49. Let $\mathcal{T}_l$ be the topology from Exercise 2.19.

(i) In the topological space $(\mathbb{R}, \mathcal{T}_l)$, find the closure, the interior and the boundary of

$$A = (0, 1) \cup [2, 3].$$

(ii) Show that $(\mathbb{R}, \mathcal{T}_l)$ and $(\mathbb{R}, \mathcal{T}_{\mathrm{eucl}})$ are not homeomorphic.

EXERCISE 2.50. Compute the interior, the closure and the boundary of

$$A = (0, 1] \times [0, 1)$$

in the topological space $X = \mathbb{R}^2$ endowed with the topology $\mathcal{T}_l \times \mathcal{T}_l$ of Exercise 2.22.

EXERCISE 2.51. Show that, in any topological space X, for any subspace $A \subset X$, one has

$$\partial(A) = \partial(X - A).$$

EXERCISE 2.52. Let A, B be two subsets of a topological space X. Recall that $\mathrm{Int}(A) = \overset{\circ}{A}$ denotes the interior of A. Prove that

1. If $A \subset B$ then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.
2. $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.
3. $\mathrm{Int}(A \cup B) \supset \mathrm{Int}(A) \cup \mathrm{Int}(B)$, but the equality may fail.

Exercise 2.53. Let A, B and $\{A_i : i \in I\}$ denote subsets of a topological space X, where i runs in a set of indexes I. Prove that

1. If $A \subset B$ then $\overline{A} \subset \overline{B}$.
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
3. If I is finite then $\overline{\cup_i A_i} = \cup_i \overline{A_i}$.
4. In general, when I is infinite, $\overline{\cup_i A_i} \supset \cup_i \overline{A_i}$, but the two may be different.
5. We say that $\{A_i : i \in I\}$ is locally finite if any point $x \in X$ admits a neighborhood V which intersects all but a finite number of A_is (i.e. such that $\{i \in I : A_i \cap V \neq \emptyset\}$ is finite). Under this assumption, show that $\overline{\cup_i A_i} = \cup_i \overline{A_i}$.

7.8. Density.

Exercise 2.54. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 2.55. Show that $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$.

Exercise 2.56. Let T be the torus. Describe (on the picture) a continuous injection $f : \mathbb{R} \to T$ whose image is dense in T. Is f an embedding?

Exercise 2.57. Show that $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$ (see Exercise 2.33).

Exercise 2.58. Show that any continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, then f must be linear, i.e. there exists $a \in \mathbb{R}$ such that

$$f(x) = ax \quad \forall\, x \in \mathbb{R}.$$

(Hint: $a = f(1)$. For what x's can you prove that $f(x) = ax$? Then use continuity and the fact $\mathbb{Q}$ is dense in $\mathbb{R}$.)

7.9. On Hausdorffness, 2nd countability, separation.

Exercise 2.59. How many from the topologies from Exercise 2.13 are Hausdorff?

Exercise 2.60. Is the topology from Exerc. 2.14 Hausdorff? But from 2.16? But from 2.17?

Exercise 2.61. Show that, in any Hausdorff space X, all the subspaces with one element (i.e. of type $A = \{x\}$ with $x \in X$) are closed.

Exercise 2.62. Given a set X, show that any Hausdorff topology on X contains $\mathcal{T}_{cf}$. When is $\mathcal{T}_{cf}$ Hausdorff? When does it exist a smallest Hausdorff topology on X (i.e. a Hausdorff topology which is contained in all other Hausdorff topologies on X)?

Exercise 2.63. Let X be a 2nd countable space. Show that from any open cover $\mathcal{U}$ of X one can extract a countable subcover. In other words, for any collection $\mathcal{U} = \{U_i : i \in I\}$ consisting of opens in X such that $X = \cup_{i\in I} U_i$, one can find $i_1, i_2, \ldots \in I$ such that $X = \cup_k U_{i_k}$.

Exercise 2.64. Consider $\mathbb{R}$ with the Euclidean topology. In each of the following cases, decide (and explain!) when A and B can be separated topologically or by continuous functions:

1. $A = [0, 1)$, $B = (1, 2]$.
2. $A = [0, 1)$, $B = [1, 2]$.
3. $A = [0, 1)$, $B = (2, 3]$.

Exercise. Prove that the sphere S^n is an n-dimensional topological manifold. Show that it can be covered by two coordinate charts. Compute the change of coordinates.

CHAPTER 3

Constructions of topological spaces

1. Constructions of topologies: quotients

We now discuss another general construction of topologies. Let's start with a surjective map $\pi : X \to Y$. Typically, (Y, π) is a quotient of X modulo an equivalence relation R on X ("gluing data"). Assume now that X is endowed with a topology $\mathcal{T}$. Then one defines

$$\pi_*(\mathcal{T}) := \{V \subset Y : \pi^{-1}(V) \in \mathcal{T}\},$$

called the quotient topology on Y induced by π. A surjective map $\pi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between two topological spaces is called a topological quotient map if $\mathcal{T}_Y = \pi_*(\mathcal{T}_X)$.

THEOREM 3.1. *$\pi_*(\mathcal{T})$ is indeed a topology on Y. Moreover, it is the largest topology on Y with the property that $\pi : X \to Y$ becomes continuous.*

PROOF. Axiom (T1) is immediate. For (T2), let $U_i \in \pi_*(\mathcal{T})$ (with $i \in \{1, 2\}$), i.e. subsets of Y satisfying $\pi^{-1}(U_i) \in \mathcal{T}$. Then

$$\pi^{-1}(U_1 \cap U_2) = \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$$

must be in $\mathcal{T}$, i.e. $U_1 \cap U_2 \in \pi_*(\mathcal{T})$. The axiom (T3) follows similarly, using the fact that $\pi^{-1}(\cup_i U_i) = \cup_i \pi^{-1}(U_i)$. The last part follows from the definition of continuity and of $\pi_*(\mathcal{T})$. $\square$

The following is a very useful recognition criteria for continuity of maps defined on quotients.

PROPOSITION 3.2. *Let X be a topological space, $\pi : X \to Y$ a surjection, and let Y be endowed with the quotient topology. Then, for any other topological space Z, a function $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is.*

PROOF. $f^{-1}(U)$ is open in Y if and only if $\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$ is open in X. $\square$

There are some variations on the previous discussion, mainly terminological, when we want to emphasize that Y is the quotient modulo an equivalence relation (see Definition 3.3).

DEFINITION 3.3. *Let R be an equivalence relation defined on a topological space $(X, \mathcal{T})$. A quotient of $(X, \mathcal{T})$ modulo R is a pair (Y, π) consisting of a topological space Y and a topological quotient map $\pi : X \to Y$ with the property that $\pi(x) = \pi(x')$ holds if and only if $(x, x') \in R$.*

If $Y = X/R$ is the abstract quotient, then the resulting topological space $(X/R, \pi_*(\mathcal{T}))$ is called the abstract quotient of $(X, \mathcal{T})$ modulo R.

With this terminology, the last proposition translates into:

COROLLARY 3.4. *Assume that (Y, π) is a quotient of the topological space X modulo R. Then, for any topological space Z, there is a 1-1 correspondence between*

(i) *continuous maps $f : Y \to Z$.*
(ii) *continuous maps $\tilde{f} : X \to Z$ such that $\tilde{f}(x) = \tilde{f}(x')$ whenever $(x, x') \in R$.*

This correspondence is characterized by $\tilde{f} = f \circ \pi$.

Finally, we would like to point out one of the notorious problems that arises when considering quotients: Hausdorffness may be destroyed! (this problem does not appear when we consider subspace or product topologies!). Hence extra-care is required when we deal with quotients.

EXERCISE 3.1. Take two copies of the interval $[0, 2]$, say $X = [0, 2] \times \{0\} \cup [0, 2] \times \{1\}$ (in the plane) and glue the points $(t, 0)$ and $(t, 1)$ for each $t \in [0, 2]$, $t \neq 1$. Show that the resulting quotient space Y is not Hausdorff.

2. Examples of quotients: the abstract torus, Moebius band, etc

In the first chapter we discussed the torus, Moebius band, etc intuitively. We can now have a more complete discussion about them, as topological spaces. Let us concentrate, for example, on the torus. Here are some remarks on the discussions from the first chapter:

1. when constructing it by gluing the opposite sides of a square, although the "shape" of the result may be predicted, the actual result (as a subset of $\mathbb{R}^3$) depends on all the movements we make while gluing. But even the "shape" is not completely clear: we could have performed the same gluing in a "clumsier way" (e.g., for the Moebius band, we could have twisted the piece of paper three times before the actual gluing).

2. when saying "torus" , we would like to think about the intrinsic space itself. The information that this space can be embedded in $\mathbb{R}^3$ is interesting and nice, but there may be many such embeddings. See Figure 1 for several different looking embeddings. The "shape" reflects the way one embeds the torus into $\mathbb{R}^3$, not only the intrinsic torus.

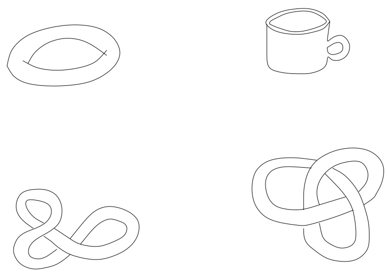


FIGURE 1.

The way to deal with all these in a more precise way is the following:

1. Consider the abstract torus, defined as the abstract quotient

$$T_{\text{abs}} := [0,1] \times [0,1]/R,$$

where R is the equivalence relation encoding the gluing. This defines T_{abs} as a topological space.

2. Embed the abstract torus: realizing the torus more concretely in $\mathbb{R}^3$, i.e. finding explicit models of it, translates now into the question of describing embeddings

$$f : T_{\text{abs}} \to \mathbb{R}^3.$$

By Corollary 3.4, continuous f's correspond to continuous maps

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^3$$

with the property that $\tilde{f}(t,s) = \tilde{f}(t',s')$ for all $((t,s),(t',s')) \in R$. The injectivity of f is equivalent to the condition that the last equality holds if and only if $((t,s),(t',s')) \in R$.

EXAMPLE 3.5. Such $\tilde{f}$'s arise in the explicit realizations of the torus (see Chapter 1):

$$\tilde{f}(t,s) = (R + rcos(2\pi t))cos(2\pi s), (R + rcos(2\pi t))sin(2\pi s), rsin(2\pi t)).$$

The induced f is a continuous injection of T_{abs} into $\mathbb{R}^3$ whose image is the geometric model $T_{R,r}$ from Section 6. But please note: while we now know that $f : T_{\text{abs}} \to T_{R,r}$ is a continuous bijection, one still has to show that the inverse is continuous. The best proof of this, which applies immediately to all examples of this type, not only to the torus, and not only to our explicit f, follows from one of the basic properties of compact spaces, which will be discussed in the next chapter.

EXERCISE 3.2. *Fill in the details; do the same for the Moebius band, Klein bottle, $\mathbb{P}^2$.*

3. Special classes of quotients I: quotients modulo group actions

In this section we discuss quotients by group actions. Let X be a topological space. We denote by Homeo(X) the set of all homeomorphisms from X to X. Together with composition of maps, this is a group. Let Γ be another group, whose operation is denoted multiplicatively.

DEFINITION 3.6. *An action of the group Γ on the topological space X is a group homomorphism*

$$\phi : \Gamma \to Homeo(X),\ \gamma \mapsto \phi_\gamma.$$

Hence, for each $\gamma \in \Gamma$, one has a homeomorphism ϕ_γ of X ("the action of γ on X"), so that

$$\phi_{\gamma\gamma'} = \phi_\gamma \circ \phi_{\gamma'} \ \ \forall\ \gamma, \gamma' \in X.$$

Sometimes $\phi_\gamma(x)$ is also denoted $\gamma(x)$, or simply $\gamma \cdot x$, and one looks at the action as a map

$$\Gamma \times X \to X,\ (\gamma, x) \to \gamma \cdot x.$$

The action induces an equivalence relation R_Γ on X defined by:

$$(x, y) \in R_\Gamma \iff \exists\ \gamma \in \Gamma \text{ s.t. } y = \gamma \cdot x.$$

The resulting topological quotient is called the quotient of X by the action of Γ, and is denoted by X/Γ. Note that the R_Γ-equivalence class of an element $x \in X$ is precisely its Γ-orbit:

$$\Gamma \cdot x := \{\gamma \cdot x : \gamma \in \Gamma\}.$$

Hence X/Γ consists of all such orbits, and the quotient map sends x to Γx.

EXAMPLE 3.7. The additive group $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ (n, r) \mapsto \phi_n(r) = n \cdot r := n + r.$$

The resulting quotient is (homeomorphic to) S^1. More precisely, one uses Corollary 3.4 again to see that the map $\tilde{f} : \mathbb{R} \to S^1$, $t \mapsto (cos(2\pi t), sin(2\pi t))$ induces a continuous bijection $f : \mathbb{R}/\mathbb{Z} \to S^1$; then one proves directly (e.g. using sequences) that f is actually a homeomorphism, or one waits again until compactness and its basic properties are discussed.

Here is a fortunate case in which Hausdorffness is preserved when passing to quotients.

THEOREM 3.8. *If X is a Hausdorff space and Γ is a finite group acting on X, then the quotient X/Γ is Hausdorff.*

PROOF. Let $\Gamma x, \Gamma y \in X/\Gamma$ be two distinct points $(x, y \in X)$. That they are distinct means that, for each $\gamma \in \Gamma$, $x \neq \gamma y$. Hence, for each $\gamma \in \Gamma$, we find disjoint opens $U_\gamma, V_\gamma \subset X$ containing x, and γy, respectively. Note that

$$W_\gamma = \phi_\gamma^{-1}(V_\gamma)$$

is an open containing y, and what we know is that

$$U_\gamma \cap \phi_\gamma(W_\gamma) = \emptyset.$$

Since Γ is finite, $U := \cap_\gamma U_\gamma$, $V := \cap_\gamma W_\gamma$ will be open neighborhoods of x and y, respectively, with the property that

$$U \cap \phi_a(V) = \emptyset,\ \ \forall\ a \in \Gamma.$$

Using the quotient map $\pi : X \to X/\Gamma$, we consider $\pi(U), \pi(V)$, and we claim that they are disjoint opens in X/Γ separating Γx and Γy. That they are disjoint follows from the previous property of U and V. To see that $\pi(U)$ is open, we have to check that $\pi^{-1}(\pi(U))$ is open, but

$$\pi^{-1}(\pi(U)) = \cup_{\gamma\in\Gamma}\phi_\gamma(U)$$

(check this!) is a union of opens, hence opens. Similarly, $\pi(V)$ is open. Clearly, $\Gamma x = \pi(x) \in \pi(U)$ and $\Gamma y = \pi(y) \in \pi(V)$. $\square$

4. Another example of quotients: the projective space $\mathbb{P}^n$

A very good illustration of the use of quotient topologies is the construction of the projective space, as a topological space (a set theoretical version of which appeared already in Exercise 1.29).

Recall that, set theoretically, $\mathbb{P}^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$:

$$\mathbb{P}^n = \{l \subset \mathbb{R}^{n+1} : l - \text{one dimensional vector subspace}\}.$$

To realize it as a topological space, we relate it to topological spaces that we already know. There are several ways to handle it.

4.1. As a quotient of $\mathbb{R}^{n+1} - \{0\}$: For this, we use a simple idea: for each point in $\mathbb{R}^{n+1} - \{0\}$ there is (precisely) one line passing through the origin and that point. This translates into the fact that there is a surjective map

$$\pi : \mathbb{R}^{n+1} \to \mathbb{P}^n, x \mapsto l_x,$$

where l_x is the line through the origin and x:

$$l_x = \mathbb{R}x = \{\lambda x : \lambda \in \mathbb{R}\} \subset \mathbb{R}^{n+1}.$$

The projective space $\mathbb{P}^n$ can now be defined as the set $\mathbb{P}^n$ endowed with the quotient topology. Note also that the equivalence relation underlying π comes from a group action. This is based on the remark that $\pi(x) = \pi(y)$, i.e. $l_x = l_y$, happens if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}^*$. Hence, taking $\Gamma = \mathbb{R}^*$ (a group with usual multiplication), it acts on $\mathbb{R}^{n+1} - \{0\}$ by:

$$\phi_\lambda(x) = \lambda x \quad \text{for } \lambda \in \mathbb{R}^*, x \in \mathbb{R}^{n+1} - \{0\}$$

and the projective space becomes

$$\mathbb{P}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*.$$

4.2. As a quotient of S^n: This is based on another simple remark: a line in $\mathbb{R}^{n+1}$ through the origin is uniquely determined by its intersection with the unit sphere $S^n \subset \mathbb{R}^{n+1}$- which is a set consisting of two antipodal points (the first picture in Figure 2). This indicates that $\mathbb{P}^n$ can be obtained from S^n by identifying (gluing) its antipodal points. Again, this is a quotient that arises from a group action: the group $\mathbb{Z}_2$ acting on S^n. Using the multiplicative description $\mathbb{Z}_2 = \{1, -1\}$, the action is: ϕ_1 is the identity map, while ϕ_{-1} is the map sending $x \in S^n$ to its antipodal point $-x$. Hence the discussion indicates:

PROPOSITION 3.9. *$\mathbb{P}^n$ is homeomorphic to $S^n/\mathbb{Z}_2$.*

PROOF. The conclusion of the previous discussion is that there is a set-theoretical bijection:

$$\phi : S^n/\mathbb{Z}_2 \to \mathbb{P}^n,$$

which sends the $\mathbb{Z}_2$-orbit of $x \in S^n$ to the line l_x through x, with the inverse

$$\psi : \mathbb{P}^n \to S^n/\mathbb{Z}_2$$

which sends the line l to $S^n \cap l$ (a $\mathbb{Z}_2$-orbit!). We have to check that they are continuous. We use Proposition 3.2 and its corollary. To see that ϕ is continuous, we have to check that the composition with the quotient map $S^n \to S^n/\mathbb{Z}_2$ is continuous. But this composition is precisely the restriction of the quotient map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n$ to S^n, hence is continuous. In conclusion, ϕ is continuous.

To see that ψ is continuous, we have to check that its composition with the quotient map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n$ is continuous. But this composition- which is a map from $\mathbb{R}^{n+1} - \{0\}$ to $S^n/\mathbb{Z}_2$ can be written as the composition of two other maps which we know to be continuous:

- The map $\mathbb{R}^{n+1} - \{0\} \to S^n$ sending x to $x/||x||$.

- The quotient map $S^n \to S^n/\mathbb{Z}_2$.

In conclusion ψ is continuous. □

COROLLARY 3.10. *The projective space $\mathbb{P}^n$ is Hausdorff.*

4.3. As a quotient of D^n: Again, the starting remark is very simple: the orbits of the action of $\mathbb{Z}_2$ on S^n always intersect the upper hemisphere S^n_+ (for notations, see Section 4 in the first chapter). Moreover, such an orbit either lies entirely in the boundary of S^n_+, or intersects its interior in a unique point. See the second picture in Figure 2. This indicates that $\mathbb{P}^n$ can be obtained from S^n_+ by gluing the antipodal points that belong to its boundary. On the other hand, the orthogonal projection onto the horizontal hyperplane defines a homeomorphism between S^n_+ and D^n (see Figure 2). Passing to D^n, we obtain an equivalence relation R on D^n given by:

$$(x, y) \in R \iff (x = y) \text{ or } (x, y \in S^{n-1} \text{ and } x = -y),$$

and we have done a part of the following:

EXERCISE 3.3. Show that $\mathbb{P}^n$ is homeomorphic to D^n/R. What happens when $n = 1$?

COROLLARY 3.11. *$\mathbb{P}^n$ for $n = 2$ is homeomorphic to the projective plane as defined in Chapter 1 (Section 8), i.e. obtained from the square by gluing the opposite sides as indicated in Figure 3.*

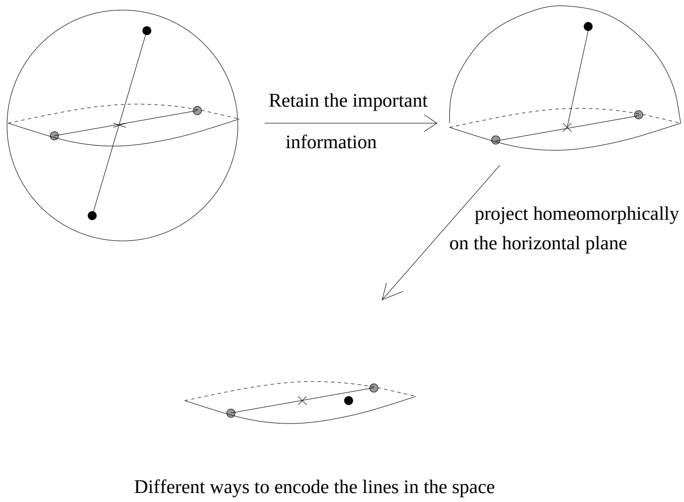


FIGURE 2.

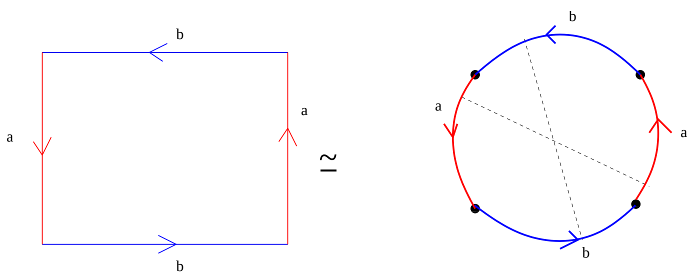


FIGURE 3.

5. Constructions of topologies: products

In this section we explain how the Cartesian product of two topological spaces is naturally a topological space itself. Given two sets X and Y we consider their Cartesian product

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

Given a topology $\mathcal{T}_X$ on X and a topology $\mathcal{T}_Y$ on Y, one defines a topology on $X \times Y$, the "product topology" $\mathcal{T}_X \times \mathcal{T}_Y$, as follows. We say that a subset $D \subset X \times Y$ is open if and only if

$$\forall \ (x, y) \in D \ \exists \ U \in \mathcal{T}_X, V \in \mathcal{T}_Y \ \text{ such that } \ x \in U, y \in V, U \times V \subset D. \tag{5.1}$$

We denote by $\mathcal{T}_X \times \mathcal{T}_Y$ the collection of all such D's and we call it the product topology.

Proposition 3.12. *Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, $\mathcal{T}_X \times \mathcal{T}_Y$ is indeed a topology on $X \times Y$. Moreover, it is the smallest topology on $X \times Y$ with the property that the two projections*

$$pr_X : X \times Y \to X, pr_Y : X \times Y \to Y$$

(sending (x, y) to x, and y, respectively) are continuous.

Proof. Axiom (T1) is clear. For (T2), let D_1, D_2 be in the product topology, and we show that $D := D_1 \cap D_2$ is as well. To check (5.1), let $(x, y) \in D$. Since D_1 and D_2 satisfy (5.1), for each $i \in \{1, 2\}$, we find $U_i \in \mathcal{T}_X$ and $V_i \in \mathcal{T}_Y$ such that

$$(x, y) \in U_i \times V_i \subset D_i.$$

Then $U := U_1 \cap U_2 \in \mathcal{T}_X$ (axiom (T2) for $\mathcal{T}_X$), and similarly $V := V_1 \cap V_2 \in \mathcal{T}_Y$, while clearly we have $x \in U$, $y \in V$, $U \times V \subset D$. The proof of the axiom (T3) is similar.

For the second part, note that a topology $\mathcal{T}$ on $X \times Y$ has the property that both projections are continuous if and only if $U \times Y \in \mathcal{T}$ and $X \times V \in \mathcal{T}$ for all $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Clearly $\mathcal{T}_X \times \mathcal{T}_Y$ has the property, hence the projections are continuous with respect to the product topology. For an arbitrary topology $\mathcal{T}$ on $X \times Y$ with the same property, since

$$U \times V = (U \times Y) \cap (X \times V),$$

we deduce that $U \times V \in \mathcal{T}$ for all $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$. To show that $\mathcal{T}_X \times \mathcal{T}_Y \subset \mathcal{T}$, let D be an open in the product topology and we show that it must belong to $\mathcal{T}$. Since D satisfies (5.1), for each $z = (x, y) \in D$ we find $U_z \in \mathcal{T}_X$, $V_z \in \mathcal{T}_Y$ such that

$$\{z\} \subset U_z \times V_z \subset D.$$

Taking the union over all $z \in D$, we deduce that

$$D = \cup_{z \in D} U_z \times V_z.$$

But, as we have already seen, all members $U_z \times V_z$ must be in $\mathcal{T}$ hence, using axiom (T3) for $\mathcal{T}$, we deduce that $D \in \mathcal{T}$. □

Exercise 3.4. Show that, if $(Z, \mathcal{T}_Z)$ is a third topological space, then a function

$$h = (f, g) : Z \to X \times Y, \ h(z) = (f(z), g(z))$$

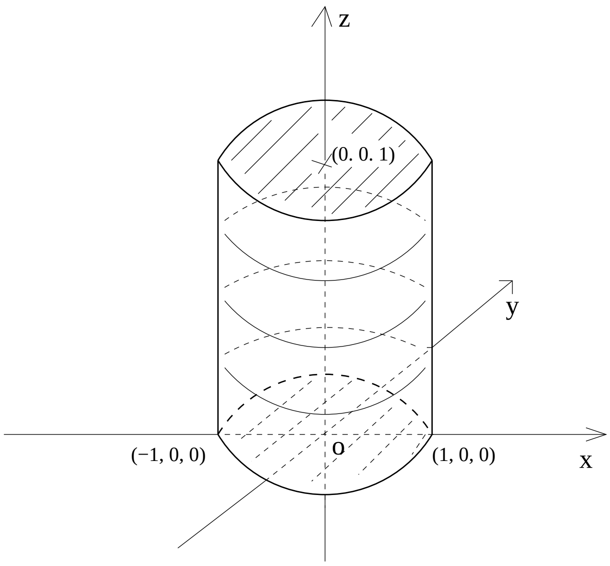
is continuous if and only if its components $f : Z \to X$ and $g : Z \to Y$ are both continuous.

Example 3.13. In $\mathbb{R}^3$ we have the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \le z \le 1\},$$

which is pictured in Figure 4. According to our conventions, C is considered with the topology induced from $\mathbb{R}^3$. On the other hand, since

$$C = S^1 \times [0, 1],$$



The cyclinder in $|\mathbb{R}^3$

FIGURE 4.

where S^1 is the unit circle in $\mathbb{R}^2$, C carries yet another natural topology, namely the product topology. These two topologies are the same. This can be proven in a much greater generality, as described in Exercise 3.27.

EXERCISE 3.5. A topological group is a group $(G, \cdot)$ endowed with a topology on G such that all the group operations, i.e.

1. the inversion map $\tau : G \to G, g \mapsto g^{-1}$,
2. the composition map $m : G \times G \to G$, $(g, h) \mapsto g \cdot h$

are continuous (where $G \times G$ is endowed with the product topology).

Note that the sets of matrices $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$ that appear in Exercise 2.33, together with multiplication of matrices, are groups. The same exercise describe natural topologies on them (induced from $M_n(\mathbb{R})$). Show that, with respect to these topologies, they are all topological groups.

6. Combining quotients with products: cones, suspensions

Another class of quotient spaces are quotients obtained by collapsing a subspace to a point.

DEFINITION 3.14. *Let X be a topological space and let $A \subset X$. We define X/A as the topological space obtained from X by collapsing A to a point (i.e. by identifying to each other all the points of A). Equivalently,*

$$X/A = X/R_A,$$

where R_A is the equivalence relation on X defined by

$$R_A = \{(x,y) : x = y \text{ or } x, y \in A\}.$$

Here are some more constructions of this type. Let X be a topological space.

The cylinder on X is defined as

$$\mathrm{Cyl}(X) := X \times [0,1],$$

endowed with the product topology (and the unit interval is endowed with the Euclidean topology). It contains two interesting copies of X: $X \times \{1\}$ and $X \times \{0\}$.

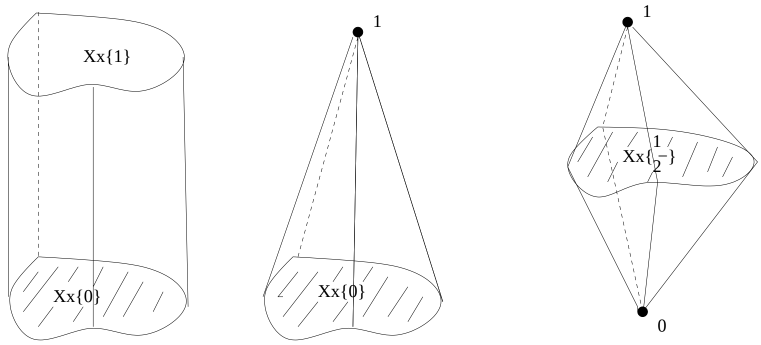
The cone on X is defined as the quotient obtained from $\mathrm{Cyl}(X)$ by collapsing $X \times \{1\}$ to a point:

$$\mathrm{Cone}(X) := X \times [0,1]/(X \times \{1\})$$

(endowed with the quotient topology). Intuitively, it looks like a cone with basis X.The cone contains the copy $X \times \{0\}$ of X (the basis of the cone).

The suspension of X is defined as the quotient obtained from $\mathrm{Cone}(X)$ by collapsing the basis $X \times \{0\}$ to a point:

$$\mathrm{S}(X) := \mathrm{Cone}(X)/(X \times \{0\}).$$

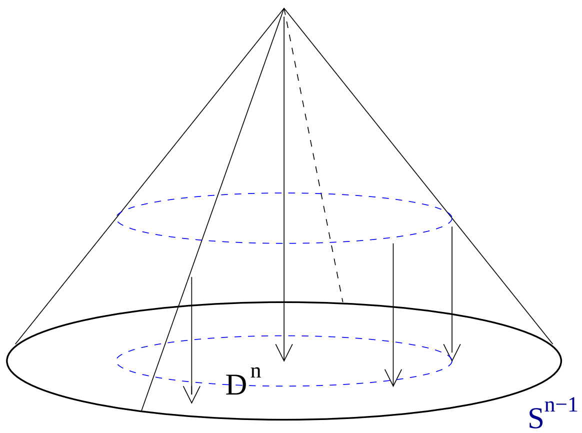


The cylinder of X The cone of X The suspension of X

FIGURE 5.

EXAMPLE 3.15. The general constructions of quotients, such as the quotient by collapsing a subspace to a point, the cone construction and the suspension construction, are nicely illustrated by the various relations between the closed unit balls $D^n \subset \mathbb{R}^n$ and the unit spheres $S^n \subset \mathbb{R}^{n+1}$. We mention here the following:

(i) D^n is homeomorphic to $\mathrm{Cone}(S^{n-1})$- the cone of S^{n-1}.
(ii) S^n is homeomorphic to $\mathrm{S}(S^{n-1})$- the suspension of S^n.
(iii) S^n is homeomorphic to D^n/S^{n-1}- the space obtained from D^n by collapsing its boundary to a point.



The cone of S^{n-1} is homeomorphic to the ball D^n

FIGURE 6.

PROOF. The first homeomorphism is indicated in Figure 6 (project the cone down to the disk). It is not difficult to make this precise: we have a map

$$\tilde{f} : S^{n-1} \times [0,1] \to D^n, \tilde{f}(x,t) = (1-t)x.$$

This is clearly continuous and surjective, and it has the property that

$$\tilde{f}(x,t) = \tilde{f}(x',t') \Longleftrightarrow (x,t) = (x',t') \text{ or } t = 1,$$

which is precisely the equivalence relation corresponding to the quotient defining the cone. Hence we obtain a continuous bijective map

$$f : \mathrm{Cone}(S^{n-1}) = S^{n-1} \times [0,1]/(S^{n-1} \times \{1\}) \to D^n.$$

After we will discuss the notion of compactness, we will be able to conclude that also f^{-1} is continuous, hence f is a homeomorphism. Note that this f sends $S^{n-1} \times \{1\}$ to the boundary of D^n, hence (ii) will follow from (iii). In turn, (iii) is clear on the picture (see Figure 11 in the previous Chapter); the map from D^n to S^n indicated on the picture can be written explicitly as

$$\tilde{g} : D^n \to S^n, x \mapsto (\frac{x_1}{||x||} sin(\pi||x||), \ldots, \frac{x_n}{||x||} sin(\pi||x||), cos(\pi||x||))$$

(well defined for $x \neq 0$) and which sends 0 to the north pole $(0, \ldots, 0, 1)$.

One can check directly that

$$\tilde{g}(x) = \tilde{g}(x') \Longleftrightarrow x = x' \text{ or } x, x' \in S^{n-1},$$

which is the equivalence relation corresponding to the quotient D^n/S^{n-1}. We deduce that we have a bijective continuous map:

$$g : D^n/S^{n-1} \to S^n$$

but, again, we leave it to after the discussion of compactness the final conclusion that g is a homeomorphism. $\square$

7. Constructions of topologies: Bases for topologies

In the construction of metric topologies, the balls were the building pieces. Similarly for the product topology, where the building pieces were the subsets of type $U \times V$ with $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$. In both cases, the collection of "building pieces" was not a topology, but "generated" a topology. The abstract notion underlying these constructions is that of topology basis.

DEFINITION 3.16. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. We say that $\mathcal{B}$ is a topology basis if it satisfies the following two axioms:*

(B1) *for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.*
(B2) *for each $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.*

In this case, we define the topology induced by $\mathcal{B}$ as the collection

$$\mathcal{T}(\mathcal{B}) := \{U \subset X : \forall\ x \in U\ \exists\ B \in \mathcal{B}\ s.t.\ \ x \in B,\ B \subset U\}.$$

EXERCISE 3.6. Show that, indeed, for any metric d on X, the collection $\mathcal{B}_d$ of all open balls is a topology basis and topology $\mathcal{T}(\mathcal{B}_d) = \mathcal{T}_d$. Prove a similar statement for the product topology.

We still have to prove that $\mathcal{T}(\mathcal{B})$ is, indeed, a topology (the next proposition). Here we point out a different description of $\mathcal{T}(\mathcal{B})$ (which we have already seen in the case of metric an product topologies- and this is a hint for the next exercise!).

EXERCISE 3.7. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Then a subset $U \subset X$ is in $\mathcal{T}(\mathcal{B})$ if and only if there exist $B_i \in \mathcal{B}$ with $i \in I$ (I-an index set) such that $U = \cup_{i \in I} B_i$.

PROPOSITION 3.17. *Given a collection $\mathcal{B}$ of subsets of a set X, the following are equivalent:*

1. *$\mathcal{B}$ is a topology basis.*
2. *$\mathcal{T}(\mathcal{B})$ is a topology on X.*

In this case $\mathcal{T}(\mathcal{B})$ is the smallest topology on X which contains $\mathcal{B}$; moreover, $\mathcal{B}$ is a basis for the topological space $(X, \mathcal{T}(\mathcal{B}))$, in the sense of Definition 2.47.

PROOF. We prove that the axioms (T1), (T2) and (T3) of a topology (applied to $\mathcal{T}(\mathcal{B})$) are equivalent to axioms (B1) and (B2) of a topology basis (applied to $\mathcal{B}$). First of all, the previous exercise shows that (T3) is satisfied without any assumption on B. Next, due to the definition of $\mathcal{T}(\mathcal{B})$, (B1) is equivalent to $X \in \mathcal{T}(\mathcal{B})$. Since clearly $\emptyset \in \mathcal{T}(\mathcal{B})$, (B1) is equivalent to (T1). Hence it suffices to prove that (T2) (for $\mathcal{T}(\mathcal{B})$) is equivalent to (B2) (for $\mathcal{B}$). That (T2) implies (B2) is immediate: given $B_1, B_2 \in \mathcal{B}$, since they are in $\mathcal{T}(\mathcal{B})$ so is their intersection, i.e. for all $x \in B_1 \cap B_2$ there exists $B \in \mathcal{B}$ such that $x \in B$, $B \subset B_1 \cap B_2$. For the converse, assume that (B2) holds. To prove $(T2)$ for $\mathcal{T}(\mathcal{B})$, we start with $U, V \in \mathcal{T}(\mathcal{B})$ and we want to prove that $U \cap V \in \mathcal{T}(\mathcal{B})$. I.e., for an arbitrary $x \in U \cap V$, we have to find $B \in \mathcal{B}$ such that $x \in B \subset U \cap V$. Since $x \in U \in \mathcal{T}(\mathcal{B})$, we find $B_1 \in \mathcal{B}$ such that $x \in B_1 \subset U$. Similarly, we find $B_2 \in \mathcal{B}$ such that $x \in B_2 \subset V$. By (B2) we find $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$. We deduce that $x \in B \subset U \cap V$, proving (T2). Finally, the last part of the proposition follows from the previous exercise, as any topology which contains $\mathcal{B}$ must contain all unions of sets in $\mathcal{B}$. □

Next, since many topologies are defined with the help of a basis, it is useful to know how to compare topologies by only looking at basis elements (see Exercises 3.29 and 3.31).

LEMMA 3.18. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two topology bases on X. Then $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$ if and if and only if: for each $B_1 \in \mathcal{B}_1$ and each $x \in B_1$, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$.*

PROOF. What we have to show is that $\mathcal{T}_1 \subset \mathcal{T}_2$ is equivalent to $\mathcal{B}_1 \subset \mathcal{T}_2$. The direct implication is clear since $\mathcal{B}_1 \subset \mathcal{T}_1$. For the converse, we use the fact that every element in $\mathcal{T}_1 = \mathcal{T}(\mathcal{B}_1)$ can be written as a union of elements of $\mathcal{B}_1$; hence, if $\mathcal{B}_1 \subset \mathcal{T}_2$, every element of $\mathcal{T}_1$ can be written as a union of elements of $\mathcal{T}_2$ hence is in itself in $\mathcal{T}_2$. □

8. General remarks on generated and initial topologies

8.1. Generated Topologies. There is a slightly more general recipe for generating topologies. What it may happen is that we have a set X, and we are looking for a topology on X which contains certain (specified) subsets of X. In other words,

- we start with a set X and a collection $\mathcal{S}$ of subsets of X

and we are looking for a (interesting) topology $\mathcal{T}$ on X which contains $\mathcal{S}$. Of course, the discrete topology $\mathcal{T}_{\text{dis}}$ on X is always a choice, but it is not a very interesting one (it does not even depend on $\mathcal{S}$). Is there a "best" one? More precisely:

- is there a smallest possible topology on X which contains $\mathcal{S}$?

EXAMPLE 3.19. If $\mathcal{S} = \mathcal{B}$ is a topology basis on the set X, Proposition 3.17 shows that the answer is positive, and the resulting topology is precisely $\mathcal{T}(\mathcal{B})$.

The answer to the question is always "yes", for any collection $\mathcal{S}$. Indeed, Exercise 2.20 of the previous chapter tells us that intersections of topologies is a topology. Hence one can just proceed abstractly and define:

$$\langle \mathcal{S} \rangle := \bigcap_{\mathcal{T}-\text{topology on } X \text{ containing } \mathcal{S}} \mathcal{T}$$

This is called the topology generated by $\mathcal{S}$. By Exercise 2.20, it is a topology. By construction, it is the smallest one containing $\mathcal{S}$. Of course, this abstract description is not the most satisfactory one. However, using exactly the same type of arguments as in the proof of Proposition 3.17:

PROPOSITION 3.20. *Let X be a set, let $\mathcal{S}$ be a collection of subsets. Define $\mathcal{B}(\mathcal{S})$ as the collection of subsets of X which can be written as finite intersections of subsets that belong to $\mathcal{S}$. Then $\mathcal{B}(\mathcal{S})$ is a topology basis and the associated topology is precisely $\langle \mathcal{S} \rangle$. In conclusion, a subset $U \subset X$ belongs to $\langle \mathcal{S} \rangle$ if and only if it is a union of finite intersections of members of $\mathcal{S}$.*

8.2. Initial topologies. Here is a general principle for constructing topologies. Many topological constructions are what we call "natural", or "canonical" (in any case, not arbitrary). Very often, when one looks for a topology, one wants certain maps to be continuous. This happens e.g. with induced and product topologies. A general setting is as follows.

- start with a set X and a collection of maps $\{f_i : X \to X_i\}_{i \in I}$ (I is an index set), where each X_i is endowed with a topology $\mathcal{T}_i$.

We are looking for (interesting) topologies $\mathcal{T}_X$ on X such that all the maps f_i become continuous. As before, this has an obvious but unsatisfactory answer: $\mathcal{T}_X = \mathcal{T}_{\text{dis}}$ (which does not reflect the functions f_i). One should also remark that the smaller $\mathcal{T}_X$ becomes, the smaller are the chances that f_i are continuous. With these, the really interesting question is to

- find the smallest topology on X such that all the functions f_i become continuous.

Now, by the definition of continuity, a topology on X makes the functions f_i continuous if and only if all subsets of type $f_i^{-1}(U_i)$ with $i \in I$, $U_i \in \mathcal{T}_i$, are open. Hence, denoting

$$\mathcal{S} := \{U \subset X : \ \exists\, i \in I, \exists\, U_i \in \mathcal{T}_i \text{ such that } U = f_i^{-1}(U_i)\}$$

the answer to the previous question is: the topology $\langle \mathcal{S} \rangle$ generated by $\mathcal{S}$. This is called the initial topology on X associated to the starting data (the topological spaces X_i and the functions f_i).

EXAMPLE 3.21. Given a subset A of a topological space $(X, \mathcal{T})$, the natural map here is the inclusion $i : A \to X$. The associated initial topology on A is the induced topology $\mathcal{T}|_A$.

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the Cartesian product $X \times Y$ comes with two natural maps: the projections $\text{pr}_X : X \times Y \to X$, $\text{pr}_Y : X \times Y \to Y$. The associated initial topology is the product topology on $X \times Y$.

9. Example: some spaces of functions

Given two sets X and Y we denote by $\mathcal{F}(X, Y)$ the set of all functions from X to Y. In many parts of mathematics, when interested in a certain problem, one deals with subsets of $\mathcal{F}(X, Y)$, endowed with a topology which is relevant to the problem; the topology is dictated by the type of convergence one has to deal with. The list of examples is huge; we will look at some topological examples, i.e. at the set of continuous functions $\mathcal{C}(X, Y) \subset \mathcal{F}(X, Y)$ between two spaces. The general setting will be discussed later. Here we treat the particular case

$$X = I \subset \mathbb{R} \text{ an interval, } Y = \mathbb{R}^n \text{ endowed with the Euclidean metric } d.$$

Here I could be any interval, open or not, closed or not, equal to $\mathbb{R}$ or not.

There are several notions of convergence on the set $\mathcal{F}(I, \mathbb{R}^n)$ of functions from I to $\mathbb{R}^n$.

Definition 3.22. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $\mathcal{F}(I, \mathbb{R}^n)$, $f \in \mathcal{F}(I, \mathbb{R}^n)$. We say that:*

- *f_n converges pointwise to f, and we write $f_n \stackrel{pt}{\to} f$, if $f_n(x) \to f(x)$ for all $x \in I$.*
- *f_n converges uniformly to f, and we write $f_n \rightrightarrows f$, if for any $\epsilon > 0$, there exists n_ϵ s.t.*
$$d(f_n(x), f(x)) < \epsilon \quad \forall\, n \geq n_\epsilon,\ \forall\, x \in I.$$
- *f_n converges uniformly on compacts to f, and we write $f_n \stackrel{cp}{\to} f$ if, for any compact sub-interval $K \subset I$, $f_n|_K \rightrightarrows f|_K$.*

We show that these convergences correspond to certain topologies on $\mathcal{F}(I, \mathbb{R}^n)$. First the pointwise convergence. For $x \in I$, $U \subset \mathbb{R}^n$ open, we define

$$S(x, U) := \{f \in \mathcal{F}(I, \mathbb{R}^n) : f(x) \in U\} \qquad \subset \mathcal{F}(I, \mathbb{R}^n).$$

These form a family $\mathcal{S}$. The topology of pointwise convergence, denoted $\mathcal{T}_{pt}$, is the topology on $\mathcal{F}(I, \mathbb{R}^n)$ generated by $\mathcal{S}$. Hence $\mathcal{S}$ defines a topology basis, consisting of finite intersections of members of $\mathcal{S}$, and $\mathcal{T}_{pt}$ is the associated topology.

Proposition 3.23. *The pointwise convergence coincides with the convergence in $(\mathcal{F}(I, \mathbb{R}^n), \mathcal{T}_{pt})$.*

Proof. Rewrite the condition that $f_n \to f$ with respect to $\mathcal{T}_{pt}$. It means that, for any neighborhood of f of type $S(x, U)$ there exists an integer N such that $f_n \in S(x, U)$ for $n \geq N$. I.e., for any $x \in I$ and any open U containing $f(x)$, there exists an integer N such that $f_n(x) \in U$ for all $n \geq N$. I.e., for any $x \in I$, $f_n(x) \to f(x)$ in $\mathbb{R}^n$. □

For uniform convergence, the situation is more fortunate: it is induced by a metric. Given two functions $f, g \in \mathcal{F}(I, \mathbb{R}^n)$, we define the sup-distance between f and g by

$$d_{\sup}(f, g) = \sup\{d(f(x), g(x)) : x \in I\}.$$

Since this supremum may be infinite for some f and g (and only for that reason!), we define

$$\hat{d}_{\sup}(f, g) = \min(d_{\sup}(f, g), 1).$$

Note that $d_{\sup}$ and $\hat{d}_{\sup}$ are morally the same when it comes to convergence ($d_{\sup}(f, g)$ is" small" if and only if $\hat{d}_{\sup}(f, g)$ is); they are actually the same on $\mathcal{C}(I, \mathbb{R}^n)$ if I is compact (why?). The associated topology is called the topology of uniform convergence.

Exercise 3.8. Show that $\hat{d}_{\sup}$ is a metric on $\mathcal{F}(I, \mathbb{R}^n)$.

Proposition 3.24. *The uniform convergence coincides with the convergence in $(\mathcal{F}(I, \mathbb{R}^n), \hat{d}_{sup})$.*

Proof. According to the definition of uniform convergence, $f_n \rightrightarrows f$ if and only if for each $\epsilon > 0$, we find n_ϵ such that $d_{\sup}(f, g) \leq \epsilon$ for all $n \geq n_\epsilon$. Of course, only ϵ's small enough matter here, hence we recover the convergence with respect to $\hat{d}_{\sup}$. □

We now move to uniform convergence on compacts. Given $K \subset I$ a compact subinterval, $\epsilon > 0$, $f \in \mathcal{F}(I, \mathbb{R}^n)$, we define

$$B_K(f, \epsilon) := \{g \in \mathcal{F}(I, \mathbb{R}^n) : d(f(x), g(x)) < \epsilon \ \ \forall\, x \in K\}.$$

The toplogy of compact convergence, denoted $\mathcal{T}_{cp}$, is the topology on $\mathcal{F}(I, \mathbb{R}^n)$ generated by the family of all the subsets $B_K(f, \epsilon)$. As above, the definitions immediately imply:

Proposition 3.25. *The uniform convergence on compacts coincides with the convergence in the topological space $(\mathcal{F}(I, \mathbb{R}^n), \mathcal{T}_{cp})$.*

In topology, we are interested in continuous functions. The situation is as follows:

Theorem 3.26. *For a sequence of continuous functions $f_n \in \mathcal{C}(I, \mathbb{R}^n)$, and $f \in \mathcal{F}(I, \mathbb{R}^n)$:*

$$(f_n \rightrightarrows f) \Longrightarrow (f_n \overset{cp}{\to} f) \Longrightarrow (f \in \mathcal{C}(I, \mathbb{R}^n)). \tag{9.1}$$

More precisely, one has an inclusion of topologies

$$(\textit{pointwise}) \subset (\textit{uniform on compacts}) \subset (\textit{uniform})$$

and $\mathcal{C}(I, \mathbb{R}^n)$ is closed in $(\mathcal{F}(I, \mathbb{R}^n), \mathcal{T}_{cp})$ (hence also in $(\mathcal{F}(I, \mathbb{R}^n), \hat{d}_{sup})$).

Proof. The comparison between the three topologies is again a matter of checking the definitions. Also, the first implication in (9.1) is trivial; the second one follows from the last part of the theorem, on which we concentrate next. We first show that $\mathcal{C}(I, \mathbb{R}^n)$ is closed in $(\mathcal{F}(I, \mathbb{R}^n), \hat{d}_{\text{sup}})$. Assume that f is in the closure, i.e. $f : I \to \mathbb{R}^n$ is the uniform limit of a sequence of continuous functions f_n. We show that f is continuous. Let $x_0 \in I$ and we show that f is continuous at x_0. I.e., we fix $\epsilon > 0$ and we look for a neighborhood V_ϵ of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for all $x \in V_\epsilon$. Since $f_n \rightrightarrows f$, we find N such that $d(f_n(x), f(x)) < \epsilon/3$ for all $n \geq N$ and all $x \in I$. Since f_N is continuous at x_0, we find a neighborhood V_ϵ such that $d(f_N(x), f_N(x_0)) < \epsilon/3$ for all $x \in V_\epsilon$. But then, for all $x \in V_\epsilon$,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < 3 \times \epsilon/3 = \epsilon.$$

Finally, we show that $\mathcal{C}(I, \mathbb{R}^n)$ is closed in $(\mathcal{F}(I, \mathbb{R}^n), \mathcal{T}_{cp})$. Assume that f is in the closure. Remark that, for f to be continuous, it suffices that $f|_K$ is continuous for any compact subinterval $K \subset I$. Fx such a K. Considering neighborhoods of type $B_K(f, 1/n)$, we find $f_n \in \mathcal{C}(I, \mathbb{R}^n)$ lying in this neighborhood. But then $f_n|_K \rightrightarrows f|_K$ hence $f|_K$ is continuous. ◻

Here is the most important property of the uniform topology ($\mathcal{T}_{cp}$ will be discussed later).

Theorem 3.27. *$(\mathcal{F}(I, \mathbb{R}^n), \hat{d}_{sup}))$ and $(\mathcal{C}(I, \mathbb{R}^n), \hat{d}_{sup}))$ are complete metric spaces.*

Proof. Using the previous theorem and the simple fact that closed subspaces of complete metric spaces are complete, we are left with showing that $(\mathcal{F}(I, \mathbb{R}^n), \hat{d}_{\text{sup}})$ is complete. So, let $(f_n)_{n \geq 1}$ be a Cauchy sequence with respect to $\hat{d}_{\text{sup}}$ (as mentioned above, for such arguments there is no difference between using d or $\hat{d}$). Since for all $x \in I$,

$$d(f_n(x), f_m(x)) \leq d_{\text{sup}}(f_n, f_m),$$

it follows that $(f_n(x))_{n \geq 1}$ is a Cauchy sequence in $(\mathbb{R}^n, d)$, for all $x \in X$. Denoting by $f(x)$ the limit, we obtain $f \in \mathcal{F}(I, \mathbb{R}^n)$ (to which f_n converges pointwise). To show that $f_n \rightrightarrows f$, let $\epsilon > 0$ and we look for n_ϵ such that $d_{\text{sup}}(f_n, f) < \epsilon$ for all $n \geq n_\epsilon$. For that, use that $(f_n)_{n \geq 1}$ is Cauchy and choose n_ϵ such that $d_{\text{sup}}(f_n, f_m) < \epsilon/2$ for all $n, m \geq n_\epsilon$. Combininig with the previous displayed inequality, we have $d(f_n(x), f_m(x)) < \epsilon/2$ for all such n, m and all $x \in I$. Taking $m \to \infty$, we find that $d(f_n(x), f(x)) \leq \epsilon/2 < \epsilon$ for all $n \geq n_\epsilon$ and $x \in I$, i.e. $d_{\text{sup}}(f_n, f) < \epsilon$ for all $n \geq n_\epsilon$. ◻

10. More exercises

10.1. Quotients.

EXERCISE 3.9. Let T be a model for the torus, with quotient map $\pi : [0,1]\times[0,1] \to T$ (choose your favourite). Give an example of a function $f : [0,1] \to [0,1] \times [0,1]$ with the property that f is not continuous, but $\pi \circ f$ is.

EXERCISE 3.10. Start with the interval $[0,2]$ and glue the points 0, 1 and 2. Describe the equivalence relation R encoding this gluing and let $X = [0,2]/R$. Describe an embedding of X in $\mathbb{R}^2$.

EXERCISE 3.11. Show that the space obtained from $\mathbb{R}$ by collapsing $[-1,1]$ to a point is homeomorphic to $\mathbb{R}$.

EXERCISE 3.12. Show that the space obtained from $\mathbb{R}$ by collapsing $(-1,1)$ to a point is not Hausdorff.

EXERCISE 3.13. Let $X = (0,\infty)$. Show that

$$\phi_n(r) := 2^n r$$

defines an action of $\mathbb{Z}$ on X and $X/\mathbb{Z}$ is homeomorphic to S^1.
(hint: see Example 3.7 and Exercise 1.2).

EXERCISE 3.14. Consider the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Let n be a positive integer, consider the n-th root of unity

$$\xi = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n}) \in \mathbb{C}$$

and let $\mathbb{Z}_n$ be the (additive) group of integers modulo n. Show that

$$\phi_{\hat{k}}(z) := \xi^k z$$

defines an action of $\mathbb{Z}_n$ on S^1, explain its geometric meaning, and show that $S^1/\mathbb{Z}_n$ is homeomorphic to S^1. What do you obtain when $n = 2$?

EXERCISE 3.15. Let R be the equivalence relation on $\mathbb{R}$ consisting of those pairs (r,s) of real numbers with the property that there exist two integers m and n such that $r - s = m + n\sqrt{2}$. Show that the resulting quotient space is not Hausdorff.

EXERCISE 3.16. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}$ with the usual group operation

$$(m,n) + (m',n') = (m + m', n + n').$$

Show that

$$\phi_{m,n}(x,y) := (x + m, y + n)$$

defines an action of Γ on $\mathbb{R}^2$ and $\mathbb{R}^2/\Gamma$ is homeomorphic to the torus.

EXERCISE 3.17. Do the same for the same group but the new action:

$$\phi_{m,n}(x,y) := (x + my + n + \frac{m(m-1)}{2}, y + m).$$

EXERCISE 3.18. Consider the following groups:

1. $\Gamma = \langle a, b; bab = a\rangle$ (the group in two generators a and b, subject to the relation $bab = a$).
2. $\Gamma' = \mathbb{Z} \times \mathbb{Z}$ with the operation

$$(m,n) \circ (m',n') = (m + m', n + (-1)^m n').$$

3. Γ'' is the subgroup of $(\mathrm{Homeo}(\mathbb{R}^2), \circ)$ (the group of homeomorphisms of the plane endowed with the composition of functions) generated by the transformations
$$\phi(x, y) = (x + 1, -y), \psi(x, y) = (x, y + 1).$$

Show that:

1. these three groups are isomorphic.
2. one obtains an action of these groups on $\mathbb{R}^2$; write it down explicitly.
3. the resulting quotient is homeomorphic to the Klein bottle.

EXERCISE 3.19. Compose ad solve a similar exercise in which the resulting quotient is homeomorphic to the Moebius band.

EXERCISE 3.20.

(i) Write the Moebius band as a union of two subspaces M and C where M is itself a Moebius band, C is a cylinder (i.e. homeomorphic to $S^1 \times [0, 1]$) and $M \cap C$ is a circle.
(ii) Similarly, decompose $\mathbb{P}^2$ as the union of a Moebius band M and another subspace Q, such that Q is a quotient of the cylinder and $M \cap Q$ is a circle.
(iii) Deduce that $\mathbb{P}^2$ can be obtained from a Moebius band and a disk D^2 by gluing them along their boundary circles.

EXERCISE 3.21. Show that

1. $\mathbb{P}^n$ is an n-dimensional topological manifold.
2. The map $f : \mathbb{P}^n \to \mathbb{R}^{\frac{(n+1)(n+2)}{2}}$ which sends the line l_x through $x = (x_0, \ldots, x_n)$ to
$$f(l_x) = (x_0x_0, x_0x_1, \ldots, x_0x_n, x_1x_1, x_1x_2, \ldots, x_1x_n, \ldots, x_nx_n)$$
is an embedding.

10.2. Product topology.

EXERCISE 3.22. Show that
$$(\mathbb{R}^n, \mathcal{T}_{\text{eucl}}) \times (\mathbb{R}^m, \mathcal{T}_{\text{eucl}}) = (\mathbb{R}^{n+m}, \mathcal{T}_{\text{eucl}}).$$

EXERCISE 3.23. Show that if X and Y are both Hausdorff, then so is $X \times Y$. Similarly for metrizability and first countability.

EXERCISE 3.24. Look at exercise 2.22 and show that there is no conflict in the notation; i.e. the topology $\mathcal{T}_l \times \mathcal{T}_l$ defined in that exercise does coincide with the product topology.

EXERCISE 3.25. Show that if n is an odd number, then $GL_n(\mathbb{R})$ is homeomorphic to $GL_1(\mathbb{R}) \times SL_n(\mathbb{R})$ (see Exercise 2.33).

EXERCISE 3.26. Show that a topological space X is Hausdorff if and only if
$$\Delta := \{(x, x) : x \in X\}$$
is closed in $X \times X$.

EXERCISE 3.27. Let X and Y be two topological spaces, $A \subset X$, $B \subset Y$. Then the following two topologies on $A \times B$ coincide:

(i) The product of the topology of A (induced from X) and that of B (induced from Y).
(ii) The topology induced on $A \times B$ from the product topology on $X \times Y$.

EXERCISE 3.28. Let X be the cone of the interval $(0, 1)$. Construct an explicit continuous injection
$$f : X \to \mathbb{R}^2$$
and let C be its image. Is f a homeomorphism?

10.3. Topology bases.

EXERCISE 3.29. Using Lemma 3.18, prove again that the Euclidean and the square metrics induce the same topology.

EXERCISE 3.30. Go back to exercise 2.17 and show that $\mathcal{B}$ there is a topology basis. Then using Lemma 3.18, show again that the resulting topology is larger than the topology from Exercise 2.16.

EXERCISE 3.31. Show that

$$\mathcal{B}_l := \{[a, b) : a, b \in \mathbb{R}\}$$

is a topology basis and $\mathcal{T}(\mathcal{B}_l)$ is the topology $\mathcal{T}_l$ from Exercise 2.19. Then use Lemma 3.18 to prove again that $\mathcal{T}_l$ is finer than $\mathcal{T}_{\text{Eucl}}$.

EXERCISE 3.32. Do again Exercise 2.22 using Lemma 3.18.

EXERCISE 3.33. Consider the unit circle S^1 and the functions

$$\alpha, \beta : S^1 \to \mathbb{R}, \alpha(x, y) = x, \beta(x, y) = y.$$

Show that:

1. S^1, endowed with the smallest topology which makes α continuous, is not Hausdorff.
2. the smallest topology on S^1 which makes both α and β continuous is the Euclidean one.

10.4. Spaces of functions.

EXERCISE 3.34. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n} sin(nx)$. Show that (f_n) is uniformly convergent.

EXERCISE 3.35. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = x^n$. Is (f_n) pointwise convergent? But uniformly on compacts? But uniformly?

EXERCISE 3.36. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. Is (f_n) pointwise convergent? But uniformly on compacts? But uniformly?

EXERCISE 3.37. *(Dini's theorem)* Let $f_n : I \to \mathbb{R}$ be an increasing sequence of continuous functions defined on an interval I, which converges pointiwse to a continuous function f. Show that $f_n \rightrightarrows f$. Is the same true if we do not assume that f is continuous?

EXERCISE 3.38. Let $p_n \in \mathcal{C}([0, 1], \mathbb{R})$ be the sequence of functions (even polynomials!) defined inductively by

$$p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n(t)^2), \quad p_1 = 0.$$

Show that (p_n) converges uniformly to the function $f(t) = \sqrt{t}$.

(Hint: first show that $p_n(t) \leq \sqrt{t}$, then that $p_n(t)$ is increasing, then that it converges pointwise to $\sqrt{t}$, then look above).

EXERCISE 3.39. Now, did you know that there are continuous surjective functions $f : [0, 1] \to [0, 1] \times [0, 1]$??? (yes, continuous curves in the plane which fill up an entire square!). Actually, you now have all the knowledge to show that (using Theorem 3.27); it is not obvious but, once you see the pictures, the proof is not too difficult; have a look at Munkres' book.