CHAPTER 5

Partitions of unity

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1. Some axioms for sets of functions

The theory of "partitions of unity" is the most important tool that allows one to pass "from local to global". As such, it is widely used in many fields of mathematics, most notably in many branches of Geometry and Analysis. The word "unity" stands for the constant function equal to 1, on some given space X. A "partition of unity" is a decomposition

$$\sum_i \eta_i = 1$$

of the constant function into a sum of continuous functions η_i. One is interested in such partitions of unity with the extra-requirement that each η_i is "concentrated in a given (usually very small) open U_i". The U_i's form a (given) open cover of X and one is interested in the existence of partitions of unity "subordinated" to the cover.

Let us also mention that, when it comes to applications to Geometry and Analysis, one deals with topological spaces that have extra-structure and the "partitions of unity" are required to be more than continuous (in most cases one can talk about differentiable functions, and the partitions are required to be so). Ironically, the existence of such "special" partitions of unity is easier to establish than the existence of the continuous partitions for general topological spaces. To include such applications, we will include in our discussion a given set $\mathcal{A}$ of continuous functions. To specify the axioms for $\mathcal{A}$, we consider the space of continuous functions on X:

$$\mathcal{C}(X) = \mathcal{C}(X, \mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}.$$

We use some of the structure present on $\mathcal{C}(X)$. First, we can take sums of continuous functions:

$$(f + g)(x) = f(x) + g(x).$$

Secondly, we can take quotients f/g, whenever g is nowhere vanishing:

$$\frac{f}{g}(x) := \frac{f(x)}{g(x)},$$

DEFINITION 5.1. *Given a topological space X, we say that a subset $\mathcal{A} \subset \mathcal{C}(X)$:*

- *<u>is closed under finite sums</u> if $f + g \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$.*
- *<u>is closed under quotients</u> if $f/g \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$ and g is nowhere vanishing.*

There are more operations that we can perform on $\mathcal{C}(X)$- multiplication by real numbers, or multiplication of continuous functions; in examples, $\mathcal{A}$ is usually closed under all these operations. However, the most important condition on $\mathcal{A}$ is the following topological one:

DEFINITION 5.2. *Given a topological space X and $\mathcal{A} \subset \mathcal{C}(X)$, we say that $\mathcal{A}$ is <u>normal</u> if for any two closed disjoint subsets $A, B \subset X$, there exists $f : X \to [0, 1]$ which belongs to $\mathcal{A}$ and such that $f|_A = 0$, $f|_B = 1$.*

As we remarked in Section 6 of Chapter 2, the existence of such continuous functions implies that X must be normal: any two closed disjoint subsets $A, B \subset X$ can be separated topologically. In what follows we will repeatedly make use of the following:

LEMMA 5.3. *In a normal space X, if $A \subset U \subset X$ with A-closed and U-open in X, then there exists an open V in X such that $A \subset V \subset \overline{V} \subset U$.*

PROOF. Since $A \subset U$, A and $X - U$ are disjoint. They are both closed, hence we know that we can find disjoint opens W and V such that $A \subset V$, $X - U \subset W$. The condition $V \cap W = \emptyset$ is equivalent to $V \subset X - W$. Since $X - W$ is a closed containing V, this implies $\overline{V} \subset X - W$. On the other hand, $X - U \subset W$ can be re-written as $X - W \subset U$. Hence $\overline{V} \subset X - W \subset U$. $\square$

2. Finite partitions of unity

In this section we give a precise meaning to the statement that a continuous function $\eta : X \to \mathbb{R}$ is "concentrated" in an open $U \subset X$. We will use the notation:

$$\{f \neq 0\} := \{x \in X : f(x) \neq 0\}.$$

DEFINITION 5.4. *Given a topological space X and $\eta : X \to \mathbb{R}$, define the support of η as*

$$supp(\eta) := \overline{\{f \neq 0\}} \subset X.$$

We say that η is supported in an open U if $supp(\eta) \subset U$.

It is important that the support is defined as the closure of $\{f \neq 0\}$. This condition allows us to perform "globalization", as the following exercise indicates.

EXERCISE 5.1. Let $(X, \mathcal{T})$ be a topological space, $U \subset X$ open and $\eta \in \mathcal{C}(X)$ supported in U. Then, for any continuous map $g : U \to \mathbb{R}$,

$$(\eta \cdot g) : X \to \mathbb{R},\ (\eta \cdot g)(x) = \begin{cases} \eta(x)g(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}.$$

is continuous. Show that this statement fails if we only assume that $\{f \neq 0\} \subset U$.

Next we discuss finite partitions of unity.

DEFINITION 5.5. *Let X be a topological space, $\mathcal{U} = \{U_1, \ldots, U_n\}$ a finite open cover of X. A partition of unity subordinated to $\mathcal{U}$ is a family of functions $\eta_i : X \to [0,1]$ satisfying:*

$$\eta_1 + \ldots + \eta_n = 1, \quad supp(\eta_i) \subset U_i.$$

Given $\mathcal{A} \subset \mathcal{C}(X)$, we say that $\{\eta_i\}$ is an $\mathcal{A}$-partition of unity if $\eta_i \in \mathcal{A}$ for all i.

EXERCISE 5.2. Show that, given $\mathcal{A} \subset \mathcal{C}(X)$, the following are equivalent:

1. any 2-open cover $\mathcal{U} = \{U_1, U_2\}$ admits an $\mathcal{A}$-partition of unity subordinated to it.
2. $\mathcal{A}$ separates the closed subsets of X.

THEOREM 5.6. *Let X be a topological space and assume that $\mathcal{A} \subset \mathcal{C}(X)$ is normal and closed under finite sums and quotients.*

Then, for any finite open cover $\mathcal{U}$, there exists an $\mathcal{A}$-partition of unity subordinated to $\mathcal{U}$.

PROOF. The main topological ingredient in the proof is the following "shrinking lemma'.'

LEMMA 5.7. *(the finite shrinking lemma) For any finite open covering $\mathcal{U} = \{U_i : 1 \leq i \leq n\}$ of a normal space X, there exists a covering $\mathcal{V} = \{V_i : 1 \leq i \leq n\}$ such that*

$$\overline{V}_i \subset U_i, \quad \forall\ i = 1, \ldots, n.$$

PROOF. Let

$$A = X - (U_2 \cup \ldots \cup U_n), D = U_1.$$

Then A is closed, D is open, and $A \subset D$. By Lemma 5.3 from the end of Chapter 2, we find V_1 open such that

$$A \subset V_1 \subset \overline{V}_1 \subset D (= U_1).$$

This means that

$$\{V_1, U_2, \ldots, U_n\}$$

is a new open cover of X with $\overline{V}_1 \subset U_1$. In other words, we have managed to "refine U_1". Applying the same argument to this new cover (to refine U_2), we find a new open cover

$$\{V_1, V_2, U_3, \ldots, U_n\}$$

with $\overline{V}_1 \subset U_1$, $\overline{V}_2 \subset U_2$. Continuing this argument, we obtain the desired open cover $\mathcal{V}$. □

We now prove the theorem. Let $\mathcal{U} = \{U_i\}$ be the given finite open cover. Apply the previous lemma twice and choose open covers $\mathcal{V} = \{V_i\}$, $\mathcal{W} = \{W_i\}$, with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. For each i, we use the separation property of $\mathcal{A}$ for the disjoint closed sets $(\overline{W}_i, X - V_i)$. We find $f_i : X \to [0,1]$ that belongs to $\mathcal{A}$, with $f_i = 1$ on $\overline{W}_i$ and $f_i = 0$ outside V_i. Note that

$$f := f_1 + \ldots + f_n$$

is nowhere zero. Indeed, if $f(x) = 0$, we must have $f_i(x) = 0$ for all i, hence, for all i, $x \notin W_i$. But this contradicts the fact that $\mathcal{W}$ is a cover of X. From the properties of $\mathcal{A}$, each

$$\eta_i := \frac{f_i}{f_1 + \ldots + f_n} : X \to [0,1]$$

is continuous. Clearly, their sum is 1. Finally, $\text{supp}(\eta_i) \subset U_i$ because $\overline{V}_i \subset U_i$ and$\{x : \eta_i(x) \neq 0\} = \{x : f_i(x) \neq 0\} \subset V_i$. □

3. Arbitrary partitions of unity

For arbitrary partitions of unity one has to deal with infinite sums $\sum_i f_i$ of continuous functions on X (indexed by some infinite set I). In such cases it is natural to require that, for each $x \in X$, the sum $\sum_i f_i(x)$ is finite (i.e. $f_i(x) = 0$ for all but a finite number of i's). Although the sum is then well defined as a function on X, to retain continuity, a slightly stronger notion is needed.

DEFINITION 5.8. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S} = \{S_i\}$ be a family of subsets of X. We say that $\mathcal{S}$ is <u>locally finite</u> (in the space X) if for any $x \in X$, there exists a neighborhood V_x of x such that V_x intersects only finitely many subsets that belong to $\mathcal{S}$.*

EXAMPLE 5.9. The collection $\mathcal{S} = \{(0, 1/n) : n \in \mathbb{Z}\}$ is locally finite in $(0,1)$, but not in $\mathbb{R}$.

DEFINITION 5.10. *Given a topological space X, a family $\{\tilde{g}_i : i \in I\}$ of continuous functions $\tilde{g}_i : X \to \mathbb{R}$ is called a <u>locally finite family of continuous functions</u> if $\{\text{supp}_X(\tilde{g}_i) : i \in I\}$ is locally finite.*

EXERCISE 5.3. Show that if $\{\tilde{g}_i : i \in I\}$ is a locally finite family of continuous functions, then

$$X \ni x \mapsto \sum_i \tilde{g}_i(x)$$

gives a well-defined continuous function $\sum_i g_i : X \to \mathbb{R}$.

DEFINITION 5.11. *Given a topological space X and $\mathcal{A} \subset \mathcal{C}(X)$, we say that $\mathcal{A}$ is <u>closed under locally finite sums</u> if for any locally finite family $\{\tilde{g}_i : i \in I\}$ of functions from $\mathcal{A}$, $\sum f_i \in \mathcal{A}$.*

DEFINITION 5.12. *Let X be a topological space, $\mathcal{U} = \{U_i : i \in I\}$ an open cover of X. A partition of unity subordinated to $\mathcal{U}$ is a locally finite family of functions $\eta_i : X \to [0,1]$ satisfying:*

$$\sum_i \eta_i = 1, \quad \text{supp}(\eta_i) \subset U_i.$$

Given $\mathcal{A} \subset \mathcal{C}(X)$, we say that $\{\eta_i\}$ is an <u>$\mathcal{A}$-partition of unity</u> if $\eta_i \in \mathcal{A}$ for all i.

The existence of partitions of unity (for arbitrary covers) forces X to have a special topological property, called "paracompactness", which we discuss next. As in the case of compactness, paracompactness is best characterized in terms of open covers.

DEFINITION 5.13. *Let X be a topological space and let $\mathcal{A}$ be a cover of X. A <u>refinement</u> of $\mathcal{A}$ is any other cover $\mathcal{B}$ with the property that any $B \in \mathcal{B}$ is contained in some $A \in \mathcal{A}$.*

EXAMPLE 5.14. For $X = \mathbb{R}$ and $\mathcal{A} = \{(0, \epsilon) : \epsilon \in (0,1)\}$, $\mathcal{B} = \{(0, 1/n) : n \in \mathbb{Z}_+\}$, $\mathcal{B}$ is subcover (hence also a refinement) of $\mathcal{A}$ but, at the same time, $\mathcal{A}$ is a refinement of $\mathcal{B}$.

As a motivation for the next definition, note that if $\{\eta_i\}$ is a partition of unity subordinated to $\mathcal{U}$, then $\{\eta_i \neq 0\}$ is an open refinement of $\mathcal{U}$ (which still covers X!), which is locally finite.

DEFINITION 5.15. *A topological space X is called paracompact if any open cover admits a locally finite refinement.*

EXAMPLE 5.16. Compact spaces are paracompact (use again that any subcover is a refinement). As we will prove in the next section, any locally compact, Hausdorff, 2nd countable space (hence also any topological manifold) is paracompact. One can also show that all metric spaces are paracompact. Hence paracompactness is shared by the most important classes of spaces.

As in the previous subsection, for partitions of unity, we will need a "shrinking lemma".

LEMMA 5.17. *(shrinking lemma) If X is a paracompact Hausdorff space then X is normal and, for any open cover $\mathcal{U} = \{U_i : i \in I\}$ there exists a locally finite open cover $\mathcal{V} = \{V_i : i \in I\}$ with the property that $\overline{V}_i \subset U_i$ for all $i \in I$.*

PROOF. We first show that X is normal. The proof is very similar to the compact case, i.e. the proof of Proposition 4.19. We use the same idea and the same notations. We see that it suffices to show that, for $Y, Z \subset X$, if Z is *closed* and $Y|\{z\}$ for all $z \in Z$, then $Y|Z$. To prove this, we first make a general remark: the condition $Y|Z$ is implies (and it is actually equivalent to) the existence of an open neighborhood V of Z such that $Y \cap \overline{V} = \emptyset$. Indeed, if $U \cap V = \emptyset$ for some open neighborhoods U of Y and V of Z, then $V \subset X - U$ where the last set is closed, hence $\overline{V} \subset X - U$, hence $\overline{V} \cap U \neq \emptyset$; since $Y \subset U$, we must have $\overline{V} \cap Y = \emptyset$ (for the converse, just take $U = X - \overline{V}$).

Hence we assume now that $Y|\{z\}$ for all $z \in Z$ and we prove $Y|Z$. For each $z \in Z$ choose an open neighborhood V_z such that $Y \cap \overline{V}_z = \emptyset$. Then $\{V_z : z \in Z\} \cup \{X - Z\}$ is an open cover of X. Let $\mathcal{U}$ be a locally finite refinement and let $\mathcal{W} = \{W_i : i \in I\}$ consisting of those members of $\mathcal{U}$ which intersect Z. Define $V = \cup_i W_i$. This is an open neighborhood of Z. Note that $Y \cap \overline{W}_i = \emptyset$ for all i (since each W_i is inside some V_z and $Y \cap \overline{V}_z = \emptyset$ by construction). Also, due to local finiteness (and Exercise 2.53),

$$\overline{V} = \cup_i \overline{W}_i.$$

Hence $\overline{V} \cap Y = \emptyset$, proving that $Y|Z$. In conclusion X must be normal.

We now prove the second part. Consider $\mathcal{A} := \{V \subset X \text{ open} : \overline{V} \subset U_i \text{ for some } i \in I\}$. Since X is normal, Lemma 5.3 implies that $\mathcal{A}$ is an open cover of X. Let $\mathcal{B} = \{B_j : j \in J\}$ be a locally finite refinement of $\mathcal{A}$ which is an open cover of X. Then, for each $j \in I$, we find an element $f(j) \in I$ such that $\overline{B}_j \subset U_{f(j)}$ (and this defines a function $f : J \to I$). We define

$$V_i := \cup_{j \in f^{-1}(i)} B_j$$

(by convention, this is empty if $f^{-1}(i)$ is empty). Using Exercise 2.53, we have $\overline{V}_i \subset U_i$ for all i. Finally, remark that $\{V_i\}$ is locally finite: if a neighborhood of a point intersects V_i then it intersects B_j for some $j \in f^{-1}(i)$, hence it intersects an infinite number of V_i's, then it would also intersect an infinite number of B_j's. ☐

THEOREM 5.18. *Let X be a paracompact Hausdorff space and assume that $\mathcal{A} \subset \mathcal{C}(X)$ is normal, closed under locally finite sums and closed under quotients.*

Then, for any open cover $\mathcal{U}$ of X, there exists an $\mathcal{A}$-partition of unity subordinated to $\mathcal{U}$.

PROOF. The proof is completely similar to the proof from the finite case. Apply the shrinking lemma twice to find coverings $\{V_i\}$ and $\{W_i\}$ with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. Then choose $\phi_i : X \to [0, 1]$ such that $\phi_i = 1$ on $\overline{W}_i$ and 0 on $X - V_i$, with $\phi_i \in \mathcal{A}$. Finally, since our families are locally finite, $\eta_i = \phi_i / \sum_j \phi_j$ makes sense and is our desired partition of unity (fill in the details!). ☐

4. The locally compact case

The locally compact Hausdorff case is nicer. First of all the condition on $\mathcal{A} \subset \mathcal{C}(X)$ to separate the closed subsets of X (which may be difficult to prove!) can be reduced to a local condition.

THEOREM 5.19. *Let X be a Hausdorff paracompact space and $\mathcal{A} \subset \mathcal{C}(X)$ closed under locally finite sums and under quotients. If X is also locally compact, then the following are equivalent:*
1. *$\mathcal{A}$ is normal.*
2. *$\forall$ ($x \in U \subset X$ with U open), $\exists$ ($f \in \mathcal{A}$ positive, supported in U, with $f(x) > 0$).*

Secondly, 2nd countability and local compactness imply paracompactness:

THEOREM 5.20. *Any Hausdorff, locally compact and 2nd countable space is paracompact.*

PROOF. (of Theorem 5.20) We use an exhaustion $\{K_n\}$ of X (Theorem 4.37). Let $\mathcal{U}$ be an open cover of X. For each $n \in \mathbb{Z}_+$ there is a finite family $\mathcal{V}_n$ which covers $K_n - \text{Int}(K_{n-1})$, consisting of opens V with the properties: $V \subset \text{Int}(K_{n+1}) - K_{n-1}$, $V \subset U$ for some $U \in \mathcal{U}$. Indeed, for any $x \in K_n - \text{Int}(K_{n-1})$ let V_x be the intersection of $\text{Int}(K_{n+1}) - K_{n-1}$ with any member of $\mathcal{U}$ containing x; since $K_n - \text{Int}(K_{n-1})$ is compact, just take a finite subcollection $\mathcal{V}_n$ of $\{V_x\}$, covering $K_n - \text{Int}(K_{n-1})$. Set $\mathcal{V} = \cup_n \mathcal{V}_n$; it covers X since each $K_n - K_{n-1} \subset K_n - \text{Int}(K_{n-1})$ is covered by $\mathcal{V}_n$. Finally, it is locally finite: if $x \in X$, choosing n and V such that $V \in \mathcal{V}_n$, $x \in V$, we have $V \subset \text{Int}(K_{n+1}) - K_{n-1}$, hence V can only intersect members of $\mathcal{V}_m$ with $m \leq n+1$ (a finite number of them!). □

PROOF. (of Theorem 5.19) That 1 implies 2 is clear: apply the separation property to $\{x\}$ and $X - V$. Assume 2. We claim that for any $C \subset X$ compact and any open U such that $C \subset U$, there exists $f \in \mathcal{A}$ supported in U, such that $f|_C > 0$. Indeed, by hypothesis, for any $c \in C$ we can find an open neighborhood V_c of c and $f_c \in \mathcal{A}$ positive such that $f_c(c) > 0$; then $\{f_c \neq 0\}_{c \in C}$ is an open cover of C in X, hence we can find a finite subcollection (corresponding to some points $c_1, \ldots, c_k \in C$) which still covers C; finally, set $f = f_{c_1} + \ldots + f_{c_k}$.

To prove 1, let $A, B \subset X$ be two closed disjoint subsets. As terminology, $D \subset X$ is called relatively compact if $\overline{D}$ is compact. Since X is locally compact, any point has arbitrarily small relatively compact open neighborhoods (why?). For each $y \in X - A$, we choose such a neighborhood $D_y \subset X - A$. For each $a \in A$, since $a \in X - B$, by Lemma 5.17 and Lemma 5.3, we find an open D_a such that $a \in D_a \subset X - B$. Again, we may assume that $\overline{D}_a$ is relatively compact. Then $\{D_x : x \in X\}$ is an open cover of X; let $\mathcal{U} = \{U_i : i \in I\}$ be a locally finite refinement. We split the set of indices as $I = I_1 \cup I_2$, where I_1 contains those i for which $U_i \cap A \neq \emptyset$, while I_2 those for which $U_i \subset X - A$. Using Lemma 5.17 we also choose an open cover of X, $\mathcal{V} = \{V_i : i \in I\}$, with $\overline{V}_i \subset U_i$. Note that, by construction, each U_i (hence also each V_i) is relatively compact. Hence, by the claim above, we can find $\eta_i \in \mathcal{A}$ such that

$$\eta_i|_{\overline{V}_i} > 0, \quad \text{supp}(\eta_i) \subset U_i.$$

Finally, we define

$$f(x) = \frac{\sum_{i \in I_1} \eta_i(x)}{\sum_{i \in I} \eta_i(x)}$$

From the properties of $\mathcal{A}$, $f \in \mathcal{A}$. Also, $f|_A = 1$. Indeed, for $a \in A$, a cannot belong to the U_i's with $i \in I_2$ (i.e. those $\subset X - A$); hence $\eta_i(a) = 0$ for all $i \in I_2$, hence $f(a) = 1$. Finally, $f|_B = 0$. To see this, we show that $\eta_i(b) = 0$ for all $i \in I_1$, $b \in B$. Assume the contrary. We find $i \in I_1$ and $b \in B \cap U_i$. Now, from the construction of $\mathcal{U}$, $U_i \subset D_x$ for some $x \in X$. There are two cases. If $x = a \in A$, then the defining property for D_a, namely $D_a \cap B = \emptyset$, is in contradiction with our assumption ($b \in B \cap U_i$). If $x = y \in X - A$, then the defining property for D_y, i.e. $D_y \subset X - A$, is in contradiction with the fact that $i \in I_1$ (i.e. $U_i \cap A \neq \emptyset$). □

5. Urysohn's lemma

This section is devoted to the proof of what is known as "the Urysohn lemma":

THEOREM 5.21. *If X is a normal space then for any two closed disjoint subsets $A, B \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$, $f|_B = 1$.*

In other words, if X is normal, then $\mathcal{C}(X)$ is normal. Hence one can construct continuous partitions of unity.

COROLLARY 5.22. *If X is Hausdorff and paracompact then, for any open cover $\mathcal{U}$ of X, there exists a continuous partition of unity subordinated to $\mathcal{U}$.*

We start with the proof. Fix A and B disjoint closed subsets. From now on, when saying that "A is closed" or "D is open", we mean that they are closed (open) in the given topological space $(X, \mathcal{T})$. We will repeatedly use Lemma 5.3 from this chapter.

Claim 1: Then there is a family of opens sets $\{U_q : q \in \mathbb{Q}\}$ such that

(C1) $U_q = \emptyset$ for $q < 0$, U_0 contains A, $U_1 = X - B$, $U_q = X$ for $q > 1$.

(C2) $\overline{U}_q \subset U_{q'}$ for all $q < q'$.

PROOF. The condition (C1) force the definition of U_q for $q < 0$ and for $q \geq 1$. For $q = 0$, we choose U_0 to be any open set such that

$$A \subset U_0 \subset \overline{U}_0 \subset U_1.$$

This is possible since $A \cap B = \emptyset$ means that $A \subset X - B = U_1$ hence we can apply Lemma 5.3.

We are left with the construction of U_q for $q \in \mathbb{Q} \cap (0,1)$. Writing

$$\mathbb{Q} \cap [0,1] = \{q_0, q_1, q_2, \ldots\},$$

with $q_0 = 0$, $q_1 = 1$, we will define U_{q_n} by induction on n such that (C2) holds for all $q = q_i$, $q' = q_j$ with $0 \leq i, j \leq n$. Assume that U_q is constructed for $q \in \{q_0, \ldots, q_n\}$ and we construct it for $q = q_{n+1}$. Looking at all intervals of type (q_i, q_j) with $0 \leq i, j \leq n$, there is a smallest one containing q_{n+1}. Call it (q_a, q_b). Since $q_a < q_b$, by the induction hypothesis we have

$$\overline{U}_a \subset U_b$$

hence, by Lemma 5.3, we find an open U such that

$$\overline{U}_a \subset U \subset \overline{U} \subset U_b.$$

Define $U_{q_{n+1}} = U$. We have to check that (C2) holds for $q, q' \in \{q_0, \ldots, q_{n+1}\}$. Fix q, q'. If $q \neq q_{n+1}$ and $q' \neq q_{n+1}$, $\overline{U}_q \subset U_{q'}$ holds by the induction hypothesis. Hence we may assume that $q = q_{n+1}$ or $q' = q_{n+1}$. We treat the case $q = q_{n+1}$, the other one being similar. Write $q' = q_j$ with $j \in \{0, 1, \ldots, n\}$. The assumption is that $q_{n+1} < q_j$ and we want to show that

$$\overline{U}_{q_{n+1}} \subset U_{q_j}.$$

But, since $q_{n+1} < q_j$ and (q_a, q_b) is the smallest interval of this type containing q_{n+1}, we must have $q_j \geq q_b$. But then

$$\overline{U}_{q_{n+1}} = \overline{U} \subset U_{q_b} \subset U_{q_j}.$$

□

Claim 2: The function $f : X \to [0,1]$, $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$ satisfies:

(1) $f(x) > q \Longrightarrow x \notin \overline{U}_q$.

(2) $f(x) < q \Longrightarrow x \in U_q$.

(in particular, $f(x) = q$ for $x \in \partial U_q$).

PROOF. For (1), we prove its negation, i.e. that $x \in \overline{U}_q$ implies $f(x) \leq q$. Hence assume that $x \in \overline{U}_q$. From (C2) we deduce that $x \in U_{q'}$ for all $q' > q$. Hence $f(x) \leq q'$ for all $q' > q$. This implies $f(x) \leq q$. For (2), we assume that $f(x) < q$. By the definition of $f(x)$ (as an infimum), there exists $q' < q$ such that $x \in U_{q'}$. But $q' < q$ implies $U_{q'} \subset U_q$, hence $x \in U_q$. $\square$

Claim 3: $f|_A = 0$, $f|_B = 1$, and f is continuous.

PROOF. The first two conditions are immediate from the definition of f and properties (C1) of the first claim. We now prove that f is continuous. We have to prove that for any open interval (a, b) in $\mathbb{R}$, and any $x \in f^{-1}((a, b))$, there exists an open U containing x such that $f(U) \subset (a, b)$. Fix (a, b) and x such that $f(x) \in (a, b)$ and look for U satisfying the desired condition. Choosing $p, q \in \mathbb{Q}$ such that

$$a < p < f(x) < q < b,$$

then $U := U_q - \overline{U}_p$ will do the job. Indeed:

1. using Claim 2, $f(x) > p$ implies $x \notin \overline{U}_p$, while $f(x) < q$ implies $x \in U_q$. Hence $x \in U$.
2. for $y \in U$ arbitrary, we have $f(y) \in (a, b)$ because:
   - $y \in U_q \subset \overline{U}_q$ which, by the previous claim, implies $f(y) \leq q < b$.
   - $y \notin \overline{U}_p$, hence $y \notin U_p$ which, by the previous claim, implies $f(y) \geq p > a$.

$\square$

6. More exercises

EXERCISE 5.4. Let $\mathcal{A}$ be the following collection of subsets of $\mathbb{R}$:

$$\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}.$$

Which of the following collections refine $\mathcal{A}$?

$$\mathcal{B} = \{(x, x+1) : x \in \mathbb{R}\},$$
$$\mathcal{C} = \{(n, n+\frac{3}{2}) : n \in \mathbb{Z}\},$$
$$\mathcal{C} = \{(x, x+\frac{3}{2}) : x \in \mathbb{R}\}.$$

EXERCISE 5.5. Which of the collections from the previous exercise is locally finite?

EXERCISE 5.6. Show that if a family $\{p_i : i \in I\}$ of non-zero polynomial functions $p_i : \mathbb{R} \to \mathbb{R}$ is locally finite, then it must be finite.

EXERCISE 5.7. Let $\mathcal{P} \subset \mathcal{C}(\mathbb{R})$ be the space of all polynomial functions on $\mathbb{R}$. Is $\mathcal{P}$ normal?

EXERCISE 5.8. Show that the space $\mathcal{C}^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$ of functions of class C^1 is normal. What do you conclude from this?

EXERCISE 5.9. Do the same for the space $\mathcal{C}^\infty(\mathbb{R})$ of smooth (i.e. infinitely differentiable) functions on $\mathbb{R}$.

EXERCISE 5.10. Now do the same for $\mathcal{C}^\infty(\mathbb{R}^n)$.

It is very tempting now to talk about smooth manifolds. These are manifolds on which we can talk about smoothness. More precisely, a smooth manifold is a topological manifold X together with a specified family of coordinate charts $\{\chi_i : U_i \to \mathbb{R}^n\}$, such that $\{U_i\}$ is an open cover of X, $c_{i,j} := \chi_i \circ \chi_j^{-1}$ is a smooth function. Here, $c_{i,j}$ plays the role of the "change of coordinates" since

$$\chi_i(x) = c_{i,j}(\chi_j(x)).$$

Also, $c_{i,j}$ is a function defined on an open in $\mathbb{R}^n$ (namely $\chi_j(U_i \cap U_j)$) with values in $\mathbb{R}^n$; hence it makes sense to talk about its smoothness. Given such a smooth manifold, a function $f : X \to \mathbb{R}$ is called smooth if its representation in each chart, i.e. each $f \circ \chi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth. Denote by $\mathcal{C}^\infty(X)$ the space of smooth functions on X; of course, $\mathcal{C}^\infty(X) \subset \mathcal{C}(X)$. Once you get used to all these definitions, the following should not be too difficult now:

EXERCISE 5.11. Show that, for any smooth manifold X, $\mathcal{C}^\infty(X)$ is normal. Deduce that any open cover admits a smooth partition of unity subordinated to it.

In this context, a map $f : X \to \mathbb{R}^N$ is called smooth if all its components are smooth. Adapting the proof of Theorem 4.30 and using Exercise 5.10 above, one can now try a more difficult exercise:

EXERCISE 5.12. Show that, for any smooth compact manifold X, there exists a smooth embedding $f : X \to \mathbb{R}^N$, for N large enough.