CHAPTER 8

Spaces of functions

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1. The algebra $\mathcal{C}(X)$ of continuous functions

We start this chapter with a discussion of continuous functions from a Hausdorff compact space to the real or complex numbers. It makes no difference whether we work over $\mathbb{R}$ or $\mathbb{C}$, so let's just use the notation $\mathbb{K}$ for one of these base fields and we call it the field of scalars. For each $z \in \mathbb{K}$, we can talk about $|z| \in \mathbb{R}$- the absolute value of z in the real case, or the norm of the complex number z in the complex case.

For a compact Hausdorff space X, we consider the set of scalar-valued functions on X:

$$\mathcal{C}(X) := \{f : X \to \mathbb{K} : f \text{ is continuous}\}.$$

When we want to make a distinction between the real and complex case, we will use the more precise notations $\mathcal{C}(X, \mathbb{R})$ and $\mathcal{C}(X, \mathbb{C})$.

In this section we look closer at the "structure" that is present on $\mathcal{C}(X)$. First, there is a topological one. When X was an interval in $\mathbb{R}$, this was discussed in Section 9, Chapter 3 (there $n = 1$ in the real case, $n = 2$ in the complex one). As there, there is a metric on $\mathcal{C}(X)$:

$$d_{\text{sup}}(f, g) := \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}.$$

Since f, g are continuous and X is compact, $d_{\text{sup}}(f, g) < \infty$. As in *loc.cit.*, d_{sup} is a metric and the induced topology is called <u>the uniform topology</u> on $\mathcal{C}(X)$. And, still as in *loc.cit.*:

THEOREM 8.1. *For any compact Hausdorff space X, $(\mathcal{C}(X), d_{sup})$ is a complete metric space.*

PROOF. Let (f_n) be a Cauchy sequence in $\mathcal{C}(X)$. The proof of Theorem 3.27 applies word by word to our X instead of the interval I, to obtain a function $f : X \to \mathbb{R}$ such that $d_{\text{sup}}(f_n, f) \to 0$ when $n \to \infty$. Then similarly, the proof of Theorem 3.26 (namely that $\mathcal{C}(I, \mathbb{R}^n)$ is closed in $(\mathcal{F}(I, \mathbb{R}^n), \hat{d}_{\text{sup}})$) applies word by word with I replaced by X to conclude that $f \in \mathcal{C}(X)$. $\square$

The metric on $\mathcal{C}(X)$ is of a special type: it comes from a norm. Namely, defining

$$||f||_{\text{sup}} := \sup\{|f(x)| : x \in X\} \in [0, \infty)$$

for $f \in \mathcal{C}(X)$, we have

$$d_{\text{sup}}(f, g) = ||f - g||_{\text{sup}}.$$

DEFINITION 8.2. *Let V be a vector space (over our $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A norm on V is a function*

$$|| \cdot || : V \to [0, \infty), \ v \mapsto ||v||$$

such that

$$||v|| = 0 \Longleftrightarrow v = 0$$

and is compatible with the vector space structure in the sense that:

$$||\lambda v|| = |\lambda| \cdot ||v|| \quad \forall \ \lambda \in \mathbb{K}, v \in V,$$

$$||v + w|| \leq ||v|| + ||w|| \quad \forall \ v, w \in V.$$

The metric associated to $|| \cdot ||$ is the metric $d_{||\cdot||}$ given by

$$d_{||\cdot||}(v, w) := ||v - w||.$$

A Banach space is a vector space V endowed with a norm $|| \cdot ||$ such that $d_{||\cdot||}$ is complete. When $\mathbb{K} = \mathbb{R}$ we talk about real Banach spaces, when $\mathbb{K} = \mathbb{C}$ about complex ones.

With these, we can now reformulate our discussion as follows:

COROLLARY 8.3. *For any compact Hausdorff space X, $(\mathcal{C}(X), || \cdot ||_{sup})$ is a Banach space.*

Of course, this already makes reference to some of the algebraic structure on $\mathcal{C}(X)$- that of vector space. But, of course, there is one more natural operation on continuous functions: the multiplication, defined pointwise:

$$(fg)(x) = f(x)g(x).$$

DEFINITION 8.4. *A $\mathbb{K}$-algebra is a vector space A over $\mathbb{K}$ together with an operation*

$$A \times A \to A, \quad (a, b) \mapsto a \cdot b$$

which is unital in the sense that there exists an element $1 \in A$ such that

$$1 \cdot a = a \cdot 1 = a \quad \forall\, a \in A,$$

and which is $\mathbb{K}$-bilinear and associative, i.e., for all $a, a', b, b', c \in A$, $\lambda \in \mathbb{K}$,

$$(a + a') \cdot b = a \cdot b + a' \cdot b, \;\; a \cdot (b + b') = a \cdot b + a \cdot b',$$
$$(\lambda a) \cdot b = \lambda(a \cdot b) = a \cdot (\lambda b), ,$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \;.$$

We say that A is commutative if $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$.

EXAMPLE 8.5. The space of polynomials $\mathbb{K}[X_1, \ldots, X_n]$ in n variables (with coefficients in $\mathbb{K}$) is a commutative algebra.

Hence the algebraic structure of $\mathcal{C}(X)$ is that of an algebra. Of course, the algebraic and the topological structures are compatible. Here is the precise abstract definition.

DEFINITION 8.6. *A Banach algebra (over $\mathbb{K}$) is an algebra A equipped with a norm $||\cdot||$ which makes $(A, ||\cdot||)$ into a Banach space, such that the algebra structure and the norm are compatible, in the sense that,*

$$||a \cdot b|| \leq ||a|| \cdot ||b|| \quad \forall\, a, b \in A.$$

Hence, with all these terminology, the full structure of $\mathcal{C}(X)$ is summarized in the following:

COROLLARY 8.7. *For any compact Hausdorff space, $\mathcal{C}(X)$ is a Banach algebra.*

There is a bit more one can say in the case when $\mathbb{K} = \mathbb{C}$: there is also the operation of conjugation, defined again pointwise:

$$\overline{f}(x) := \overline{f(x)}.$$

As before, this comes with an abstract definition:

DEFINITION 8.8. *A *-algebra is an algebra A over $\mathbb{C}$ together with an operation*

$$(-)^* : A \to A, \;\; a \mapsto a^*,$$

which is an involution, i.e.

$$(a^*)^* = a \quad \forall\, a \in A,$$

and which satisfies the following compatibility relations with the rest of the structure:

$$(\lambda a)^* = \overline{\lambda} a^* \quad \forall\, a \in A, \lambda \in \mathbb{C},$$
$$1^* = 1, (a + b)^* = a^* + b^*, \;\; (ab)^* = b^* a^* \quad \forall\, a, b \in A.$$

*Finally, a C^*-algebra is a Banach algebra $(A, ||\cdot||)$ endowed with a *-algebra structure, s.t.*

$$||a^*|| = ||a||, \quad ||a^* a|| = ||a||^2 \quad \forall\, a \in A.$$

Of course, $\mathbb{C}$ with its norm is the simplest example of C^*-algebra. To summarize our discussion in the complex case:

COROLLARY 8.9. *For any compact Hausdorff space, $\mathcal{C}(X, \mathbb{C})$ is a C^*-algebra.*

2. Approximations in $\mathcal{C}(X)$: the Stone-Weierstrass theorem

The Stone-Weierstrass theorem is concerned with density in the space $\mathcal{C}(X)$ (endowed with the uniform topology); the simplest example is Weierstrass's approximation theorem which says that, when X is a compact interval, the set of polynomial functions is dense in the space of all continuous functions. The general criterion makes use of the algebraic structure on $\mathcal{C}(X)$.

DEFINITION 8.10. *Given an algebra A (over the base field $\mathbb{R}$ or $\mathbb{C}$), a subalgebra is any vector subspace $B \subset A$, containing the unit 1 and such that*

$$b \cdot b' \in B \quad \forall\, b, b' \in B.$$

When we want to be more specific about the base field, we talk about real or complex subalgebras.

EXAMPLE 8.11. When $X = [0,1]$, the set of polynomial functions on $[0,1]$ is a unital subalgebra of $\mathcal{C}([0,1])$. Here the base field can be either $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 8.12. *Given a topological space X and a subset $\mathcal{A} \subset \mathcal{C}(X)$, we say that $\mathcal{A}$ is point-separating if for any $x, y \in X$, $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.*

EXAMPLE 8.13. When $X = [0,1]$, the subalgebra of polynomial functions is point-separating.

Here is the Stone-Weierstrass theorem in the real case ($\mathbb{K} = \mathbb{R}$).

THEOREM 8.14. *(Stone-Weierstrass) Let X be a compact Hausdorff space. Then any point-separating real subalgebra $\mathcal{A} \subset \mathcal{C}(X, \mathbb{R})$ is dense in $(\mathcal{C}(X,\mathbb{R}), d_{sup})$.*

PROOF. We first show that there exists a sequence $(p_n)_{n\geq 1}$ of real polynomials which, on the interval $[0,1]$, converges uniformly to the function $\sqrt{t}$. We construct p_n inductively by

$$p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n(t)^2), \quad p_1 = 0.$$

We first claim that $p_n(t) \leq \sqrt{t}$ for all $t \in [0,1]$. This follows by induction on n since

$$\sqrt{t} - p_{n+1}(t) = (\sqrt{t} - p_n(t))(1 - \frac{\sqrt{t} + p_n(t)}{2}).$$

and $p_n(t) \leq \sqrt{t}$ implies that $\sqrt{t} + p_n(t) \leq 2\sqrt{t} \leq 2$ for all $t \in [0,1]$ (hence the right hand side is positive). Next, the recurrence relation implies that $p_{n+1}(t) \geq p_n(t)$ for all t. Then, for each $t \in [0,1]$, $(p_n(t))_{n\geq 1}$ is increasing and bounded above by 1, hence convergent; let $p(t)$ be its limit. By passing to the limit in the recurrence relation we find that $p(t) = \sqrt{t}$.

We still have to show that p_n converges uniformly to p on $[0,1]$. Let $\epsilon > 0$ and we look for N such that $p(t) - p_n(t) < \epsilon$ for all $n \geq N$ (note that $p - p_n$ is positive). Let $t \in [0,1]$. Since $p_n(t) \to p(t)$, we find $N(t)$ such that $p(t) - p_n(t) < \epsilon/3$ for all $n \geq N(t)$. Since p and $p_{N(t)}$ are continuous, we find an open neighborhood $V(t)$ of t such that $|p(s) - p(t)| < \epsilon/3$ and similarly for $p_{N(t)}$, for all $s \in V(t)$. Note that, for each $s \in V(t)$ we have the desired inequality:

$$p(s) - p_{N(t)}(s) = (p(s) - p(t)) + (p(t) - p_{N(t)}(t)) + (p_{N(t)}(t) - p_{N(t)}(s)) < 3\frac{\epsilon}{3} = \epsilon.$$

Varying t, $\{V(t) : t \in [0,1]\}$ will be an open cover of $[0,1]$ hence we can extract an open subcover $\{V(t_1), \ldots, V(t_k)\}$. Then $N(t) := \max\{N(t_1), \ldots, N(t_k)\}$ does the job: for $n \geq N(t)$ and $t \in [0,1]$, t belongs to some $V(t_i)$ and then

$$p(s) - p_n(s) \leq p(s) - p_{N(t_i)}(s) < \epsilon.$$

We now return to the theorem and we denote by $\overline{\mathcal{A}}$ the closure of $\mathcal{A}$. We claim that:

$$f, g \in \overline{\mathcal{A}} \Longrightarrow \sup(f,g), \inf(f,g) \in \overline{\mathcal{A}},$$

where $\sup(f, g)$ is the function $x \mapsto \max\{f(x), g(x)\}$, and similarly $\inf(f, g)$. Since any $f \in \overline{\mathcal{A}}$ is the limit of a sequence in $\mathcal{A}$, we may assume that $f, g \in \mathcal{A}$. Since $\sup(f, g) = (f + g + |f - g|)/2$, $\inf(f, g) = (f + g - |f - g|)/2$ and $\mathcal{A}$ is a vector space, it suffices to show that, for any $f \in \mathcal{A}$, $|f| \in \overline{\mathcal{A}}$. Since any continuous f is bounded (X is compact!), by dividing by a constant, we may assume that $f \in \mathcal{A}$ takes values in $[-1, 1]$. Using the polynomials p_n, since $\mathcal{A}$ is a subalgera, $f_n := p_n(f^2) \in \mathcal{A}$, and it converges uniformly to $p(f^2) = |f|$. Hence $|f| \in \overline{\mathcal{A}}$.

We need one more remark: for any $x, y \in X$ with $x \neq y$ and any $a, b \in \mathbb{R}$, there exists $f \in \mathcal{A}$ such that $f(x) = a$, $f(y) = b$. Indeed, by hypothesis, we find $g \in \mathcal{A}$ such that $g(x) \neq g(y)$; since $\mathcal{A}$ contains the unit (hence all the constants),

$$f := a + \frac{b - a}{g(y) - g(x)}(g - g(x))$$

will be in $\mathcal{A}$ and it clearly satisfies $f(x) = a$, $f(y) = b$.

Let now $h \in \mathcal{C}(X, \mathbb{R})$. Let $\epsilon > 0$ and we look for $f \in \mathcal{A}$ such that $d_{\sup}(h, f) \leq \epsilon$.

We first show that, for any $x \in X$, there exists $f_x \in \mathcal{A}$ such that $f_x(x) = h(x)$ and $f_x(y) < h(y) + \epsilon$ for all $y \in X$. For this, for any $y \in X$ we choose a function $f_{x,y} \in \mathcal{A}$ such that $f_{x,y}(x) = h(x)$ and $f_{x,y}(y) \leq h(y) + \epsilon/2$ (possible due to the previous step). Using continuity, we find a neighborhood $V(y)$ of y such that $f_{x,y}(y') < h(y') + \epsilon$ for all $y' \in V(y)$. From the cover $\{V(y) : y \in X\}$ we extract a finite subcover $\{V(y_1), \ldots, V(y_k)\}$ and put

$$f_x := \inf\{f_{x,y_1}, \ldots, f_{x,y_k}\}.$$

From the previous steps, $f_x \in \overline{\mathcal{A}}$; by construction, $f_x(y) < h(y) + \epsilon$ for all $y \in X$ and $f_x(x) = h(x)$. Due to the last equality, we find an open neighborhood $W(x)$ of x such that $f_x(x') > h(x') - \epsilon$ for all $x' \in W(x)$. We now let x vary and choose $x_1, \ldots, x_l$ such that $\{W(x_1), \ldots, W(x_l)\}$ cover X. Finally, we put

$$f := \sup\{f_{x_1}, \ldots, f_{x_l}\}.$$

By the discussion above, it belongs to $\overline{\mathcal{A}}$ while, by construction, $d_{\sup}(h, f) \leq \epsilon$. $\square$

The previous theorem does not hold (word by word) over $\mathbb{C}$ instead of the reals. The appropriate complex-version of the Stone-Weierstrass theorem requires an extra-condition which refers precisely to the extra-structure present in the complex case: conjugation (hence the *-algebra structure on $\mathcal{C}(X, \mathbb{C})$).

Definition 8.15. *Given a unital *-algebra A, a subalgebra $B \subset A$ is called a <u>*- subalgebra</u> if*

$$b^* \in B \quad \forall\, b \in B.$$

With this, we have:

Corollary 8.16. *Let X be a compact Hausdorff space. Then any point-separating *-subalgebra $\mathcal{A} \subset \mathcal{C}(X, \mathbb{C})$ is dense in $(\mathcal{C}(X, \mathbb{C}), d_{sup})$.*

Proof. Let $\mathcal{A}_{\mathbb{R}} := \mathcal{A} \cap \mathcal{C}(X, \mathbb{R})$. Since for any $f \in \mathcal{F}$,

$$\mathrm{Re}(f) = \frac{f + \overline{f}}{2}, \ \mathrm{Im}(f) = \frac{f - \overline{f}}{2i}$$

belong to $\mathcal{A}_{\mathbb{R}}$, it follows that $\mathcal{A}_{\mathbb{R}}$ separates points and is a unital subalgebra of $\mathcal{C}(X, \mathbb{R})$. From the previous theorem, $\mathcal{A}_{\mathbb{R}}$ is dense in $\mathcal{C}(X, \mathbb{R})$. Hence $\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}}$ is dense in $\mathcal{C}(X, \mathbb{C})$. $\square$

3. Recovering X from $\mathcal{C}(X)$: the Gelfand Naimark theorem

The Gelfand-Naimark theorem says that a compact Hausdorff space can be recovered from its algebra $\mathcal{C}(X)$ of continuous functions (using only the algebra structure!!!) . Again, it makes no difference whether we work over $\mathbb{R}$ or $\mathbb{C}$; so let's just fix $\mathbb{K}$ to be one of them and that we work over $\mathbb{K}$. The key ingredient in recovering X from $\mathcal{C}(X)$ is the notion of maximal ideal.

DEFINITION 8.17. *Let A be an algebra. A ideal of A is any vector subspace $I \subset A$ satisfying*

$$a \cdot x,\ x \cdot a \in I \quad \forall\, a \in A,\ x \in I.$$

The ideal I is called maximal if there is no other ideal J strictly containing I and different from A. We denote by M_A the set of all maximal ideals of A.

For instance, for $A = \mathcal{C}(X)$ (X a topological space), any subspace $A \subset X$ defines an ideal:

$$I_A := \{f \in \mathcal{C}(X) : f|_A = 0\}.$$

When $A = \{x\}$ is a point, we denote this ideal simply by I_x. Note that $I_A \subset I_x$ for all $x \in A$.

PROPOSITION 8.18. *If X is a compact Hausdorff space, then I_x is a maximal ideal of $\mathcal{C}(X)$ for all $x \in X$, and any maximal ideal is of this type. In other words, one has a bijection*

$$\phi : X \xrightarrow{\sim} M_{\mathcal{C}(X)}, \quad x \mapsto I_x.$$

PROOF. Fix $x \in X$ and we show that I_x is maximal. Let I be another ideal strictly containing I_x; we prove $I = \mathcal{C}(X)$. Since $I \neq I_x$, we find $f \in I$ such that $f(x) \neq 0$. Since f is continuous, we find an open U such that $x \in U$, $f \neq 0$ on U. Now, $\{x\}$ and $X - U$ are two disjoint closed subsets of X hence, by Urysohn lemma, there exists $\eta \in \mathcal{C}(X)$ such that $\eta(x) = 0$, $\eta = 1$ outside U. Clearly, $\eta \in I_x \subset I$. Since I is and ideal containing f and η, $g := |f|^2 + \eta^2 \in I$. Note that $g > 0$: for $x \in U$, $f(x) \neq 0$, while for $x \in X - U$, $\eta(x) = 1$. But then any $h \in \mathcal{C}(X)$ is in I since it can be written as $g\frac{h}{g}$ with $g \in I$, $\frac{h}{g} \in \mathcal{C}(X)$. Hence $I = \mathcal{C}(X)$.

We still have to show that, if I is a maximal ideal, then $I = I_x$ for some x. It suffices to show that $I \subset I_x$ for some $x \in X$. Assume the contrary. Then, for any $x \in X$, we find $f_x \in I$ s.t. $f_x(x) \neq 0$. Since f_x is continuous, we find an open U_x s.t. $x \in U_x$, $f_x \neq 0$ on U_x. Now, $\{U_x : x \in X\}$ is an open cover of X. By compactness, we can select a finite subcover $\{U_{x_1}, \ldots, U_{x_k}\}$. But then

$$g := |f_{x_1}|^2 + \ldots + |f_{x_k}|^2 \in I$$

and $g > 0$ on X. By the same argument as above, we get $I = \mathcal{C}(X)$- contradiction! □

The proposition shows how to recover X from $\mathcal{C}(X)$ as a set. To recover the topology, it is useful to slightly change the point of view and look at characters instead of maximal ideals.

DEFINITION 8.19. *Given an algebra A, a character of A is any $\mathbb{K}$-linear function $\chi : A \to \mathbb{K}$ which is not identically zero and satisfies*

$$\chi(a \cdot b) = \chi(a)\chi(b) \quad \forall\ \ a, b \in A.$$

The set of characters of A is denoted by X_A and is called the spectrum of A. When we want to be more precise about $\mathbb{K}$, we talk about the real or the complex spectrum of A.

The previous proposition can be reformulated into:

COROLLARY 8.20. *If X is a compact Hausdorff space then, for any $x \in X$,*

$$\chi_x : \mathcal{C}(X) \to \mathbb{K},\ \chi_x(f) = f(x)$$

is a character of $\mathcal{C}(X)$, and any character is of this type. In other words, one has a bijection

$$\phi : X \xrightarrow{\sim} X_{\mathcal{C}(X)}, \quad x \mapsto \chi_x.$$

PROOF. The main observation is that characters correspond to maximal ideals. It should be clear that any χ_x is a character. Let now χ be an arbitrary character. Let $I := \{f \in \mathcal{C}(X) : \chi(f) = 0\}$ (an ideal- check that!). We will make use of the remark that

$$f - \chi(f) \cdot 1 \in I \tag{3.1}$$

for all $f \in \mathcal{C}(X)$ (indeed, all these elements are killed by χ). We show that I is maximal. Let J be another ideal strictly containing I. Choosing $f \in J$ not belonging to I (i.e. $\chi(f) \neq 0$) and using (3.1) and $I \subset J$, we find that $1 \in J$ hence, as above, $J = \mathcal{C}(X)$. This proves that I is maximal. We deduce that it is of type I_x for some $x \in X$. But then, using (3.1) again we deduce that $(f - \chi(f) \cdot 1)(x) = 0$ for all f, i.e. $\chi = \chi_x$. $\square$

The advantage of characters is that there is a natural topology on X_A for any algebra A.

DEFINITION 8.21. *Let A be an algebra A and let X_A be its spectrum. For any $a \in A$, define*

$$f_a : X_A \to \mathbb{K}, \; f_a(\chi) := \chi(a).$$

We define $\mathcal{T}$ as the smallest topology on X_A with the property that all the functions $\{f_a : a \in A\}$ are continuous. The resulting topological space $(X_A, \mathcal{T})$ is called the <u>*topological spectrum of A*</u>.

THEOREM 8.22. *Any compact Hausdorff space X is homeomorphic to the topological spectrum of its algebra $\mathcal{C}(X)$ of continuous functions.*

PROOF. Let $\mathcal{T}_X$ be the topology of X. We still have to show that the bijection ψ is a homeomorphism. Equivalently: ψ induces a topology $\mathcal{T}'$ on X which is the smallest topology with the property that all $f \in \mathcal{C}(X)$ are continuous as functions with respect to this new topology $\mathcal{T}'$. We have to show that $\mathcal{T}'$ coincides with the original topology $\mathcal{T}_X$. From the defining property of $\mathcal{T}'$, the inclusion $\mathcal{T}' \subset \mathcal{T}_X$ is tautological. For the other inclusion, we have to use the more explicit description of $\mathcal{T}'$: it is the topology generated by the subsets of X of type $f^{-1}(V)$ with $f \in \mathcal{C}(X)$ and $V \subset \mathbb{K}$ open. We have to show that any $U \in \mathcal{T}_X$ is in $\mathcal{T}'$. Fixing U, it suffices to show that for any $x \in U$ we find f and V such that

$$x \in f^{-1}(V) \subset U.$$

But this follows again by the Urysohn lemma: we find $f : X \to [0, 1]$ continuous such that $f(x) = 0$ and $f = 1$ outside U. Taking $V = (-1, 1)$, we have the desired property. $\square$

REMARK 8.23. *(for the curious reader)* In this remark we work over $\mathbb{K} = \mathbb{C}$. An interesting question that we did not answer is: which algebras A are of type $\mathcal{C}(X)$ for some compact Hausdorff X? What we did show is that the space must be X_A. Note also that the map ψ makes sense for any algebra A:

$$\psi_A : A \to \mathcal{C}(X_A), \quad a \mapsto (f_a : X_A \to \mathbb{C} \text{ given by } f_a(\chi) = \chi(a)).$$

Hence a possible answer is: algebras with the property that X_A is compact and Hausdorff, and ψ_A is an isomorphism (bijection). But this is clearly far from satisfactory.

The best answer is given by the full version of the Gelfand-Naimark theorem: it is the commutative C^*-algebras! This answer may seem a bit unfair since the notion of C^*-algebras seem to depend on data which is not algebraic (the norm!). However, a very special feature of C^*-algebras is that their norm can be recovered from the algebraic structure by the formula:

$$||a||^2 = \sup\{|\lambda| : \lambda \in \mathbb{C} \text{ such that } \lambda 1 - a^*a \text{ is not invertible}\}.$$

One remark about the proof: one first shows that X_A is compact and Hausdorff; then that $||\psi_A(a)|| = ||a||$ for all $a \in A$; this implies that ψ_A is injective and the image is closed in $\mathcal{C}(X_A)$; finally, the Stone-Weierstrass implies that the image is dense in $\mathcal{C}(X_A)$; hence ψ_A is bijective.

4. General function spaces $\mathcal{C}(X, Y)$

For any two topological spaces X and Y we denote by $\mathcal{C}(X, Y)$ the set of continuous functions from X to Y- a subset of the set $\mathcal{F}(X, Y)$ of all functions from X to Y. In general, there are several interesting topologies on $\mathcal{C}(X, Y)$. So far, in this chapter we were concerned with the uniform topology on $\mathcal{C}(X, Y)$ when X is compact and Hausdorff and $Y = \mathbb{R}$ or $\mathbb{C}$. In Section 9 , Chapter 3, in the case when $X \subset \mathbb{R}$ was an interval and $Y = \mathbb{R}^n$, we looked at the three topologies: of pointwise convergence, of uniform convergence, and of uniform convergence on compacts.

In this section we look at generalizations of these topologies to the case when X and Y are more general topological spaces. We assume throughout this entire section that

$$X - \text{is a locally compact topological space}, \; Y - \text{is a metric space with a fixed metric } d.$$

These assumptions are not needed everywhere (e.g. for the pointwise topology on $\mathcal{C}(X, Y)$, the topology of X is completely irrelevant, etc etc). They are made in order to simplify the presentation.

4.1. Pointwise convergence, uniform convergence, compact convergence.

Almost the entire Section 9, Chapter 3 goes through in this generality without any trouble ("word by word" most of the times). For instance, given a sequence $\{f_n\}_{n\geq 1}$ in $\mathcal{F}(X, Y)$, $f \in \mathcal{F}(X, Y)$, we will say that:

- f_n converges pointwise to f, and we write $f_n \stackrel{pt}{\to} f$, if $f_n(x) \to f(x)$ for all $x \in X$.
- f_n converges uniformly to f, and we write $f_n \rightrightarrows f$, if for any $\epsilon > 0$, there exists n_ϵ s.t.
$$d(f_n(x), f(x)) < \epsilon \quad \forall\, n \geq n_\epsilon, \; \forall\, x \in X.$$
- f_n converges uniformly on compacts to f, and we write $f_n \stackrel{\text{cp}}{\to} f$ if, for any $K \subset X$ compact, $f_n|_K \rightrightarrows f|_K$.

And, as in *loc.cit* (with exactly the same proof), these convergences correspond to convergences with respect to the following topologies on $\mathcal{F}(X, Y)$:

- the pointwise topology, denoted $\mathcal{T}_{pt}$, is the topology generated by the family of subsets
$$S(x, U) := \{f \in \mathcal{F}(X, Y) : f(x) \in U\} \subset \mathcal{F}(X, Y),$$
with $x \in X$, $U \subset Y$ open.
- the uniform topology is induced by a sup-metric. For $f, g \in \mathcal{F}(X, Y)$, we define
$$d_{\text{sup}}(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$
Again, to overcome the problem that this supremum may be infinite (for some f and g) and to obtain a true metric, once considers
$$\hat{d}_{\text{sup}}(f, g) = \min(d_{\text{sup}}(f, g), 1).$$
The uniform topology is the topology associated to $\hat{d}_{\text{sup}}$; it is denoted by $\mathcal{T}_{\text{unif}}$.
- the topology of compact convergence, denoted $\mathcal{T}_{cp}$, is the topology generated by the family of subsets
$$B_K(f, \epsilon) := \{g \in \mathcal{F}(X, Y) : d(f(x), g(x)) < \epsilon \;\; \forall\, x \in K\},$$
with $K \subset X$ compact, $\epsilon > 0$.

We will be mainly concerned with the restrictions of these topologies to the set $\mathcal{C}(X, Y)$ of continuous functions from X to Y. So, the part of Theorem 3.27 concerning continuous functions, with exactly the same proof, gives us the following:

THEOREM 8.24. *If (Y, d) is complete, then $(\mathcal{C}(X, Y), \hat{d}_{sup})$ is complete.*

4.2. Equicontinuity. A useful concept regarding function spaces is equicontinuity. Recall that X is a locally compact space and (Y,d) is a metric space. Then a function $f : X \to Y$ is continuous if it is continuous at each point, i.e. if for each $x_0 \in X$ and any $\epsilon > 0$ there exists a neighborhood V of x_0 such that

$$d(f(x), f(x_0)) < \epsilon \quad \forall\, x \in U. \tag{4.1}$$

DEFINITION 8.25. *A subset $\mathcal{F} \subset \mathcal{C}(X,Y)$ is called equicontinuous if for any $x_0 \in X$ and any $\epsilon > 0$ there exists a neighborhood V of x_0 such that (4.1) holds for all $f \in \mathcal{F}$.*

When X is itself a metric space, then there is a "uniform" version of continuity and equicontinuity.

DEFINITION 8.26. *Assume that both (X,d) and (Y,d) are metric spaces. Then*

1. *A map $f : X \to Y$ is called uniformly continuous if for all $\epsilon > 0$ there exists $\delta > 0$ s.t.*

$$d(f(x), f(y)) < \epsilon \quad \forall\, x, y \in X \text{ with } d(x,y) < \delta. \tag{4.2}$$

2. *A subset $\mathcal{F} \subset \mathcal{C}(X,Y)$ is called uniformly equicontinuous if for all $\epsilon > 0$ there exists $\delta > 0$ s.t. (4.2) holds for all $f \in \mathcal{F}$.*

From the definitions we immediately see that, in general, the following implications hold:

$$\begin{tikzcd}
\mathcal{F} - \text{is uniformly equicontinuous} \arrow[r] \arrow[d] & \text{each } f \in \mathcal{F} \text{ is uniformly continuous} \arrow[d] \\
\mathcal{F} - \text{is equicontinuous} \arrow[r] & \text{each } f \in \mathcal{F} \text{ is continuous}
\end{tikzcd}$$

As terminology, we say that a sequence $(f_n)_{n\geq 1}$ is equicontinuous if the set $\{f_n : n \geq 1\}$ is.

PROPOSITION 8.27. *A sequence $\{f_n\}_{n\geq 1}$ is convergent in $(\mathcal{C}(X,Y), \mathcal{T}_{cp})$ if and only if it is convergent in $(\mathcal{C}(X,Y), \mathcal{T}_{pt})$ and it is equicontinuous.*

PROOF. For the direct implication, we still have to show that, if $f_n \overset{\text{cp}}{\to} f$, then $\{f_n\}$ is equicontinuous. Let $x_0 \in X$. Since f is continuous, we find a neighborhood V of x_0 such that $d(f(x), f(x_0)) < \epsilon/3$ for all $x \in V$. Since X is locally compact, we may assume V to be compact. Then $f_n|_V \rightrightarrows f|_V$ hence we find n_ϵ such that $d(f_n(x), f(x)) < \epsilon/3$ for all $n \geq n_\epsilon$, $x \in V$. Then

$$d(f_n(x), f_n(x_0)) \leq d(f_n(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), f_n(x_0)) < \epsilon.$$

for all $x \in V$ and $n \geq n_\epsilon$. By making V smaller if necessary, the previous inequality will also hold for all $n < n_\epsilon$ (since there are a finite number of such n's, and each f_n is continuous).

We now prove the converse. Assume equicontinuity and assume that $f_n \to f$ pointwise. Let $K \subset X$ be compact; we prove that $f_n|_K \rightrightarrows f|_K$. Let $\epsilon > 0$. For each $x \in K$, there is an open V_x containing x, such that

$$d(f_n(y), f_n(x)) < \epsilon/3 \quad \forall\, y \in V_x,\ \forall\, n.$$

Since $\{V_x\}$ covers the compact K, we find a finite number of points $x_i \in K$ (with $1 \leq i \leq k$) such that the opens $V_i = V_{x_i}$ cover K. Since $f_n(x_i) \to f(x_i)$, we find n_ϵ such that

$$d(f_n(x_i), f(x_i)) < \epsilon/3 \quad \forall\, n \geq n_\epsilon\ \forall\, i \in \{1, \ldots, k\}.$$

Now, for all $n \geq n_\epsilon$ and $x \in K$, choosing i such that $x \in V_i$, we have

$$d(f_n(x), f(x)) \leq d(f_n(x), f_n(x_i)) + d(f_n(x_i), f(x_i)) + d(f(x_i), f(x)) < 3\frac{\epsilon}{3} = \epsilon.$$

(here we used that $d(f(x_i), f(x)) = \lim_n d(f_n(x_i), f_n(x)) \leq \epsilon/3$). $\square$

4.3. Boundedness. Let's also briefly discuss boundedness. For $\mathcal{F} \subset \mathcal{C}(X, Y)$ and $x \in X$ we use the notation

$$\mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\}.$$

As we have already discussed, in a metric space, there is also the notion of "totally bounded", which is an improvement of the notion of "bounded". Also, in our case we can talk about boundedness (and totally boundedness) with respect to $\hat{d}_{\text{sup}}$, or we can have pointwise versions (with respect to the metric d of Y). In total, four possibilities:

DEFINITION 8.28. *We say that $\mathcal{F} \subset \mathcal{C}(X, Y)$ is:*

- *Bounded if it is bounded in $(\mathcal{C}(X, Y), \hat{d}_{sup})$.*
- *Totally bounded if it is totally bounded in $(\mathcal{C}(X, Y), \hat{d}_{sup})$.*
- *Pointwise bounded if $\mathcal{F}(x)$ is bounded in (Y, d) for all $x \in X$.*
- *Pointwise totally bounded if $\mathcal{F}(x)$ is totally bounded in (Y, d) for all $x \in X$.*

From the definitions we immediately see that, in general, the following implications hold:

$$\begin{tikzcd} \text{totally bounded} \arrow[r] \arrow[d] & \text{bounded} \arrow[d] \\ \text{pointwise totally bounded} \arrow[r] & \text{pointwise bounded} \end{tikzcd}$$

EXAMPLE 8.29. For $Y = \mathbb{R}^n$ with the Euclidean metric d, since totally boundedness and boundedness in $(\mathbb{R}^n, d)$ are equivalent, we see that a subset $\mathcal{F} \subset \mathcal{C}(X, \mathbb{R}^n)$ is pointwise totally bounded if and only if it is pointwise bounded. However, it is not true that $\mathcal{F}$ is totally bounded if and only if it is bounded. In general, totally boundedness implies equicontinuity:

PROPOSITION 8.30. *If $\mathcal{F} \subset \mathcal{C}(X, Y)$ is totally bounded then it must be equicontinuous. Moreover, if each $f \in \mathcal{F}$ is uniformly continuous, then $\mathcal{F}$ is even uniformly equicontinuous.*

PROOF. Fix $\epsilon > 0$ and $x_0 \in X$. By assumption, we find $f_1, \ldots, f_k \in \mathcal{F}$ such that

$$\mathcal{F} \subset B(f_1, \epsilon/3) \cup \ldots \cup B(f_k, \epsilon/3),$$

where the balls are the ones corresponding to $\hat{d}_{\text{sup}}$. Since each f_i is continuous, we find a neighborhood U_i of f_i such that

$$d(f_i(x), f_i(x_0)) < \epsilon/3, \quad \forall\, x \in U_i.$$

Then $U = \cap_i U_i$ is a neighborhood of x_0. For $x \in U$, $f \in \mathcal{F}$, choosing i s.t. $f \in B(f_i, \epsilon/3)$:

$$d(f(x), f(x_0)) \leq d(f(x), f_i(x_0)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0)),$$

which is $< \epsilon$ (for all $x \in U$, $f \in \mathcal{F}$). This proves equicontinuity. For the second part the argument is completely similar (even simpler as all U_i will become X), where we use that each f_i is uniformly continuous. $\square$

Let us also recall that the notion of totally boundedness was introduced in order to characterize compactness. Since $\hat{d}_{\text{sup}}$ is complete whenever (Y, d) is (Theorem 8.24) we deduce:

PROPOSITION 8.31. *A subset $\mathcal{F} \subset \mathcal{C}(X, Y)$ is totally bounded if and only if it is relatively compact in $(\mathcal{C}(X, Y), \hat{d}_{sup})$.*

4.4. The case when X is a compact metric space. When X is a compact metric space the situation simplifies quite a bit. In some sense, the pointwise conditions imply the uniform ones (in the vertical implications from the previous two diagrams). To be more precise, let us combine the two diagrams and Proposition 8.30 together into a diagram of implications:

$$\begin{tikzcd}
\text{totally bounded} \arrow[r] \arrow[d] & \text{bounded and uniformly equic.} \arrow[r] \arrow[d] & \text{uniformly equic.} \arrow[d] \\
\text{pt. totally bounded and equic.} \arrow[r] & \text{pt. bounded and equic.} \arrow[r] & \text{equic.}
\end{tikzcd}$$

where "pt." stands for pointwise and "equic." for equicontinuous. Of course, we could have continued to the right with "each $f \in \mathcal{F}$ is (uniformly) continuous". We are looking at the converses of the vertical implications. The first one is of the next subsection. For the rest:

THEOREM 8.32. *If (X,d) is a compact metric space, $f : X \to Y$, $\mathcal{F} \subset \mathcal{C}(X,Y)$, then*
1. *(f is continuous) $\iff$ (f is uniformly continuous).*
2. *($\mathcal{F}$ is equicontinuous) $\iff$ ($\mathcal{F}$ is uniformly equicontinuous). In this case, moreover,*
   a. *($\mathcal{F}$ is pointwise bounded) $\iff$ ($\mathcal{F}$ is bounded).*
   b. *$\mathcal{T}_{unif}$ and $\mathcal{T}_{pt}$ induce the same topology on $\mathcal{F}$.*

In particular, if a sequence $(f_n)_{n\geq 1}$ is equicontinuous, then it is uniformly convergent (or bounded) iff it is pointwise convergent (or pointwise bounded, respectively).

PROOF. For 1, 2 and (a) the nontrivial implications are the direct ones. For 1, assume that f is continuous. Let $\epsilon > 0$. For each $x \in X$ choose V_x such that

$$d(f(y), f(x)) < \epsilon/2 \quad \forall\, y \in V_x.$$

Apply now the Lebesgue lemma (Proposition 6.10) and let $\delta > 0$ be a resulting Lebesgue number. Then, for each $y, z \in X$ with $d(y,z) < \delta$ we find $x \in X$ such that $y, z \in V_x$, hence

$$d(f(y), f(z)) \leq d(f(y), f(x)) + d(f(z), f(x)) < \epsilon.$$

This proves that f is uniformly continuous. Exactly the same proof applies to 2 (just add "for all $f \in \mathcal{F}$" everywhere). For (a), assume that $\mathcal{F}$ is equicontinuous and pointwise bounded. From the first condition we find an open cover $\{V_x\}_{x\in X}$ with $x \in V_x$ and $d(f(y), f(x)) < 1$ for all $y \in V_x$. Choose a finite subcover corresponding to $x_1, \ldots, x_k \in X$. Using that $\mathcal{F}(x_i)$ is bounded for each i, we find $M > 0$ such that

$$d(f(x_i), g(x_i)) < M \quad \forall\, f, g \in \mathcal{F}, \;\; \forall\, 1 \leq i \leq k.$$

Then, for arbitrary $x \in X$, choosing i such that $x \in V_{x_i}$, we have

$$d(f(x), g(x)) \leq d(f(x), f(x_i)) + d(f(x_i), g(x_i)) + d(g(x_i), g(x)) < M + 2,$$

for all $f, g \in \mathcal{F}$, showing that $\mathcal{F}$ is bounded. For (b), the non-obvious part is to show that $\mathcal{T}_{\mathrm{unif}}|_{\mathcal{F}} \subset \mathcal{T}_{pt}|_{\mathcal{F}}$. Due to the definitions of these topologies, we start with $f \in \mathcal{F}$ and a ball

$$B_{\mathcal{F}}(f, \epsilon) = \{g \in \mathcal{F} : d_{\sup}(g, f) < \epsilon\},$$

and we are looking for $x_1, \ldots, x_k \in X$ and $\epsilon_1, \ldots, \epsilon_k > 0$ such that

$$\cap_i \{g \in \mathcal{F} : d(g(x_i), f(x_i)) < \epsilon_i\} \subset B_{\mathcal{F}}(f, \epsilon).$$

For that, choose as before a finite open cover $\{V_{x_i}\}$ of X such that

$$d(f(x), f(x_i)) < \epsilon/6 \quad \forall\, f \in \mathcal{F}, \; x \in V_{x_i}$$

and, by the same inequalities as above, we find that the x_i and $\epsilon_i = \epsilon/3$ have the desired properties. $\square$

4.5. The Arzela-Ascoli theorem. The Arzela-Ascoli theorem has quite a few different looking versions. They all give compactness criteria for subspaces of $\mathcal{C}(X,Y)$ in terms of equicontinuity; sometimes the statement is a sequential one (giving criteria for an equicontinuous sequence to admit a convergent subsequence). The difference between the several versions comes either from the starting hypothesis on X and Y, or from the topologies one considers on $\mathcal{C}(X,Y)$. As in the last subsections, we restrict ourselves to the case that X and Y are metric and X is compact; the interesting topology on the space of functions will then be the uniform one.

THEOREM 8.33. *(Arzela-Ascoli) Assume that (X,d) is a compact metric space, (Y,d) is complete. Then, for a subset $\mathcal{F}\subset\mathcal{C}(X,Y)$, the following are equivalent:*

1. *$\mathcal{F}$ is relatively compact in $(\mathcal{C}(X,Y),d_{sup})$.*
2. *$\mathcal{F}$ is equicontinuous and pointwise totally bounded.*

(note: when $Y=\mathbb{R}^n$ with the Euclidean metric, "pointwise totally bounded"= "pointwise bounded").

COROLLARY 8.34. *Let X and Y be as above. Then any sequence $(f_n)_{n\geq 1}$ which is equicontinuous and pointwise totally bounded admits a subsequence which is uniformly convergent.*

PROOF. The direct implication is clear now: if $\overline{\mathcal{F}}$ is compact, it must be totally bounded (cf. Theorem 6.7); this implies that $\overline{\mathcal{F}}$ (hence also $\mathcal{F}$) is equicontinuous and pointwise bounded. We now prove the converse. Let us assume for simplicity that $\mathcal{F}$ is also closed with respect to the uniform topology (otherwise replace it by its closure and, by the same arguments as before, show that equicontinuity and pointwise totally boundedness hold for the closure as well). We show that $\mathcal{F}$ is compact. Using Theorem 6.7, it suffices to show that $\mathcal{F}$ is sequentially compact. So, let $(f_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ and we will show that it contains a convergent subsequence. Use Corollary 6.9 and consider

$$A=\{a_1,a_2,\ldots\}\subset X$$

which is dense in X. Since $(f_n(a_1))_{n\geq 1}$ is totally bounded, using Corollary 6.8, it follows that it has a convergent subsequence $(f_n(a_1))_{n\in I_1}$, where $I_1\subset\mathbb{Z}_+$. Let n_1 be the smallest element of I_1. Similarly, since $(f_n(a_2))_{n\in I_1}$ is totally bounded, we find a convergent subsequence $(f_n(a_2)_{n\in I_2}$ where $I_2\subset I_1$. Let n_2 be the smallest element of I_2. Continue inductively to construct I_j and its smallest element n_j for all j. Choosing $g_k=f_{n_k}$, this will be a subsequence of (f_n) which has the property that $(g_k(a_i))_{k\geq 1}$ is convergent for all i. We will show that (g_k) is Cauchy (hence convergent). Let $\epsilon>0$. Since $\mathcal{F}$ is uniformly equicontinuous, we find δ such that

$$d(g_k(x),g_k(y))<\epsilon/3\quad\forall\ k \text{ and whenever } d(x,y)<\delta.$$

Since A is dense in X, the balls $B(a_i,\delta)$ cover X; since X is compact, we find some integer N such that X is covered by $B(a_i,\delta)$ with $1\leq i\leq N$. Since each of the sequences $(g_k(a_i))_{k\geq 1}$ is convergent for all $1\leq i\leq N$, we find n_ϵ such that

$$d(g_j(a_i),g_k(a_i))<\epsilon/3\quad\forall\ j,k\geq n_\epsilon\quad\forall\ 1\leq i\leq N.$$

Then, for all $x\in X$, $j,k\geq n_\epsilon$, choosing $i\leq N$ such that $x\in B(a_i,\delta)$, we have

$$d(g_j(x),g_k(x))\leq d(g_j(x),g_j(a_i))+d(g_j(a_i),g_k(a_i))+d(g_k(a_i),g_k(x))<\epsilon.$$

□

Finally, let us also mention the following more general version of the Arzela-Ascoli (see Munkres' book).

THEOREM 8.35. *(Arzela-Ascoli) Assume that X is a locally compact Hausdorff space, (Y,d) is a complete metric space, $\mathcal{F}\subset\mathcal{C}(X,Y)$. Then $\mathcal{F}$ is relatively compact in $(\mathcal{C}(X,Y),\mathcal{T}_{cp})$ if and only if $\mathcal{F}$ is equicontinuous and pointwise totally bounded.*

In particular, any sequence $(f_n)_{n\geq 1}$ in $\mathcal{C}(X,\mathbb{R}^N)$ which is equicontinuous and pointwise totally bounded admits a subsequence which is uniformly convergent on compacts.

5. More exercises

5.1. On Stone-Weierstrass.

EXERCISE 8.1. Show that $\mathcal{C}^{\infty}([0,1])$ (the space of real-valued functions on $[0,1]$ which are infinitely many times differentiable) is dense in $\mathcal{C}([0,1])$.

EXERCISE 8.2. On the sphere S^2 we consider the real-valued functions

$$f(x,y,z) = x+y+z, g(x,y,z) = xy+yz+zx.$$

Does $\{f,g\}$ separate points? What if we add the function $h(x,y,z) = xyz$?

EXERCISE 8.3. Let X be a compact topological space. Show that if a finite set of continuous functions

$$\mathcal{A} = \{f_1, \dots, f_k\} \subset \mathcal{C}(X)$$

separates points, then X can be embedded in $\mathbb{R}^k$.

EXERCISE 8.4. Consider the 3-dimensional sphere interpreted as:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

Consider the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, viewed as a group with respect to the multiplication of complex numbers, and we consider the action of S^1 on S^3 given by

$$z \cdot (z_1, z_2) := (zz_1, \overline{z}z_2).$$

Let $X := S^3/S^1$. Consider $\tilde{f}, \tilde{g}, \tilde{h} : S^3 \to \mathbb{R}$ given by

$$\tilde{f}(z_1,z_2) = z_1z_2 + \overline{z}_1\overline{z}_2, \tilde{g}(z_1,z_2) = i(z_1z_2 - \overline{z}_1\overline{z}_2), \tilde{h}(z_1,z_2) = |z_1|^2 - |z_2|^2.$$

Show that

1. $\tilde{f}, \tilde{g}, \tilde{h}$ induce functions $f, g, h : X \to \mathbb{R}$.
2. $\{f, g, h\}$ separates points.
3. The image of the resulting embedding $(f, g, h) : X \to \mathbb{R}^3$ is S^2.

EXERCISE 8.5. If K is a compact subspace of $\mathbb{R}^n$, show that the space $\mathrm{Pol}(K)$ of polynomial functions on K is dense in $\mathcal{C}(K)$ in the uniform topology.
(recall that a function $f : K \to \mathbb{R}^n$ is polynomial if there exists a polynomial $P \in \mathbb{R}[X_1, \dots, X_n]$ such that $f(x) = P(x)$ for all $x \in K$).

EXERCISE 8.6. Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic of period 2π, then f can be realized as the limit of a sequence of functions of type

$$T(x) = a_0 + \sum_{k=1}^{N}(a_k cos(kx) + b_k sin(kx)).$$

EXERCISE 8.7. Show that, for any compact smooth manifold X, $\mathcal{C}^{\infty}(X)$ is dense in $\mathcal{C}(X)$.

5.2. On Gelfand-Naimark.

EXERCISE 8.8. Consider the algebra $A = \mathbb{R}[t]$ of polynomials in one variable t. Show that X_A is homeomorphic to $\mathbb{R}$. What if you take more variables?

EXERCISE 8.9. Consider

$$A = \mathbb{R}[X,Y]/(X^2 + Y^2 - 1)$$

(the quotient of the ring of polynomials in two variables, modulo the ideal generated by $f = X^2 + Y^2 - 1$ or, equivalently, the ring of remainders modulo f). Interpret it as an algebra over $\mathbb{R}$ and show that X_A is homeomorphic to S^1. (Hint: Let $\alpha, \beta \in A$ corresponding to X and Y.

Then a character χ is determined by $u = \chi(\alpha)$ and $v = \chi(\beta)$. What do they must satisfy? Also, have a look back at Exercise 3.33).

(if you do not understand the above definition of A, take as definition A :=the algebra of polynomial functions on $S^1 \subset \mathbb{R}^2$ and then α and β in the hint are the first and second projection).

Exercise 8.10. Consider the algebra $A = \mathbb{R}[t]/(t^3)$ (remainders modulo t^3). Compute X_A.

Exercise 8.11. Consider the algebra

$$A = \{f \in \mathcal{C}(S^n) : f(z) = f(-z) \ \ \forall \ z \in S^n\}.$$

Show that X_A is homeomorphic to $\mathbb{P}^n$.

Exercise 8.12. More generally, if a finite group Γ acts on a compact space X, consider

$$A = \mathcal{C}(X)^{\Gamma} := \{f \in \mathcal{C}(X) : f(\gamma \cdot x) = f(x) \ \ \forall \ x \in X, \gamma \in \Gamma\}$$

and compute X_A.