CHAPTER 6

Metric properties versus topological ones

1. Completeness and the Baire property

2. Boundedness and totally boundedness

3. Compactness

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5. More exercises

1. Completeness and the Baire property

Probably the most important metric property is that of completeness which we now recall.

DEFINITION 6.1. *Given a metric space (X, d) and a sequence $(x_n)_{n\geq 1}$ in X, we say that $(x_n)_{n\geq 1}$ is a* <u>*Cauchy sequence*</u> *if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

i.e., for each $\epsilon > 0$, there exists an integer n_ϵ such that

$$d(x_n, x_m) < \epsilon$$

for all $n, m \geq n_\epsilon$. One says that (X, d) is <u>*complete*</u> *if any Cauchy sequence is convergent.*

Very simple examples (see e.g Exercise 1.33 from the first chapter) show that completeness is not a topological property. However, it does have topological consequences. The first one is a relative topological property for complete spaces.

PROPOSITION 6.2. *If (X, d) is a complete metric space then $A \subset X$ is complete (with respect to the restriction of d to A) if and only if A is closed in X.*

PROOF. Assume first that A is complete and show that $\overline{A} = A$. Let $x \in \overline{A}$. Then we find a sequence (a_n) in A converging (in (X, d)) to x. In particular, (a_n) is Cauchy. But the completeness of A implies that the sequence is convergent (in A!) to some $a \in A$. Hence $x = a \in A$. This proves that A is closed. For the converse, assume A is closed and let (a_n) be a Cauchy sequence in A. Of course, the sequence is Cauchy also in X. Since X is complete, it will be convergent to some $x \in X$. Since A is closed, $x \in A$, i.e. (a_n) is convergent in A. □

The next topological property that complete metric spaces automatically have is:

PROPOSITION 6.3. *Any complete metric space (X, d) has the Baire property, i.e. for any countable family $\{U_n\}_{n\geq 1}$ consisting of open sets $U_n \subset X$, if U_n is dense in X for all n, then $\cap_n U_n$ is dense in X.*

PROOF. Assume now that $\{U_n\}_{n\geq 1}$ consists of open dense subsets of X. We show that any $x \in X$ is in the closure of $\cap_n U_n$. Let U be an open containing x; we have to show that U intersects $\cap_n U_n$. First, since U_1 is dense in X, $U \cap U_1 \neq \emptyset$; choosing x_1 in this intersection, we find $r_1 > 0$ such that $B[x_1, r_1] \subset U \cap U_1$. We may assume $r_1 < 1$. Next, since U_2 is dense in X, $B(x_1, r_1) \cap U_2 \neq \emptyset$; choosing x_2 in this intersection, we find $r_2 > 0$ such that $B[x_2, r_2] \subset B(x_1, r_1) \cap U_2$. We may assume $r_2 < 1/2$. Similarly, we find x_3 and $r_3 < 1/3$ such that $B[x_3, r_3] \subset B(x_2, r_2) \cap U_3$ and we continue inductively. Then the resulting sequence (x_n) is Cauchy because $d(x_n, x_m) < r_n$ for $n \leq m$. This implies that (x_n) is convergent to some $y \in X$ and $d(x_n, y) \leq r_n$ for all n. Hence $y \in B[x_n, r_n] \subset U_n$, i.e. $y \in \cap_n U_n$. Also, since $B[x_1, r_1] \subset U$, $y \in U$. Hence $U \cap (\cap_n U_n) \neq \emptyset$, as we wanted. □

2. Boundedness and totally boundedness

Another notion that strongly depends on a metric is the notion of boundedness.

DEFINITION 6.4. *Given a metric space (X, d), we say that $A \subset X$ is*
1. *bounded in (X, d) (or with respect to d) if there exists $x \in X$ and $R > 0$ such that $A \subset B(x, R)$.*
2. *totally bounded in (X, d) if, for any $\epsilon > 0$, there exist a finite number of balls in X of radius ϵ covering A.*

When $A = X$, we say that (X, d) is bounded, or totally bounded, respectively.

You should convince yourself that, when $X = \mathbb{R}^n$ and d is the Euclidean metric, total boundedness with respect to d is equivalent to the usual notion of boundedness.

A few remarks are in order here. First of all, these properties are not really relative properties (i.e. they did not depend on the way that A sits inside X), but properties of the metric space (A, d_A) itself, where d_A is the induced metric on A.

EXERCISE 6.1. Given a metric space (X, d) and $A \subset X$, A is bounded in (X, d) if and only if (A, d_A) is bounded. Similarly for totally bounded.

Another remark is that the property of "totally bounded" is an improvement of that of "bounded". The following exercise shows that, by a simple trick, a metric d can always be made into a bounded metric $\hat{d}$ without changing the induced topology; although the notion of boundedness is changed, totally boundedness with respect to d and $\hat{d}$ is the same.

EXERCISE 6.2. As in Exercise 1.34, for a metric space (X, d) we define $\hat{d} : X \times X \to \mathbb{R}$ by

$$\hat{d}(x, y) = \min\{d(x, y), 1\}.$$

We already know that $\hat{d}$ is a metric inducing the same topology on X as d, and that $(X, \hat{d})$ is complete if and only if (X, d) is. Also, it is clear that $(X, \hat{d})$ is always bounded. Show now that $(X, \hat{d})$ is totally bounded if and only if (X, d) is.

Finally, here is a lemma that we will use later on:

LEMMA 6.5. *Given a metric space (X, d) and $A \subset X$, then A is totally bounded if and only if $\overline{A}$ is.*

PROOF. Let $\epsilon > 0$. Choose $x_1, \ldots, x_k$ such that A is covered by the balls $B(x_i, \epsilon/2)$. Then $\overline{A}$ will be covered by the balls $B(x_i, \epsilon)$. Indeed, if $y \in \overline{A}$, we find $x \in A$ such that $d(x, y) < \epsilon/2$; also, we find x_i such that $x \in B(x_i, \epsilon/2)$; from the triangle inequality, $y \in B(x_i, \epsilon)$. ∎

3. Compactness

The main criteria to recognize when a subspace $A \subset \mathbb{R}^n$ is compact is by checking whether it is closed and bounded in $\mathbb{R}^n$. For general metric spaces:

THEOREM 6.6. *A subset A of a complete metric space (X, d) is compact if and only if it is closed (in X) and totally bounded (with respect to d).*

This theorem will actually be an immediate consequence of another theorem, which also clarifies the relationship between compactness and sequential compactness for metric spaces.

THEOREM 6.7. *For a metric space (X, d), the following are equivalent:*

1. *X is compact.*
2. *X is sequentially compact.*
3. *X is complete and totally bounded.*

PROOF. We first prove Theorem 6.7. The implication 1$\Longrightarrow$ 2 is Corollary 4.33. For 2$\Longrightarrow$ 3, assume that X is sequentially compact. We first prove that X is complete. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence. By hypothesis, we find a convergent subsequence $(x_{n_k})_{k\geq 1}$. Let x be its limit. We prove that the entire sequence (x_n) converges to x. Let $\epsilon > 0$. We look for an integer N_ϵ such that $d(x_n, x) < \epsilon$ for all $n > N_\epsilon$. Since (x_n) is Cauchy we find N'_ϵ such that

$$d(x_n, x_m) < \epsilon/2$$

for all $n, m \geq N'_\epsilon$. Since $(x_{n_k})_{k\geq 1}$ converges to x, we find k_ϵ such that

$$d(x_{n_k}, x) < \epsilon/2$$

for all $k \geq k_\epsilon$. Choose $N_\epsilon = \max\{N'_\epsilon, n_{k_\epsilon}\}$. Then, for $n > N_\epsilon$, choosing k such that $n_k > n$ (such a k exists since $n_1 < n_2 < \ldots$ is a sequence that tends to ∞), we must have $k > k_\epsilon$ and $n_k > N'_\epsilon$, hence

$$d(x_n, x_{n_k}) < \epsilon/2,\ d(x_{n_k}, x) < \epsilon/2.$$

Using the triangle inequality, we obtain $d(x_n, x) < \epsilon$, and this holds for all $n \geq N_\epsilon$. This proves that (x_n) converges to x. We now prove that X is totally bounded. Assume it is not. Then we find $r > 0$ such that X cannot be covered by a finite number of balls of radius r. Construct a sequence $(x_n)_{n\geq 1}$ as follows. Start with any $x_1 \in X$. Since $X \neq B(x_1, r)$, we find $x_2 \in X - B(x_1, r)$. Since $X \neq B(x_1, r) \cup B(x_2, r)$, we find $x_3 \in X - B(x_1, r) \cup B(x_2, r)$. Continuing like this we find a sequence with $x_n \notin B(x_m, r)$ for $n > m$. Hence $d(x_n, x_m) > r$ for all $n \neq m$. But by hypothesis, (x_n) has a subsequence $(x_{n_k})_{k\geq 1}$ which converges to some $x \in X$. But then we find N such that $d(x_{n_k}, x) < r/2$ for all $k \geq N$, hence

$$d(x_{n_k}, x_{n_l}) \leq d(x_{n_k}, x) + d(x, x_{n_l}) < r$$

for all $k, l \geq N$, and this contradicts the condition "$d(x_n, x_m) > r$ for all $n \neq m$".

3$\Longrightarrow$ 1: Assume that (X, d) is complete and totally bounded. The last condition ensures that for each integer $n \geq 0$, there is a finite set $F_n \subset X$ such that

$$X = \bigcup_{x\in F_n} B(x, \frac{1}{2^n}).$$

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X, and we want to prove that we can extract a finite subcover of $\mathcal{U}$. Assume this is not possible. We construct a sequence $(x_n)_{n\geq 1}$ inductively as follows. Since $\cup_{x\in F_1} B(x, \frac{1}{2}) = \cup_i U_i$, and the first union is a finite union (F_1 is finite), we find $x_1 \in F_1$ such that $B(x_1, \frac{1}{2})$ cannot be covered by a finite number of opens from $\mathcal{U}$. Now, since

$$B(x_1, \frac{1}{2}) = \bigcup_{x\in F_2} (B(x_1, \frac{1}{2}) \cap B(x, \frac{1}{4}))$$

we find $x_2 \in F_2$ such that

$$B(x_1, \frac{1}{2}) \cap B(x_2, \frac{1}{4}) \neq \emptyset$$

and $B(x_2, \frac{1}{4})$ cannot be covered by a finite numbers of opens from $\mathcal{U}$. Continuing this, at step n we find $x_n \in F_n$ such that

$$B(x_{n-1}, \frac{1}{2^{n-1}}) \cap B(x_n, \frac{1}{2^n}) \neq \emptyset$$

and $B(x_n, \frac{1}{2^n})$ cannot be covered by a finite number of opens from $\mathcal{U}$. Note that, choosing an element y in the (non-empty) intersection above, the triangle inequality implies that

$$d(x_{n-1}, x_n) < \frac{1}{2^{n-1}} + \frac{1}{2^n} = \frac{3}{2^n},$$

from which we deduce that $(x_n)_{n\geq 1}$ is a Cauchy sequence (why?). By hypothesis, it will converge to en element $x \in X$. Choose $U \in \mathcal{U}$ such that $x \in U$. Since U is open, we find $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Since $x_n \to x$, we find n_ϵ such that $d(x_n, x) < \epsilon/2$ for all $n > n_\epsilon$. Using the triangle inequality, we deduce that $B(x_n, \epsilon/2) \subset U$ for all $n \geq n_\epsilon$. Choosing n so that also $1/2^n < \epsilon/2$, we deduce that $B(x_n, 1/2^n) \subset U$, which contradicts the fact that $B(x_n, \frac{1}{2^n})$ cannot be covered by of finite number of opens from $\mathcal{U}$.

This ends the proof of Theorem 6.7. For Theorem 6.6, one uses the equivalence between 1 and 3 above, applied to the metric space (A, d_A), and Proposition 6.2. $\square$

We now derive some more properties of compactness in the metric case. In what follows, given $F \subset X$, we say that F is relatively compact in X if the closure $\overline{F}$ in X is compact.

Corollary 6.8. *For a subset F of a complete metric space (X, d), the following are equivalent*

1. *F is relatively compact in X.*
2. *any sequence in F admits a convergent subsequence (with some limit in X).*
3. *F is totally bounded.*

Proof. We apply Theorem 6.7 to $\overline{F}$. We know that 3 is equivalent to the same condition for $\overline{F}$ (Lemma 6.5). We prove the same for 2; the non-obvious part is to show that $\overline{F}$ satisfies 2 if F does. So, let (y_n) be a sequence in $\overline{F}$. For each n we find $x_n \in F$ such that $d(x_n, y_n) < 1/n$. After eventually passing to a subsequence, we may assume that (x_n) is convergent to some $x \in X$ and $d(x_n, y_n) \to 0$ as $n \to \infty$. But this implies that (y_n) itself must converge to x. $\square$

Corollary 6.9. *Any compact metric space (X, d) is separable, i.e. there exists $A \subset X$ which is at most countable and which is dense in X.*

Proof. For each n choose a finite set A_n such that X is covered by $B(a, \frac{1}{n})$ with $a \in A_n$. Then $A := \cup A_n$ is dense in X: for $x \in X$ and ϵ we have to show that $B(x, \epsilon) \cap A \neq \emptyset$; but we find n with $\frac{1}{n} < \epsilon$ and $a \in A_n$ such that $x \in B(a, \frac{1}{n})$; then $a \in B(x, \epsilon) \cap A$. $\square$

Proposition 6.10. *(the Lebesgue lemma) If (X, d) is a compact metric space then, for any open cover $\mathcal{U}$ of X, there exists $\delta > 0$ such that*

$$A \subset X,\ diam(A) < \delta \Longrightarrow\ \exists\ U \in \mathcal{U} \text{ such that } A \subset U.$$

(δ is called a Lebesgue number for the cover $\mathcal{U}$).

Proof. It suffices to show that there exists δ such that each ball $B(x, \delta)$ is contained in some $U \in \mathcal{U}$. If no such δ exists, we find $\delta_n \to 0$ such that $B(x_n, \delta_n)$ is not inside any $U \in \mathcal{U}$. Using (sequential) compactness we may assume that (x_n) is convergent, with some limit $x \in X$ (if not, pass to a convergent subsequence). Let $U \in \mathcal{U}$ with $x \in U$ and let $r > 0$ with $B(x, r) \subset U$. Since $\delta_n \to 0$, $x_n \to x$, we find n s.t. $\delta_n < r/2$, $d(x_n, x) < r/2$. From the triangle inequality, $B(x_n, \delta_n) \subset B(x, r)$ $(\subset U)$ which contradicts the choice of x_n and δ_n. $\square$

4. Paracompactness

Finally, we show that:

THEOREM 6.11. *Any metric space is paracompact.*

PROOF. Start with an arbitrary open cover $\mathcal{U} = \{U_i : i \in I\}$ of X. We consider an order relation "$\leq$" on I, which makes I into a well-ordered set (i.e. so that any subset of I has a smallest element). We will construct a locally finite refinement of type $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$ where, for each n, the family $\mathcal{V}(n)$ will have one member for each $i \in I$; i.e. it is of type:

$$\mathcal{V}(n) = \{V_i(n) : i \in I\}.$$

We set $X(n) = \cup_i V_i(n)$. The definition of $\mathcal{V}(n)$ is by induction on n. For $n = 1$:

$$V_i(1) := \bigcup_{a\in U_i - (\cup_{j<i}U_j) \text{ with } B(a,\frac{3}{2})\subset U_i} B(a, \frac{1}{2}).$$

Assuming that $\mathcal{V}(1), \ldots, \mathcal{V}(n-1)$ have been constructed, we define, for each $i \in I$:

$$V_i(n) = \bigcup_{a\in U_i - (\cup_{j<i}U_j) \text{ with } B(a,\frac{3}{2^n})\subset U_i,\ a\notin X(1)\cup\ldots\cup X(n-1)} B(a, \frac{1}{2^n}).$$

It is clear that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Next, we claim that $X = \cup_n X(n)$ (i.e. $\mathcal{V}$ is a cover): for $x \in X$, choose the smallest i such that $x \in U_i$ and choose n such that $B(x, 3/2^n) \subset U_i$; then either $x \in X(1) \cup \ldots \cup X(n-1)$ and we are done, or x can serve as an index in the definition of $V_i(n)$, hence $x \in X(n)$. Before showing local finiteness, we remark that, for each n:

$$d(V_i(n), V_j(n)) \geq \frac{1}{2^n} \quad \forall\ i \neq j. \tag{4.1}$$

To see this, assume that $i < j$ and let $x \in V_i(n)$, $y \in V_j(n)$. Then $x \in B(a, \frac{1}{2^n})$ for some $a \in X$ with $B(a, \frac{3}{2^n}) \subset U_i$ and $y \in B(b, \frac{1}{2^n})$ for some $b \in X$ with $b \notin U_i$. These imply that $b \notin B(a, \frac{3}{2^n})$, i.e. $d(a, b) \geq \frac{3}{2^n}$. From the triangle inequality:

$$d(x, y) \geq d(a, b) - d(a, x) - d(b, y) > \frac{3}{2^n} - \frac{1}{2^n} - \frac{1}{2^n} = \frac{1}{2^n}.$$

We now show local finiteness. Let $x \in X$. Fix $n_0 \geq 1$ integer, $i_0 \in I$ with $x \in V_{i_0}(n_0)$. Also, choose $n_1 \geq 1$ integer with

$$B(x, \frac{1}{2^{n_1}}) \subset V_{i_0}(n_0). \tag{4.2}$$

We claim that

$$V := B(x, r) \quad \text{where } r = \frac{1}{2^{n_0+n_1}}$$

intersects only a finite number of members of $\mathcal{V}$. This follows from the following two remarks

1. For $n < n_0 + n_1$, V intersects at most one member of the family $\mathcal{V}(n)$.
2. For $n \geq n_0 + n_1$, V intersects no member of the family $\mathcal{V}(n)$.

Part 1 follows from (4.1): if V intersects both $V_i(n)$ and $V_j(n)$ with $i \neq j$, we would find $a, b \in V$ with $d(a, b) \geq \frac{1}{2^n}$ but $d(a, b) \leq d(a, x) + d(x, b) < 2r \leq \frac{1}{2^n}$ for all $a, b \in V$.

For part 2, assume that $n \geq n_0 + n_1$. Assume that $V \cap V_i(n) = \emptyset$ for some $i \in I$. From the definition of $V_i(n)$, we then find $B(a, \frac{1}{2^n}) \cap V \neq \emptyset$ for some $a \in U_i - (\cup_{j<i}U_j)$, with $B(a, \frac{3}{2^n}) \subset U_i$, $a \notin X(1) \cup \ldots \cup X(n-1)$. Since $n > n_0$, we have $a \notin X(n_0)$, hence $a \notin V_{i_0}(n_0)$. From the choice of n_1 (see (4.2) above), $a \notin B(x, \frac{1}{2^{n_1}})$, hence $d(a, x) \geq \frac{1}{2^{n_1}}$. But, by the triangle inequality again, this implies that $B(a, \frac{1}{2^n}) \cap B(x, r) = \emptyset$. I.e., for any a which contributes to the definition of $V_i(n)$, its contribution $B(a, \frac{1}{2^n})$ does not intersect V. Hence $V \cap V_i(n) = \emptyset$. $\square$

5. More exercises

EXERCISE 6.3. Let (X, d) be a metric space. Show that any sequence $(x_n)_{n\geq 1}$ in X with the property that

$$d(x_{n+1}, x_n) \leq \frac{d(x_n, x_{n-1})}{2}$$

for all n, is Cauchy.

EXERCISE 6.4. Let (x, d) be a complete metric space and let $f : X \to X$ be a map with the property that there exists $\lambda \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Show that f has a unique fixed point (i.e. $a \in X$ with $f(a) = a$).
(Hint: the difficult part is the existence. Start with any x_0 and consider $x_{n+1} = f(x_n)$).

EXERCISE 6.5. We say that a topological space X is separable if there exists $A \subset X$ countable and dense in X.

1. Show that if X is 2nd countable, then it is separable.
2. Show that a metric space is 2nd countable if and only if it is separable.
3. Deduce that $(\mathbb{R}, \mathcal{T}_l)$ is not metrizable (see exercise 2.19).