CHAPTER 4

Topological properties

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1. Connectedness

1.1. Definitions and examples.

DEFINITION 4.1. *We say that a topological space* $(X, \mathcal{T})$ *is* connected *if* X *cannot be written as the union of two disjoint non-empty opens* $U, V \subset X$.

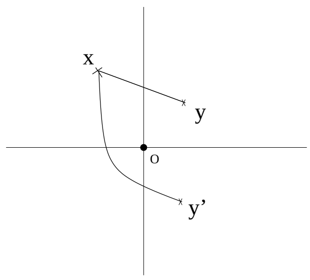
We say that a topological space $(X, \mathcal{T})$ *is* path connected *if for any* $x, y \in X$, *there exists a path* γ *connecting* x *and* y, *i.e. a continuous map* $\gamma : [0,1] \to X$ *such that* $\gamma(0) = x$, $\gamma(1) = y$.

Given $(X, \mathcal{T})$, *we say that a subset* $A \subset X$ *is connected (or path connected) if* A, *together with the induced topology, is connected (path connected).*

As we shall soon see, path connectedness implies connectedness. This is good news since, unlike connectedness, path connectedness can be checked more directly (see the examples below).

EXAMPLE 4.2.

(1) $X = \{0, 1\}$ with the discrete topology is not connected. Indeed, $U = \{0\}$, $V = \{1\}$ are disjoint non-empty opens (in X) whose union is X.
(2) Similarly, $X = [0, 1) \cup [2, 3]$ is not connected (take $U = [0, 1)$, $V = [2, 3]$). More generally, if $X \subset \mathbb{R}$ is connected, then X must be an interval. Indeed, if not, we find $r, s \in X$ and $t \in (r, s)$ such that $t \notin X$. But then $U = (-\infty, t) \cap X, V = (t, \infty) \cap X$ are opens in X, nonempty (as $r \in U$, $s \in V$), disjoint, with $U \cup V = X$ (as $t \notin X$).
(3) However, although true, the fact that any interval $I \subset \mathbb{R}$ is connected is not entirely obvious. In contrast, the path connectedness of intervals is clear: for any $x, y \in I$,
$$\gamma : [0, 1] \to \mathbb{R}, \gamma(t) = (1 - t)x + ty$$
takes values in I (since I is an interval) and connects x and y.
(4) Similarly, any convex subset $X \subset \mathbb{R}^n$ is path connected (recall that X being convex means that for any $x, y \in X$, the whole segment $[x, y]$ is contained in X).
(5) $X = \mathbb{R}^2 - \{0\}$, although not convex, is path connected: if $x, y \in X$ and the segment $[x, y]$ does not contain the origin, we use the linear path from x to y. But even if $[x, y]$ contains the origin, we can join them by a path going around the origin (see Figure 1).



|R^2−{0} is path connected

FIGURE 1.

LEMMA 4.3. *The unit interval* $[0, 1]$ *is connected.*

PROOF. We assume the contrary: $\exists$ disjoint non-empty U, V, opens in $[0, 1]$ such that $U \cup V = [0, 1]$. Since $U = [0, 1] - V$, U must be closed in $[0, 1]$. Hence, as a limit of points in U, $R := \sup U$ must belong to U. We claim that $R = 1$. If not, we find an interval $(R - \epsilon, R + \epsilon) \subset U$ and then $R + \frac{1}{2}\epsilon$ is an element in U strictly greater than its supremum- which is impossible. In conclusion, $1 \in U$. But exactly the same argument shows that $1 \in V$, and this contradicts the fact that $U \cap V = \emptyset$. $\square$

1.2. Basic properties.

Proposition 4.4.

(i) *If $f : X \to Y$ is a continuous map and X is connected, then $f(X)$ is connected.*
(ii) *Given $(X, \mathcal{T})$, if for any two points $x, y \in X$, there exists $\Gamma \subset X$ connected such that $x, y \in \Gamma$, then X is connected.*

Proof. For (i), replacing Y by $f(X)$, we may assume that f is surjective, and we want to prove that Y is connected. If it is not, we find $U, V \subset Y$ disjoint nonempty opens whose union is Y. But then $f^{-1}(U), f^{-1}(V) \subset X$ are disjoint (since U and V are), nonempty (since U and V are and f is surjective) opens (because f is continuous) whose union is X- and this contradicts the connectedness of X. For (ii) we reason again by contradiction, and we assume that X is not connected, i.e. $X = U \cup V$ for some disjoint nonempty opens U and V. Since they are non-empty, we find $x \in U$, $y \in V$. By hypothesis, we find Γ connected such that $x, y \in \Gamma$. But then

$$U' = U \cap \Gamma, \; V' = V \cap \Gamma$$

are disjoint non-empty opens in Γ whose union is Γ- and this contradicts the connectedness of Γ. □

Theorem 4.5. *Any path connected space X is connected.*

Proof. We use (ii) of the proposition. Let $x, y \in X$. We know there exists $\gamma : [0, 1] \to X$ joining x and y. But then $\Gamma = \gamma([0, 1])$ is connected by using (i) of the proposition and the fact that $[0, 1]$ is connected; also, $x, y \in \Gamma$. □

Since we have already remarked that a connected subset of $\mathbb{R}$ must be an interval, and that any interval is path connected, the theorem implies:

Corollary 4.6. *The only connected subsets of $\mathbb{R}$ are the intervals.*

Combining with part (i) of Proposition 4.4 we deduce the following:

Corollary 4.7. *If X is connected and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is an interval.*

Corollary 4.8. *If X is connected, then any quotient of X is connected.*

Example 4.9. There are a few more consequences that one can derive by combining connectedness properties with the "removing one point trick" (Exercise 2.39 in Chapter 2).

(1) $\mathbb{R}$ cannot be homeomorphic to S^1. Indeed, if we remove a point from $\mathbb{R}$ the result is disconnected, while if we remove a point from S^1, the result stays connected.
(2) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$. The argument is similar to the previous one (recall that $\mathbb{R}^2 - \{0\}$ is path connected, hence connected).
(3) The more general statement that $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic if $n \neq m$ is much more difficult to prove. One possible proof is a generalization of the argument given above (when $n = 1$, $m = 2$)- but that is based on "higher versions of connectedness", a notion which is at the core of algebraic topology.

Exercise 4.1. Show that $[0, 1)$ and $(0, 1)$ are not homeomorphic.

1.3. Connected components.

DEFINITION 4.10. *Let $(X, \mathcal{T})$ be a topological space. A connected component of X is any maximal connected subset of X, i.e. any connected $C \subset X$ with the property that, if $C' \subset X$ is connected and contains C, then C' must coincide with C.*

PROPOSITION 4.11. *Let $(X, \mathcal{T})$ be a topological space. Then*
(i) *Any point $x \in X$ belongs to a connected component of X.*
(ii) *If C_1 and C_2 are connected components of X then either $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$.*
(iii) *Any connected component of X is closed in X.*

PROOF. To prove the proposition we will use the following

EXERCISE 4.2. If $A, B \subset X$ are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

For (i) of the proposition, let $x \in X$ and define $C(x)$ as the union of all connected subsets of X containing x. We claim that $C(x)$ is a connected component. The only thing that is not clear is the connectedness of $C(x)$. To prove that, we will use the criterion given by (ii) of Proposition 4.4. For $y, z \in C(x)$ we have to find $\Gamma \subset X$ connected such that $y, z \in \Gamma$. Due to the definition of $C(x)$, we find C_y and C_z- connected subsets of X, both containing x, such that $y \in C_y$, $z \in C_z$. The previous exercise implies that $\Gamma := C_y \cup C_z$ is connected containing both y and z.

To prove (ii), we use again the previous exercise applied to $A = C_1$ and $B = C_2$, and the maximality property of connected components.

To prove (iii), due to the maximality of connected components, it suffices to prove that if $C \subset X$ is connected, then $\overline{C}$ is connected. Assume that $Y := \overline{C}$ is not connected. We find D_1, D_2 nonempty opens in Y, disjoint, such that $Y = D_1 \cup D_2$. Take $U_i = C \cap D_i$. Clearly, U_1 and U_2 are disjoint opens in C, with $C = U_1 \cup U_2$. To reach a contradiction (with the fact that C is connected) it suffices to show that U_1 and U_2 are nonempty. To show that U_i is non-empty ($i \in \{1, 2\}$), we use the fact that D_i is non-empty. We find a point $x_i \in D_i$. Since x_i is in the closure of C in Y, we have $U \cap C \neq \emptyset$ for each neighborhood U of x_i in Y. Choosing $U = D_i$, we have $C \cap D_i \neq \emptyset$. □

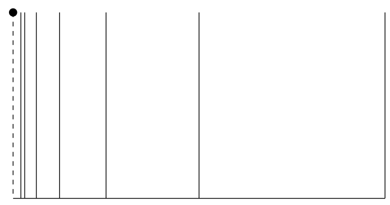
REMARK 4.12. From the previous proposition we deduce that, for any topological space $(X, \mathcal{T})$, the family $\{C_i\}_{i \in I}$ of connected components of $(X, \mathcal{T})$ (where I is an index set) give a partition of X:
$$X = \cup_{i \in I} C_i, \quad C_i \cap C_j = \emptyset \ \forall \ i \neq j,$$
called the partition of X into connected components.

EXERCISE 4.3. Let X be a topological space and assume that $\{X_1, \ldots, X_n\}$ is a finite partition of X, i.e.
$$X = X_1 \cup \ldots \cup X_n, \quad X_i \cap X_j = \emptyset \quad \forall \ i \neq j.$$
Then the following are equivalent:
(i) All X_i's are closed.
(ii) All X_i's are open.

If moreover each X_i is connected, then $\{X_1, \ldots, X_n\}$ coincides with the partition of X into connected components.

1.4. Connected versus path connected, again. Theorem 4.5 shows that path connectedness implies connectedness. However, the converse does not hold in general. The standard example is "the flea and comb", drawn in Figure 2. Explicitely, $X = C \cup \{f\}$, where

$$C = [0,1] \cup \{(\frac{1}{n}, y) : y \in [0,1], n \in \mathbb{Z}_{>0}\}, \quad f = (0,1).$$



The flea and comb

FIGURE 2.

EXERCISE 4.4. Show that, indeed, X from the picture is connected but not path connected.

However, there is a partial converse to Theorem 4.5.

THEOREM 4.13. *Let $(X, \mathcal{T})$ be a topological space with the property that any point of X has a path connected neighborhood. Then X is path connected if and only if it is connected.*

PROOF. We still have to prove that, if X is connected, then it is also path connected. We define on X the following relation $\sim$: $x \sim y$ if and only if x and y can be joined by a (continuous) path. This is an equivalence relation. Indeed, for $x, y, z \in X$:

- $x \sim x$: consider the constant path.
- if $x \sim y$ then $y \sim x$: if γ is a path from x to y, then $\gamma^-(t) = \gamma(1-t)$ is a path from y to x.
- if $x \sim y$ and $y \sim z$, then $x \sim z$. Indeed, if γ_1 is a path from x to y, while γ_2 from y to z, then

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } x \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

is a path from x to z.

For $x \in X$, we denote by $C(x)$ the equivalence class of x:

$$C(x) = \{y \in X : y \sim x\}.$$

Note that each $C(x)$ is path connected. We claim that $C(x) = X$ for any $x \in X$. The fact that $\sim$ is an equivalence relation implies that

$$\{C(x) : x \in X\}$$

is a partition of X: if $C(x) \cap C(x') \neq \emptyset$, then $C(x) = C(x')$. What we want to prove is that this partition consists of one set only. Since X is connected, it is enough to prove that $C(x)$ is open for each $x \in X$. Fixing x, we want to prove that for any $y \in C(x)$, there is an open U such that $y \in U \subset C(x)$. To see this, we use the hypothesis and we choose any path connected neighborhood V of y. Since $y \sim x$ and $z \sim y$ fro any $z \in V$, we deduce that $z \sim x$ fro any $z \in V$, hence $V \subset C(x)$. Since V is a neighborhood of y, we find U open such that $y \in U \subset V$, and this U clearly has the desired properties. □

EXERCISE 4.5. Find a subspace $X \subset \mathbb{R}^2$ which is connected but not path connected.

2. Compactness

2.1. Definition and first examples. Probably many of you have seen the notion of compact space in the context of subsets of $\mathbb{R}^n$, as sets which are closed and bounded. Although not obviously at all, this is a topological property (it can be defined using open sets only).

DEFINITION 4.14. *Given a topological space $(X, \mathcal{T})$ an open cover of X is a family $\mathcal{U} = \{U_i : i \in I\}$ (I-some index set) consisting of open sets $U_i \subset X$ such that*

$$X = \cup_{i \in I} U_i.$$

A subcover is any cover $\mathcal{V}$ with the property that $\mathcal{V} \subset \mathcal{U}$.

We say that a topological space $(X, \mathcal{T})$ is compact if from any open cover $\mathcal{U} = \{U_i : i \in I\}$ of X one can extract a finite open subcover, i.e. there exist $i_1, \ldots, i_k \in I$ such that

$$X = U_{i_1} \cup \ldots \cup U_{i_k}.$$

REMARK 4.15. Given a topological space $(X, \mathcal{T})$ and $A \subset X$, the compactness of A (viewed as a topological space with the topology induced from X (cf. Example 2.8 in Chapter 2) can be expressed using "open coverings of A in X", i.e. families $\mathcal{U} = \{U_i : i \in I\}$ (I-some index set) consisting of open sets $U_i \subset X$ such that

$$A \subset \cup_{i \in I} U_i.$$

A subcover is any cover (of A in X) $\mathcal{V}$ with the property that $\mathcal{V} \subset \mathcal{U}$. With these, A is compact if and only if from any open cover of A in X one can extract a finite open subcover. This follows immediately from the fact that the opens for the induced topology on A are of type $A \cap U$ with $U \in \mathcal{T}$ and from the fact that, for a family $\{U_i : i \in I\}$, we have

$$\bigcup_i (A \cap U_i) = A \Longleftrightarrow A \subset \bigcup_i U_i,$$

EXAMPLE 4.16.

(1) $(X, \mathcal{T}_{\text{discr}})$ is compact if and only if X is finite (use the cover of X by the open-point opens).

(2) $\mathbb{R}$ is not compact. Indeed,

$$\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (-k, k)$$

is an open cover from which we cannot extract a finite open subcover. By the same argument, any compact $A \subset \mathbb{R}^n$ must be bounded (e.g., when $n = 1$, write $A \subset \cup_k (-k, k)$).

(3) $[0, 1)$ is not compact. Indeed,

$$[0, 1) \subset \bigcup_k (-\infty, 1 - \frac{1}{k})$$

defines an open cover of $[0, 1)$ in $\mathbb{R}$, from which we cannot extract a finite open subcover. By a similar argument, any compact $A \subset \mathbb{R}^n$ must be closed in $\mathbb{R}^n$. To show this, assume for simplicity that $n = 1$. We proceed by contradiction and assume that there exists $a \in \overline{A} - A$. Since $a \notin A$,

$$U_\epsilon := \mathbb{R} - [a - \epsilon, a + \epsilon]$$

form an open cover of A in $\mathbb{R}$ indexed by $\epsilon > 0$. Extracting a finite subcover, we find

$$A \subset U_{\epsilon_1} \cap \ldots \cap U_{\epsilon_k} = U_\epsilon \quad \text{where } \epsilon = \min\{\epsilon_1, \ldots, \epsilon_k\}.$$

But this implies that $A \cap (a - \epsilon, a + \epsilon) = \emptyset$ which contradicts $a \in \overline{A}$.

(4) $[0, 1]$ is compact.

PROOF. Assume the contrary, i.e. there exists a "bad cover of $[0,1]$", i.e. an open cover $\mathcal{U}$ of $[0,1]$ in $\mathbb{R}$ which does not admit any finite open subcover (of $[0,1]$ in $\mathbb{R}$). Divide $[0,1]$ into two intervals of equal length. Then $\mathcal{U}$ will be a bad cover for at least one of the two intervals, call it I_1. Divide now I_1 into two intervals of equal length and, again, choose one of them, call it I_2, so that $\mathcal{U}$ is a bad cover of I_2. Continuing this process, we find intervals I_k with

$$I_{k+1} \subset I_k, \mathrm{diam}(I_k) = \frac{1}{2^k},$$

(where the diameter of an interval $[a,b]$ is $(b-a)$) and such that $\mathcal{U}$ is a bad open cover of I_k in $\mathbb{R}$. Choosing $x_k \in I_k$, we have

$$d(x_k, x_{k+1}) \leq \frac{1}{2^k}$$

for all k, where d denotes the Euclidean metric ($d(a,b) = |a-b|$). We claim that the sequence of real numbers $(x_k)_{k\geq 1}$ is a Cauchy sequence, i.e. $d(x_k, x_{k+n})$ can be made arbitrarily small for k big enough and all n. To see this, we use the triangle inequality and the previous inequality repeatedly:

$$d(x_k, x_{k+n}) \leq \sum_{i=k}^{k+n-1} d(x_i, x_{i+1}) \leq \sum_{i=k}^{k+n-1} \frac{1}{2^i} = \frac{1}{2^{k-1}}(1 - \frac{1}{2^n}),$$

and deduce

$$d(x_k, x_{k+n}) < \frac{1}{2^{k-1}}$$

for all $k, n \geq 0$ integers. This shows that $(x_k)_{k\geq 1}$ is a Cauchy sequence, so, from the known properties of the real line, the sequence will be convergent. Let x be its limit. Since $x_k \in [0,1]$ for all k, we deduce that $x \in [0,1]$. Taking $n \to \infty$ in the previous inequality we also deduce that

$$d(x_k, x) \leq \frac{1}{2^{k-1}}.$$

We deduce that, for each k,

$$I_k \subset [x - \frac{3}{2^k}, x + \frac{3}{2^k}].$$

Indeed, if $y \in I_k$, since $x_k \in I_k$ and the diameter of I_k is $1/2^k$, we have $d(y, x_k) \leq 1/2^k$, hence

$$d(y,x) \leq d(y, x_k) + d(x_k, x) \leq \frac{1}{2^k} + \frac{1}{2^{k-1}} = \frac{3}{2^k}.$$

We now use the cover $\mathcal{U}$. Let $U \in \mathcal{U}$ such that $x \in U$. Since U is open, we find $r > 0$ such that

$$(x - r, x + r) \subset U.$$

But then, choosing k such that $\frac{3}{2^k} < r$, we will have $I_k \subset (x-r, x+r) \subset U$, i.e. the open cover $\mathcal{U}$ of I_k in $\mathbb{R}$ will have a finite subcover- namely the one consisting of the open set U alone. This contradicts the fact that $\mathcal{U}$ was a bad cover for I_k. □

EXERCISE 4.6. Show that the set

$$\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$$

is compact in $\mathbb{R}$.

2.2. Topological properties of compact spaces. In this section we point out some topological properties of compact spaces.

The first one says that "closed inside compact is compact".

PROPOSITION 4.17. *If $(X, \mathcal{T})$ is a compact space, then any closed subset $A \subset X$ is compact.*

The second one says that "compact inside Hausdorff is closed".

THEOREM 4.18. *In a Hausdorff space $(X, \mathcal{T})$, any compact set is closed.*

The third one says that "disjoint compacts inside a Hausdorff can be separated".

PROPOSITION 4.19. *In a Hausdorff space $(X, \mathcal{T})$, any two disjoint compact sets A and B can be separated topologically, i.e. there exist opens $U, V \subset X$ such that*

$$A \subset U, \ B \subset V, \ U \cap V = \emptyset.$$

COROLLARY 4.20. *Any compact Hausdorff space is normal.*

PROOF. (of Proposition 4.17) If $\mathcal{U}$ is an open cover of A in X, then adding $X - A$ to $\mathcal{U}$ (which is open since A is closed), we get an open cover of X. Extracting a finite subcover (which may or may not contain $X - A$), denoting by $U_1, \ldots, U_n$ the elements of this finite subcover which are different from $X - A$, this will define a finite subcover of the original cover of A in X. □

PROOF. (of Proposition 4.19): We introduce the following notation: given $Y, Z \subset X$ we write $Y|Z$ if Y and Z can be separated, i.e. if there exist opens U and V (in X) such that $Y \subset U$, $Z \subset V$ and $U \cap V = \emptyset$. We claim that, if Z is compact and $Y|\{z\}$ for all $z \in Z$, then $Y|Z$.

PROOF. (of the claim) We know that for each $z \in Z$ we find opens U_z and V_z such that $Y \subset U_z$, $z \in V_z$ and $U_z \cap V_z = \emptyset$. Note that $\{V_z : z \in Z\}$ is an open cover of Z in X: indeed, any $z \in Z$ belongs at least to one of the opens in the cover (namely V_z). By compactness of Z, we find a finite number of points $z_1, \ldots, z_n \in Z$ such that

$$Z \subset V_{z_1} \cup \ldots \cup V_{z_n}.$$

Denoting by V the last union, and considering

$$U = U_{z_1} \cap \ldots \cap U_{z_n},$$

V is an open containing Z, U is an open containing Y. Moreover, $U \cap V = \emptyset$: indeed, if x is in the intersection, since $x \in V$ we find k such that $x \in V_{z_k}$; but $x \in U$ hence $x \in U_z$, which is impossible since $U_{z_k} \cap V_{z_k} = \emptyset$. In conclusion, U and V show that Y and Z can be separated. □

Back to the proof of the proposition, let $a \in A$ arbitrary. Now, since X is Hausdorff, we have $\{a\}|\{b\}$ for all $b \in B$. Since B is compact, the claim above implies that $\{a\}|B$, or, equivalently, $B|\{a\}$. This holds for all $a \in A$, hence using again the claim (and the fact that A is compact) we deduce that $A|B$. □

PROOF. (of Theorem 4.18) For Theorem 4.18, assume that $A \subset X$ is compact, and we prove that $\overline{A} \subset A$: if $x \in \overline{A}$, then, for any neighborhood U of x, $U \cap A \neq \emptyset$, and this shows that x cannot be separated from A; using the proposition, we conclude that $x \in A$. □

EXERCISE 4.7. Deduce that a subset $A \subset \mathbb{R}$ is compact if and only if it is closed and bounded. What is missing to prove the same for subsets of $\mathbb{R}^n$?

2.3. Compactness of products, and compactness in $\mathbb{R}^n$. Next, we are interested in the compactness of the product of two compact spaces. We will use the following:

LEMMA 4.21. *(the Tube Lemma)* Let X and Y be two topological spaces, $x_0 \in X$, and let $U \subset X \times Y$ be an open (in the product topology) such that

$$\{x_0\} \times Y \subset U.$$

If Y is compact, then there exists $W \subset X$ open containing x_0 such that

$$W \times Y \subset U.$$

PROOF. Due to the definition of the product topology, for each $y \in Y$, since $(x_0, y) \in U$, there exist opens $W_y \subset X$, $V_y \subset Y$ such that

$$W_y \times V_y \subset U.$$

Now, $\{V_y : y \in Y\}$ will be an open cover of Y, hence we find $y_1, \ldots, y_n \in Y$ such that

$$Y = V_{y_1} \cup \ldots \cup V_{y_n}.$$

Choose $W = W_{y_1} \cap \ldots \cap W_{y_n}$, which is an open containing x_0, as a finite intersection of such. To check $W \times Y \subset U$, let $(x, y) \in W \times Y$. Since the V_{y_i}'s cover Y, we find i s.t. $y \in V_{y_i}$. But $x \in W$ implies $x \in W_{y_i}$, hence $(x, y) \in W_{y_i} \times V_{y_i}$. But $W_z \times V_z \subset U$ for all $z \in Y$, hence $(x, y) \in U$. □

THEOREM 4.22. *If X and Y are compact spaces, then $X \times Y$ is compact.*

PROOF. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $x \in X$,

$$\{x\} \times Y \subset X \times Y$$

is compact (why?), hence we find a $\mathcal{U}_x \subset \mathcal{U}$ finite such that

$$\{x\} \times Y \subset \bigcup_{U \in \mathcal{U}_x} U. \tag{2.1}$$

Using the previous lemma, we find W_x open containing x such that

$$W_x \times Y \subset \bigcup_{U \in \mathcal{U}_x} U. \tag{2.2}$$

Now, $\{W_x : x \in X\}$ is an open cover of X, hence we find a finite subcover

$$X = W_{x_1} \cup \ldots \cup W_{x_p}.$$

Then

$$\mathcal{V} = \mathcal{U}_{x_1} \cup \ldots \cup \mathcal{U}_{x_p}$$

is finite union of finite collections, hence finite. Moreover, $\mathcal{V}$ still covers $X \times Y$: given (x, y) arbitrary, using (2.2), we find i such that $x \in W_{x_i}$. Hence $(x, y) \in W_{x_i} \times Y$, and using (2.1) we find $U \in \mathcal{U}_{x_i}$ such that $(x, y) \in U$. Hence we found $U \in \mathcal{V}$ such that $(x, y) \in U$, □

COROLLARY 4.23. *A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

PROOF. The direct implication was already mentioned in Example 4.16 (and that A is closed follows also from Theorem 4.18). For the converse, since A is bounded, we find $R, r \in \mathbb{R}$ such that $A \subset [r, R]^n$. The intervals $[r, R]$ are homeomorphic to $[0, 1]$, hence compact. The previous theorem implies that $[r, R]^n$, hence A must be compact as closed inside a compact. □

EXAMPLE 4.24. In particular, spaces like the spheres S^n, the closed disks D^n, the Moebius band, the torus, etc, are compact.

2.4. Compactness and continuous functions.

THEOREM 4.25. *If $f : X \to Y$ is a continuous function and $A \subset X$ is compact, then $f(A) \subset Y$ is compact.*

PROOF. If $\mathcal{U}$ is an open cover of $f(A)$ in Y, then $f^{-1}(\mathcal{U}) := \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of A in X, hence it has a finite subcover $\{f^{-1}(U_i) : 1 \leq i \leq n\}$ with $U_i \in \mathcal{U}$. But then $\{U_i\}$ will be a finite subcover of $\mathcal{U}$. □

Finally, we can state the property of compact spaces that we referred to several times when having to prove that certain continuous injections are homeomorphisms.

THEOREM 4.26. *If $f : X \to Y$ is continuous and bijective, and if X is compact and Y is Hausdorff, then f is a homeomorphism.*

PROOF. We have to show that the inverse g of f is continuous. For this we show that if U is open in X then the pre-image $g^{-1}(U)$ is open in Y. Since the open (or closed) sets are just the complements of closed (respectively open) subsets, and $g^{-1}(Y - U) = X - g^{-1}(U)$, we see that the continuity of g is equivalent to: if A is closed in X then the pre-image $g^{-1}(A)$ is closed in Y. To prove this, let A be a closed subset of X. Since g is the inverse of f, we have $g^{-1}(A) = f(A)$, hence we have to show that $f(A)$ is closed in Y. Since X is compact, Proposition 4.17 implies that A is compact. By the previous theorem, $f(A)$ must be compact. Since Y is Hausdorff, Theorem 4.18 implies that $f(A)$ is closed in Y. □

COROLLARY 4.27. *If $f : X \to Y$ is a continuous injection of a compact space into a Hausdorff one, then f is an embedding.*

EXAMPLE 4.28. Here is an example which shows the use of compactness. We will show that there is no injective continuous map $f : S^1 \to \mathbb{R}$.

PROOF. Assume there is such a map. Since S^1 is connected and compact, its image is a closed interval $[m, M]$; f becomes a continuous bijection $f : S^1 \to [m, M]$, hence a homeomorphism (cf. the previous theorem). Then use the "removing a point trick". □

EXAMPLE 4.29. *(back to the torus, Moebius band, etc)* In the previous chapter we produced several continuous injective maps without being able to give a simple proof of the fact that they are embeddings: when embedding the abstract torus into $\mathbb{R}^n$ (Example 3.5 in subsection 2), when realizing S^1 as $\mathbb{R}/\mathbb{Z}$ (Example 3.7), or in our examples of cones and suspensions discussed in Example 3.15 (all the references are to the previous chapter). In all these cases, the previous theorem and its corollary immediately complete the proofs.

For clarity, let's now give a final overview of our discussions on the torus. First, in Chapter 1, Section 6, we introduced the torus intuitively, by gluing the opposite sides of a square. The result was a subspace of $\mathbb{R}^3$ (or rather a shape). After the definition of topological spaces, we learned that these subspaces of $\mathbb{R}^3$ are topological spaces on their own- endowed with the induced topology. In the previous chapter, in section 2, we gave a precise meaning to the process of gluing and introduced the abstract torus T_{abs}, endowed with the quotient topology. In the same example we also produced one (of the many possible) continuous injections

$$f : T_{\text{abs}} \to \mathbb{R}^3$$

whose image was one of the explicit models $T_{R,r} \subset \mathbb{R}^3$ of the torus, described already in the first chapter. Hence the previous corollary implies that f defines a homeomorphisms between the abstract T_{abs} and the explicit model $T_{R,r}$.

EXERCISE 4.8. Have a similar discussion for the Moebius band, Klein bottle, etc.

EXERCISE 4.9. Prove that the torus is homeomorphic to $S^1 \times S^1$.

2.5. Embeddings of compact manifolds.

THEOREM 4.30. *Any n-dimensional compact topological manifold can be embedded in $\mathbb{R}^N$, for some integer N.*

PROOF. We use the Euclidean distance in $\mathbb{R}^n$ and we denote by B_r and $\overline{B}_r$ the resulting open and closed balls of radius r centered at the origin. We choose a function

$$\eta : \mathbb{R}^n \to [0, 1] \text{ such that } \eta|_{B_1} = 1, \quad \eta|_{\mathbb{R}^n - B_2} = 0.$$

For instance, we could choose

$$\eta(x) = \frac{d(x, \mathbb{R}^n - B_2)}{d(x, \overline{B}_1) + d(x, \mathbb{R}^n - B_2)}.$$

For a coordinate chart

$$\chi : U \to \mathbb{R}^n$$

and any radius $r > 0$, we consider:

$$U(r) := \chi^{-1}(B_r), \quad U[r] = \overline{U(r)} = \chi^{-1}(\overline{B}_r).$$

Since X is compact, we find a finite number of coordinate charts

$$\chi_i : U_i \to \mathbb{R}^n, \quad 1 \leq i \leq k,$$

such that $\{U_i(1) : 1 \leq i \leq k\}$ cover X. For each i, consider $\eta \circ \chi_i : U_i \to [0, 1]$; since it vanishes on $U_i - U_i(2)$, extending it to be 0 outside U_i will give us a continuous map

$$\eta_i : X \to [0, 1].$$

Similarly, since the product $\eta_i \cdot \chi_i : U_i \to \mathbb{R}^n$ vanishes on $U_i - U_i(2)$, extending it by 0 gives us continuous maps

$$\tilde{\chi}_i : X \to \mathbb{R}^n.$$

Finally, we define

$$f = (\eta_1, \ldots, \eta_k, \tilde{\chi}_1, \ldots, \tilde{\chi}_k) : X \to \mathbb{R}^{(1+k)n}.$$

It is continuous by construction hence, by Theorem 4.26, it suffices to show that f is injective. Assume that $f(x) = f(y)$ with $x, y \in X$. From the choice of the charts, we find i such that $x \in U_i(1)$. Then $\eta_i(x) = 1$. But $f(x) = f(y)$ implies that $\eta_i(y) = \eta_i(x) = 1$. On one hand, this implies that $\eta_i(y) \neq 0$, hence y must be inside U_i (even inside $U_i(2)$). But these imply

$$\tilde{\chi}_i(x) = \eta_i(x)\chi_i(x) = \chi_i(x)$$

and similarly for y. Finally, $f(x) = f(y)$ also implies that $\tilde{\chi}_i(x) = \tilde{\chi}_i(y)$. Hence x and y are in the domain of χ_i and are send by χ_i into the same point. Hence $x = y$. □

COROLLARY 4.31. *Any n-dimensional compact topological manifold is metrizable.*

2.6. Sequential compactness. When one deals with sequences, one often sees statements of type "we now consider a subsequence with this property". Compactness is related to the existence of convergent subsequences. In general, given a topological space $(X, \mathcal{T})$, one says that X is sequentially compact if any sequence $(x_n)_{n\geq 1}$ of elements of X has a convergent subsequence. Recall that a subsequence of $(x_n)_{n\geq 1}$ is a sequence $(y_k)_{k\geq 1}$ of type

$$y_k = x_{n_k}, \text{ with } n_1 < n_2 < n_3 < \dots .$$

However, we have already mentioned (and seen in various other cases) that topological properties involving sequences usually require the axiom of first countability.

THEOREM 4.32. *Any first countable compact space is sequentially compact.*

PROOF. Let X be first countable and compact, and assume that $(x_n)_{n\geq 1}$ is an arbitrary sequence in X. For each integer $n \geq 1$ we put

$$U_n = X - \overline{\{x_n, x_{n+1}, \dots\}}.$$

Note that these define an increasing sequence of open subsets of X:

$$U_1 \subset U_2 \subset U_3 \subset \dots .$$

We now claim that $\cup_n U_n \neq X$. If this is not the case, $\{U_n : n \geq 1\}$ is an open cover of X hence we find a finite set F such that $\{U_i : i \in F\}$ covers X. Since our cover is increasing, we find that $U_p = X$ where $p = \max F$, and this is clearly impossible. In conclusion, $\cup_n U_n \neq X$. Hence there exists $x \in X$ such that, for all $n \geq 1$, $x \notin U_n$. Choose a countable basis of neighborhoods of x

$$V_1, V_2, V_3, \dots .$$

Since $x \notin U_1$, we have $V \cap \{x_1, x_2, \dots\} \neq \emptyset$ for all neighborhoods V of x. Choosing $V = V_1$, we find n_1 such that

$$x_{n_1} \in V_1.$$

Next, we use the fact that $x \notin U_n$ for $n = n_1 + 1$. This means that $V \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset$ for all neighborhoods V of x. Choosing $V = V_2$, we find $n_2 > n_1$ such that

$$x_{n_2} \in V_2.$$

We continue this process inductively (e.g. the next step uses $x \notin U_n$ for $n = n_1 + n_2 + 1$) and we find $n_1 < n_2 < \dots$ such that

$$x_{n_k} \in V_k$$

for all $k \geq 1$. Then $(x_{n_k})_{k\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$ converging to x. □

COROLLARY 4.33. *Any compact metrizable space is sequentially compact.*

Actually, as we shall see in the next subsection, for metric spaces compactness is equivalent to sequential compactness.

3. Local compactness and the one-point compactification

3.1. Local compactness.

DEFINITION 4.34. *A topological space X is called locally compact if any point of X admits a compact neighborhood.*

We will be mainly interested in locally compact spaces which are Hausdorff.

EXERCISE 4.10. Prove that, in a locally compact Hausdorff space X, for each $x \in X$ the collection of all compact neighborhoods of x is a basis of neighborhoods of x (i.e. for any open neighborhood U of x, there exists a compact neighborhood N of x such that $N \subset U$).

EXAMPLE 4.35.

1. Any compact Hausdorff space $(X, \mathcal{T})$ is locally compact Hausdorff.
2. $\mathbb{R}^n$ is locally compact (use closed balls as compact neighborhoods).
3. any open $U \subset \mathbb{R}^n$ is locally compact (use small enough closed balls). In general, any open subset of a locally compact Hausdorff space is locally compact (use the previous exercise).
4. any closed $A \subset \mathbb{R}^n$ is locally compact. Indeed, for any $a \in A$, $B[a, 1] \cap A$ is a neighborhood of a (in A) which is compact (use again that closed inside compact is compact). Similarly, a closed subset of a locally compact Hausdorff space is locally compact.
5. the interval $(0, 1]$ is locally compact (combine the arguments from (2) and (3)).
6. $\mathbb{Q}$ is not locally compact. To show this, assume that 0 has a compact neighborhood N. Then $(-\epsilon, \epsilon) \cap \mathbb{Q} \subset N$, for some $\epsilon > 0$. Passing to closures in $\mathbb{R}$ we find $[-\epsilon, \epsilon] \subset \overline{N} = N$, where we used that N is compact (hence closed). This contradicts $N \subset \mathbb{Q}$.

Locally compact Hausdorff spaces which are 2nd countable deserve special attention: they include topological manifolds and, as we shall see later on, they are easier to handle. The most basic property (to be used several times) is that they can be "exhausted" by compact spaces.

DEFINITION 4.36. *Let $(X, \mathcal{T})$ be a topological space. An exhaustion of X is a family $\{K_n : n \in \mathbb{Z}_+\}$ of compact subsets of X such that $X = \cup_n K_n$ and $K_n \subset \mathring{K}_{n+1}$ for all n.*

THEOREM 4.37. *Any locally compact, Hausdorff, 2nd countable space admits an exhaustion.*

PROOF. Let $\mathcal{B}$ be a countable basis and consider $\mathcal{V} = \{B \in \mathcal{B} : \overline{B} - \text{compact}\}$. Then $\mathcal{V}$ is a basis: for any open U and $x \in X$ we choose a compact neighborhood N inside U; since $\mathcal{B}$ is a basis, we find $B \in \mathcal{B}$ s.t. $x \in B \subset N$; this implies $\overline{B} \subset N$ and then $\overline{B}$ must be compact; hence we found $B \in \mathcal{V}$ s.t. $x \in B \subset U$. In conclusion, we may assume that we have a basis $\mathcal{V} = \{V_n : n \in \mathbb{Z}_+\}$ where $\overline{V}_n$ is compact for each n. We define the exhaustion $\{K_n\}$ inductively, as follows. We put $K_1 = \overline{V}_1$. Since $\mathcal{V}$ covers the compact K_1, we find i_1 such that

$$K_1 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_1}.$$

Denoting by D_1 the right hand side of the inclusion above, we put

$$K_2 = \overline{D}_1 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_1}.$$

This is compact because it is a finite union of compacts. Since $D_1 \subset K_2$ and D_1 is open, we must have $D_1 \subset \mathring{K}_2$; since $K_1 \subset D_1$, we have $K_1 \subset \mathring{K}_2$. Next, we choose $i_2 > i_1$ such that

$$K_2 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_2},$$

we denote by D_2 the right hand side of this inclusion, and we put

$$K_3 = \overline{D}_2 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_2}.$$

As before, K_3 is compact, its interior contains D_2, hence also K_2. Continuing this process, we construct the family K_n, which clearly covers X. □

3.2. The one-point compactification. Intuitively, the idea of the one-point compactification of a space is to "add a point at infinity" to achieve compactness.

DEFINITION 4.38. *Let $(X, \mathcal{T})$ be a topological space. A* <u>*one-point compactification*</u> *of X is a compact Hausdorff space $(\tilde{X}, \tilde{\mathcal{T}})$ together with an embedding $i : X \to \tilde{X}$, with the property that $\tilde{X} - X$ consists of one point only.*

From the remarks above it follows that, if X admits a one-point compactification, then it must be locally compact and Hausdorff. Conversely, we have:

THEOREM 4.39. *If X is a locally compact Hausdorff space, then*
1. *It admits a one-point compactification X^+.*
2. *Any two one-point compactifications of X are homeomorphic.*

Moreover, if X is 2nd countable, then so is X^+.

EXAMPLE 4.40.
1. If $X = (0, 1]$, then X^+ is (homeomorphic to) $[0, 1]$. Indeed, $\tilde{X} = [0, 1]$ is compact, and the inclusion $i : (0, 1] \to [0, 1]$ satisfies the properties from the previous proposition.
2. If $X = (0, 1)$, then X^+ is (homeomorphic to) the circle S^1. Indeed, $i : (0, 1) \to S^1$ as in Figure 3 (e.g. $i(t) = (cos(2\pi t), sin(2\pi t))$ has the properties from the previous proposition.

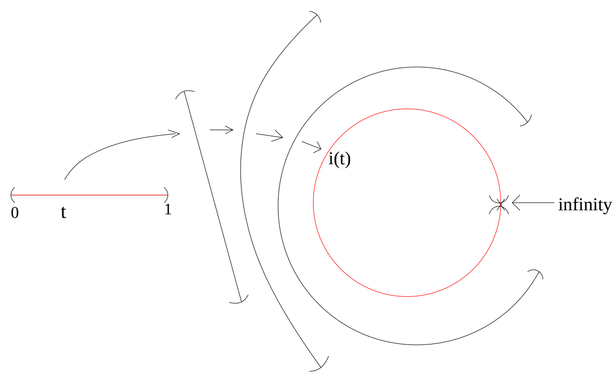


FIGURE 3.

3. If $X = [-1, 0) \cup (1, 2) \subset \mathbb{R}$, X^+ is shown in Figure 4.

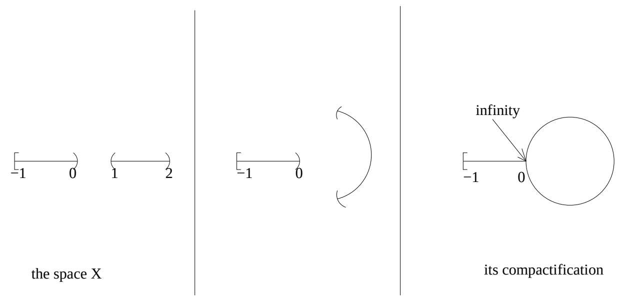


FIGURE 4.

4. If $X = \overset{\circ}{D}{}^n$ the one-point compactification is S^n.

PROOF. (of Theorem 4.39) For the existence, choose a symbol $\infty \notin X$ and consider

$$X^+ = X \cup \{\infty\}.$$

Since $X \subset X^+$, any subset of X is a subset of X^+. We consider the family of subsets of X^+:

$$\mathcal{T}^+ = \mathcal{T} \cup \mathcal{T}(\infty), \quad \text{where} \quad \mathcal{T}(\infty) = \{X^+ - K : K \subset X, K - \text{compact}\}.$$

We claim that $\mathcal{T}^+$ is a topology on X^+. First, we show that $U \cap V \in \mathcal{T}^+$ whenever $U, V \in \mathcal{T}^+$. We have three cases. If $U, V \in \mathcal{T}$, we know that $U \cap V \in \mathcal{T}$. If U and V are both in $\mathcal{T}(\infty)$, then so is their intersection because union of two compacts is compact (show this!). Finally, if $U \in \mathcal{T}$ and $V = X^+ - K \in \mathcal{T}(\infty)$, then $U \cap V = U \cap (X - K)$ is open in X because V and $X - K$ are. Next, we show that arbitrary union of sets from $\mathcal{T}^+$ is in $\mathcal{T}^+$. This property holds for $\mathcal{T}$, and also for $\mathcal{T}(\infty)$ since intersection of compacts is compact (why?). Hence it suffices to show that $U \cup V \in \mathcal{T}^+$ whenever $U \in \mathcal{T}$, $V \in \mathcal{T}(\infty)$. Writing $V = X^+ - K$ with $K \subset X$ compact, we have

$$U \cup V = X^+ - K',$$

where $K' = K \cap (X - U)$. Since $K - K' = K \cap U$, $K - K'$ is open in K, i.e. K' is closed in the compact K, hence K' is compact (Proposition 4.17 again!). Hence $U \cup V \in \mathcal{T}^+$.

We show that $(X^+, \mathcal{T}^+)$ is compact. Let $\mathcal{U}$ be an open cover of X^+. Choose $U = X^+ - K \in \mathcal{U}$ containing ∞ and let $\mathcal{U}' = \{V \cap X : V \in \mathcal{U}, V \neq U\}$. Then $\mathcal{U}'$ is an open cover of the compact K in X^+. Choosing $\mathcal{V} \subset \mathcal{U}'$ finite which covers K, $\mathcal{V} \cup \{U\} \subset \mathcal{U}$ is finite and covers X^+.

Next, we show that X^+ is Hausdorff. So, let $x, y \in X^+$ distinct, and we are looking for $U, V \in \mathcal{T}^+$ such that $U \cap V = \emptyset$, $x \in U, y \in V$. When $x, y \in X$, just use the Hausdorffness of X. So, let's assume $y = \infty$. Then choose a compact neighborhood K of x and we consider $U \subset X$ open such that $x \in U \subset K$. Then $x \in U$, $\infty \in X - K$ and $U \cap (X^+ - K) = \emptyset$.

Next, we show that the inclusion $i : X \to X^+$ is an embedding, i.e. that $\mathcal{T}^+|_X = \mathcal{T}$. Now,

$$\mathcal{T}^+|_X = \mathcal{T} \cup \{U \cap X : U \in \mathcal{T}(\infty)\}$$

(just apply the definition!), and just remark that, for $U = X^+ - K \in \mathcal{T}(\infty)$, $U \cap X = X - K \in \mathcal{T}$.

This concludes the proof of 1. For 2, let $\tilde{X}$ be another one-point compactification and we prove that it is homeomorphic to X. Choose $y_\infty \in \tilde{X}$ such that $\tilde{X} = i(X) \cup \{y_\infty\}$ and define

$$f : \tilde{X} \to X^+, \ f(y) = \begin{cases} x & \text{if } y = i(x) \in i(X) \\ \infty & \text{if } y = y_\infty \end{cases}$$

Since f is bijective, $\tilde{X}$ is compact and X^+ is Hausdorff, it suffices to show that f is continuous (Theorem 4.26). Let $U \in \mathcal{T}^+$; we prove $f^{-1}(U) \in \tilde{\mathcal{T}}$. If $U \in \mathcal{T}$, then $f^{-1}(U) \subset i(X)$ and then

$$f^{-1}(U) = \{y = i(x) : f(y) \in U\} = \{i(x) : x \in U\} = i(U)$$

is open in $i(X)$ since i is an embedding. But, since $\tilde{X}$ is Hausdorff, $i(X) = \tilde{X} - \{y_\infty\}$ is open in $\tilde{X}$, hence so is $f^{-1}(U)$. The other case is when $U = X^+ - K$ with $K \subset X$ compact. Then

$$f^{-1}(U) = f^{-1}(y_\infty) \cup f^{-1}(X - K) = \{\infty\} \cup (i(X) - i(K)) = X^+ - i(K)$$

is again open in X^+ ($i(K)$ is compact as the image of a compact by a continuous function).

Finally, we prove the last part of the theorem. Let $\{K_n : n \in \mathbb{Z}_+\}$ be an exhaustion of X, $\mathcal{B}$ a countable basis of X and we claim that the following is a basis of X^+:

$$\mathcal{B}^+ := \mathcal{B} \cup \mathcal{B}(\infty), \text{ where } \mathcal{B}(\infty) = \{X^+ - K_n : n \in \mathbb{Z}_+\}.$$

To show: for any $U \in \mathcal{T}^+$ and any $x \in U$, there exists $B \in \mathcal{B}^+$ such that $x \in B \subset U$. If $U \in \mathcal{T}$, just use that $\mathcal{B}$ is a basis. Similarly, if $U = X^+ - K \in \mathcal{T}(\infty)$, the interesting case is when $x = \infty$. Then we look for $B = X - K_n$ such that $B \subset U$ (i.e. $K \subset K_n$). But $\{\overset{\circ}{K}_n\}$ is an open cover of X, hence also of K; since K is compact and $K_n \subset K_{n+1}$, we find n such that $K \subset \overset{\circ}{K}_n$. □

4. More exercises

4.1. Connectedness.

EXERCISE 4.11. In Exercise 1.4 you showed that four of the spaces in the picture are homeomorphic to each other, but they did not seem to be homeomorphic to the fifth one. You now have the tool to prove the last assertion (so do it!).

EXERCISE 4.12. Prove that the following spaces are not homeomorphic:

1. S^1 and $[0, 1)$.
2. $[0, 1)$ and $\mathbb{R}$.
3. S^1 and S^2.
4. S^1 and a bouquet of two circles (the space from Figure 5).

EXERCISE 4.13. Show that there exists a real number $r \in (2, 3)$ such that

$$r^7 - 2r^4 - r^2 - 2r = 2011.$$

EXERCISE 4.14. Show that any continuous fuction $f : [0, 1] \to [0, 1]$ admits at least one fixed point (i.e. there exists $t \in [0, 1]$ such that $f(t) = t$).
(Hint: $g(t) = f(t) - t$ positive or negative?)

EXERCISE 4.15. Assuming that the temperature on the surface of the earth is a continuous function, prove that, at any moment in time, on any great circle of the earth, there are two antipodal points with the same temperature.

EXERCISE 4.16. Assuming that you know that $S^1 \times (0, 1)$ is not homeomorphic to $\mathbb{R}^2$, show that the sphere S^2 is not homeomorphic to the space obtained from S^2 by gluing two antipodal points.

EXERCISE 4.17. Is $(\mathbb{R}, \mathcal{T}_l)$ from Exercise 2.19 connected? But $[0, 1)$ with the induced topology? But $(0, 1]$?

EXERCISE 4.18. Recall that by a circle we mean any topological space which is homeomorphic to S^1, and by a circle embedded in a topological space X we mean any subset $A \subset X$ which, when endowed with the induced topology, is a circle. Similarly, by a bouquet of two circles we mean any space which is homeomorphic to the space drawn in Figure 5, and we talk about embedded bouquets of circles. Let T be a a torus.

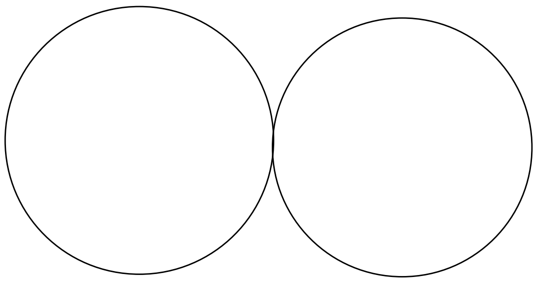


FIGURE 5.

1. Describe a circle C embedded in T such that the complement of $T - C$ is connected.
2. Describe a bouquet of two circle B embedded in T such that $T - B$ is connected.

EXERCISE 4.19. Show that the group $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices with real coefficients (see Exercise 2.33) is not connected.

EXERCISE 4.20. Show that any two of the three spaces drawn in Figure 6 are not homeomorphic.

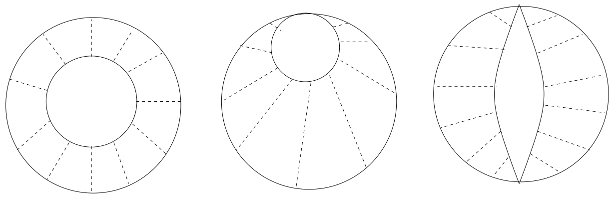


FIGURE 6.

EXERCISE 4.21. Prove that if $X \subset \mathbb{R}^n$ is connected then its closure is also connected but its interior may fail to be connected.

EXERCISE 4.22. Show that the cone of any space is connected.

EXERCISE 4.23. For a topological space $(X, \mathcal{T})$, show that the following are equivalent:
(1) X is connected.
(2) $\emptyset$ and X are the only subsets of X which are both open and closed.
(3) the only continuous functions $f : X \to \{0,1\}$ are the constant ones (where $\{0,1\}$ is endowed with the discrete topology).

EXERCISE 4.24. Prove that if X and Y are homeomorphic, then they have the same number of connected components.

4.2. Compactness.

EXERCISE 4.25. Assume that X is a topological space and $(x_n)_{n\geq 1}$ is a sequence in X, convergent to $x \in X$. Show that

$$A = \{x, x_1, x_2, x_3, \ldots\}$$

is compact.

EXERCISE 4.26. Let $\mathcal{T}_l$ be the topology from Exercise 2.19. With the topology induced from $(\mathbb{R}, \mathcal{T}_l)$ is $[0,1)$ compact? But $[0,1]$?

EXERCISE 4.27. Let $(X, \mathcal{T})$ be a Hausdorff space, $A, B \subset X$. If A and B are compact, show that $A \cap B$ is compact. Is the Hausdorffness assumption on X essential?

EXERCISE 4.28. Let X be a topological space and $A, B \subset X$. If A and B are compact, show that $A \cup B$ is compact.

EXERCISE 4.29. Given a set X, when is $(X, \mathcal{T}_{\text{cf}})$ compact?

EXERCISE 4.30. Show that the sequence $x_n = 2011^{sin(5n)}$ (in $\mathbb{R}$ with the Euclidean topology) has a convergent subsequence.

EXERCISE 4.31. Recall that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$Gr(f) = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

If f is bounded, show that f is continuous if and only if $Gr(f)$ is closed in $\mathbb{R}^2$. What if f is not bounded.
(hint: for the first part, use sequences; for the last part: $1/x$).

Exercise 4.32. Find a topological space $(X, \mathcal{T})$ which is compact but is not Hausdorff. In the example that you found, exhibit a compact set $A \subset X$ such that A is not closed.

Exercise 4.33. Find an example which shows that the Tube Lemma fails if Y is not compact.

Exercise 4.34. Let $(X, \mathcal{T})$ be a second countable topological space. Show that X is compact if and only if for any decreasing sequence of nonempty closed subsets

$$\ldots \subset F_3 \subset F_2 \subset F_1 \subset X,$$

one has $\cap_{n=1}^{\infty} F_n \neq \emptyset$.

(Hint: restate the property in terms of opens; then try to use Exercise 2.63).

Exercise 4.35. If X is a Hausdorff space, and $A \subset X$ is compact, show that the quotient X/A (obtained from X by collapsing A to a point) is Hausdorff.

Exercise 4.36. Show that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$ (see Exercise 1.22 in the first chapter). Then do again Exercise 3.3 from Chapter 3.

Exercise 4.37. Show that the cone and the suspension of any compact topological space are compact.

Exercise 4.38. If X is connected and compact and $f : X \to \mathbb{R}$ is continuous, show that $f(X) = [m, M]$ for some $m, M \in \mathbb{R}$.

Exercise 4.39.

(i) Is it true that any continuous surjective map $f : S^1 \to S^1$ is a homeomorphism?
(ii) Show that S^1 cannot be embedded in $\mathbb{R}$.
(iii) Show that any continuous injective map $f : S^1 \to S^1$ is a homeomorphism.

4.3. Local compactness and the one-point compactification.

Exercise 4.40. What is the one-point compactification of $X = (0, 1) \cup (2, 3)$?

Exercise 4.41. What is the one-point compactification of $\mathbb{R}^2$?

Exercise 4.42. Show that the following subspace of $\mathbb{R}^2$ is locally compact and find its one-point compactification:

$$X = S^1 \cup ((0, 2) \times \{0\}) \subset \mathbb{R}^2.$$

Exercise 4.43. Show that, for any circle C embedded in the Klein bottle K, $K - C$ is locally compact. Then describe such a circle C such that $(K - C)^+$ is homeomorphic to $\mathbb{P}^2$.

Exercise 4.44. Show that there exists a Hausdorff space X and an embedded circle C in X, such that the one-point compactification of $X - C$ is homeomorphic to X.

Exercise 4.45. Consider

$$X = [0, 1] \times [0, 1)$$

with the topology induced from $\mathbb{R}^2$. Prove that X is a locally compact Hausdorff space and describe its one-point compactification (use a picture). What happens if we replace X by $Y = X \cup \{(1, 1)\}$?

Exercise 4.46. For any continuous map $f : S^1 \to T^2$ we define

$$X_f := T^2 - f(S^1).$$

(i) Is it true that, for any continuous function f, X_f is compact? But locally compact? But metrizable? But connected?
(ii) Describe two embeddings $f_1, f_2 : S^1 \to T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic.

(iii) Describe the one-point compactifications $X_{f_1}^+$ and $X_{f_2}^+$.
(iv) Describe $f : S^1 \to T^2$ continuous such that X_f^+ is homeomorphic to S^2.

EXERCISE 4.47. Describe an embedding of the cylinder $S^1 \times [0, 1]$ into the space X, show that its complement is locally compact, and find the one-point compactification of the complement, in each of the cases:

(i) X is a torus.
(ii) X is the plane $\mathbb{R}^2$.

EXERCISE 4.48. Let X be a connected, locally compact, Hausdorff space. Show that X is compact if and only if X^+ is not connected.

EXERCISE 4.49. Let X be a Hausdorff compact space and $A \subset X$ be a closed subset. Show that

1. $X - A$ is locally compact.
2. X/A is compact and Hausdorff.
3. X/A is (homeomorphic to) the one-point compactification of $X - A$.