CHAPTER 7

Metrizability theorems

1. The Urysohn metrization theorem

In following is known as the Urysohn metrization theorem.

THEOREM 7.1. *Any topological space which is normal and second countable is metrizable.*

The rest of this section is devoted to the proof of this theorem.

Claim 1: $\exists$ a countable family $(f_n)_{n\geq 0}$ of continuous functions $f_n : X \to [0,1]$ satisfying:

(1.1) $\quad (\forall\ U - \text{open},\ x \in U), \quad (\exists\ N \in \mathbb{Z}) \text{ such that: } (f_N(x) = 1, f_N = 0 \text{ outside } U).$

PROOF. Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable basis for the topology $\mathcal{T}$. Consider

$$I = \{(n,m) : \overline{B}_n \subset B_m\} \subset \mathbb{N} \times \mathbb{N}.$$

This is countable (subset of countable is countable), hence we can enumerate it as (n_0, m_0), (n_1, m_1), ... For each i, using Urysohn's lemma, we find a continuous function $f_i : X \to [0,1]$ such that $f_i|_{\overline{B}_{n_i}} = 1$, $f_i|_{X - B_{m_i}} = 0$. Then $(f_i)_{i\geq 0}$ has the desired properties: for $U \in \mathcal{T}$ and $x \in U$, we can choose m such that $x \in B_m \subset U$. By Lemma 5.3, we find V-open containing x such that $x \in V \subset \overline{V} \subset B_m$. Since $\mathcal{B}$ is a basis, we find n such that $x \in B_n \subset V$. Then $\overline{B}_n \subset \overline{V} \subset B_m$, hence $(n,m) \in I$. Writing $(n,m) = (n_N, m_N)$ with $N \in \mathbb{N}$, since $x \in B_n$ we have $f_N(x) = 1$, and since $B_m \subset U$, we have $f_N = 0$ outside B_m hence also outside U. ☐

Claim 2: The following is a metric on X inducing the topology of X:

$$d : X \times X \to \mathbb{R}, \quad d(x,y) = \sup\{\frac{|f_n(x) - f_n(y)|}{n} : n \geq 1 \text{ integer}\}.$$

PROOF. Note that $d(x,y)$ is finite since $0 \leq f_n \leq 1$. For the triangle inequality, we use:

$$\frac{|f_n(x) - f_n(y)|}{n} \leq \frac{|f_n(x) - f_n(z)|}{n} + \frac{|f_n(z) - f_n(y)|}{n} \leq d(x,z) + d(z,y)$$

for all n. To see that $d(x,y) \neq 0$ whenever $x \neq y$, choose $U \in \mathcal{T}$ containing x and not containing y, choose N as in (1.1) and remark that $|f_N(x) - f_N(y)| = 1$ hence $d(x,y) \geq 1/N > 0$.

Next, we show that $\mathcal{T} \subset \mathcal{T}_d$. Let $U \in \mathcal{T}$ and we have to show that:

$$\forall\ x \in U,\ \exists\ \epsilon > 0 : B_d(x,\epsilon) \subset U.$$

Let $x \in U$ and choose N as in (1.1). Then $\epsilon := \frac{1}{N}$ does the job. Indeed, if $y \in B_d(x,\epsilon)$, then

$$\frac{|1 - f_N(y)|}{N} = \frac{|f_N(x) - f_N(y)|}{N} \leq d(x,y) < \frac{1}{N},$$

hence $f_N(y) \neq 0$ and this can only happen if $y \in U$.

Finally, we show that $\mathcal{T}_d \subset \mathcal{T}$. It suffices to prove that, for each ball $B(x,\epsilon)$, there exists $U = U_{x,\epsilon} \in \mathcal{T}$ such that $x \in U \subset B(x,\epsilon)$. This will imply that $B(x,\epsilon)$ is open in X: indeed, for any $y \in B(x,\epsilon)$ we can choose $r > 0$ such that $B(y,r) \subset B(x,\epsilon)$ (e.g. take $r = \epsilon - d(x,y)$ and use the triangle inequality), and then $U_{y,r}$ will be an open in X contained in $B(x,\epsilon)$.

So, let us fix $x \in X$, $\epsilon > 0$ and look for $U \in \mathcal{T}$ with $x \in U \subset B(x,\epsilon)$. Choose $n > 2/\epsilon$ and set

$$U := \bigcap_{n=1}^{n_0} \{y \in X : \frac{|f_n(y) - f_n(x)|}{n} < \epsilon\}.$$

Since this is a finite intersection and each f_n is continuous, we have $U \in \mathcal{T}$. Clearly, $x \in U$. Note also that, from the choice of n_0 and the fact that $0 \leq |f_n| \leq 1$,

$$\frac{|f_n(x) - f_n(y)|}{n} \leq \frac{2}{n_0} < \epsilon \quad \forall\ n \geq n_0.$$

We deduce that $d(x,y) < \epsilon$ for all $y \in U$, i.e. $U \subset B(x,\epsilon)$. ☐

2. The Smirnov Metrization Theorem

In following is known as the Smirnov Metrization Theorem.

THEOREM 7.2. *A space X is metrizable iff it is Hausdorff, paracompact and locally metrizable.*

Theorem 6.11 takes care of the direct implication. Here we prove the converse. The proof is very similar to the proof of the Urysohn metrization theorem.

Claim 1: There exists a basis $\mathcal{B}$ for the topology of X, of type $\mathcal{B} = \cup_{n \in \mathbb{N}^*} \mathcal{B}_n$, where each $\mathcal{B}_n$ is a locally finite family. Moreover, for each $B \in \mathcal{B}$, there is a continuous function

$$f_B : X \to [0, 1] \quad \text{such that} \quad B = \{x \in X : f_B(x) \neq 0.\}.$$

PROOF. From the hypothesis it follows that there is a cover $\mathcal{U} = \{U_i : i \in I\}$ of X by opens in X, on which the topology is induced by a metric d_i; we may assume that $d_i \leq 1$ (cf. e.g. Exercise 1.34). For each $i \in I$, we denote by $B_i(x, r)$ the balls induced by d_i. They are open subsets of U_i, hence also open in X. By the shrinking lemma (Lemma 5.17), we can find another locally finite cover $\{V_i : i \in I\}$ with $\overline{V}_i \subset U_i$. For each integer n, we consider the open cover of X

$$\{B_i(x, \frac{1}{n}) \cap V_i : i \in I,\ x \in U_i\}.$$

Let $\mathcal{B}_n$ be a locally finite refinement of it and $\mathcal{B} = \cup_n \mathcal{B}_n$. For each $B \in \mathcal{B}$, we find i such that $B \subset V_i$ and then $f_B(x) := d_i(x, U_i - B)$ is a well-defined continuous function on U_i with which is zero outside B; since $\overline{B} \subset \overline{V}_i \subset U_i$ (where all the closures are in X), extending f_B by zero outside U_i, it will give us a function with the desired properties.

Finally, we show that $\mathcal{B}$ is a basis. Consider $U \subset X$ open, $x \in U$; we show that $x \in B \subset U$ for some $B \in \mathcal{B}$. Since $\mathcal{U}$ is locally finite, there is only a finite set of indices i with $x \in U_i$; call it F_x. For each $i \in F_x$, $U \cap U_i$ is open in (U_i, d_i) hence we find ϵ_i such that $B_i(x, \epsilon_i) \subset U \cap U_i$. Choose m with $2/m < \epsilon_i$ for all $i \in F_x$. Choose $B \in \mathcal{B}_m$ such that $x \in B$; due to the definition of $\mathcal{B}_m$, we have $B \subset B_i(y, 1/m)$ for some $i \in I$, $y \in U_i$. In particular, $x \in U_i$, hence $i \in F_x$. From the choice of m, we have $B_i(y, 1/m) \subset B_i(x, \epsilon_i)$; from the choice of ϵ_i, these are inside U. ☐

Claim 2: The following is a metric on X inducing the topology $\mathcal{T}$ of X.

$$d : X \times X \to \mathbb{R},\ d(x, y) = \sup\{\frac{1}{n}|f_B(x) - f_B(y)| : n \geq 1 \text{ integer}, B \in \mathcal{B}_n\}.$$

PROOF. By the same argument as in the Urysohn metrization theorem, d is a metric. Next, we show that $\mathcal{T} \subset \mathcal{T}_d$. Let $U \subset X$ open, $x \in U$. We have to find $r > 0$ such that $B_d(x, r) \subset U$. Since $\mathcal{B}$ is a basis, we find $B \in \mathcal{B}_n$ for some n, with $x \in B \subset U$. We claim that $r = \frac{1}{n}|f_B(x)|$ does the job. Indeed, if $y \in B_d(x, r)$, we have $\frac{1}{n}|f_B(y) - f_B(x)| < \frac{1}{n}|f_B(x)|$, hence $f_B(y) \neq 0$, hence $y \in B$, hence $y \in U$.

Finally, we show that $\mathcal{T}_d \subset T$. It suffices to show that, for any $x \in X$, $r > 0$, there exists $U \in \mathcal{T}$ such that $x \in U \subset B(x, r)$. Let $n_0 > 2/r$ be an integer. Since each $\mathcal{B}_n$ is locally finite, we find a neighborhood V of x which intersects only a finite number of Bs with $B \in \mathcal{B}_n$, $n \leq n_0$. Call these members $B_1, \ldots, B_k$. Choose $U \subset V$ such that

$$|f_{B_i}(y) - f_{B_i}(x)| < r \quad \forall\, y \in U,\ \forall\, i \in \{1, \ldots, k\}. \tag{2.1}$$

We claim that $U \subset B(x, r)$. That means that, for any $y \in U$, we have $\frac{1}{n}|f_B(y) - f_B(x)| < r$ for all $n \geq 1$ and $B \in \mathcal{B}_n$. If $n \geq n_0$ this is automatically satisfied since $|f_B| \leq 1$ and $2/n \leq 2/n_0 < r$. Assume now that $n \leq n_0$. If B is not one of the $B_1, \ldots, B_k$, then $U \cap B = \emptyset$ hence $f_B(y) = f_B(x) = 0$ and we are done. Finally, if $B = B_i$ for some i, then the desired inequality follows from (2.1). ☐

3. Consequences: the compact case, the locally compact case, manifolds

Here are some consequences of the metrization theorems from the previous sections. First of all, since topological manifolds are paracompact (see e.g. 5.20), the Smirnov metrization theorem immediately implies

THEOREM 7.3. *Any topological manifold is metrizable.*

This theorem follows also from the Urysohn metrization theorem (but note that the proof base on Smirnov's result is somehow more satisfactory: it uses paracompactness to pass from the local information to the global one; in particular, the Urysohn lemma is not used!). The Urysohn metrization theorem has however two more interesting consequences. First, for the compact case, we obtain:

THEOREM 7.4. *If X is a compact Hausdorff space, then the following are equivalent*
1. *X is metrizable.*
2. *X is second countable.*

Using the one-point compactification, for locally compact spaces we will obtain the following (which provides another proof to Theorem 7.3).

THEOREM 7.5. *Any locally compact Hausdorff and 2nd countable space is metrizable.*

In what follows, we will provide the missing proofs.

PROOF. (of Theorem 7.4) The reverse implication follows from the Urysohn metrization theorem since compact spaces are normal (Corollary 4.20). We now prove $1 \Longrightarrow 2$. Let d be a metric inducing the topology of X. Since X is totally bounded (cf. Theorem 6.7), for each n we find a finite set F_n such that

$$X = \bigcup_{x \in F_n} B(x, \frac{1}{n}).$$

The set $A = \cup_n F_n$ is a countable union of finite sets, hence it is countable. We deduce that

$$\mathcal{B} = \{B(a, \frac{1}{n}) : a \in A, n \geq 1 \text{ integer}\}$$

is a countable family of open sets of X. We claim it is a basis for the topology of X. Let U be an arbitrary open and $x \in U$. We have to prove that there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Since $x \in U$, we find an integer n such that $B(x, \frac{1}{n}) \subset U$. Using the defining property for F_{2n}, we see that there exists $a \in A$ such that $x \in B(a, \frac{1}{2n})$. Using the triangle inequality, we deduce that for each $y \in B(a, \frac{1}{2n})$,

$$d(x, y) \leq d(x, a) + d(a, y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

hence $y \in B(x, \frac{1}{n}) \subset U$. In conclusion, $B = B(a, \frac{1}{2n}) \in \mathcal{B}$ satisfies $x \in B \subset U$. $\square$

PROOF. (of Theorem 7.5) We apply the Theorem 7.4 to the one-point compactification (see Theorem 4.39) to deduce that X^+ is metrizable. Since X is a subspace of X^+, it is itself metrizable. $\square$