CHAPTER 9

Embedding theorems

In this chapter we will describe a general method for attacking embedding problems. We will establish several results but, as the main final result, we state here the following:

THEOREM 9.1. *Any compact n-dimensional topological manifold can be embedded in $\mathbb{R}^{2n+1}$.*

1. Using function spaces

2. Using covers and partitions of unity

3. Dimension and open covers

4. More exercises

1. Using function spaces

Throughout this section (X, d) is a metric space which is assumed to be compact and Hausdorff, and (Y, d) is a complete metric space (which, for the purpose of the chapter, youy may assume to be $\mathbb{R}^n$ with the Euclidean metric). The associated embedding problem is: can X be embedded in $(Y, \mathcal{T}_d)$. Since X is compact, this is equivalent to the existence of a continuous injective function $f : X \to Y$.

DEFINITION 9.2. *Given $f \in \mathcal{C}(X, Y)$, the injectivity defect of f is defined as*

$$\delta(f) := sup \ \{d(x, x') : x, x' \in X \ such \ that \ f(x) = f(x')\}.$$

For each $\epsilon > 0$, we defined the space of ϵ-approximately embeddings of X in Y as:

$$Emb_\epsilon(X, Y) := \{f \in \mathcal{C}(X, Y) : \delta(f) < \epsilon\}$$

endowed with the topology of uniform convergence.

PROPOSITION 9.3. *If $Emb_\epsilon(X, Y)$ is dense in $\mathcal{C}(X, Y)$ with respect to the uniform topology, for all $\epsilon > 0$, then there exists an embedding of X in Y.*

PROOF. The space $\mathrm{Emb}(X, Y)$ of all embeddings of X in Y can be written as

$$\mathrm{Emb}(X, Y) = \cap_n \mathrm{Emb}_{1/n}(X, Y)$$

where the intersection is over all positive integers. Since (Y, d) is complete, Theorem 8.24 implies that $(\mathcal{C}(X, Y), d_{\sup})$ is complete. By Proposition 6.3, it will have the Baire property. Hence, it suffices to show that the spaces $\mathrm{Emb}_\epsilon(X, Y)$ are open in $\mathcal{C}(X, Y)$ (and then it follows not only that $\mathrm{Emb}(X, Y)$ is non-empty, but actually dense in $\mathcal{C}(X, Y)$).

So, let $\epsilon > 0$ and we show that $\mathrm{Emb}_\epsilon(X, Y)$ is open. Let $f \in \mathrm{Emb}_\epsilon(X, Y)$ arbitrary; we are looking for δ such that

$$B_{d_{\sup}}(f, \delta) = \{g \in \mathcal{C}(X, Y) : d_{\sup}(g, f) < \delta\}$$

is inside $\mathrm{Emb}_\epsilon(X, Y)$. We first claim that there exists δ such that

$$d(f(x), f(y)) < 2\delta \Longrightarrow d(x, y) < \epsilon. \tag{1.1}$$

If no such δ exists, we would find sequences (x_n) and (y_n) in X with

$$d(f(x_n), f(y_n)) \to 0, \quad d(x_n, y_n) \geq \epsilon.$$

Hence (as we have already done several times by now), after eventually passing to convergent subsequences, we may assume that (x_n) and (y_n) are convergent, with limits denoted x and y. It follows that

$$d(f(x), f(y)) = 0, \quad d(x, y) \geq \epsilon,$$

which is in contradiction with $f \in \mathrm{Emb}_\epsilon(X, Y)$. Hence we do find δ satisfying (1.1). We claim that δ has the desired property; hence let $g \in B_{d_{\sup}}(f, \delta)$ and we prove that $g \in \mathrm{Emb}_\epsilon(X, Y)$. Note that

$$\delta(g) = \sup \ \{d(x, x') : x, x' \in K(g)\}$$

where $K(f) \subset X \times X$ consists of pairs (x, x') with $g(x) = g(x')$. Since g is continuous, $K(f)$ is closed in $X \times X$; since X is compact, it follows that $K(f)$ is compact; hence the above supremum will be attained at some $x, x' \in K(g)$. But for such x and x':

$$d(f(x), f(x')) \leq d(f(x), g(x)) + d(g(x'), f(x')) + d(g(x), g(x')) < 2\delta$$

hence, by (1.1), $d(x, x') < \epsilon$; hence $\delta(g) < \epsilon$. $\square$

2. Using covers and partitions of unity

In this section we assume that (X, d) is a compact metric space and $Y = \mathbb{R}^N$ is endowed with the Euclidean metric (where $N \geq 1$ is some integer). For the resulting embedding problem, we use the result of the previous section. We fix

$$f \in \mathcal{C}(X, \mathbb{R}^N), \ \epsilon, \delta > 0$$

and we search for $g \in \mathcal{C}(X, Y)$ with $\delta(g) < \epsilon$, $d_{\sup}(f, g) < \delta$. The idea is to look for g of type

$$g(x) = \sum_{i=1}^{p} \eta_i(x) z_i, \tag{2.1}$$

where $\{\eta_i\}$ is a continuous partition of unity and $z_i \in \mathbb{R}^N$ some points. To control $\delta(g)$, the points z_i have to be chosen in "the most general" position.

DEFINITION 9.4. *We say that a set $\{z_1, \ldots, z_p\}$ of points in $\mathbb{R}^N$ is in the general position if, for any $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ from which at most $N + 1$ are non-zero, one has:*

$$\sum_{i=1}^{p} \lambda_i z_i = 0, \ \sum_{i=1}^{p} \lambda_i = 0 \implies \lambda_i = 0 \quad \forall \ i \in \{1, \ldots, p\}.$$

We now return to our problem. Recall that, for a subset A of a metric space (X, d), $\mathrm{diam}(A)$ is $\sup\{d(a, b) : a, b \in A\}$. For a family $\mathcal{A} = \{A_i : i \in I\}$, denote by $\mathrm{diam}(\mathcal{A})$ the the supremum of $\{\mathrm{diam}(A_i)$ with $i \in I\}$. In the following, we control $\delta(g)$.

LEMMA 9.5. *Let $\mathcal{U} = \{U_i\}$ be an open cover of X, $\{\eta_i\}$ a partition of unity subordinated to $\mathcal{U}$ and $\{z_i\}$ a set of points in $\mathbb{R}^N$ in general position, all indexed by $i \in \{1, \ldots, p\}$. Assume that, for some integer m, each point in X lies in at most $m + 1$ members of $\mathcal{U}$. If $N \geq 2m + 1$ then the resulting function g given by (2.1) satisfies $\delta(g) \leq diam(\mathcal{U})$.*

PROOF. Assume that $g(x) = g(y)$, i.e. $\sum_{i=1}^{p}(\eta_i(x) - \eta_i(y))z_i = 0$. Now, x lies in at most $m+1$ members of $\mathcal{U}$, so at most $m+1$ numbers from $\{\eta_i(x) : 1 \leq i \leq p\}$ are non-zero. Similarly for y. Hence at most $2(m + 1)$ coefficients $\eta_i(x) - \eta_i(y)$ are non-zero. Note also that the sum of these coefficients is zero. Hence, since $\{z_1, \ldots, z_p\}$ is in general position and $2(m + 1) \leq N + 1$, it follows that $\eta_i(x) = \eta_i(y)$ for all i. Choosing i such that $\eta_i(x) > 0$, it follows that $x, y \in U_i$, hence $d(x, y) \leq \mathrm{diam}(U_i)$. □

Next, we control $d_{\sup}(f, g)$. We use the notation $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$.

LEMMA 9.6. *Let $\mathcal{U} = \{U_i\}$ be an open cover of X, $\{\eta_i\}$ a partition of unity subordinated to $\mathcal{U}$ and $\{z_i\}$ a set of points in $\mathbb{R}^N$, all indexed by $i \in \{1, \ldots, p\}$. Assume that, for some $r > 0$,*

$$diam(f(\mathcal{U})) < r, \quad d(z_i, f(U_i)) < r \quad \forall \ i \ \in \{1, \ldots, p\}.$$

Then the resulting function g given by (2.1) satisfies $d_{sup}(f, g) < 2r$.

PROOF. Since $d(z_i, f(U_i)) < r$ we find $x_i \in U_i$ with $||z_i - f(x_i)|| < r$. Writing

$$g(x) - f(x) = \sum_i \eta_i(x)(z_i - f(x_i)) + \sum_i \eta_i(x)(f(x_i) - f(x)),$$

$$||g(x) - f(x)|| \leq \sum_i \eta_i(x)||z_i - f(x_i)|| + \sum_i \eta_i(x)||f(x_i) - f(x)||.$$

Here each $||z_i - f(x_i)|| < r$ by hypothesis, hence the first sum is $< r$. For the second sum note that, whenever $\eta_i(x) \neq 0$, we must have $x \in U_i$ hence, $||f(x_i) - f(x)|| < r$. Hence also the second sum is $< r$, proving that $||g(x) - f(x)|| < 2r$ for all $x \in X$. Since X is compact, we have $d_{\sup}(f, g) < 2r$. □

Next, we show the existence of "small enough" covers of X and points in $\mathbb{R}^N$ in general position.

PROPOSITION 9.7. *For ϵ, $\delta > 0$ there exists an open cover $\mathcal{U} = \{U_i : 1 \leq i \leq p\}$ of X with*

$$diam(\mathcal{U}) < \epsilon, \quad diam(f(\mathcal{U})) < \delta/2.$$

Moreover, for any such cover, there exist points $\{z_1, \ldots, z_p\}$ in $\mathbb{R}^N$ in general position such that

$$d(z_i, f(U_i)) < \delta/2 \quad \forall \;\; i \in \{1, \ldots, p\}.$$

In particular, g given by (2.1) satisfies $\delta(g) < \epsilon$ and $d_{sup}(f, g) < \delta$, provided $\mathcal{U}$ has the property that each point in X lies in at most $m+1$ members of $\mathcal{U}$, where m satisfies $N \geq 2m+1$.

PROOF. For the first part we use that f is uniformly continuous and choose $r < \epsilon$ such that

$$d(x, y) < r \Longrightarrow d(f(x), f(y)) < \frac{\delta}{2}.$$

Consider then the open cover of X by balls of radius r (or any other arbitrarily smaller radius) and choose a finite subcover. For the second part, we choose $x_i \in U_i$ arbitrary and set $y_i = f(x_i) \in \mathbb{R}^N$. We prove that, in general, for any finite set $\{y_1, \ldots, y_p\}$ of points in $\mathbb{R}^N$ and any $r > 0$, there exists a set $\{z_1, \ldots, z_p\}$ of points in general position such that $d(z_i, y_i) < r$ for all i. We proceed by induction on p. Assume the statement holds up to p and we prove it for $p+1$. So, let $\{y_1, \ldots, y_{p+1}\}$ be points in $\mathbb{R}^N$. From the induction hypothesis, we may assume that $\{y_1, \ldots, y_p\}$ is already in general position. For each $I \subset \{1, \ldots, p\}$ of cardinality at most N we consider the "hyperplane"

$$\mathcal{H}_I := \{\sum_{i \in I} \lambda_i y_i : \lambda_i \in \mathbb{R}, \sum_{i \in I} \lambda_i = 1\}.$$

Since $|I| \leq N$, each such hyperplane has empty interior (why?), hence so does their union $\cup_I \mathcal{H}_I$ taken over all Is as above. Hence $B(y_{p+1}, r)$ will contain an element z_{p+1} which is not in this intersection. It is not difficult to check now that $\{y_1, \ldots, y_p, z_{p+1}\}$ is in general position. $\square$

3. Dimension and open covers

As in the previous section, we fix a compact metric space (X, d) and $Y = \mathbb{R}^N$ with the Euclidean metric. We assume that $N = 2m + 1$ for some integer m. Proposition 9.7 almost completes the proof of the existence of an embedding of X in $\mathbb{R}^{2m+1}$; what is missing is to make sure that the covers $\mathcal{U}$ from the proposition can be chosen so that each point in X lies in at most $m + 1$ members of $\mathcal{U}$. Note however that this is an important demand. After all, all that we have discussed applies to any compact metric space X (e.g. S^5) and any $\mathbb{R}^N$ (e.g $\mathbb{R}$!); this extra-demand is the only one placing a condition on N in terms of the topology of X. Actually, this is about "the dimension" of X.

Definition 9.8. *Let X be a topological space, $m \in \mathbb{Z}_+$. We say that X has dimension less or equal to m, and we write $dim(X) \leq m$, if any open cover $\mathcal{U}$ admits an open refinement $\mathcal{V}$ of multiplicity $mult(\mathcal{V}) \leq m + 1$, i.e. with the property that each $x \in X$ lies in at most $m + 1$ members of $\mathcal{V}$.*

The dimension of X is the smallest m with this property.

With this, Proposition 9.7 and Proposition 9.3 give us immediately:

Corollary 9.9. *Any compact metric space X with $dim(X) \leq m$ can be embedded in $\mathbb{R}^{2m+1}$.*

Of course, this nice looking corollary is rather cheap at this point: it looks like we just defined the dimension of a space, so that the corollary holds. However, the definition of dimension given above is not at all accidental. By the way, did you ever think how to define the (intuitively clear) notion of dimension by making use only of the topological information? What are the properties of the opens that make $\mathbb{R}$ one-dimensional and $\mathbb{R}^2$ two-dimensional? You may then discover yourself the previous definition. Of course, one should immediately prove that $\dim(\mathbb{R}^N)$ is indeed N or, more generally, that any m-dimensional topological manifold X has $\dim(X) = m$. These are all true, but they are not easy to prove right away. What we will show here is that:

Theorem 9.10. *Any compact m-dimensional manifold X satisfies $dim(X) \leq m$.*

This will be enough to apply the previous corollary and deduce Theorem 9.1 from the beginning of this chapter. The rest of this section is devoted to the proof of this theorem. First, we have the following metric characterization of dimension:

Lemma 9.11. *Let (X, d) be a compact metric space and m an integer. Then $dim(X) \leq m$ if and only if, for each $\delta > 0$, there exists an open cover $\mathcal{V}$ with $diam(\mathcal{V}) < \delta$ and $mult(\mathcal{V}) \leq m+1$.*

Proof. For the direct implication, start with the cover by balls of radius $\delta/2$ and choose any refinement $\mathcal{V}$ as in Definition 9.8. For the converse, let $\mathcal{U}$ be an arbitrary open cover. It then suffices to consider an open cover as in the statement, with δ a Lebesgue number for the cover $\mathcal{U}$ (see Proposition 6.10). $\square$

Lemma 9.12. *Any compact subspace $K \subset \mathbb{R}^N$ has $dim(K) \leq N$.*

Proof. For simplicity in notations, we assume that $N = 2$. We will use the previous lemma. First, we consider the following families of opens in the plane:

- $\mathcal{U}_0$ consisting of the open unit squares with vertices in the integral points (m, n) $(m, n \in \mathbb{Z})$.
- $\mathcal{U}_1$ consisting of the open balls of radius $\frac{1}{2}$ with centers in the integral points.
- $\mathcal{U}_2$ consisting of the open balls of radius $\frac{1}{4}$ with centers in the middles of the edges of the integral lattice.

Make a picture!Note that the members of each of the families $\mathcal{U}_i$ are disjoint. Hence

$$\mathcal{U} := \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$$

is an open cover of $\mathbb{R}^2$ of multiplicity 3 with $\mathrm{diam}(\mathcal{U}) = \sqrt{2}$. To obtain similar covers of smaller diameter, we rescale. For each $\lambda > 0$, $\phi_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto \lambda \mapsto \lambda v$ is a homeomorphism. The rescaling of $\mathcal{U}$ is

$$\mathcal{U}^\lambda = \{\phi_\lambda(U) : U \in \mathcal{U}\},$$

it has multiplicity 3 and diameter $\lambda\sqrt{2}$. Now, for $K \subset \mathbb{R}^2$ compact, we use the covers $\mathcal{U}^\lambda$ (and the compactness o K) to apply the previous lemma. □

LEMMA 9.13. *If X is a topological space and $X = \cup_{i=1}^p X_i$ where each X_i is closed in X with dim$(X_i) \leq m$, then dim$(X) \leq m$.*

PROOF. Proceeding inductively, we may assume $m = 2$, i.e. $X = Y \cup Z$ with Y, Z-closed in X of dimension $\leq m$. Let $\mathcal{U}$ be an arbitrary open cover of X; we prove that it has a refinement of multiplicity $\leq m + 1$. First we claim that $\mathcal{U}$ has a refinement $\mathcal{V}$ such that each $y \in Y$ lies in at most $m + 1$ members of $\mathcal{V}$. To see this, note that $\{U \cap Y : U \in \mathcal{U}\}$ is an open cover of Y, hence it has a refinement (covering Y) $\{Y_a : a \in A\}$ (for some indexing set A). For each $a \in A$, write $Y_a = Y \cap V_a$ with $V_a \subset X$ open, and choose $U_a \in \mathcal{U}$ such that $Y_a \subset U_a$. Then

$$\mathcal{V} := \{V_a \cap U_a : a \in A\} \cup \{U - Y : U \in \mathcal{U}\}$$

is the desired refinement. Re-index it as $\mathcal{V} = \{V_i : i \in I\}$ (we assume that there are no repetitions, i.e. $V_i \neq V_{i'}$ whenever $i \neq i'$). Similarly, let $\mathcal{W} = \{W_j : j \in J\}$ be a refinement of $\mathcal{V}$ with the property that each $z \in Z$ belongs to at most $m + 1$ members of $\mathcal{W}$. For each $j \in J$, choose $\alpha(j) \in I$ such that $W_j \subset V_{\alpha(j)}$. For each $i \in I$, define

$$D_i = \cup_{j\in\alpha^{-1}(i)} W_j.$$

Consider $\mathcal{D} = \{D_i : i \in I\}$. Since for each $j \in J$, $i \in I$

$$W_j \subset D_{\alpha(j)}, \quad D_i \subset V_i,$$

$\mathcal{D}$ is an open cover of X, which refines $\mathcal{V}$ (hence also $\mathcal{U}$). It suffices to show that $\mathrm{mult}(\mathcal{D}) \leq m+1$. Assume that there exist k distinct indices $i_1, \ldots, i_k$ with

$$x \in D_{i_1}, \ldots, D_{i_k}.$$

We have to show that $k \leq m + 1$. If $x \in Y$, since $D_i \subset V_i$ for all i, the defining property of $\mathcal{V}$ implies that $k \leq m + 1$. On the other hand, for each $a \in \{1, \ldots, k\}$, since $x \in D_{i_a}$, we find $j_a \in \alpha^{-1}(i_a)$ such that $x \in W_{j_a}$; hence, if $x \in Z$, then the defining property of $\mathcal{W}$ implies that $k \leq m + 1$. □

PROOF. (end of the proof of Theorem 9.10) Since X is a manifold, around each $x \in X$ we find a homeomorphism $\phi_x : U_x \to \mathbb{R}^n$ defined on an open neighborhood U_x of x. Let $V_x \subset U_x$ corresponding (by ϕ_x) to the open ball of radius 1. From the open cover $\{V_x : x \in X\}$, extract an open subcover, corresponding to $x_1, \ldots, x_k \in X$. Then $X = \cup_i X_i$, and each X_i is a closed subset of X homeomorphic to a closed ball of radius 1, hence has $\dim(X_i) \leq m$. □