Inleiding Topologie

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- Utrecht Lecture Notes 2026-2027 -

PART II: EXERCISES

Contents of Part II

10 Exercises on the chapter "Topological spaces" 129
10.1 On topologies 129
10.2 On induced topologies 130
10.3 On continuity 130
10.4 On homeomorphisms and embeddings 132
10.5 The "removing a point trick" 133
10.6 On convergence 133
10.7 On closure, interior, etc 134
10.8 Density 135
10.9 On Hausdorffness, 2nd countability, separation 136
10.10Mixed exercises 136

11 Exercises on the chapter "Constructions of topological spaces" 141
11.1 General quotients/gluings 141
11.2 Some special quotients 143
11.3 Quotients modulo group actions 143
11.4 Product topology 145
11.5 Topology bases 146
11.6 Spaces of functions 148

12 Exercises on the chapter "Topological properties" 149
12.1 Connectedness 149
12.2 Compactness 152
12.3 Connectedness and compactness for some exotic topologies 154
12.4 Local compactness and the one-point compactification 155

13 Exercises on the chapter "Topological properties versus metric ones" 161

14 Exercises on the chapter "The algebra of observables/spaces of functions" 163
14.1 On Stone-Weierstrass 163
14.2 On Gelfand-Naimark-more theoretical 164
14.3 On Gelfand-Naimark- "computing" spectra 166
14.4 On Gelfand-Naimark- a few more exercises 168

15 Exercises on the chapter "Partitions of unity" 171
15.1 More exercises 171

16 Previous exams 175
16.1 The 2025 retake exam (April 17, 2025) 175
16.2 The 2025 exam (January 30, 2025) 177
16.3 The 2024 retake exam (April 15, 2024) 179
16.4 The 2024 exam (February 2nd, 2024), English 181
16.5 The 2023 retake exam (March 16 , 2023) 183
16.6 The 2023 exam (February 2 , 2023) 185
16.7 The 2022 retake exam (April 2022) 187
16.8 The 2022 exam (January 2022) 188
16.9 The 2021 retake exam (April 2021) 189
16.10 The 2021 exam (January 2021) 190
16.11 The 2020 retake exam 191
16.12 The 2020 exam (January 28, 2020) 193
16.13 The 2019 retake exam (April 15, 2019) 195
16.14 The 2019 exam (January 29, 2019) 197
16.15 The 2018 retake exam (April 16, 2018) 199
16.16 The 2018 exam (January 29, 2018) 201
16.17 The 2017 retake exam (April 18, 2017) 203
16.18 The 2017 exam (January 30, 2017) 205
16.19 The 2016 exam (January 24, 2016) 207
16.20 The 2015 exam (January 28, 2015) 209
16.21 The 2015 retake exam (March 11, 2015) 211
16.22 The 2014 exam (January 29, 2014) 213
16.23 The 2014 retake exam (March 12, 2014) 215
16.24 The 2013 exam A (April 17, 2013) 217
16.25 The 2013 exam B (June 26, 2013) 219
16.26 The 2013 retake exam (August, 2013) 221
16.27 The 2012 exam A (April 18, 2012) 223
16.28 The 2012 exam B (June 27, 2012) 224
16.29 The 2012 retake exam (August 22, 2012) 226
16.30 The 2011 exam A (April 20, 2011) 227
16.31 The 2011 exam B (June 29, 2011) 228
16.32 The 2011 retake exam (August 24, 2011) 230
16.33 The 2010 exam (April 21, 2010) 232
16.34 The 2010 retake exam (August 27, 2010) 234
16.35 The 2009 exam part A (April, 2009) 235
16.36 The 2009 retake exam (August, 2009) 236
16.37 The 2008 exam A (April 15, 2008) 237
16.38 The 2007 exam A (April 16, 2007) 240
16.39 The 2006 exam A (April 16, 2006) 241

Index 243

Chapter 10
Exercises on the chapter "Topological spaces"

10.1 On topologies

Exercise 10.1. How many distinct topologies can there be defined on a set with two elements? But with three?

Exercise 10.2. Prove that, for any set X, $\mathscr{T}_{\text{cf}}$ and $\mathscr{T}_{\text{dis}}$ are indeed topologies.

Exercise 10.3. Given a topology $\mathscr{T}$ on $\mathbb{R}$, show that $\mathscr{T}$ is the discrete topology iff $\{r\} \in \mathscr{T}$ for all $r \in \mathbb{R}$.

Exercise 10.4. On $\mathbb{R}$ consider the family $\mathscr{T}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of type $(-\infty, r)$ with $r \in \mathbb{R}$. Show that $\mathscr{T}$ is a topology on $\mathbb{R}$ and compare it with the Euclidean topology.

Exercise 10.5. On $\mathbb{R}$ consider the family $\mathscr{B}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of type $(-\infty, r]$ with $r \in \mathbb{R}$. Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$ and find the smallest topology containing $\mathscr{B}$. Is it larger or smaller than the topology from the previous exercise? But than the Euclidean topology?

Exercise 10.6. This is to recap the lower limit topology $\mathscr{T}_l$ on $\mathbb{R}$ from Example 2.9. Recall that it was defined as follows: a subset $D \subset \mathbb{R}$ belongs to $\mathscr{T}_l$ if and only if the following condition holds: for any $x \in D$ there exists an interval $[a,b)$ such that

$$x \in [a,b) \subset D.$$

Show that:

(a) $\mathscr{T}_l$ the smallest topology on $\mathbb{R}$ which contains all the intervals of type $[a,b)$ with $a,b \in \mathbb{R}$
(b) replacing the intervals of type $[a,b)$ by intervals of type $(a,b]$, one obtains a similar topology $\mathscr{T}_u$ (why u?).
(c) $\mathscr{T}_l$ and $\mathscr{T}_u$ are finer than $\mathscr{T}_{\text{eucl}}$, but $\mathscr{T}_l$ and $\mathscr{T}_u$ are not comparable.
(d) $\mathscr{T}_{\text{dis}}$ is the only topology on $\mathbb{R}$ which contains both $\mathscr{T}_l$ and $\mathscr{T}_u$.

Exercise 10.7. On $\mathbb{R}^2$ we define the topology $\mathscr{T}_l \times \mathscr{T}_l$ as the smallest topology which contains all subsets of type

$$[a,b) \times [c,d)$$

with $a,b,c,d \in \mathbb{R}$. Define similarly $\mathscr{T}_u \times \mathscr{T}_u$, $\mathscr{T}_l \times \mathscr{T}_u$ and $\mathscr{T}_u \times \mathscr{T}_l$. Show that any two of these are not comparable.

Exercise 10.8. Consider the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (the set of strictly positive integers to which we add the infinity) and, for each $n \in \mathbb{N}$ put

$$U_n := \left\{k \in \overline{\mathbb{N}} : k \geq n\right\} = \{n, n+1, \ldots\} \cup \{\infty\}.$$

Show that the following is a topology on $\overline{\mathbb{N}}$:

$$\mathscr{T}_{\text{seq}} := \{\emptyset, U_1, U_2, U_3, \ldots\}.$$

Exercise 10.9. Let (X,d) be a metric space, A, B subspaces of X such that

$$d(A,B) = 0.$$

(where $d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$). Is it true that A and B must have a common point (i.e. $A \cap B \neq \emptyset$)? What if we assume that both A and B are closed?

Exercise 10.10. Consider a set X, a set of indices I and, for each $i \in I$, a topology $\mathscr{T}_i$ on X. Show that $\mathscr{T} := \cap_i \mathscr{T}_i$ (i.e. the family consisting of subsets $U \subset X$ with the property that $U \in \mathscr{T}_i$ for all $i \in I$) is a topology on X.

Exercise 10.11. Given a set X and a family $\mathscr{S}$ of subsets of X, prove that there exists a topology $\mathscr{T}(\mathscr{S})$ on X which contains $\mathscr{S}$ and is the smallest with this property. (Hint: use the axioms to see what other subsets of X, besides the ones from $\mathscr{S}$, must $\mathscr{T}(\mathscr{S})$ contain.)

10.2 On induced topologies

Exercise 10.12. Consider the real line $\mathbb{R}$ as a subset of the plane $\mathbb{R}^2$. Show that the induced topology on $\mathbb{R}$ coincides with the Euclidean topology on the real line.

Exercise 10.13. To remove ambiguities regarding the Convention 2.13 show that, inside any topological space $(X, \mathscr{T})$, for all $A \subset Y \subset X$ one has

$$(\mathscr{T}|_Y)|_A = \mathscr{T}|_A.$$

To remove ambiguities regarding Convention 2.8 and 2.13 show that, for $A \subset \mathbb{R}^n$, the Euclidean topology on A coincides with the restriction to A of the Euclidean topology of $\mathbb{R}^n$.

Exercise 10.14. Let $(X, \mathscr{T})$ be a topological space and $B \subset A \subset X$.

(a) If A is open in X, show that B is open in X if and only if it is open in A.
(b) If A is closed in X, show that B is closed in X if and only if it is closed in A.

Exercise 10.15. Find an example of a topological space $(X, \mathscr{T})$ and $A \subset B \subset X$ such that A is closed in B, B is open in X, and A is neither open nor closed in X.

10.3 On continuity

Exercise 10.16. Consider

$$D := \{(x,y) : e^x > \sin(y)\cos(x)\},$$

$$A := \left\{(x,y) : x^7 - \sin(y^7) \geq \frac{1}{x^2+y^2+1}\right\}.$$

Show that D is open in $\mathbb{R}^2$, while A is closed in $\mathbb{R}^2$.

Exercise 10.17. Let $\mathbb{R}$ be endowed with the lower limit topology $\mathscr{T}_l$. Which one of the following functions $f:\mathbb{R}\to\mathbb{R}$ is continuous:

(a) $f(x)=x+1$
(b) $f(x)=-x$.
(c) $f(x)=x^2$.

Exercise 10.18. If (X,d) is a metric space and A is a subspace of X, then the function

$$d_A : X\to\mathbb{R},\ \ d_A(x)=d(x,A)$$

is continuous. Deduce that, for any closed subset A of a metric space X, there exists a continuous function $f:X\to[0,1]$ such that $A=f^{-1}(0)$.

Exercise 10.19. (from the 2021 retake exam) Let $(X,\mathscr{T}_X)$ and $(Y,\mathscr{T}_Y)$ be topological spaces and $A\subset Y$ be a subset equipped with the subspace topology; we denote by $i:A\to Y$ the inclusion. Show that a function $f:X\to A$ is continuous if and only if the composition $i\circ f:X\to Y$ is continuous.

Exercise 10.20. Given a topological space $(X,\mathscr{T})$ and $A\subset X$, show that the induced topology $\mathscr{T}|_A$ is the smallest topology on A with the property that the inclusion map $i:A\to X$ is continuous.

Exercise 10.21. (from the 2021 exam) Given a topological space $(X,\mathscr{T})$, and we use $[0,1]$ endowed with the Euclidean topology, show that if $\gamma_1,\gamma_2:[0,1]\to X$ are continuous and $\gamma_1(1)=\gamma_2(0)$, then

$$\gamma:[0,1]\to X,\ \ \gamma(t)=\begin{cases}\gamma_1(2t) & \text{if } t\in[0,\frac{1}{2}]\\ \gamma_2(2t-1) & \text{if } t\in(\frac{1}{2},1]\end{cases}$$

is continuous.

Exercise 10.22. (from the 2024 exam) Assume that $(X,\mathscr{T})$ is a topological space. We say that $X=A_1\cup A_2$ is a *decomposition* of $(X,\mathscr{T})$ if A_1,A_2 are subsets of X whose union is the entire X. For any such decomposition we endow A_1 and A_2 with the induced topology and we associate to it the following collection of subsets of X:

$$\mathscr{T}(A_1,A_2):=\{U\subset X : U\cap A_1 \text{ is open in } A_1 \text{ and } U\cap A_2 \text{ is open in } A_2\}.$$

(a) Show that $\mathscr{T}(A_1,A_2)$ is a topology on X and $\mathscr{T}\subset\mathscr{T}(A_1,A_2)$.

Next, a decomposition $X=A_1\cup A_2$ will be called *convenient* if $\mathscr{T}=\mathscr{T}(A_1,A_2)$. Please show that:

(b) If $X=A_1\cup A_2$ is a convenient decomposition of X then, for any function $f:X\to Y$ with values in another topological space Y, one has:

$$f \text{ is continuous} \iff f|_{A_1}:A_1\to Y \text{ and } f|_{A_2}:A_2\to Y \text{ are continuous.}$$

(c) If A_1 and A_2 are both open, or both closed, in X, then $X=A_1\cup A_2$ is a convenient decomposition.
(d) When $X=\mathbb{R}$ endowed with the Euclidean topology $\mathscr{T}_{\text{Eucl}}$, $\mathbb{R}=(-\infty,0)\cup[0,\infty)$ is not a convenient decomposition.

Similar exercises:

- *Exercise 16.26 from the 2021 exam*
- *Exercise 16.39 from the 2022 exam.*

Exercise 10.23. * Show that any continuous map

$$f : (\mathbb{R}, \mathscr{T}_{\text{Eucl}}) \to (\mathbb{R}, \mathscr{T}_l)$$

must be constant (recall that $\mathscr{T}_l$ is the lower limit topology- see Exercise 10.6).

10.4 On homeomorphisms and embeddings

Exercise 10.24. From the topologies that you found in Exercise 10.1, how many non homeomorphic ones are there?

Exercise 10.25. On $\mathbb{R}$ consider the topologies $\mathscr{T}_l$ and $\mathscr{T}_u$ (see Exercise 10.6). Show that $(\mathbb{R}, \mathscr{T}_l)$ and $(\mathbb{R}, \mathscr{T}_u)$ are homeomorphic.

Exercise 10.26. Show that $\mathbb{R}$ endowed with the Euclidean topology is not homeomorphic to $\mathbb{R}$ endowed with the topology from Exercise 10.4.

Exercise 10.27. Let $f\colon (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a continuous map between topological spaces. We say that f is

- *open* if for every open subset $U \subseteq X$, the subset $f(U) \subseteq Y$ is open.
- *closed* if for every closed subset $A \subseteq X$, the subset $f(A) \subseteq Y$ is closed.
- *local homeomorphism* if for each $x \in X$ there exists an open neighborhood U of x in X, and V of $f(x)$ in Y such that the restriction of f to U becomes a homeomorphism $f|_U : U \to V$.

Show that if $f\colon (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a continuous bijective map then

$$f \text{ is a homeomorphism} \iff f \text{ is an open map} \iff f \text{ is a closed map} \iff f \text{ is a local homeomorphism.}$$

Exercise 10.28. Show that, for any $n \geq 1$ integer,

$$\underbrace{S^1 \times \ldots \times S^1}_{n-\text{times}} \subset \underbrace{\mathbb{R}^2 \times \ldots \times \mathbb{R}^2}_{n-\text{times}} = \mathbb{R}^{2n}$$

can be embedded in $\mathbb{R}^{n+1}$. First prove it without using explicit formulas, then by explicit formulas.

Exercise 10.29. Exhibit an embedding $f : M \to M$ of the Moebius band into itself which is not surjective. What is the boundary of $f(M)$ in M?

Exercise 10.30. (from the 2016 exam) Consider

$$X = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, \text{ or } x = 0\right\},$$

$$Y = \left\{(x,y) \in \mathbb{R}^2 : xy = 0\right\},$$

both endowed with the Euclidean topology. Show that Y can be embedded in X, You do not have to provide explicit formulas for the embedding but please use pictures to describe the embedding and please provide explanations to support your claims.

Exercise 10.31. Show that for any $A \subset S^2$ nonempty, $S^2 \setminus A$ can be embedded in $\mathbb{R}^2$.

Exercise 10.32. Show that if n is an odd number, then $GL_n(\mathbb{R})$ is homeomorphic to $GL_1(\mathbb{R}) \times SL_n(\mathbb{R})$ (see Exercise 10.72).

10.5 The "removing a point trick"

The first part of the following exercise is extremely useful for some of the later exercises.

Exercise 10.33. If X and Y are homeomorphic, prove that for any $x \in X$ there exists $y \in Y$ such that $X - \{x\}$ is homeomorphic to $Y - \{y\}$.

Also explain that, "there exists $y \in Y$" cannot be replaced with "for any $y \in Y$".

10.6 On convergence

Exercise 10.34. Let $X = (0,1)$ endowed with the Euclidean topology. Is the sequence

$$x_n = \frac{1}{n+1}$$

convergent in the topological space X?

Exercise 10.35. Let $\mathbb{R}$ be endowed with the lower limit topology $\mathscr{T}_l$. Study the convergence of the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ where

$$x_n = \frac{1}{n},\ y_n = -\frac{1}{n}.$$

Exercise 10.36. On $X = \mathbb{R}$ consider the topology $\mathscr{T} = \{(-a,a) : 0 \leq a \leq \infty\}$. Is the sequence

$$x_n = (-1)^n + \frac{1}{n}$$

convergent in $(X, \mathscr{T})$? To what?

Exercise 10.37. What about convergence in $(X, \mathscr{T}_{\text{cf}})$. But about $(X, \mathscr{T}_{\text{cc}})$?

Exercise 10.38. We say that two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on X have the same convergence of sequences if, for a sequence $(x_n)_{n\geq 1}$ of elements of X, and $x \in X$, one has:

$$x_n \xrightarrow{\mathscr{T}_1} x \Longleftrightarrow x_n \xrightarrow{\mathscr{T}_2} x.$$

Show that

- If $\mathscr{T}_1$ and $\mathscr{T}_2$ satisfy the first countability axiom and have the same convergence of sequences, then $\mathscr{T}_1 = \mathscr{T}_2$.
- This is no longer true if one gives up the first countability axiom.

(Hint for the second part: This is difficult. Try $\mathscr{T}_{\text{discr}}$ and $\mathscr{T}_{\text{cf}}$. If it does not work, try to change one of them).

10.7 On closure, interior, etc

Exercise 10.39. Let $X = (-\infty,1)\cup(1,4)\cup[5,\infty)$. Find the closure, the interior and the boundary (in X) of $A = [0,1)\cup(1,2)\cup[3,4)\cup(5,6)$.

Exercise 10.40. Find the closure, the interior and the boundary of the following subsets of the plane:

(a) $\{(x,y) : x \geq 0, y \neq 0\}$.
(b) $\{(x,y) : x \in \mathbb{Q}, y > 0\}$.

Exercise 10.41. Show that the following three spaces are homeomorphic (giving the explicit homeomorphisms):

$$X = \left\{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 \leq 1\right\} \subset \mathbb{R}^2$$

$$Y = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2+y^2 \leq 1\right\} \subset \mathbb{R}^2$$

$$Z = S^1 \times (0,1] = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, 0 < z \leq 1\right\} \subset \mathbb{R}^3.$$

Then compute the interiors and the boundaries of X, Y (in $\mathbb{R}^2$) and Z (in $\mathbb{R}^3$). How comes that, although X, Y and Z are homeomorphic, their interiors and boundaries are quite different?

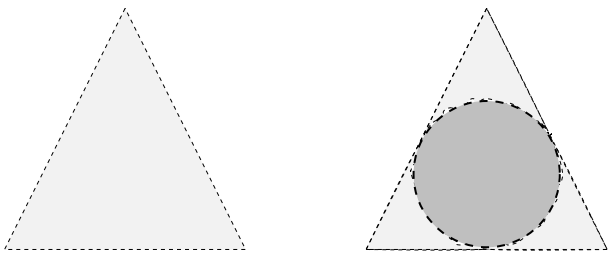
Exercise 10.42. Find the interior and the closure of $\mathbb{Q}$ in $\mathbb{R}$ in each of the cases: when $\mathbb{R}$ is endowed with

- the Euclidean topology.
- the discrete topology.
- the cofinite topology.
- the co-countable topology.

Exercise 10.43. Let $\mathscr{T}_l$ be the lower limit topology on $\mathbb{R}$. In the topological space $(\mathbb{R},\mathscr{T}_l)$, find the closure, the interior and the boundary of each of the intervals

$$[0,1), (0,1], (0,1), [0,1], (0,1)\cup[2,3].$$

Exercise 10.44. (from a 2010 exam) Let X be the (interior of an) open triangle, as drawn in the picture (the edges are not part of X!), viewed as a topological space with the topology induced from $\mathbb{R}^2$. Let $A \subset X$ be the open disk drawn in the picture (tangent to the edges of the closed triangle). Compute the closure and the boundary of A in X.



X= an open triangle A= an open disk inside X

Fig. 10.1

Exercise 10.45. Compute the interior, the closure and the boundary of

$$A = (0,1] \times [0,1)$$

in the topological space $X = \mathbb{R}^2$ endowed with the topology $\mathscr{T}_l \times \mathscr{T}_l$ of Exercise 10.7.

Exercise 10.46. Let A, B and $\{A_i : i \in I\}$ denote subsets of a topological space X, where i runs in a set of indexes I. Prove that

(a) If $A \subset B$ then $\overline{A} \subset \overline{B}$.
(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) If I is finite then $\overline{\cup_i A_i} = \cup_i \overline{A}_i$.
(d) In general, when I is infinite, $\overline{\cup_i A_i} \supset \cup_i \overline{A}_i$, but the two may be different.
(e) We say that $\{A_i : i \in I\}$ is locally finite if any point $x \in X$ admits a neighborhood V which intersects only finitely many subsets A_is (i.e. such that $\{i \in I : A_i \cap V \neq \emptyset\}$ is finite). Under this assumption, show that $\overline{\cup_i A_i} = \cup_i \overline{A}_i$.

Exercise 10.47. Let A, B be two subsets of a topological space X. Recall that $\mathrm{Int}(A) = \overset{\circ}{A}$ denotes the interior of A. Prove that

(a) If $A \subset B$ then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.
(b) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.
(c) $\mathrm{Int}(A \cup B) \supset \mathrm{Int}(A) \cup \mathrm{Int}(B)$, but the equality may fail.

Then show that the closure and the interior are related as follows:

$$X \setminus \overline{A} = \mathrm{Int}(X \setminus A), \quad X \setminus \mathrm{Int}(A) = \overline{X \setminus A}$$

and, using this, relate the first items of this exercise to the ones from the previous exercise.

10.8 Density

Exercise 10.48. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 10.49. Show that $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$.

Exercise 10.50. Let T be the torus. Describe (on the picture) a continuous injection $f : \mathbb{R} \to T$ whose image is dense in T. Is f an embedding?

Exercise 10.51. Show that $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$ (see Exercise 10.72).

Exercise 10.52. Show that any continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, then f must be linear, i.e. there exists $a \in \mathbb{R}$ such that

$$f(x) = ax \quad \forall\, x \in \mathbb{R}.$$

(Hint: $a = f(1)$. For what x's can you prove that $f(x) = ax$? Then use continuity and the fact $\mathbb{Q}$ is dense in $\mathbb{R}$.)

10.9 On Hausdorffness, 2nd countability, separation

Exercise 10.53. How many from the topologies from Exercise 10.1 are Hausdorff?

Exercise 10.54. Is the topology from Exerc. 10.8 Hausdorff? But from 10.4? But from 10.5?

Exercise 10.55. Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Which of the following holds true:

(a) if $(X, \mathscr{T}_1)$ is Hausdorff then so is $(X, \mathscr{T}_2)$.
(b) if $(X, \mathscr{T}_2)$ is 2nd countable then so is $(X, \mathscr{T}_1)$.
(c) if $(X, \mathscr{T}_2)$ is 1st countable then so is $(X, \mathscr{T}_1)$.

Exercise 10.56. Show that, in any Hausdorff space X, all the subspaces with one element (i.e. of type $A = \{x\}$ with $x \in X$) are closed.

Example 10.57. Consider $\mathbb{R}$ endowed with the lower limit topology (see (2.1.2)). Show that

$$\mathscr{B}' := \{[p,q) : p,q \in \mathbb{Q}\}$$

is not a basis for $(\mathbb{R}, \mathscr{T}_l)$. Then try to prove that $(\mathbb{R}, \mathscr{T}_l)$ is not 2nd countable. Prove however that it is 1st countable.

Exercise 10.58. Consider $\mathbb{R}$ with the Euclidean topology. In each of the following cases, decide (and explain!) when A and B can be separated topologically or by continuous functions:

(a) $A = [0,1)$, $B = (1,2]$.
(b) $A = [0,1)$, $B = [1,2]$.
(c) $A = [0,1)$, $B = (2,3]$.

Exercise 10.59. (from the 2012 retake exam) Let X be a Hausdorff normal space. Show that, for $A, B \subset X$, A and B have the same closure in X if and only if, for any continuous function $f : X \to \mathbb{R}$, one has the equivalence

$$f|_A = 0 \Longleftrightarrow f|_B = 0.$$

Exercise 10.60. Prove that the sphere S^n is an n-dimensional topological manifold. Show that it can be covered by two coordinate charts. Compute the change of coordinates.

Exercise 10.61. Let X be a 2nd countable space. Show that from any open cover $\mathscr{U}$ of X one can extract a countable subcover. In other words, for any collection $\mathscr{U} = \{U_i : i \in I\}$ consisting of opens in X such that $X = \cup_{i\in I} U_i$, one can find $i_1, i_2, \ldots \in I$ such that $X = \cup_k U_{i_k}$.

10.10 Mixed exercises

Exercise 10.62. Consider the following subsets of the plane:

(a) $A = \{(x,y) : x \geq 0\}$.

(b) $B = \{(x,y) : x = 0\}$.
(c) $C = \{(x,y) : x > 0, y < 5\}$.
(d) $D = \{(x,y) : xy < 1, x \geq 0\}$.
(e) $E = \{(x,y) : 0 \leq x < 5\}$.

Please answer the following:

(a) Which one of these is open in the plane?
(b) For each one of them, find the interior, the closure and the boundary in the plane.
(c) Notice that all of them are contained in A. Which ones are also open in A?
(d) For each one of them, find the interior, the closure and the boundary in A.

Exercise 10.63. Assume that $(X, \mathscr{T})$ is a topological space, $A \subset X$ and we endow A with the induced topology $\mathscr{T}|_A$. Show that:

1. if $(X, \mathscr{T})$ is Hausdorff, then so is $(A, \mathscr{T}|_A)$.
2. if $(X, \mathscr{T})$ is metrizable, then so is $(A, \mathscr{T}|_A)$.
3. if $(X, \mathscr{T})$ is second countable, then so is $(A, \mathscr{T}|_A)$.

Exercise 10.64. (from the 2010 exam) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathscr{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$.
- $\mathscr{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n, n+1, \ldots\}$ with $n \in \mathbb{N}$.

Questions:

1. Show that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on $\mathbb{N}$.
2. Show that the spaces $(\mathbb{N}, \mathscr{T}_1)$ and $(\mathbb{N}, \mathscr{T}_2)$ are not homeomorphic.

Exercise 10.65. (from the 2014 exam) For $\mathbb{R}$ we consider the family $\mathscr{S}$ of subsets consisting of all the intervals of type (m, M) with $m < M < 0$, all intervals of type (m, M) with $0 < m < M$ (m and M real numbers) and the interval $[-1, 1)$. Denote by $\mathscr{T}$ the smallest topology on $\mathbb{R}$ containing $\mathscr{S}$.

a. describe a basis $\mathscr{B}$ of $(\mathbb{R}, \mathscr{T})$ with $\mathscr{B} \neq \mathscr{T}$. Why taking $\mathscr{B} = \mathscr{S}$ does not work?
b. is $(\mathbb{R}, \mathscr{T})$ Hausdorff? Is it second countable?
c. find an interval of type $[a, b]$ whose closure inside $(\mathbb{R}, \mathscr{T})$ is not an interval.
d. find an interval of type (a, b) whose interior inside $(\mathbb{R}, \mathscr{T})$ is not an interval.
g. (+) consider

$$f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}),\ f(x) = \begin{cases} 0 \text{ if } x < -1 \\ 1 \text{ if } x \geq -1 \end{cases}$$

Is f continuous? Is f sequentially continuous?

Exercise 10.66. (from the 2024 exam) Let $\mathscr{T}_{\text{new}}$ be the topology on $\mathbb{R}$ consisting of those subsets $U \subset \mathbb{R}$ with the property that $U \cap (-\infty, 0)$ is open in $(-\infty, 0)$ and $U \cap [0, \infty)$ is open in $[0, \infty)$.

(a) Compute the closure of $A = (-1, 0)$ and the interior of $B = [0, 1]$ in $(\mathbb{R}, \mathscr{T}_{\text{new}})$.
(b) Is the sequence given by $x_n := \frac{(-1)^n}{n}$ convergent in the topological space $(\mathbb{R}, \mathscr{T}_{\text{new}})$?
(c) For what values of $\lambda \in \mathbb{R}$ is the following function

$$f_\lambda : (\mathbb{R}, \mathscr{T}_{\text{new}}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \quad f_\lambda(x) = \begin{cases} x & \text{if } x < 0 \\ \lambda & \text{if } x = 0 \\ x + \lambda^2 & \text{if } x > 0 \end{cases}$$

continuous?

(d) Does there exists λ such that f_λ is an embedding?

More exercises similar to the last two:

- *Exercise 16.13 from the 2024 exam, without items a and c.*
- *Exercise 16.43 from the 2016 exam, without items f, g and h.*
- *Exercise 11.41, without the first part of item a and without item c.*
- *Exercise 16.48 from the 2015 exam without item d.*
- *Exercise 16.83 from the 2012 exam*
- *Exercise 16.71 from the 2013 exam.*
- *Exercise 16.53 from the 2015 retake, where instead of items a and b you do: "Prove that for $k \geq 1$ the smallest topology on $\mathbb{R}$ containing $\mathscr{B}_k$ is the Euclidean topology" and $\mathscr{T}_0$ is the smallest topology containing $\mathscr{B}_0$. Also, ignore h.*
- *Exercise 16.100 from the 2011 exam A, all items except the first two (replace those by the question: prove that $\mathscr{T}$ is a topology) and except the question on connectedness.*

Exercise 10.67. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and let

$$f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$$

be a continuous map. Define $f^* \mathscr{T}_Y$ as the smallest topology on X with the property that

$$f : (X, f^* \mathscr{T}_Y) \to (Y, \mathscr{T}_Y)$$

is continuous. Show that

$$f^* \mathscr{T}_Y = \left\{ f^{-1}(V) : V \in \mathscr{T}_Y \right\}$$

and then, assuming that $(X, \mathscr{T}_X)$ is Hausdorff, show that

$$(f^* \mathscr{T}_Y = \mathscr{T}_X) \iff (f \text{ is an embedding}).$$

Similar exercise: Exercise 16.64 from the 2014 retake.

Exercise 10.68. Given a set X, show that any Hausdorff topology on X contains $\mathscr{T}_{\text{cf}}$. When is $\mathscr{T}_{\text{cf}}$ Hausdorff? When does it exist a smallest Hausdorff topology on X (i.e. a Hausdorff topology which is contained in all other Hausdorff topologies on X)?

Exercise 10.69. Show that if a space $(X, \mathscr{T})$ can be embedded in some Euclidean space $\mathbb{R}^n$, then $(X, \mathscr{T})$ is metrisable and must have the following three topological properties: it is Hausdorff, 2nd countable and normal.

Exercise 10.70. Let X be a Hausdorff normal space and let $A \subset X$ be a subspace with the property that any two continuous functions $f, g : X \to \mathbb{R}$ which coincide on A must coincide everywhere on X. Show that A is dense in X (i.e. the closure of A in X coincides with X).

Exercise 10.71. (from the 2022 exam) Give an example of a topological space $(X, \mathscr{T})$ such that for each $x \in X$ there exists a neighborhood U of x such that U is homeomorphic to $\mathbb{R}^2$, but X is not Hausdorff.

Exercise 10.72. The space $M_n(\mathbb{R})$, of $n \times n$ matrices with real coefficients can be identified with $\mathbb{R}^{n^2}$ and in this way has a natural topology (coming from the Euclidean metric). Prove that:

(a) the subspace $GL_n(\mathbb{R})$ of invertible matrices is open in $M_n(\mathbb{R})$.
(b) the subspace $SL_n(\mathbb{R})$ consisting of invertible matrices of determinant equal to 1 is closed in $GL_n(\mathbb{R})$.
(c) the subspace

$$O(n) := \{A \in GL_n(\mathbb{R}) : AA^* = \mathrm{Id}\}$$

is also closed (where A^* denotes the transpose of A and Id is the identity matrix).

More (similar) exercises can be found in previous exams as follows:

- On *topologies*: Exercise 16.37 from the 2022 exam.
- On *topologies, Hausdorffness*: Exercise 16.22 from the 2021 retake, item (a) and first half of item (b).
- On *topologies, homeomorphisms, interiors and closures*: Exercise 16.38 from the 2022 exam, except for the bulleted items.
- On *continuity, induced topologies*: Exercise 16.39 from the 2022 exam.

Chapter 11
Exercises on the chapter "Constructions of topological spaces"

11.1 General quotients/gluings

Exercise 11.1. Recall that a map $f\colon X \to Y$ is said to be open if for every open subset $U \subseteq X$, $f(U)$ is open in Y. Show that any surjective continuous open map is automatically a topological quotient map.

Exercise 11.2. Give an example of a topological quotient map which is not an open map.

Exercise 11.3. Start with the interval $[0,2]$ and glue the points 0, 1 and 2. Describe the equivalence relation R encoding this gluing and let $X = [0,2]/R$. Describe an embedding of X in $\mathbb{R}^2$.

Exercise 11.4. Let $X \subset \mathbb{R}^2$ like in Figure 11.1 (the shaded region). Describe an equivalence relation on X such that the result of the gluing is a sphere S^2. Then another one to get a torus. Then one that gives a disjoint union of a sphere and a torus.

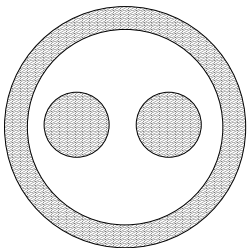


Fig. 11.1

Exercise 11.5. (part of an exercise from a 2010 exam) Let X be obtained by taking two disjoint copies of the interval $[0,2]$ (with the Euclidean topology) and gluing each t in the first copy with the corresponding t in the second copy, for all $t \in [0,2]$ different from the middle point. Explicitly, one may take the space

$$Y = [0,2] \times \{0\} \cup [0,2] \times \{1\} \subset \mathbb{R}^2$$

with the topology induced from the Euclidean topology, and X is the space obtained from Y by gluing $(t,0)$ to $(t,1)$ for all $t \in [0,2]$, $t \neq 1$. We endow X with the quotient topology.

(a) Is X Hausdorff?
(b) Show that X can also be obtained as a quotient of the circle S^1.

Exercise 11.6. (from the 2012 exam A) Let X be the space obtained from the sphere S^2 by gluing the north and the south pole (with the quotient topology). Show that X can be obtained from a square $[0,1]\times[0,1]$ by glueing some of the points on the *boundary* (note: you are not allowed to glue a point in the *interior* of the square to any other point). More precisely:

(a) Describe the equivalence relation R_0 on S^2 encoding the glueing that defines X.
(b) Make a picture of X in $\mathbb{R}^3$.
(c) Describe an equivalence relation R on $[0,1]\times[0,1]$ encoding a glueing with the required properties.
(d) Show that, indeed, X is homeomorphic to $[0,1]\times[0,1]/R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture).

Exercise 11.7. The aim of this exercise is to explain and show that $\mathbb{P}^2$ can be obtained from a Moebius band by gluing along its boundary circle a disk.

(a) Write the Moebius band as a union of two subspaces M and C where M is itself a Moebius band, C is a cylinder (i.e. homeomorphic to $S^1\times[0,1]$) and $M\cap C$ is a circle.
(b) Similarly, decompose $\mathbb{P}^2$ as the union of a Moebius band M and another subspace Q, such that Q is a quotient of the cylinder and $M\cap Q$ is a circle.
(c) Deduce that $\mathbb{P}^2$ can be obtained from a Moebius band and a disk D^2 by gluing them along their boundary circles.

Exercise 11.8. The aim of this exercise is to explain and show that the 3-sphere can be obtained by gluing two solid tori along their boundaries.

(a) First of all, by a solid torus we mean any space homeomorphic to $S^1\times D^2$, the product of the unit circle with the unit disk. Show that

$$f: S^1\times D^2\to\mathbb{R}^3,\ f((\alpha,\beta),(x,y))=((2-x)\alpha,(2-x)\beta,y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name solid torus).

Next, consider the 3-sphere S^3 viewed as a subspace $\mathbb{C}^2$,

$$S^3=\left\{(u,v): u,v\in\mathbb{C}, |u|^2+|v|^2=1\right\},$$

and, inside the sphere, we consider

$$X_1=\left\{(u,v)\in S^3: |u|<|v|\right\},\quad X_2=\left\{(u,v)\in S^3: |u|>|v|\right\},\quad A:=\left\{(u,v)\in S^3: |v|=\frac{\sqrt{2}}{2}\right\}.$$

(b) Show that $\overline{X}_i$ is a solid torus for $i\in\{1,2\}$.
(c) Show that $\overline{X}_1\cap\overline{X}_2=\partial(X_1)=\partial(X_2)=A$. (where the closures and boundaries are inside the space S^3).
(d) Show that the 3-sphere S^3 can be obtained from two disjoint copies of $S^1\times D^2$ (i.e. two solid tori) by gluing each point $(z_1,z_2)\in S^1\times S^1$ in the boundary of the first copy to the point (z_2,z_1) in the boundary of the second.

Exercise 11.9. Show that

(a) $\mathbb{P}^n$ is an n-dimensional topological manifold.
(b) The map $f:\mathbb{P}^n\to\mathbb{R}^{\frac{(n+1)(n+2)}{2}}$ which sends the line l_x through $x=(x_0,\ldots,x_n)\in S^n$ to

$$f(l_x)=(x_0x_0,x_0x_1,\ldots,x_0x_n,x_1x_1,x_1x_2,\ldots,x_1x_n,\ldots,x_nx_n)$$

is an embedding.

11.2 Some special quotients

Exercise 11.10. (from the 2021 exam) Consider the equivalence relation $\sim$ on $\mathbb{R}$, where $x \sim y$ iff x and y are both positive or both negative or both zero. Let $X = \mathbb{R}/\sim$ with the quotient topology. List (with proof) all closed subsets of X.

Exercise 11.11. Show that the space obtained from $\mathbb{R}$ by collapsing $[-1,1]$ to a point is homeomorphic to $\mathbb{R}$.

Exercise 11.12. Let X be the cone of the interval $(0,1)$. Construct an explicit continuous injection

$$f : X \to \mathbb{R}^2$$

and let C be its image. Is f a homeomorphism between X and C?

Exercise 11.13. Show that the space obtained from $\mathbb{R}$ by collapsing $(-1,1)$ to a point is not Hausdorff.

Exercise 11.14. Let $f : X \to Y$ be a surjective continuous map between two topological spaces, and consider the equivalence relation induced by f, $R_f = \{(x,x') \in X \times X : f(x) = f(x')\}$. By the general theory, we obtain a continuous map

$$\phi : X/R_f \to Y.$$

What are we using here, and what is the map explicitly? Then show that ϕ is a homeomorphism if and only if f is a topological quotient map.

11.3 Quotients modulo group actions

Exercise 11.15. (from the 2024 exam) Consider the action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \left\{\widehat{0},\ \widehat{1}\right\}$ (the groups of integers modulo 2) on the unit two-dimensional sphere S^2 given by

$$\widehat{k} \cdot (x,y,z) = \left(x,y,(-1)^k z\right) \qquad \left(\text{for } \widehat{k} \in \mathbb{Z}_2,\ (x,y,z) \in S^2\right).$$

Show that the resulting quotient space $S^2/\mathbb{Z}_2$ is homeomorphic to the closed unit disk D^2.

Similar exercise: Exercise 16.14 from the 2024 retake.

Exercise 11.16. (from the 2013 exam) Let $X = (-1,\infty)$.

(a) Find all the numbers $a,b \in \mathbb{R}$ with the property that

$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group $(\mathbb{Z},+)$ on X.

(b) For the a and b that you found, show that the resulting quotient space $X/\mathbb{Z}$ is homeomorphic to S^1.

Similar exercise: Exercise 16.61 from the 2014 exam.

Exercise 11.17. Let $X = (0,\infty)$. Show that

$$\phi_n(r) := 2^n r$$

defines an action of $\mathbb{Z}$ on X and $X/\mathbb{Z}$ is homeomorphic to S^1. (hint: see Example 3.34 and Exercise 1.3).

Exercise 11.18. Show that, for any action of a group Γ on a space X, the quotient map is automatically open. (Note: to appreciate this, recall that continuous surjective open maps are automatically topological quotient maps, but there exist topological quotient maps that are not open maps- see Exercise 11.1 and Exercise 11.2).

Exercise 11.19. Show that if Γ is a finite group acting on a Hausdorff space X, and if the action is free, then the quotient map $\pi : X \to X/\Gamma$ is a local homeomorphism.

Exercise 11.20. (from the 2011 exam) Consider the map

$$\pi\colon \mathbb{R}^2 \to S^1 \times \mathbb{R}, \qquad (x,y) \mapsto \Big(\big(\cos(x+y),\sin(x+y)\big), x-y\Big) \in S^1 \times \mathbb{R}.$$

- Describe an equivalence relation R on $\mathbb{R}^2$ such that $(S^1 \times \mathbb{R}, \pi)$ is a quotient of $\mathbb{R}^2$ modulo R.
- Find a group Γ and an action of Γ on $\mathbb{R}^2$ such that R is the equivalence relation induced by this action.
- Show that, indeed, $\mathbb{R}^2/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}$.

Exercise 11.21. Consider the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Let n be a positive integer, consider the n-th root of unity

$$\xi = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \in \mathbb{C}$$

and let $\mathbb{Z}_n$ be the (additive) group of integers modulo n. Show that

$$\phi_k(z) := \xi^k z$$

defines an action of $\mathbb{Z}_n$ on S^1, explain its geometric meaning, and show that $S^1/\mathbb{Z}_n$ is homeomorphic to S^1. What do you obtain when $n = 2$?

Exercise 11.22. Consider the 3-dimensional sphere interpreted as:

$$S^3 = \left\{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}.$$

Consider the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ viewed as a group with respect to the multiplication of complex numbers. Consider the action of S^1 on S^3 given by $z \cdot (z_1,z_2) := (zz_1, \bar{z}z_2)$. We are interested in the resulting quotient S^3/S^1.

(a) Show that the maps $\tilde{f},\tilde{g},\tilde{h} : S^3 \to \mathbb{R}$ given by

$$\tilde{f}(z_1,z_2) = z_1z_2 + \bar{z}_1\bar{z}_2, \quad \tilde{g}(z_1,z_2) = i(z_1z_2 - \bar{z}_1\bar{z}_2), \quad \tilde{h}(z_1,z_2) = |z_1|^2 - |z_2|^2$$

induces continuous maps $f,g,h : S^3/S^1 \to \mathbb{R}$.

(b) Show that the maps $\tilde{f},\tilde{g},\tilde{h}$ put together, define a continuous surjective map

$$\text{Hopf} := (\tilde{f},\tilde{g},\tilde{h}) : S^3 \to S^2.$$

(c) Show that the Hopf is a topological quotient map.

(d) Deduce that we obtain a homeomorphism $(f,g,h) : S^3/S^1 \xrightarrow{\sim} S^2$.

Exercise 11.23. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}$ with the usual group operation

$$(m,n)+(m',n')=(m+m',n+n').$$

Show that

$$\phi_{m,n}(x,y) := (x+m, y+n)$$

defines an action of Γ on $\mathbb{R}^2$ and $\mathbb{R}^2/\Gamma$ is homeomorphic to the torus.

Exercise 11.24. Do the same for the same group but the new action:

$$\phi_{m,n}(x,y) := (x+my+n+\frac{m(m-1)}{2}, y+m).$$

Exercise 11.25. Consider the following groups:

(1) $\Gamma = \langle a,b; bab=a\rangle$ (the group in two generators a and b, subject to the relation $bab=a$).
(2) $\Gamma' = \mathbb{Z}\times\mathbb{Z}$ with the operation

$$(m,n)\circ(m',n') = (m+m', n+(-1)^m n').$$

(3) Γ'' is the subgroup of $(\mathrm{Homeo}(\mathbb{R}^2),\circ)$ (the group of homeomorphisms of the plane endowed with the composition of functions) generated by the transformations

$$\phi(x,y) = (x+1,-y), \psi(x,y) = (x,y+1).$$

Show that:

(a) these three groups are isomorphic.
(b) one obtains an action of these groups on $\mathbb{R}^2$; write it down explicitly.
(c) the resulting quotient is homeomorphic to the Klein bottle.

Exercise 11.26. Compose and solve a similar exercise in which the resulting quotient is homeomorphic to the Moebius band.

11.4 Product topology

Exercise 11.27. Show that

$$(\mathbb{R}^n, \mathscr{T}_{\mathrm{eucl}})\times(\mathbb{R}^m, \mathscr{T}_{\mathrm{eucl}}) = (\mathbb{R}^{n+m}, \mathscr{T}_{\mathrm{eucl}}).$$

Exercise 11.28. Look at exercise 10.7 and show that there is no conflict in the notation; i.e. the topology $\mathscr{T}_l\times\mathscr{T}_l$ defined in that exercise does coincide with the product topology.

Exercise 11.29. Show that, if $(Z,\mathscr{T}_Z)$ is a third topological space, then a function

$$h=(f,g): Z\to X\times Y,\ h(z)=(f(z),g(z))$$

is continuous if and only if its components $f: Z\to X$ and $g: Z\to Y$ are both continuous.

Exercise 11.30. Show that if X and Y are both Hausdorff, then so is $X\times Y$. Similarly for metrizability and first countability.

Exercise 11.31. (from the 2022 retake exam) Let $X = \{a,b\}$ be a set with two elements, endowed with the topology

$$\mathscr{T}_X = \{\emptyset, \{a\}, X\}.$$

Consider $X \times X$ endowed with the product topology.

1. Describe all the open subsets of $X \times X$.
2. Compute the interior and the closure of $A = \{(a,b)\} \subset X \times X$.

Exercise 11.32. Let X and Y be two topological spaces, $A \subset X$, $B \subset Y$. Then the following two topologies on $A \times B$ coincide:

(a) The product of the topology of A (induced from X) and that of B (induced from Y).
(b) The topology induced on $A \times B$ from the product topology on $X \times Y$.

Exercise 11.33. Show that a topological space X is Hausdorff if and only if

$$\Delta := \{(x,x) : x \in X\}$$

is closed in $X \times X$.

11.5 Topology bases

Exercise 11.34. Using Lemma 3.48, prove again that the Euclidean and the square metrics induce the same topology.

Exercise 11.35. Show that

$$\mathscr{B}_l := \{[a,b) : a,b \in \mathbb{R}\}$$

is a topology basis and $\mathscr{T}(\mathscr{B}_l)$ is the lower limit topology $\mathscr{T}_l$. Then use Lemma 3.48 to prove again that $\mathscr{T}_l$ is finer than $\mathscr{T}_{\text{Eucl}}$.

Exercise 11.36. Do again Exercise 10.7 using Lemma 3.48.

Exercise 11.37. Go back to exercise 10.5 and show that $\mathscr{B}$ there is a topology basis. Then using Lemma 3.48, show again that the resulting topology is larger than the topology from Exercise 10.4.

Exercise 11.38. Let $(X, \mathscr{T})$ be a topological space. For $f \in \mathscr{C}(X)$ we define

$$V(f) := \Big\{x \in X : f(x) \neq 0\Big\}.$$

Show that:

(a) The collection $\mathscr{B} := \{V(f) : f \in C(X)\}$ is a topology basis on X.
(b) In general, $\mathscr{T}(\mathscr{B}) \subset \mathscr{T}$.
(c) If X is Hausdorff and normal then $\mathscr{T}(\mathscr{B}) = \mathscr{T}$.

Exercise 11.39. Let k be a natural number ($k \in \{0,1,2,\ldots\}$). Let $\mathscr{B}_k$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and all the open intervals (a,b) which contain at most k integers (when $k = 0$, that means: intervals (a,b) that contain no integers). Show that:

1. each $\mathscr{B}_k$ is a topology basis on $\mathbb{R}$.

2. for $k \geq 1$, the topology induced by $\mathscr{B}_k$ on $\mathbb{R}$ is the Euclidean topology.
3. for $k = 0$, the topology induced by $\mathscr{B}_0$ is strictly smaller than the Euclidean topology.
4. Compute the interior and the closure of

$$D := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\} \subset \mathbb{R}^2$$

in $\mathbb{R}^2$ endowed with the product topology $\mathscr{T}_0 \times \mathscr{T}_{\text{Eucl}}$.

Exercise 11.40. (from the 2012 exam) Let $\mathscr{B}$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets

$$[n,a) := \{r \in \mathbb{R} : n \leq r < a\} \quad \text{with } n \in \mathbb{Z}, a \in \mathbb{R}.$$

1. Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $\mathscr{T}$ the associated topology.
2. Is $(\mathbb{R}, \mathscr{T})$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)?
3. compute the closure, the interior and the boundary of $A = [-\frac{1}{2}, \frac{1}{2}]$ in $(\mathbb{R}, \mathscr{T})$.

Similar exercises:

- *Exercise 16.43 from the 2016 exam, without items e, and g.*
- *Exercise 16.13 without item c.*
- *Exercise 16.100 from the 2011 exam A, without the question on connectedness.*
- *Exercise 16.48 from the 2015 exam, without item (d).*
- *Exercise 16.13 from the 2024 retake exam.*
- *Exercise 16.71 from the 2013 exam.*

Exercise 11.41. (from the 2014 retake exam) Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ with the property that

$$f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \;\; f(x) = x^2$$

is continuous.

a. Describe a basis of $(\mathbb{R}, \mathscr{T})$ and show that any $U \in \mathscr{T}$ has the property that

$$-x \in U \quad \forall \; x \in U.$$

b. Find the closure and the interior of $(-1,2)$ in $(\mathbb{R}, \mathscr{T})$.
c. Is $(\mathbb{R}, \mathscr{T})$ Hausdorff? Can you find a sequence with two (distinct) limits?
g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R}, \mathscr{T})$?

Exercise 11.42. Consider the unit circle S^1 and the functions

$$\alpha, \beta : S^1 \to \mathbb{R}, \quad \alpha(x,y) = x, \beta(x,y) = y.$$

Show that:

(a) S^1, endowed with the smallest topology which makes α continuous, is not Hausdorff.
(b) the smallest topology on S^1 which makes both α and β continuous is the Euclidean one.

11.6 Spaces of functions

Exercise 11.43. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(nx)$. Show that (f_n) is uniformly convergent.

Exercise 11.44. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = x^n$. Is (f_n) pointwise convergent? But uniformly on compacts? But uniformly?

Exercise 11.45. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. Is (f_n) pointwise convergent? But uniformly on compacts? But uniformly?

Exercise 11.46. (from the 2011 exam) Consider the space $\mathscr{C}([0,1])$ of all continuous maps $f : [0,1] \to \mathbb{R}$, endowed with the sup-metric. Show that

$$A := \left\{ f \in \mathscr{C}([0,1]) : x^2 \leq e^{f(x)} + \sin(f(x)) \leq x \ \ \forall\, x \in [0,1] \right\}$$

is a closed subset of $\mathscr{C}([0,1])$, and that it is not bounded.

Exercise 11.47. (Dini's theorem) Let $f_n : I \to \mathbb{R}$ be an increasing sequence of continuous functions defined on a compact interval I, which converges pointwise to a continuous function f. Show that $f_n \rightrightarrows f$. Is the same true if we do not assume that f is continuous?

Exercise 11.48. Let $p_n \in \mathscr{C}([0,1],\mathbb{R})$ be the sequence of functions (even polynomials!) defined inductively by

$$p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n(t)^2), \quad p_1 = 0.$$

Show that (p_n) converges uniformly to the function $f(t) = \sqrt{t}$.

(Hint: first show that $p_n(t) \leq \sqrt{t}$, then that $p_n(t)$ is increasing, then that it converges pointwise to $\sqrt{t}$, then look above).

Exercise 11.49. Now, did you know that there are continuous surjective functions $f : [0,1] \to [0,1] \times [0,1]$??? (yes, continuous curves in the plane which fill up an entire square!). Actually, you now have all the knowledge to show that (using Theorem 3.58); it is not obvious but, once you see the pictures, the proof is not too difficult; have a look at Munkres' book.

Chapter 12
Exercises on the chapter "Topological properties"

12.1 Connectedness

Exercise 12.1. Prove that connectedness is a topological property: if two spaces are homeomorphic, then one of them is connected if and only if the other one is connected.

Exercise 12.2. Let $\mathscr{T}_l$ be the lower limit topology on $\mathbb{R}$.

(a) Is $(\mathbb{R}, \mathscr{T}_l)$ connected? Deduce that $(\mathbb{R}, \mathscr{T}_l)$ and $(\mathbb{R}, \mathscr{T}_{\text{Eucl}})$ are not homeomorphic.
(b) But $[0,1]$ with the induced topology? Deduce that $[0,1]$ with the topology induced from $\mathscr{T}_l$ and $[0,1]$ with the topology induced from the Euclidean topology are not homeomorphic.
(c) But $(0,1]$? Can you deduce whether $(0,1]$ with the topology induced from $\mathscr{T}_l$ and $(0,1]$ with the topology induced from the Euclidean topology are not homeomorphic or not?

Exercise 12.3. Recall that by a circle we mean any topological space which is homeomorphic to S^1, and by a circle embedded in a topological space X we mean any subset $A \subset X$ which, when endowed with the induced topology, is a circle. Similarly, by a bouquet of two circles we mean any space which is homeomorphic to the space drawn in Figure 12.1, and we talk about embedded bouquets of circles. Let T be a torus.

1. Describe a circle C embedded in T such that its complement $T - C$ is connected.
2. Describe a bouquet of two circle B embedded in T such that $T - B$ is connected.

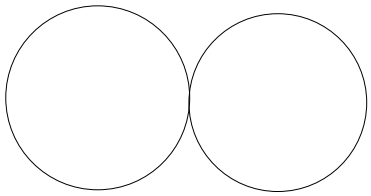


Fig. 12.1

Exercise 12.4. In Exercise 1.5 you showed that four of those spaces are homeomorphic to each other, but they did not seem to be homeomorphic to the fifth one. You now have the tool to prove the last assertion (so do it!).

Exercise 12.5. Prove that the following spaces are not homeomorphic:

1. S^1 and $[0,1)$.
2. $[0,1)$ and $\mathbb{R}$.
3. S^1 and S^2.
4. S^1 and a bouquet of two circles (the space from Figure 12.1).

Exercise 12.6. Show that the group $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices with real coefficients (see Exercise 10.72) is not connected.

Exercise 12.7. (from the 2022 retake exam) Consider the following three subsets of $\mathbb{R}^2$ equipped with the subspace topology:

- $A_1 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$
- $A_2 = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{4}x^2 + y^2 = 1\right\}$
- $A_3 = \left\{(x,y) \in \mathbb{R}^2 : y^2 = 1\right\}$

Give a sketch of all three subspaces. Then decide for each pair of subspaces whether they are homeomorphic or not: if they are, give a homeomorphism; if they are not, give an argument why they are not homeomorphic.

Exercise 12.8. (from the 2016 exam) Return to the spaces X and Y from Exercise 10.30. Show now that X cannot be embedded in Y.

Exercise 12.9. Show that any two of the three spaces drawn in Figure 12.2 are not homeomorphic.

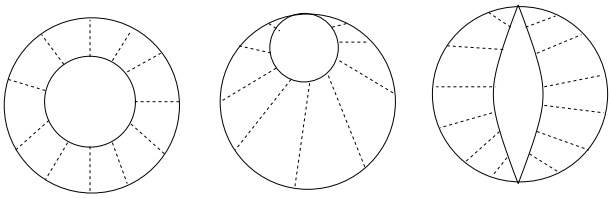


Fig. 12.2

Exercise 12.10. Show that there exists a real number $r \in (2,3)$ such that

$$r^7 - 2r^4 - r^2 - 2r = 2009.$$

Exercise 12.11. Show that any continuous function $f : [0,1] \to [0,1]$ admits at least one fixed point (i.e. there exists $t \in [0,1]$ such that $f(t) = t$). (Hint: $g(t) = f(t) - t$ positive or negative?)

Exercise 12.12. Assuming that the temperature on the surface of the earth is a continuous function, prove that, at any moment in time, on any great circle of the earth, there are two antipodal points with the same temperature.

Exercise 12.13. For a topological space $(X, \mathscr{T})$, show that the following are equivalent:

(1) X is connected.
(2) $\emptyset$ and X are the only subsets of X which are both open and closed.
(3) the only continuous functions $f : X \to \{0,1\}$ are the constant ones (where $\{0,1\}$ is endowed with the discrete topology).

Exercise 12.14. Prove that if $X \subset \mathbb{R}^n$ is connected then so is its closure, but its interior may fail to be connected.

Exercise 12.15. Show that the cone of any space is connected.

Exercise 12.16. (from the 2015 exam) Let X be a topological space.

a. If $\gamma_1, \gamma_2 : [0,1] \to X$ are continuous and $\gamma_1(1) = \gamma_2(0)$, show that

$$\gamma : [0,1] \to X, \quad \gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is continuous.

b. If there exist $A, B \subset X$ path connected such that $X = A \cup B$ and $A \cap B \neq \emptyset$, then X is connected.

Exercise 12.17. Prove that if X and Y are homeomorphic, then they have the same number of connected components.

Exercise 12.18. (from the 2021 exam) Let $X = \mathscr{C}([0,1], \mathbb{R})$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. We consider on it the topology induced by the metric $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$. Show that X is path-connected.

12.2 Compactness

Exercise 12.19. Prove that compactness is a topological property: if two spaces are homeomorphic, then one of them is compact if and only if the other one is compact.

Exercise 12.20. Let $\mathscr{T}_l$ be the lower limit topology on $\mathbb{R}$.

(a) Is $[0,1]$, with the topology induced from $\mathscr{T}_l$, compact?
(b) Answer again the question from Exercise 12.2, (b), on homeomorphisms, with a different proof now.
(c) Is $(0,1]$ with the topology induced from $\mathscr{T}_l$ compact? Try now again the very last question from Exercise 12.2.

Exercise 12.21. (from the 2022 retake exam) Consider

$$X = \left\{(x,y) \in \mathbb{R}^2 : xy = 1\right\}, \quad Y = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1.\right\}$$

endowed with the induced by the Euclidean topology on $\mathbb{R}^2$.

1. Draw X and Y.
2. Show that X and Y are not homeomorphic. Provide two proofs, each one using another topological property.

Exercise 12.22. Show that the boundary of any compact subspace $X \subset \mathbb{R}^n$ is compact.

Exercise 12.23. Find a topological space $(X, \mathscr{T})$ which is compact but is not Hausdorff. In the example that you found, exhibit a compact set $A \subset X$ such that A is not closed.

Exercise 12.24. Find an example which shows that the Tube Lemma (Lemma 4.31) fails if Y is not compact.

Exercise 12.25. Let X be a topological space and $A, B \subset X$. If A and B are compact, show that $A \cup B$ is compact.

Exercise 12.26. Let $(X, \mathscr{T})$ be a Hausdorff space, $A, B \subset X$. If A and B are compact, show that $A \cap B$ is compact. Is the Hausdorffness assumption on X essential?

Exercise 12.27. (from the 2014 retake exam) Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Show that:

(a) if $(X, \mathscr{T}_2)$ is connected then so is $(X, \mathscr{T}_1)$;
(b) if $(X, \mathscr{T}_2)$ is compact then so is $(X, \mathscr{T}_1)$;
(c) if $(X, \mathscr{T}_2)$ is sequentially compact then so is $(X, \mathscr{T}_1)$;

Exercise 12.28. For the notion of closed map we refer to Exercise 10.27. Show that, if X is a compact space and Y is a Hausdorff space, then any continuous map $f : X \to Y$ is automatically a closed map. What do you obtain if you combine this with item Exercise 10.27?

Exercise 12.29. Assume that X is a topological space and $(x_n)_{n\geq 1}$ is a sequence in X, convergent to $x \in X$. Show that

$$A = \{x, x_1, x_2, x_3, \ldots\}$$

is compact.

Exercise 12.30. Given a set X, when is $(X, \mathscr{T}_{\text{cf}})$ compact?

Exercise 12.31. Recall that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$Gr(f) = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

If f is bounded, show that f is continuous if and only if $Gr(f)$ is closed in $\mathbb{R}^2$. What if f is not bounded. (hint: for the first part, use sequences; for the last part: $1/x$).

Exercise 12.32. Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Show that if $(X, \mathscr{T}_2)$ is compact and $(X, \mathscr{T}_1)$ is Hausdorff then $\mathscr{T}_1$ and $\mathscr{T}_2$ must coincide.

Exercise 12.33. Let $(X, \mathscr{T})$ be a second countable topological space. Show that X is compact if and only if for any decreasing sequence of nonempty closed subsets

$$\ldots \subset F_3 \subset F_2 \subset F_1 \subset X,$$

one has $\cap_{n=1}^{\infty} F_n \neq \emptyset$. (Hint: restate the property in terms of opens; then try to use Exercise 10.61).

Exercise 12.34. If X is a Hausdorff space, and $A \subset X$ is compact, show that the quotient X/A (obtained from X by collapsing A to a point) is Hausdorff.

Exercise 12.35. Show that the cone and the suspension of any compact topological space are compact.

Exercise 12.36. Go back to exercise 11.8 and provide all the arguments (making use of compactness) and solve part (a) as well.

Exercise 12.37. Show that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$ (see Exercise 1.27 in the first chapter). Then do again Exercise 3.39 from Chapter 3.

Exercise 12.38. (from the 2015 exam) Consider the group of integers modulo 2, $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$. Define the following action of $\mathbb{Z}_2$ on the closed unit disk $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$:

$$\hat{0} \cdot z = z, \ \hat{1} \cdot z = -z \quad (\text{for all } z \in D^2).$$

Prove that the resulting quotient $D^2/\mathbb{Z}_2$ is homeomorphic to D^2.

Exercise 12.39. If X is connected and compact and $f : X \to \mathbb{R}$ is continuous, show that $f(X) = [m, M]$ for some $m, M \in \mathbb{R}$.

Exercise 12.40.

(a) Is it true that any continuous surjective map $f : S^1 \to S^1$ is a homeomorphism?
(b) Show that S^1 cannot be embedded in $\mathbb{R}$.
(c) Show that any continuous injective map $f : S^1 \to S^1$ is a homeomorphism.

Exercise 12.41. (from the 2011 exam) Prove that there is no continuous injective map $f : S^1 \vee S^1 \to S^1$, where $S^1 \vee S^1$ is a bouquet of two circles (two copies of S^1, tangent to each other).

Exercise 12.42. (from the 2012 exam) Prove that there is no continuous map $g : \mathbb{C} \to \mathbb{C}$ with the property that $g(z)^2 = z$ for all $z \in \mathbb{C}$.

Exercise 12.43. (from the 2013 exam) Prove that there is no continuous injective map $f : S^2 \to S^1$. *(1p)*
(warning: there are injective maps from S^2 to S^1!).

Exercise 12.44. Show that the sequence $x_n = 2011^{\sin(5n)}$ (in $\mathbb{R}$ with the Euclidean topology) has a convergent subsequence.

Exercise 12.45. Show that for any sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$, there exists a subsequence $(x_{n_k})_{k\geq 1}$ and a sequence of integers $(a_k)_{k\geq 1}$ such that $(x_{n_k} + a_k)_{k\geq 1}$ is convergent.

Exercise 12.46. (from the 2012 exam) Let X be a topological space and let $\gamma : [0,1] \to X$ be a continuous function. Assume that γ is locally injective, i.e. that, for any $t \in [0,1]$, there exists a neighborhood V of t in $[0,1]$ such that

$$\gamma|_V : V \to X$$

is injective. Show that, for any $x \in X$, the set

$$\gamma^{-1}(x) := \{t \in [0,1] : \gamma(t) = x\}$$

is finite.

Exercise 12.47. (from the 2011 exam) Let X be the space of continuous maps $f : [0,1] \to [0,1]$ with the property that $f(0) = f(1)$. We endow it with the sup-metric $d_{\sup}$ and the induced topology (recall that $d_{\sup}(f,g) = sup\{|f(t) - g(t)| : t \in [0,1]\}$). Prove that:

(a) X is bounded and complete.
(b) X is not compact.

12.3 Connectedness and compactness for some exotic topologies

Many of the concepts that we introduced were tested on $\mathbb{R}$ endowed with various exotics topologies, starting with the lower limit topology $\mathscr{T}_l$. For connectedness and compactness on $(\mathbb{R}, \mathscr{T}_l)$ we already had Exercise 12.2 and Exercise 12.20. It is now time to test them on a couple of other exotic (i.e., non Euclidean) topologies that we have seen.

Exercise 12.48. Go back to Exercise 10.65 and please continue with the following:

e. find an interval $[a,b]$ with the property that, together with the topology induced from $(\mathbb{R}, \mathscr{T})$, is not compact.
f. find an interval (a,b) with the property that, together with the topology induced from $(\mathbb{R}, \mathscr{T})$, is not connected.

Exercise 12.49. Go back to Exercise 11.41 and answer also:

d. Is $[-1,1]$ (with the topology induced from $\mathscr{T}$) compact? But connected?
e. Is $[-1,1)$ (with the topology induced from $\mathscr{T}$) compact?
f. Is $[-3,1)\cup(1,3]$ (with the topology induced from $\mathscr{T}$) connected?
g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R},\mathscr{T})$?

Exercise 12.50. Go back to Exercise 10.66 and answer also:

d. Is $C=[-1,1]$, with the induced topology $\mathscr{T}_{\text{new}}|_C$, compact? But connected?

Here are a couple more similar exercises:

- in Exercise 11.39 add the item: "Is $(\mathbb{R},\mathscr{T}_0)$ compact?"
- in Exercise 10.64: for each of the spaces $(\mathbb{N},\mathscr{T}_1)$ and $(\mathbb{N},\mathscr{T}_2)$, decide whether the space is Hausdorff, connected or compact.

Similar items appear also in some of the exam exercises that did not include in the main text, such as:

- in Exercise 16.48 from the 2015 exam: look at part of item (d).
- in Exercise 16.43 from the 2016 retake: look at items (f), (g) and (h). Then try to use those to do (i).
- in Exercise 16.13 from the 2024 retake: look at item (c)
- in Exercise 16.22 from the 2021 retake: look at item (b).
- in Exercise 16.100 from the 2011 exam A: look at item 4.

12.4 Local compactness and the one-point compactification

Exercise 12.51. What is the one-point compactification of $X=(0,1)\cup(2,3)$?

Exercise 12.52. What is the one-point compactification of $\mathbb{R}^2$?

Exercise 12.53. Show that the following subspace of $\mathbb{R}^2$ is locally compact and find its one-point compactification:

$$X=S^1\cup((0,2)\times\{0\})\subset\mathbb{R}^2.$$

Exercise 12.54. (from the 2010 exam) Let X, Y and Z be the spaces drawn in Figure 12.3.

(a) Show that any two of them are not homeomorphic.
(b) Compute their one-point compactifications X^+, Y^+ and Z^+.
(c) Which two of the spaces X^+, Y^+ and Z^+ are homeomorphic and which are not?

Exercise 12.55. (from the 2024 exam) Which of the following spaces are homeomorphic to each other (and why):

(a) $C_0=(S^1\times[0,1))^+$, the one-point compactification of $S^1\times[0,1)$.
(b) $C_1=\text{Cone}(S^1)$, the abstract cone associated to the space S^1.
(c) the concrete cone $C_2=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=(1-z)^2,\ 0\le z\le 1\right\}$.

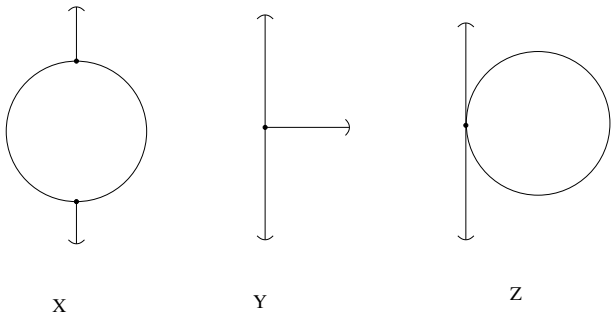


Fig. 12.3

Exercise 12.56. Consider

$$X = [0,1] \times [0,1)$$

with the topology induced from $\mathbb{R}^2$. Prove that X is a locally compact Hausdorff space and describe its one-point compactification (use a picture). What happens if we replace X by $Y = X \cup \{(1,1)\}$?

Exercise 12.57. (from the 2016 exam) Return to the spaces X and Y from Exercise 10.30:

$$X = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, \text{ or } x = 0\right\},$$

$$Y = \left\{(x,y) \in \mathbb{R}^2 : xy = 0\right\}.$$

In exercises 10.30 and 12.8 we have seen that Y can be embedded in X but X cannot be embedded in Y. Show that:

c) the one-point compactifications of X and Y are homeomorphic.

Again, please use pictures but back it up by explanations (please explain why the picture you draw is indeed the one-point compactification).

Exercise 12.58. (from the 2024 retake) Given a topological space X, by *a two-point compactification* of X we mean a pair (Y,i) consisting of a compact Hausdorff space together with a map $i : X \to Y$ which is an embedding and such that Y contains exactly two elements that not belong to the image of i. For $X = (0,1) \cup (2,3)$ endowed with the Euclidean topology:

(a) Give an example of such a two-point compactification, call it Y_1 (I am going to ignore the map i_1 from the notation, although you should indicate it on the picture).
(b) Find now another two-point compactification Y_2 of X such that Y_1 and Y_2 are not homeomorphic.
(c) Find two more, Y_3 and Y_4, not homeomorphic to Y_1 or Y_2 and not homeomorphic to each other.

Clarification: you can describe the spaces Y_1, Y_2, etc and the maps i_1, i_2, etc using pictures. But please provide the arguments/propositions that justify/support the claims that you make- e.g: "Y_1 is compact because ...", or "to see that Y_1 and Y_2 are not homeomorphic we use the X-method as follows ...".

Exercise 12.59. (from the 2011 exam) Consider

$$X_1 = \left\{(x,y,z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = y = 0, z \geq 0)\right\},$$

$$X_2 = \left\{(x,y,z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = 0, y^2 + z^2 = 1, z \geq 0)\right\}.$$

$$X_3 = \left\{(x,y,z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 = 1) \text{ or } (y = 0, z = 0, \frac{1}{2} < |x| < 1)\right\},$$

(a) Show that X_1, X_2, X_3 are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures).
(b) Show that the one-point compactifications of X_1, X_2 and X_3 are homeomorphic to each other.

Exercise 12.60. (from the 2013 exam) Consider

$$X = \left\{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 = 1\right\} \setminus \{(0,0,1),(0,0,-1)\},$$

$$Y = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 2\sqrt{x^2+y^2}\right\} \setminus \{(0,0,0)\}.$$

$$\pi : X \to \mathbb{R}^3,\ \pi(u,v,w) = (2u\sqrt{u^2+v^2}, 2v\sqrt{u^2+v^2}, 2w\sqrt{u^2+v^2}).$$

(a) Compute the closure $\overline{Y}$ of Y in $\mathbb{R}^3$ and prove that it is compact.
(b) Prove that the one-point compactification of X is homeomorphic to $\overline{Y}$.
(c) Draw a picture of Y and explain the map π on the picture.

Exercise 12.61. (from the 2012 exam) Let X be a bouquet of two circles:

$$X = \left\{(x,y) \in \mathbb{R}^2 : ((x-1)^2+y^2-1)((x+1)^2+y^2-1) = 0\right\}.$$

We say that a space Y is an *exam space* if there exist three distinct points $p,q,r \in X$ such that Y is homeomorphic to the one-point compactification of $X - \{p,q,r\}$.

Find the largest number l with the property that there exist exam spaces $Y_1,\ldots,Y_l$ with the property that any two of them are not homeomorphic (prove all the statements that you make!).

Exercise 12.62. Describe an embedding of the cylinder $S^1 \times [0,1]$ into the space X, show that its complement is locally compact, and find the one-point compactification of the complement, in each of the cases:

(a) X is a torus.
(b) X is the plane $\mathbb{R}^2$.

Exercise 12.63. (from the 2021 exam) Let $p,q \in S^2$ be the points $(1,0,0)$ and $(-1,0,0)$. Let $X = S^2/\{p,q\}$ with the quotient topology.

(a) Show that X is not a 2-dimensional manifold.
(b) Show that $S^2 \setminus \{p,q\}$ is not homeomorphic to $\mathbb{R}^2$.

Exercise 12.64. (from the 2014 retake exam) Assume that X is a sphere S^2 minus n points and Y is a sphere minus m points, where $m,n \geq 0$ are integers. Show that if X is homeomorphic to Y, then $n = m$.

Exercise 12.65. For each natural number n we consider a space X_n that is obtained by removing n distinct points from $\mathbb{R}^2$. We consider the 1-point compactification X_n^+ and we denote by $\infty_n \in X_n^+$ the point at infinity (so that $X_n^+ = X_n \cup \{\infty_n\}$). Show that

a. X_n^+ can be embedded in $\mathbb{R}^3$ (here you **do not** have to write down explicit formulas for the embedding, but please explain your reasoning using pictures and mention what result(s) you use in order to reach the final conclusion). *(0.5 points)*
b* If X_n and X_m are homeomorphic, then $n = m$. *(1 point)*

Exercise 12.66. Show that, for any circle C embedded in the Klein bottle K, $K-C$ is locally compact. Then describe such a circle C such that $(K-C)^+$ is homeomorphic to $\mathbb{P}^2$.

Exercise 12.67. Find two locally compact Hausdorff spaces X and Y which are not homeomorphic but which have homeomorphic one-point compactifications.

Exercise 12.68. For any continuous map $f : S^1 \to T^2$ we define

$$X_f := T^2 - f(S^1).$$

(a) Is it true that, for any continuous function f, X_f is compact? But locally compact? But metrizable? But connected?
(b) Describe two embeddings $f_1, f_2 : S^1 \to T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic.
(c) Describe the one-point compactifications $X_{f_1}^+$ and $X_{f_2}^+$.
(d) Describe $f : S^1 \to T^2$ continuous such that X_f^+ is homeomorphic to S^2.

Exercise 12.69. (from the 2010 exam) Let M be the Moebius band. For any continuous function $f : S^1 \to M$ we denote by M_f the complement of its image:

$$M_f := M - f(S^1)$$

and we denote by M_f^+ the one-point compactification of M_f.

(a) Show that for any f, M_f is open in M, it is locally compact but, if nonempty, it is not compact.
(b) Find an example of f such that M_f^+ is homeomorphic to D^2. Then one for which it is homeomorphic to S^2. And then one for $\mathbb{P}^2$.

Exercise 12.70. Show that there exists a Hausdorff space X and an embedded circle C in X, such that the one-point compactification of $X-C$ is homeomorphic to X.

Exercise 12.71. (from the 2022 retake exam) Let $N \in \mathbb{Z}_{>0}$ and let $(X, \mathscr{T})$ be a compact topological space together with an embedding $\Phi : X \to \mathbb{R}^N$, where $\mathbb{R}^N$ is endowed with the Euclidean topology. Prove that $\Phi(X) \subset \mathbb{R}^N$ cannot be open if X is non-empty.

Exercise 12.72. Let X be a connected, locally compact, Hausdorff space, $X \neq \emptyset$. Show that X is compact if and only if X^+ is not connected.

Exercise 12.73. (from the 2014 retake exam) Assume that $(X, \mathscr{T}_X)$ is a compact Hausdorff space and A is nonempty a closed subset of X. We consider the complement of A in X,

$$Y := X - A = \{x \in X : x \notin A\},$$

we denote by X/A the space obtained from X by collapsing A to a point and we consider the canonical projection:

$$\pi : X \to X/A$$

(recall that X/A is endowed with the topology $\pi_* \mathscr{T}_X$). Show that:

a. Y is locally compact and Hausdorff.

b. For any open U in X with the property that $U \cap A = \emptyset$ or $A \subset U$, one has that $\pi(U)$ is open in X/A.
c. X/A is a compact Hausdorff space.
d. The one-point compactification of Y is homeomorphic to X/A.

Exercise 12.74. (from the 2014 exam) Note that this exercise shows, in particular, that the open Moebius band and the open cylinder are not homeomorphic.

Let X be the open cylinder $(-1,1) \times S^1$ and let Y be the open Moebius band (i.e. the Moebius band discussed in the lectures, from which the boundary circle was removed).

a. Describe the 1-point compactification X^+ as a subspace of $\mathbb{R}^3$.
b. (+) Describe $X^+ \subset \mathbb{R}^3$ by explicit formulas and write down an explicit embedding

$$f : X \to \mathbb{R}^3$$

so that X^+ is the image of f together with the extra-point $(0,0,0)$.
c. Show that the 1-point compactification of Y is homeomorphic to the projective plane $\mathbb{P}^2$.
d. Show that X and Y are not homeomorphic.

Chapter 13
Exercises on the chapter "Topological properties versus metric ones"

Exercise 13.1. Let (X,d) be a metric space. Show that any sequence $(x_n)_{n\geq 1}$ in X with the property that

$$d(x_{n+1},x_n) \leq \frac{d(x_n,x_{n-1})}{2}$$

for all n, is Cauchy.

Exercise 13.2. Let (X,d) be a complete metric space and let $f : X \to X$ be a map with the property that there exists $\lambda \in (0,1)$ such that

$$d(f(x),f(y)) \leq \lambda d(x,y)$$

for all $x,y \in X$. Show that f has a unique fixed point (i.e. $a \in X$ with $f(a) = a$).
(Hint: the difficult part is the existence. Start with any x_0 and consider $x_{n+1} = f(x_n)$).

Exercise 13.3. We say that a topological space X is separable if there exists $A \subset X$ countable and dense in X.

1. Show that if X is 2nd countable, then it is separable.
2. Show that a metric space is 2nd countable if and only if it is separable.
3. Deduce that $(\mathbb{R}, \mathscr{T}_l)$ is not metrizable.

Chapter 14
Exercises on the chapter "The algebra of observables/spaces of functions"

14.1 On Stone-Weierstrass

Exercise 14.1. Show that $\mathscr{C}^\infty([0,1])$ (the space of real-valued functions on $[0,1]$ which are infinitely many times differentiable) is dense in $\mathscr{C}([0,1])$.

Exercise 14.2. On the sphere S^2 we consider the real-valued functions

$$f(x,y,z) = x+y+z, \quad g(x,y,z) = xy+yz+zx.$$

Does $\{f,g\}$ separate points? What if we add the function $h(x,y,z) = xyz$?

Exercise 14.3. Let X be a compact topological space. Show that if a finite set of continuous functions

$$\{f_1,\dots,f_n\} \subset \mathscr{C}(X)$$

separates points, then X can be embedded in $\mathbb{R}^n$.

A nice instance of this situation is the one from Exercise 11.22: then $X = S^3/S^1$, $n = 3$, with the set of functions $\{f,g,h\}$ from that exercise. You should check that, indeed, this set does separate points. The resulting embedding (f,g,h) allows us to identify S^3/S^1 with the 2-sphere $S^2 \subset \mathbb{R}^3$. Another instance is described in the next exercise.

Exercise 14.4. Consider the projective plane $\mathbb{P}^2$ interpreted as $S^2/\mathbb{Z}_2$ and we use the notation $[x,y,z] \in S^2/\mathbb{Z}_2$ for the orbit through (=equivalence class of) points $(x,y,z) \in S^2$ (recall that each such class encodes the line in $\mathbb{R}^3$ through the origin and (x,y,z)). Consider the maps

$$g,h,k : \mathbb{P}^2 \to \mathbb{R}, \quad g([x,y,z]) = xy, \quad h([x,y,z]) = xz, \quad k([x,y,z]) = yz.$$

Does $\{g,h,k,\}$ separate points? What if we add $f([x,y,z]) = x^2 - y^2$? Combine now with Exercise 14.3

Exercise 14.5. (from the 2015 retake exam) Let X be a compact Hausdorff space. Let $f_1,\dots,f_n : X \to \mathbb{R}$ be continuous functions, and consider "the sub-algebra of $\mathscr{C}(X)$ generated by $f_1,\dots,f_n$":

$$\mathscr{A}(f_1,\dots,f_n) := \Big\{ f \in \mathscr{C}(X) : f = P(f_1,\dots,f_n) \ \text{ for some polynomial } P \in \mathbb{R}[X_1,\dots,X_n] \Big\}. \tag{14.1.1}$$

(we use quotes because calling it a (sub)-algebra doesn't make it into an actual (sub)-algebra before you actually prove it is!). Show that $\mathscr{A}(f_1,\dots,f_n)$ is dense in $\mathscr{C}(X)$ if and only if

$$(f_1,\dots,f_n): X \to \mathbb{R}^n$$

is an embedding.

Exercise 14.6. Combine the previous exercises 14.3 and 14.5 into a single one and aim at formulating an equivalence of four items, one related to embeddings, one related to densities, and two related to point separation. Try to first guess such a complete statement, then find the optimal way to prove it.

Exercise 14.7. If K is a compact subspace of $\mathbb{R}^n$, show that the subspace $\mathrm{Pol}(K) \subset \mathscr{C}(K)$ of polynomial functions on K is dense in $\mathscr{C}(K)$ in the uniform topology. (Here, a function $f: K \to \mathbb{R}$ is called polynomial if there exists a polynomial $P \in \mathbb{R}[X_1,\dots,X_n]$ such that $f(x) = P(x)$ for all $x \in K$).

Exercise 14.8. Show that, if $f: \mathbb{R} \to \mathbb{R}$ is continuous and periodic of period 2π, then f can be realized as the limit of a sequence of functions of type

$$T(x) = a_0 + \sum_{k=1}^{N} (a_k \cos(kx) + b_k \sin(kx)).$$

Exercise 14.9. (from the 2015 exam) Let X be compact Hausdorff space and let $C(X)$ be the space of real-valued continuous functions on X, endowed with the topology induced by the *sup*-metric.

Prove that if $\mathscr{A} \subset C(X)$ is a dense subset of $C(X)$, then $\mathscr{A}$ must be point-separating.

Exercise 14.10. Show that, for any compact smooth manifold X, $\mathscr{C}^\infty(X)$ is dense in $\mathscr{C}(X)$.

Exercise 14.11. This is about adapting Stone-Weierstrass to the non-compact spaces. More precisely, we assume that X is a locally compact Hausdorff space. The first adaptation is to replace $\mathscr{C}(X)$ by $\mathscr{C}_0(X)$, which is the collection of continuous functions $f: X \to \mathbb{R}$ which vanish at ∞ in the sense that for each $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that

$$|f(x)| < \varepsilon, \quad \forall\ x \in X - K_\varepsilon.$$

This condition ensures that the sup-norm of f is finite. The second adaptation is to allow for non-unital algebras and non-unital sub-algebras, which means that we are giving up on any requirement regarding units. Show that:

(a) $C_0(X)$ is a non-unital algebra.
(b) The sup-norm (7.1.1) is a well-defined norm on $C_0(X)$ and makes $C_0(X)$ into a non-unital Banach algebra.
(c) If $\mathscr{A}$ is a non-unital subalgebra of $C_0(X)$ which is point separating and satisfies the condition that for any $x \in X$ there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$, then $\mathscr{A}$ is dense in $C_0(X)$.

(Note: if only interested in (c), item (b) can be skipped.)

14.2 On Gelfand-Naimark-more theoretical

Exercise 14.12. (from the 2015 retake exam) Let A be an algebra over $\mathbb{R}$ with unit denoted $\mathbf{1}_A$, assume that $\chi: A \to \mathbb{R}$ is a character and let $I := \mathrm{Ker}(\chi) = \{a \in A : \chi(a) = 0\}$. Show that:

(a) I is an ideal of A.
(b) $a - \chi(a) \cdot \mathbf{1}_A \in I$ for all $a \in A$.
(c) I is a maximal ideal.

Exercise 14.13. This exercise is about "geometric ideals" in the algebra $\mathscr{C}(X)$ of a space X, i.e., those of type

$$I_Y := \{f \in \mathscr{C}(X) : f|_Y = 0\} \subset \mathscr{C}(X),$$

associated to subspaces $Y \subset X$. Assume now that X is a compact Hausdorff space and show that:

(a) For any $Y \subset X$, I_Y is indeed an ideal in $\mathscr{C}(X)$.
(b) Any ideal of this type corresponds to a unique closed subset of X. More precisely:

a. for any $Y \subset X$, closed or not, one has $I_Y = I_{\overline{Y}}$, where $\overline{Y}$ is the closure in X.
b. If $Y_1, Y_2 \subset X$ are closed and $I_{Y_1} = I_{Y_2}$, then $Y_1 = Y_2$.

(c*) An ideal $I \subset \mathscr{C}(X)$ is of this type if and only if it is closed in $\mathscr{C}(X)$.

Exercise 14.14. Show that the spectrum X_A of any commutative algebra A over $\mathbb{R}$ is Hausdorff.
(Note: all you need to know about the topology on X_A is: for any $a \in A$, $f_a : X_A \to \mathbb{R}$, $f_a(\chi) = \chi(a)$, is continuous!)

Exercise 14.15. Let A be a commutative algebra over $\mathbb{R}$. Recall that, for any $a \in A$ we consider that function

$$f_a : X_A \to \mathbb{R},\ f_a(\chi) = \chi(a),$$

and the topology on X_A was defined as the smallest topology on X_A with the property that all the functions f_a, with $a \in A$, are continuous. Show that, for any other topological space X and function $f : X \to X_A$, one has:

$$f : X \to X_A \text{ is continuous} \iff f_a \circ f : X \to \mathbb{R} \text{ is continuous } \forall\, a \in A$$

Exercise 14.16. Let X be a compact, Hausdorff topological space, and consider a sub-algebra $\mathscr{A} \subset C(X)$.

(a) Show that for any $x \in X$, $\chi_x|_{\mathscr{A}} : \mathscr{A} \to \mathbb{R}$, $a \mapsto a(x)$, is a character on $\mathscr{A}$. Consider now the resulting map

$$F : X \to X_{\mathscr{A}}, \quad F(x) = \chi_x|_{\mathscr{A}}$$

(b) Show that F injective if and only if $\mathscr{A}$ is point separating.
(c) Find example of X and $\mathscr{A}$ for which F is not surjective.
(d) Show that F is always continuous.
(e*) Show that if $\mathscr{A}$ is separating and satisfies: $\sqrt{f} \in \mathscr{A}$ for all $f \in \mathscr{A}$ with $f > 0$, then F is a homeomorphism.

(Note: If you do not manage to do item (d), have a look at Exercise 14.15. For item (e) see Exercise 14.30)

Exercise 14.17. Let X and Y be two compact Hausdorff spaces and assume that

$$\Phi : \mathscr{C}(Y) \to \mathscr{C}(X)$$

is an algebra homomorphism in the sense that it is an $\mathbb{R}$-linear map, sends the constant function 1 to the constant function 1, and $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in \mathscr{C}(Y)$.

(a) For $x \in X$ show that

$$\widehat{\chi}_x : \mathscr{C}(Y) \to \mathbb{R}, \quad \widehat{\chi}_x(g) = \Phi(g)(x) \quad \forall\, g \in \mathscr{C}(Y),$$

or, equivalently, the composition $\mathscr{C}(Y) \xrightarrow{\Phi} \mathscr{C}(X) \xrightarrow{\chi_x} \mathbb{R}$, is a character on $\mathscr{C}(Y)$.
(b) Using the Gelfand-Naimark bijection (7.3.1), deduce that for any $x \in X$ there exists $y \in Y$ such that

$$\Phi(g)(x) = g(y) \quad \forall\, g \in \mathscr{C}(Y).$$

c) deduce that there exists a map $\varphi : X \to Y$ such that

$$\Phi(f) = f \circ \varphi \quad \forall\, f \in \mathscr{C}(Y).$$

(d) Prove that ϕ it is continuous.

(Hint: one way to do the last item is to try to use Exercise 14.15 ... for $A = \mathscr{C}(Y)$)

14.3 On Gelfand-Naimark- "computing" spectra

Exercise 14.18. Consider the algebra $A = \mathbb{R}[t]$ of polynomials in one variable t. Show that X_A is homeomorphic to $\mathbb{R}$. What if you take more variables?

Exercise 14.19. Consider the algebra $A = \mathbb{R}[t]/(t^3)$ (remainders modulo t^3). Compute X_A.

Exercise 14.20. Consider the quotient of the ring of polynomials in two variables, modulo the ideal generated by $f = X^2 + Y^2 - 1$

$$A = \mathbb{R}[X,Y]/(X^2 + Y^2 - 1).$$

Show that A an algebra over $\mathbb{R}$ and show that X_A is homeomorphic to S^1. (Hint: Let $\alpha, \beta \in A$ corresponding to X and Y. Then a character χ is determined by $u = \chi(\alpha)$ and $v = \chi(\beta)$. What do they must satisfy? Also, have a look back at Exercise 11.42).

Note: If you are not comfortable yet with such quotients, take as definition A :=the algebra of polynomial functions on $S^1 \subset \mathbb{R}^2$ and then α and β in the hint above are the first and second projection.

Exercise 14.21. (from the 2011exams) Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathscr{R}_p$ the set of remainders modulo p. In other words,

$$\mathscr{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) :\ \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

For $q \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denoted by $[q] \in \mathscr{R}_p$ the induced equivalence class. Show that:

(a) The following three operations are well defined and make $\mathscr{R}_p$ into an algebra.

$$[q_1] + [q_2] := [q_1 + q_2], \quad [q_1] \cdot [q_2] := [q_1 \cdot q_2], \quad \lambda [q] := [\lambda q]$$

(b) For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathscr{R}_p$ has only one point.
(c) For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathscr{R}_p$ is homeomorphic to S^n.
(d) What is the spectrum when $p = x_0 x_1 \ldots x_n$?

Exercise 14.22. (from the 2024 exam) On S^2 we consider the functions $f, g, h : S^2 \to \mathbb{R}$ given by:

$$f(x,y,z) = x, \quad g(x,y,z) = y, \quad h(x,y,z) = z.$$

We also consider the collection of all polynomial expressions generated on f, g and h:

$$\mathscr{A}(f,g,h) := \left\{ P(f,g,h) :\ P \in \mathbb{R}[X,Y,Z] \right\} = \left\{ \sum_{0 \le i,j,k \le d} c_{i,j,k} f^i g^j h^k :\ d \in \mathbb{N},\ c_{i,j,k} \in \mathbb{R} \right\} \subset C(S^2).$$

(a) Show that $\mathscr{A}(f,g,h)$ is dense in $C(S^2)$.
(b) Show that for any character $\chi : \mathscr{A}(f,g,h) \to \mathbb{R}$, the numbers $u := \chi(f), v := \chi(g), w := \chi(h)$ must satisfy

$$u^2 + v^2 + w^2 = 1.$$

(c) Show that, conversely, for any $u,v,w \in \mathbb{R}$ satisfying this equation, there exists a character $\chi : \mathscr{A}(f,g,h) \to \mathbb{R}$ with $\chi(f) = u$, $\chi(g) = v$, $\chi(h) = w$, and this character is unique.
(d) The previous two items tell us that we have a bijection

$$\Psi : X_{\mathscr{A}(f,g,h)} \to S^2, \quad \chi \mapsto (\chi(f), \chi(g), \chi(h)).$$

Show that this is actually a homeomorphism.

Exercise 14.23. (from the 2012 retake) Let A be a commutative algebra over $\mathbb{R}$ and assume that $a_1, \ldots, a_n \in A$ generate A, i.e. that any $a \in A$ can be written as $a = P(a_1, \ldots, a_n)$ for some $P \in \mathbb{R}[X_1, \ldots, X_n]$. Let X_A be the topological spectrum of A. The first aim is to show that

$$f : X_A \to \mathbb{R}^n, \;\; f(\chi) = (\chi(a_1), \ldots, \chi(a_n))$$

is an embedding. To that end we denote the components of f by $f_i : X_A \to \mathbb{R}$, and we ask you to show that:

(a) f is continuous.
(b) $\chi(P(a_1, \ldots, a_n)) = P(\chi(a_1), \ldots, \chi(a_n))$ for any character $\chi \in X_A$ and any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$.
(c) f is injective.
(d) the topology of X_A is the smallest topology on X_A with the property that all the functions f_i are continuous.
(e) f is an embedding.

We now take for A the algebra $A_K = \mathrm{Pol}(K)$ of real-valued polynomial functions on a subset $K \subset \mathbb{R}^n$ and for a_i the polynomial functions

$$a_i : K \to \mathbb{R}, \;\; a_i(x_1, \ldots, x_n) = x_i, \quad (1 \leq i \leq n),$$

and we consider the resulting function $f : X_{\mathrm{Pol}(K)} \to \mathbb{R}^n$. Please do the following:

(f) Show that $\mathrm{Pol}(K)$ is finitely generated with generators $a_1, \ldots, a_n$.
(g) Show that the image of f contains K.
(h) For the sphere $S^n \subset \mathbb{R}^{n+1}$, deduce that the spectrum of the algebra $\mathrm{Pol}(S^n)$ is homeomorphic to S^n.
(i) For which subspaces $K \subset \mathbb{R}^n$ can one use a similar argument to deduce that $X_{\mathrm{Pol}(K)}$ is homeomorphic to K?

Exercise 14.24. If A and B are two commutative algebras over $\mathbb{R}$, a map

$$\Phi : A \to B$$

is called algebra homomorphism if it is $\mathbb{R}$-linear, $\Phi(a_1 a_2) = \Phi(a_1)\Phi(a_2)$ for all $a_1, a_2 \in A$, and $\Phi(1_A) = 1_B$, where $1_A \in A$ and $1_B \in B$ are the units. It is called an isomorphism of algebras if it is also bijective.

(a) Show that for any algebra homomorphism $\Phi : A \to B$ between two algebras,

$$\Phi^* : X_B \to X_A, \;\; \Phi^*(\chi) = \chi \circ \Phi$$

is a continuous map,
(b) If Φ is an isomorphism of algebras then Φ^* is a homeomorphism.
(c) Compute the spectrum of

$$A := \left\{ f : \mathbb{R}^2 \to \mathbb{R} \text{ continuous} : f(x+1,y) = f(x,y+1) = f(x,y), \; \forall\, x,y \in \mathbb{R} \right\}.$$

Exercise 14.25. (from the 2024 retake) Consider

$$\mathscr{A} = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous such that } f(x) = f(x+1) \text{ for all } x \in \mathbb{R}\}.$$

(a) Show that $\mathscr{A}$ is a sub-algebra of $C(\mathbb{R})$.
(b) Is $\mathscr{A}$ dense in $C(\mathbb{R})$?
(c) Show that the spectrum of $\mathscr{A}$ is homeomorphic to S^1.

Exercise 14.26. (This is from the retake exam of 2014) Let $\mathscr{C}(D^2)$ be the algebra of real-valued continuous functions on the unit disk D^2 and let $\mathscr{A}$ be the subset consisting of those $f \in \mathscr{C}(D^2)$ with the property that they are constant on the boundary circle S^1.

a. is $\mathscr{A}$ a sub-algebra of $\mathscr{C}(D^2)$?
b. is $\mathscr{A}$ dense in $\mathscr{C}(D^2)$?
c. show that the spectrum of $\mathscr{A}$ is homeomorphic to S^2.

Exercise 14.27. Consider the algebra

$$A = \{f \in \mathscr{C}(S^n) : f(z) = f(-z) \ \ \forall\, z \in S^n\}.$$

Show that X_A is homeomorphic to $\mathbb{P}^n$.

14.4 On Gelfand-Naimark- a few more exercises

Exercise 14.28. While our discussion of Gelfand-Naimark focused on relating topological spaces with algebras, the discussion can be improved by showing that it interacts nicely with the concept of "morphism" on each side: continuous maps on the side of topological spaces, and algebra homomorphism on the side of algebras. In some sense, this is what Exercises 14.17 and 14.24 were about. Let us put everything together into one single story (and one single, hopefully simpler, exercise): show that:

(a) for any continuous map $\varphi : X \to Y$ between two topological spaces,

$$\varphi^* : \mathscr{C}(Y) \to \mathscr{C}(X), \quad \varphi^*(f) := f \circ \varphi$$

is an algebra homomorphism. Moreover, φ^* is an isomorphism if φ is a homeomorphism.
(b) for any algebra homomorphism $F : A \to B$ between two algebras,

$$F^* : X_B \to X_A,\ F^*(\chi) = \chi \circ F$$

is a continuous map. Moreover, F^* is a homeomorphism if F is an isomorphism.
(c) If X and Y are two compact Hausdorff spaces, than any algebra homomorphism

$$F : \mathscr{C}(Y) \to \mathscr{C}(X)$$

must be of type $F = \varphi^*$, for some continuous map $\varphi : X \to Y$, and φ is unique.
(d) Two compact Hausdorff spaces X and Y are homeomorphic iff the algebras $\mathscr{C}(X)$ and $\mathscr{C}(Y)$ are isomorphic.

Exercise 14.29. This is a reverse of Exercise 14.13, starting on the algebra side. Show that:

(a) For any algebra A and any ideal $I \subset A$, the following is a closed subspace of X_A:

$$X_{A,I} := \{\chi \in X_A : \chi(a) = 0\ \forall\, a \in I\}$$

(b) When $A = \mathscr{C}(X)$ with X compact and Hausdorff, and $I = I_Y$ corresponding to some closed subset $Y \subset X$, then the (Gelfand-Naimark) homeomorphism between X_A and X, restricts to one between $X_{A,I}$ and Y.
(c) On the other hand, in (a), notice that $X_{A,I}$ is just a description of the spectrum of the quotient A/I without any reference to the quotient itself. Make this precise and explicit! After that, re-think item (b).

Exercise 14.30. In this exercise we provide a step-by-step way to tackle item (e) from Exercise 14.16. With those notations, and under those assumptions, do the following:

(e1) Show that if $f, g \in \mathscr{A}$ and $f \geq g$, then $\chi(f) \geq \chi(g)$.
(e2) Show that for all $f \in \mathscr{A}$ one has $|\chi(f)| \leq ||f||_{\sup}$.
(e3) Deduce that for any sequence $(f_n)_{n\geq 1}$ of elements in $\mathscr{A}$, convergent to some $f \in C(X)$, the sequence $(\chi(f_n))_{n\geq 1}$ is convergent.
(e4) Deduce that there exists an extension of $\chi : \mathscr{A} \to \mathbb{R}$ to a continuous map

$$\tilde{\chi} : C(X) \to \mathbb{R}.$$

(e5) Then show that $\tilde{\chi}$ is a character.
(e6) Conclude that F is a homeomorphism between X and X_A.

Exercise 14.31. (from the 2014 exam) In this exercise we work over $\mathbb{R}$ (hence we consider real-valued functions and real algebras). Let X and Y be compact Hausdorff spaces. For $u \in \mathscr{C}(X)$, $v \in \mathscr{C}(Y)$, define $u \otimes v \in \mathscr{C}(X \times Y)$ given by

$$u \otimes v : X \times Y \to \mathbb{R}, \ \ (u \otimes v)(x,y) := u(x)v(y),$$

and we denote by $\mathscr{A} \subset \mathscr{C}(X \times Y)$ the set of functions of type

$$\sum_{i=1}^{k} u_i \otimes v_i \quad \text{with } k \in \mathbb{N}, u_1, \ldots, u_k \in \mathscr{C}(X), v_1, \ldots, v_k \in \mathscr{C}(Y)$$

Show that:

a. (+) $\mathscr{A}$ is a dense subalgebra of $\mathscr{C}(X \times Y)$.
b. For any $\chi \in X_{\mathscr{A}}$, $\chi_1 : \mathscr{C}(X) \to \mathbb{R}$ and $\chi_2 : \mathscr{C}(Y) \to \mathbb{R}$ given by

$$\chi_1(u) := \chi(u \otimes 1), \ \ \chi_2(v) := \chi(1 \otimes v)$$

are characters.
c. (+) $X_{\mathscr{A}}$ is homeomorphic to $X \times Y$.

Exercise 14.32. (from the 2016 exam) Here we work over the field $\mathbb{R}$ of real numbers. Let A be an arbitrary algebra, assume that A is commutative (i.e. $a_0 a_1 = a_1 a_0$ for any $a_0, a_1 \in A$) and denote by $\mathbf{0_A}$ and $\mathbf{1_A}$ the zero and the unit element of A, respectively. Fix an element $\alpha \in A$. Consider

$$B = A \times A = \{(a_0, a_1) : a_o, a_1 \in A\}$$

with the operations

$$(a_0, a_1) + (a_0', a_1') = (a_0 + a_0', a_1 + a_1'),$$
$$\lambda \cdot (a_0, a_1) = (\lambda a_0, \lambda a_1),$$
$$(a_0, a_1) \bullet (a_0', a_1') = (a_0 a_0' + \alpha a_1 a_1', a_0 a_1' + a_0' a_1).$$

a) Show that B, together with the previous operations (written in red), is an algebra (don't forget the unit!).

b) Assume that $\chi : B \to \mathbb{R}$ is a character and consider

$$\overline{\chi} : A \to \mathbb{R}, \quad \overline{\chi}(a) = \chi(a, \mathbf{0_A}),$$

$$r := \chi(\mathbf{0_A}, \mathbf{1_A}) \in \mathbb{R}.$$

Show that:

b1) $\overline{\chi}$ is a character on A.
b2) $\chi(a_0, a_1) = \overline{\chi}(a_0) + r\,\overline{\chi}(a_1)$ for any $(a_0, a_1) \in B$.
b3) $\overline{\chi}(\alpha) = r^2$.

c) Conversely, for any character $\overline{\chi}$ on A and any $r \in \mathbb{R}$ satisfying the equation from b3), show that χ defined by the formula from b2) is a character on B.
d) When $A := \mathbb{R}[X_0, X_1]$ is the algebra of polynomials in two variables and $\alpha := 1 - (X_0)^2 - (X_1)^2$, show that the topological spectrum of B is *homeomorphic* to S^2.

Chapter 15
Exercises on the chapter "Partitions of unity"

15.1 More exercises

Exercise 15.1. Let $\mathscr{A}$ be the following collection of subsets of $\mathbb{R}$:

$$\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}.$$

Which of the following collections refines $\mathscr{A}$?

$$\mathscr{B} = \{(x, x+1) : x \in \mathbb{R}\},$$

$$\mathscr{C} = \left\{(n, n+\frac{3}{2}) : n \in \mathbb{Z}\right\},$$

$$\mathscr{D} = \left\{(x, x+\frac{3}{2}) : x \in \mathbb{R}\right\}.$$

Exercise 15.2. Which of the collections from the previous exercise is locally finite?

Exercise 15.3. (from the 2012 exam) Consider the following open cover of $\mathbb{R}$:

$$\mathscr{U} := \left\{(r,s) : r,s \in \mathbb{R}, |r-s| < \frac{1}{3}\right\}.$$

Describe a locally finite subcover of $\mathscr{U}$.

Exercise 15.4. (from the 2011 exam) Show that the family of open intervals

$$\mathscr{U} := \{(q, q+1) : q \in \mathbb{R}\}$$

forms an open cover of $\mathbb{R}$ which admits no finite sub-cover, but which admits a locally finite sub-cover.

Exercise 15.5. (from the 2013 exam) Let $X = (0,\infty)$ and consider the open cover of X

$$\mathscr{U} = \{U_1, U_2, U_3, \ldots\} \quad \text{with } U_n = (0, n).$$

(a) Show that $\mathscr{U}$ does not admit a subcover which is locally finite.
(b) Describe a locally finite refinement of $\mathscr{U}$.

Exercise 15.6. Show that if a family $\{p_i : i \in I\}$ of non-zero polynomial functions $p_i : \mathbb{R} \to \mathbb{R}$ is locally finite in (cf. Definition 8.9), then it must be finite.

Exercise 15.7. Let $\mathscr{P} \subset \mathscr{C}(\mathbb{R})$ be the space of all polynomial functions on $\mathbb{R}$. Is $\mathscr{P}$ normal?

Exercise 15.8. Show that the space $\mathscr{C}^1(\mathbb{R}) \subset \mathscr{C}(\mathbb{R})$ of functions of class C^1 is normal. What do you conclude from this?

Exercise 15.9. Do the same for the space $\mathscr{C}^\infty(\mathbb{R})$ of smooth (i.e. infinitely differentiable) functions on $\mathbb{R}$.

Exercise 15.10. Now do the same for $\mathscr{C}^\infty(\mathbb{R}^n)$.

It is very tempting now to talk about smooth manifolds . These are manifolds on which we can talk about smoothness. More precisely, a smooth manifold is a topological manifold X together with a specified family of coordinate charts $\{\chi_i : U_i \to \mathbb{R}^n\}$, such that $\{U_i\}$ is an open cover of X, $c_{i,j} := \chi_i \circ \chi_j^{-1}$ is a smooth function. Here, $c_{i,j}$ plays the role of the "change of coordinates" since

$$\chi_i(x) = c_{i,j}(\chi_j(x)).$$

Also, $c_{i,j}$ is a function defined on an open in $\mathbb{R}^n$ (namely $\chi_j(U_i \cap U_j)$) with values in $\mathbb{R}^n$; hence it makes sense to talk about its smoothness. Given such a smooth manifold, a function $f : X \to \mathbb{R}$ is called smooth if its representation in each chart, i.e. each $f \circ \chi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth. Denote by $\mathscr{C}^\infty(X)$ the space of smooth functions on X; of course, $\mathscr{C}^\infty(X) \subset \mathscr{C}(X)$. Once you get used to all these definitions, the following should not be too difficult now:

Exercise 15.11. Show that, for any smooth manifold X, $\mathscr{C}^\infty(X)$ is normal. Deduce that any open cover admits a smooth partition of unity subordinated to it.

In this context, a map $f : X \to \mathbb{R}^N$ is called smooth if all its components are smooth. Adapting the proof of Theorem 4.41 and using Exercise 15.10 above, one can now try a more difficult exercise:

Exercise 15.12. Show that, for any smooth compact manifold X, there exists a smooth embedding $f : X \to \mathbb{R}^N$, for N large enough.

Exercise 15.13. Let X and Y be two topological spaces and assume that Y is Hausdorff and normal. Show that a function

$$f : X \to Y$$

is continuous if and only if, for each continuous function $\phi : Y \to \mathbb{R}$, the function $\phi \circ f : X \to \mathbb{R}$ is continuous.

Exercise 15.14. (from the 2021 exam)

(a) Consider the open cover $\mathscr{U} = \{(-2,1),(-1,2)\}$ of the interval $(-2,2)$ with the Euclidean topology. Give an example of a partition of unity subordinate to $\mathscr{U}$.
(b) Let $X = \{x,y\}$ be a set with two elements. Give an example of a topology on X such that every open cover[1] has a subordinate partition of unity, but the topology is not Hausdorff.

Exercise 15.15. Let X be a Hausdorff paracompact space and let $f : X \to [0,\infty)$ be a function which is assumed to be locally bounded i.e., which satisfies:

$$\forall\ a \in X,\ \exists \text{ neighborhood } V \text{ of } a, \exists\, M > 0 \text{ s.t. } \ f|_V \leq M.$$

Show that there exists $g : X \to (0,\infty)$ continuous such that $f \leq g$ on X. What can you add if X is compact? (Hint: partition of unity).

Exercise 15.16. Let X be a Hausdorff, paracompact space. Show that for any open U in $X \times \mathbb{R}$ containing $X \times \{0\}$,

$$X \times \{0\} \subset U \subset X \times \mathbb{R},$$

there exists a continuous function $r : X \to (0, \infty)$ such that U contains the r-tube, i.e.,

$$\Big\{(x,t) \in X \times \mathbb{R} : t \in (-r(x), r(x))\Big\} \subset U. \tag{15.1.1}$$

When X is compact deduce the tube lemma (which says that r can be chosen to be constant!).
(Hint start with small enough opens V_i and numbers r_i such that $V_i \times (-r_i, r_i) \subset U$. And a partition of unity.)

Exercise 15.17. Let X be a Hausdorff paracompact space. We denote by $\mathscr{C}_c(X)$ the subset of $\mathscr{C}(X)$ consisting of continuous functions $f : X \to \mathbb{R}$ with compact support. A **positive integral** on X is any linear map

$$I : \mathscr{C}_c(X) \to \mathbb{R}$$

with the property that, for any $f \in \mathscr{C}_c(X)$, one has $f \geq 0 \Longrightarrow I(f) \geq 0$. If, furthermore, the right hand side is zero only when $f = 0$, we say that I is strictly positive. The set of positive integrals on X and the set of strictly positive ones are denoted, respectively,

$$\mathscr{I}(X),\ \mathscr{I}^*(X).$$

Show that if X admits an open cover $\{U_i : i \in I\}$ such that $\mathscr{I}^*(U_i) \neq \emptyset$ for all $i \in I$, then $\mathscr{I}^*(X) \neq \emptyset$.

(Hint: notice first that $I_\eta(f) := I(\eta \cdot f)$ defines an element $I_\eta \in \mathscr{I}(X)$ whenever $I \in \mathscr{I}(U)$ and $\eta \in \mathscr{C}(X)$ positive and supported in U. And use a partition of unity.)

Exercise 15.18. As in the previous exercise, we assume that X is a Hausdorff paracompact space and we denote by $\mathscr{I}(X)$ and $\mathscr{I}^*(X)$ the set of positive integrals on X, respectively the set of strictly positive ones. Show that:

(a) For $U \subset X$ open and $f \in \mathscr{C}_c(U)$, after extending f by zero outside U one obtains a continuous function $\tilde{f} \in \mathscr{C}_c(X)$.
(b) For $U \subset X$ open and $I \in \mathscr{I}(X)$ one obtains a positive integral on U, denoted $I|_U \in \mathscr{I}(U)$ and defined as follows:

$$I|_U : \mathscr{C}_c(U) \to \mathbb{R}, \quad I|_U(f) := I(\tilde{f}).$$

(c) If $I \in \mathscr{I}(X)$ and there exists an open cover $\{U_i : i \in I\}$ of X such that $I|_{U_i} = 0$ for all i, then $I = 0$.
(d) Given an open cover $\{U_i : i \in I\}$ of X together with positive integrals $I_i \in \mathscr{I}(U_i)$, the following are equivalent:

   a. there exists $I \in \mathscr{I}(X)$ such that $I|_{U_i} = I_i$ for all $i \in I$.
   b. the I_is coincide on the overlaps, i.e., $I_i|_{U_i \cap U_j} = I_j|_{U_i \cap U_j}$ for all $i, j \in I$.

Chapter 16
Previous exams

16.1 The 2025 retake exam (April 17, 2025)

Notes: Please write down your name and the student no. Your solutions can be in Dutch or English. PLEASE MOTIVATE ALL YOUR ANSWERS!!!! EXPLAIN WHAT YOU DO, WHAT YOU USE, WHY YOUR CLAIMS ARE TRUE. There are 14 collectable points, the final mark is at most 10.

Exercise 16.1 (10x0.5= 5pts). On $X = \mathbb{R}$ we consider the family of subsets consisting of all intervals of type $(-\infty, -r]$ with $r \geq 0$, all intervals of type $[r, 2r]$ with $r \geq 0$ and the set $\{0\}$:

$$\mathscr{S} := \Big\{(-\infty, -r] : r \geq 0\Big\} \cup \Big\{[r, 2r] : r \geq 0\Big\} \cup \Big\{\{0\}\Big\}.$$

We denote by $\mathscr{T}$ the smallest topology on X containing $\mathscr{S}$.

(a) Show that $\mathscr{S}$ is not a topology basis and describe a basis for $\mathscr{T}$.

(b) Show that $\mathscr{T}$ is neither larger nor smaller than the Euclidean topology.

(c) Is $(X, \mathscr{T})$ connected?

(d) Is $(X, \mathscr{T})$ compact?

(e) Is $(X, \mathscr{T})$ Hausdorff?

(f) Is $(X, \mathscr{T})$ second countable?

(g) Compute the interior and the closure of $(-1, 2)$ in $(X, \mathscr{T})$.

(h) Is the sequence $(x_n)_{n\geq 1}$ given by $x_n = -\frac{1}{n}$ convergent in $(X, \mathscr{T})$?

(i) What about $y_n = 1 + \frac{1}{n}$?

(j) What about $z_n = -1 - \frac{1}{n}$?

Exercise 16.2 (5x0.5= 2.5pts). In this exercise X, Y and Z are topological spaces, and the set $\{0, 1\}$ will be endowed with the discrete topology, i.e. consisting of $\emptyset$, $\{0\}$, $\{1\}$ and $\{0, 1\}$.

(a) Show that if $f : X \to Y$ and $g : Y \to Z$ are continuous maps then so is $g \circ f$.

(b) Show that if $f : X \times Y \to Z$ is continuous then for any $y_0 \in Y$ the function

$$f^{y_0} : X \to Z, \quad f^{y_0}(x) := f(x, y_0)$$

is continuous.

(c) Show that if X is connected then any continuous function $f : X \to \{0, 1\}$ must be constant.

(d) Formulate and prove the converse of the previous item.

(e) Using the previous items prove that a product of two connected spaces is again connected.

Exercise 16.3 (1+1.5= 2.5pts). Consider $X := \mathbb{R} \times [-1, 1]$ endowed with the Euclidean topology.

(a) Show that the following defines an action of $(\mathbb{Z},+)$ on X:

$$n\cdot(x,t) := (x+2n\pi,(-1)^n t) \qquad (n\in\mathbb{Z}).$$

(b) Show that $X/\mathbb{Z}$ can be embedded in $\mathbb{R}^3$.

Exercise 16.4 (5x0.5= 2.5pts). Given a compact Hausdorff space X and A a closed nonempty subset,

(a) Show that $X\setminus A$ is locally compact;
(here $X\setminus A$ is the complement of A in X)

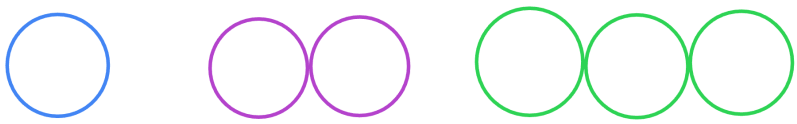
(b) Show that the one-point compactification of $X\setminus A$ is homeomorphic to X/A;
(here X/A is the quotient of X obtained by collapsing A to a point)

(c) Show that $\mathscr{A} := \{f\in C(X) : f|_A \text{ is constant}\}$ is a sub-algebra of $C(X)$;
(here $C(X)$ denotes the algebra of real-valued continuous functions on X)

(d) Show that the spectrum of $\mathscr{A}$ is homeomorphic to X/A;

(e) Show that if A has at least two points, then $\mathscr{A}$ is not dense in $C(X)$.

Exercise 16.5 (1.5pts). Which of the three spaces in the figure is/are homeomorphic to the one-point compactification of a subset of $\mathbb{R}$ (with the Euclidean topology)?



NOTES:

- Your solutions can be in Dutch or English (or a mixture of those).
- Please write down your name and the student no, and make sure your name is on each sheet.
- In total you can collect up to 14 points. The final mark is $\min\{10, \text{number of collected points}\}$.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! Please do not just write down the final result (that will not count!) but explain it/provide the details. In particular:
  - in Exercise 1, questions (c), (d), (e), (f), (h), (i), (j) please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)).
  - the same in Exercise 5: if the answer is yes describe the subspace of $\mathbb{R}$ and prove that its one-point compactification is what you claimed; if the answer is no, prove it.
  - if you do understand some parts intuitively but cannot write it down properly, try to use pictures (that is not enough for a proof, but it may be worth points, especially if the pictures are backed up by some theoretical explanations).
- To motivate your answers you can use any of the theorems, propositions, etc that we discussed in the class or that are in the dictaat.

16.2 The 2025 exam (January 30, 2025)

Notes: In total you can collect up to 13 points. The final mark is $\min\{10, \text{number of collected points}\}$. Your solutions can be in Dutch or English. PLEASE MOTIVATE ALL YOUR ANSWERS!!!! More notes on the other side.

Exercise 16.6 (7x0.5= 3.5pts). Let $\mathscr{B}$ be the collection of subsets of S^1 of type

$$S^1_{[a,b)} := \left\{(\cos 2\pi t, \sin 2\pi t) \in S^1 : t \in [a,b)\right\},$$

one for each $a, b \in \mathbb{R}$. Let $\mathscr{T} = \mathscr{T}(\mathscr{B})$ be the induced topology.

(a) Show that $\mathscr{B}$ is not a topology on S^1, but it is a topology basis.

(b) Show that $\mathscr{T}$ is strictly larger than the Euclidean topology on S^1.

(c) Is $(S^1, \mathscr{T})$ Hausdorff?

(d) Is $(S^1, \mathscr{T})$ connected?

(e) Is $(S^1, \mathscr{T})$ compact?

(f) Is the sequence $x_n = \left(\cos\frac{(n-1)\pi}{n}, \sin\frac{(n-1)\pi}{n}\right)$ convergent in $(S^1, \mathscr{T})$?

(g) Compute the interior and the closure, in the topological space $(S^1, \mathscr{T})$, of

$$A := \left\{(x,y) \in S^1 : y > 0\right\}.$$

Exercise 16.7 (1pt). Prove that the space $(S^1, \mathscr{T})$ from the previous exercise is homeomorphic to a quotient of $(\mathbb{R}, \mathscr{T}_l)$ modulo the action of a group. [Recall: $\mathscr{T}_l$ stands for the lower limit topology on $\mathbb{R}$.]

Exercise 16.8 (2pts). Show that any open continuous map $f : X \to Y$ from a compact space $X \neq \emptyset$ to a Hausdorff connected space Y is surjective. [Recall: f open means: $\forall\, U \subset X$ open, $f(U)$ is open in Y.]

Exercise 16.9 (1pt). Prove that if X and Y are path-connected spaces then $X \times Y$ is connected.

Exercise 16.10 (1+ 1+ 1+ 0.5= 3.5 pts). We consider the closed unit disk

$$D^2 := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$$

and, inside it, the subset:

$$A := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/2\right\}.$$

We endow these spaces with the Euclidean topology.

(a) Prove D^2/A, i.e. the space obtained from D^2 by collapsing A to a point, is homeomorphic to D^2. Provide explicit formulas.

(b) Prove that the 1-point compactification of the complement $D^2 - A$ of A in D^2 is homeomorphic to D^2. Provide explicit formulas.

(c) Show that $\mathscr{A} := \left\{f \in C(D^2) : f|_A \text{ is constant}\right\}$ is a sub-algebra of $C(D^2)$ and the spectrum of $\mathscr{A}$ is homeomorphic to D^2.

(d) Show that $\mathscr{A}$ is not dense in $C(D^2)$.

Exercise 16.11 (2pt). Prove that the product $S^1 \times D^2$ of the unit circle with the unit disk (with their Euclidean topologies) can be embedded in $\mathbb{R}^3$. [Hint: $S^1 \times D^2$ is also called "a full torus".]

NOTES:

- Your solutions can be in Dutch or English (or a mixture of those).
- Please write down your name and the student no, and make sure your name is on each sheet.
- In total you can collect up to 13 points. The final mark is $\min\{10, \text{number of collected points}\}$.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! Please do not just write down the final result (that will not count!) but explain it/provide the details. In particular:
  - in Exercise 1, questions (c), (d), (e), (f), please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)).
  - in Exercise 2, please specify the group, the action, and don't forget to prove your claims.
  - etc
  - in Exercise 6, please prove that the map you describe (explicitly) is actually an embedding.

16.3 The 2024 retake exam (April 15, 2024)

Exercise 16.12 (2.5=0.5+ 1+ 1 pts). Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Show that:

(a) if $(X, \mathscr{T}_1)$ is Hausdorff then so is $(X, \mathscr{T}_2)$;
(b) if $(X, \mathscr{T}_2)$ is compact then so is $(X, \mathscr{T}_1)$;
(c) if $(X, \mathscr{T}_2)$ is compact and $(X, \mathscr{T}_1)$ is Hausdorff then $\mathscr{T}_1$ and $\mathscr{T}_2$ must coincide.

Exercise 16.13 (4.5= 1+ 1+ 0.5+ 1+ 1 pts). On $X = \mathbb{R}$ we consider the family of subsets consisting of all intervals of type $[r, \infty)$ with $r \geq 0$, all intervals of type $(-\infty, -r]$ with $r \geq 0$ and the set $\{0\}$:

$$\mathscr{B} := \{(-\infty, -r] : r \geq 0\} \cup \{[r, \infty) : r \geq 0\} \cup \{\{0\}\}.$$

We denote by $\mathscr{T}$ the smallest topology on X containing $\mathscr{B}$.

(a) Show that $\mathscr{B}$ is a topology basis but not a topology.
(b) Compute the interior and the closure of $(-1, 2)$ in $(X, \mathscr{T})$.
(c) Is $(X, \mathscr{T})$ connected?
(d) Is the sequence $(x_n)_{n \geq 1}$ given by $x_n = 1 + \frac{(-1)^n}{n}$ convergent in $(X, \mathscr{T})$? To what?
(e) Describe a non-constant continuous function from $(X, \mathscr{T})$ to $\mathbb{R}$ endowed with the Euclidean topology. Then try to describe all of them (don't forget to prove all your claims).

Exercise 16.14 (2= 0.5+ 1.5 pts). Let $X = [-1, 1] \times [-1, 1]$ endowed with the Euclidean topology. Consider the group Γ that is the product of the group of integers modulo two, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \left\{\widehat{0},\ \widehat{1}\right\}$, with itself:

$$\Gamma := \mathbb{Z}_2 \times \mathbb{Z}_2 = \left\{(\widehat{0}, \widehat{0}),\ (\widehat{0}, \widehat{1}), (\widehat{1}, \widehat{0}), (\widehat{1}, \widehat{1})\right\}.$$

(a) show that the following defines an action of Γ on X:

$$(\widehat{k}, \widehat{l}) \cdot (x, y) := \left((-1)^k x, (-1)^l y\right)$$

(b) show that the resulting quotient space X/Γ is homeomorphic to X.

Exercise 16.15 (2.5= 0.5+1+1). Given a topological space X, by *a two-point compactification* of X we mean a pair (Y, i) consisting of a compact Hausdorff space together with a map $i : X \to Y$ which is an embedding and such that Y contains exactly two elements that not belong to the image of i. For $X = (0, 1) \cup (2, 3)$ endowed with the Euclidean topology:

(a) Give an example of such a two-point compactification, call it Y_1 (I am going to ignore the map i_1 from the notation, although you should indicate it on the picture).
(b) Find now another two-point compactification Y_2 of X such that Y_1 and Y_2 are not homeomorphic.
(c) Find two more, Y_3 and Y_4, not homeomorphic to Y_1 or Y_2 and not homeomorphic to each other.

Clarification: you can describe the spaces Y_1, Y_2, etc and the maps i_1, i_2, etc using pictures. But please provide the arguments/propositions that justify/support the claims that you make- e.g: "Y_1 is compact because ...", or "to see that Y_1 and Y_2 are not homeomorphic we use the X-method as follows ...".

Exercise 16.16 (2= 0.5+ 0.5+ 1 pts). Consider

$$\mathscr{A} = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous such that } f(x) = f(x+1) \text{ for all } x \in \mathbb{R}\}.$$

(a) Show that $\mathscr{A}$ is a sub-algebra of $C(\mathbb{R})$.
(b) Is $\mathscr{A}$ dense in $C(\mathbb{R})$?
(c) Show that the spectrum of $\mathscr{A}$ is homeomorphic to S^1.

NOTES:

- Please do not panic because of the length of the exam: in total you can collect up to 13.5 points, but you only need 10 to obtain the maximum mark.
- However, if you do feel overwhelmed by the length, please do not jump too much from one exercise to another trying to answer all of them, but rather focus on a couple that you choose carefully.
- I am kindly asking you to pay some attention to the handwriting. E.g., please make sure you use a proper pen/pencil/etc, and please try to avoid writing in very very small letters.
- Please write down your name and the student no, and make sure your name is on each sheet.
- your solutions can be in Dutch or English (or a mixture of those).
- Reminder: you are allowed to use four A3 pages (e.g. two sheets front and back) that you prepared in advance (handwritten or typed). No other material is allowed.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please do not just write down the final result (that will not count!) but explain it/provide the details, and please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well).

16.4 The 2024 exam (February 2nd, 2024), English

NOTES:

- Reminder: you are allowed to use four A3 pages (e.g. two sheets front and back) that you prepared in advance (handwritten or typed). No other material is allowed.
- your solutions can be in Dutch or English (or a mixture of those).
- Please write down your name and the student no, and make sure your name is on each sheet.
- In total you can collect up to 12 points. The final mark is $\min\{10, \text{number of collected points}\}$.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please do not just write down the final result (that will not count!) but explain it/provide the details, and please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)).

Exercise 16.17 (***This exercise is worth 6 points: 1pt for each of the items (1), (5) and (6), the others 0.5pts***)**.** Assume that $(X,\mathscr{T})$ is a topological space. We say that $X = A_1 \cup A_2$ is a *decomposition* of $(X,\mathscr{T})$ if A_1, A_2 are subsets of X whose union is the entire X. For any such decomposition we endow A_1 and A_2 with the induced topology and we associate to it the following collection of subsets of X:

$$\mathscr{T}(A_1,A_2) := \{U \subset X : U \cap A_1 \text{ is open in } A_1 \text{ and } U \cap A_2 \text{ is open in } A_2\}.$$

(1) Show that $\mathscr{T}(A_1,A_2)$ is a topology on X and $\mathscr{T} \subset \mathscr{T}(A_1,A_2)$.

Next, a decomposition $X = A_1 \cup A_2$ will be called *convenient* if $\mathscr{T} = \mathscr{T}(A_1,A_2)$. Please show that:

(2) If $X = A_1 \cup A_2$ is a convenient decomposition of X then, for any function $f : X \to Y$ with values in another topological space Y, one has:

$$f \text{ is continuous} \iff f|_{A_1} : A_1 \to Y \text{ and } f|_{A_2} : A_2 \to Y \text{ are continuous.}$$

(3) If A_1 and A_2 are both open, or both closed, in X, then $X = A_1 \cup A_2$ is a convenient decomposition.
(4) When $X = \mathbb{R}$ endowed with the Euclidean topology $\mathscr{T}_{\text{Eucl}}$,

$$\mathbb{R} = (-\infty, 0) \cup [0, \infty)$$

is not a convenient decomposition.

Next, we denote by $\mathscr{T}_{\text{new}}$ the topology on $\mathbb{R}$ associated to the decomposition from item (4); hence

$$\mathscr{T}_{\text{new}} := \{U \subset \mathbb{R} : U \cap (-\infty,0) \text{ is open in } (-\infty,0) \text{ and } U \cap [0,\infty) \text{ is open in } [0,\infty)\},$$

where we emphasise that "open" refers to the Euclidean topology on $(-\infty,0)$ and $[0,\infty)$.

(5) Compute the closure of $A = (-1,0)$ and the interior of $B = [0,1]$ in $(\mathbb{R}, \mathscr{T}_{\text{new}})$.
(6) Is $C = [-1,1]$, with the induced topology $\mathscr{T}_{\text{new}}|_C$, compact? But connected?
(7) Is the sequence given by $x_n := \frac{(-1)^n}{n}$ convergent in the topological space $(\mathbb{R}, \mathscr{T}_{\text{new}})$?
(8) For what values of $\lambda \in \mathbb{R}$ is the following function

$$f_\lambda : (\mathbb{R}, \mathscr{T}_{\text{new}}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \quad f_\lambda(x) = \begin{cases} x & \text{if } x < 0 \\ \lambda & \text{if } x = 0 \\ x + \lambda^2 & \text{if } x > 0 \end{cases}$$

continuous?
(9) Does there exists λ such that f_λ is an embedding?

Exercise 16.18. *(2 pts)* Consider the action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \left\{\widehat{0},\ \widehat{1}\right\}$ (the groups of integers modulo 2) on the unit two-dimensional sphere S^2 given by

$$\widehat{k}\cdot(x,y,z) = \left(x,y,(-1)^k z\right) \qquad \left(\text{for } \widehat{k}\in\mathbb{Z}_2,\ (x,y,z)\in S^2\right).$$

Show that the resulting quotient space $S^2/\mathbb{Z}_2$ is homeomorphic to the closed unit disk D^2.

Exercise 16.19. *(2 pts)* Which of the following spaces are homeomorphic to each other (and why):

(a) $C_0 = (S^1\times[0,1))^+$, the one-point compactification of $S^1\times[0,1)$.
(b) $C_1 = \mathrm{Cone}(S^1)$, the abstract cone associated to the space S^1.
(c) the concrete cone $C_2 = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 = (1-z)^2,\ 0\le z\le 1\right\}$.

Exercise 16.20. *(2 pts)* On S^2 we consider the functions $f,g,h : S^2\to\mathbb{R}$ given by:

$$f(x,y,z) = x, \quad g(x,y,z) = y, \quad h(x,y,z) = z.$$

We also consider the collection of all polynomial expressions generated on f, g and h:

$$\mathscr{A} := \left\{ \sum_{0\le i,j,k\le d} c_{i,j,k} f^i g^j h^k : d\in\mathbb{N}, c_{i,j,k}\in\mathbb{R} \right\} \subset C(S^2).$$

1. Show that $\mathscr{A}$ is dense in $C(S^2)$.
2. Show that for any character $\chi : \mathscr{A}\to\mathbb{R}$, the numbers

   $$u := \chi(f), \quad v := \chi(g), \quad w := \chi(h)$$

   must satisfy $u^2+v^2+w^2 = 1$.
3. Show that for any $u,v,w\in\mathbb{R}$ satisfying $u^2+v^2+w^2=1$, there exists a character $\chi : \mathscr{A}\to\mathbb{R}$ with $\chi(f) = u$, $\chi(g) = v$, $\chi(h) = w$, and this character is unique.
4. The previous two items tell us that we have a bijection

   $$\Psi : X_{\mathscr{A}} \to S^2, \quad \chi\mapsto(\chi(f),\chi(g),\chi(h)).$$

   Show that this is actually a homeomorphism.
   (Recall here that $X_{\mathscr{A}}$ (the topological spectrum of the algebra $\mathscr{A}$) is the set of all characters on $\mathscr{A}$ and its topology is the smallest one with the property that, for each $a\in\mathscr{A}$, the map $F_a : X_{\mathscr{A}}\to\mathbb{R}$, $F_a(\chi) = \chi(a)$ is continuous.)

AGAIN: PLEASE EXPLAIN /GIVE AS MANY DETAILS AS POSSIBLE ...

... REMEMBER THAT SOMEBODY ELSE (WHO CANNOT READ YOUR MIND) WILL EVALUATE YOUR SOLUTIONS

... and: GOOD LUCK!

16.5 The 2023 retake exam (March 16 , 2023)

Retake Exam Inleiding Topologie, March 16, 2023

- Every claim you make must be justified, either by argument or by citing a result from class. For example, if you are asked whether a certain space is a manifold, you have to prove that it is a manifold or prove that it is not a manifold – unless the example was explicitly shown in class to be or not to be a manifold, in which case you can cite it. Likewise: if you are asked for an example with a certain property, you have to prove that your example has this property, etc.
- Even if you cannot solve (a part of) a problem, you are allowed to use it in a later (part of a) problem.
- You can answer in Dutch or English. In any case, write readable!

Problem 1 (3+4 points)**.** *Consider the set $\mathbb{Z}$ of integers. Define $\mathcal{T}$ as the collection of all subsets of $\mathbb{Z}$ containing 0.*

(i) Show that $\mathcal{T}$ is not a topology, but one can make it into a topology by adding a single subset of $\mathbb{Z}$.

(ii) Decide whether the resulting topological space is Hausdorff and whether it is second-countable. (If it is second-countable, you should give explictly a countable basis of topology; if it is not, you should show that this is impossible.)

Problem 2 (3+3 points)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Recall that $X \sqcup Y$ denotes their disjoint union. This is a set such that X and Y are subsets of $X \sqcup Y$, and we have $X \cup Y = X \sqcup Y$ and $X \cap Y = \varnothing$. Recall further that we equip $X \sqcup Y$ with the disjoint union topology $\mathcal{T}_{X \sqcup Y}$, which is defined as $\{U \subseteq X \sqcup Y : U \cap X \in \mathcal{T}_X \text{ and } U \cap Y \in \mathcal{T}_Y\}$.*

(i) Let $(Z, \mathcal{T}_Z)$ be a further topological space and $f \colon X \sqcup Y \to Z$ a function. Show that f is continuous if and only if $f|_X \colon X \to Z$ and $f|_Y \colon Y \to Z$ are continuous. (Use nothing but the definition of continuity and the axioms of a topological space.)

(ii) Show that if X and Y are compact, so is $X \sqcup Y$.

2

Problem 3 (1+4+12). *Consider the subspaces*

- $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,
- $A_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$,
- $A_3 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$,
- $A_4 = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1$ *or* $(x+1)^2 + y^2 \leq 1\}$.

of $\mathbb{R}^2$ *with the Euclidean topology.*

(i) Sketch $A_1, \ldots, A_4$.

(ii) Describe the interiors of $A_1, \ldots, A_4$. *Which of these are connected?*

(iii) For every $1 \leq i < j \leq 4$ *decide whether* A_i *and* A_j *are homeomorphic.*

Problem 4 (6 points). *Give examples of two manifolds* X *and* Y *such that the one-compactification* X^+ *of* X *is a manifold, while the one-point compactification* Y^+ *of* Y *is not a manifold.*

Problem 5 (2+6). *Let* M *be the Möbius strip, i.e. the quotient of* $[0,1]^2$ *by the smallest equivalence relation* $\sim$ *on* $[0,1]^2$ *such that* $(0,t) \sim (1,1-t)$.

(i) Define an embedding $f\colon S^1 \to M$ *such that the image* $f(S^1)$ *equals the set of equivalence classes of the points* (x,y) *with* $x = 0$ *or* 1, *and show that it is an embedding.*

(ii) Consider $X = (S^1 \times [0,1]) \sqcup M$ *with the disjoint union topology and let* $\simeq$ *be the smallest equivalence relation on* X *such that* $(x,0) \simeq f(x)$ *for* $(x,0) \in S^1 \times [0,1]$. *Denote by* Y *the quotient* $X/\simeq$ *with the quotient topology. (Informally, this is glueing a cylinder and a Möbius strip along a boundary curve.) Show that* Y *is homeomorphic to* M.

Problem 6 (6 points). *Let* (X,d) *be a compact metric space. Let* $U_1, \ldots, U_n \subseteq X$ *be an open cover of* X. *Show that there exists a* $\delta > 0$ *such that for every subset* $A \subseteq X$ *with* $d(a,b) < \delta$ *for all* $a, b \in A$, *there exists an index* i *with* $A \subseteq U_i$.

16.6 The 2023 exam (February 2 , 2023)

Exam Inleiding Topologie, February 2, 2023

- Every claim you make must be justified, either by argument or by citing a result from class. For example, if you are asked whether a certain space is a manifold, you have to prove that it is a manifold or prove that it is not a manifold – unless the example was explicitly shown in class to be or not to be a manifold, in which case you can cite it. Likewise: if you are asked for an example with a certain property, you have to prove that your example has this property, etc.
- Even if you cannot solve part (i) of a problem, you are allowed to use it in part (ii).
- You can answer in Dutch or English. In any case, write readable!

Problem 1 (1+5 points). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.*

(i) For $A \subseteq X$, state the definition of the subspace topology on A.

(ii) Let $A, B \subseteq X$ be open subsets with $A \cup B = X$. Let $f\colon X \to Y$ be a map such that $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous, where we equip A and B with the subspace topology. Prove that f is continuous.

Problem 2 (3+4 points). *Consider the set $\mathbb{N}$ of positive integers. Define $\mathcal{T}$ as the collection of all subsets of $\mathbb{N}$ of the form $U_n = \{k \in \mathbb{N} : k \geq n\}$ for $n \in \mathbb{N}$.*

(i) Show that $\mathcal{T}$ is not a topology, but one can make it into a topology by adding a single subset of $\mathbb{N}$.

(ii) Decide whether the resulting topological space is Hausdorff and whether it is second-countable (i.e. has a countable basis of topology).

2

Problem 3 (1+9+5 points). *Consider the subspaces*

- $A_1 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$,
- $A_2 = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$,
- $A_3 = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$,
- $A_4 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

of $\mathbb{R}^2$ with the Euclidean topology.

(i) Sketch $A_1, \dots, A_4$.

(ii) Which A_i are manifolds?

(iii) For every $1 \leq i < j \leq 4$ decide whether A_i and A_j are homeomorphic.

Problem 4 (6 points). *Equip the set $\mathcal{C}([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the topology of uniform convergence, induced by the metric*

$$d(f,g) = \max_{x\in[0,1]} (|f(x) - g(x)|).$$

Show that there is no compact subset $K \subseteq \mathcal{C}([0,1],\mathbb{R})$ such that the functions

$$f_n \colon [0,1] \to \mathbb{R}, \qquad x \mapsto nx$$

are contained in K for all $n \in \mathbb{N}$.

Problem 5 (4+4 points). *Let $\mathbb{P}^2 = D^2/\sim$ be the projective plane, with $\sim$ being the smallest equivalence relation such that $x \sim (-x)$ for all $x \in S^1 = \partial D^2$.*

(i) Give an example of an embedding $f \colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus f(S^1)$ is path-connected.

(ii) Give examples of embeddings $f, g \colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus f(S^1)$ is not homeomorphic to $\mathbb{P}^2 \setminus g(S^1)$.

Problem 6 (8 points). *Let M be the Möbius strip, i.e. the quotient of $[0,1]^2$ by the smallest equivalence relation $\sim$ on $[0,1]^2$ such that $(0,t) \sim (1,1-t)$. Define an action of the group $(\mathbb{Z},+)$ on $\mathbb{R}\times[0,1]$ such that the quotient of $\mathbb{R}\times[0,1]$ by this group action is homeorphic to M. (Here, we equip $\mathbb{R}\times[0,1] \subset \mathbb{R}^2$ with the subspace topology of the Euclidean topology.)*

16.7 The 2022 retake exam (April 2022)

Below you find first an English, then a Dutch version of the exam. You can write all solutions in Dutch or English.
Notes:

- Write your solutions by hand. Write legible. Don't write close to the margins.
- After the exam you have 40 minutes to scan and upload your solutions **as a pdf in one file**. Check their readability and that you have scanned all pages.
- All claims have to be proven (if they are not known from the lecture or exercises). In particular if you are asked for an example you have to **prove** that the example satisfies the requirements.
- Read through all problems before you start to work.
- If you are unable to solve the first part of a question, you can still use the result in the second part.

Exercise 16.21. Consider the following three subsets of $\mathbb{R}^2$ equipped with the subspace topology:

- $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$
- $A_2 = \{(x,y) \in \mathbb{R}^2 : \frac{1}{4}x^2+y^2 = 1\}$
- $A_3 = \{(x,y) \in \mathbb{R}^2 : y^2 = 1\}$

Give a sketch of all three subspaces. Then decide for each pair of subspaces whether they are homeomorphic or not: if they are, give a homeomorphism; if they are not, give an argument why they are not homeomorphic.

Exercise 16.22. Let $\mathbb{N} = \{0,1,2,\dots\}$ be the natural numbers and $X = \{p \in \mathbb{N} : p = 0 \text{ or } p \text{ is prime}\}$. For every $k \in \mathbb{N}$ with $k > 0$ define $D_k = \{p \in X : p \text{ does not divide } k\}$, where we use the convention that 0 only divides itself. Let $\mathscr{T}$ be the smallest topology on X such that all D_k are in $\mathscr{T}$.

(a) Show a subspace $A \subset X$ is closed if and only if $A = X$ or A is finite and does not contain 0.
(b) Decide (with argument) whether $\mathscr{T}$ is Hausdorff and whether $\mathscr{T}$ is compact.

Exercise 16.23. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $A \subset Y$ be a subset equipped with the subspace topology; we denote by $i\colon A \to Y$ the inclusion. Show that a function $f\colon X \to A$ is continuous if and only if the composition $i \circ f\colon X \to Y$ is continuous.

Exercise 16.24. [6 points] Give examples $i, j\colon S^1 \to M$ of embeddings of the circle into the Möbius strip such that $M \setminus i(S^1)$ is not homeomorphic to $M \setminus j(S^1)$.

The rest of the exercises are missing.

16.8 The 2022 exam (January 2022)

Below you find first an English, then a Dutch version of the exam. You can write all solutions in Dutch or English.
Notes:

- Write your solutions by hand. Write legible. Don't write close to the margins.
- After the exam you have 40 minutes to scan and upload your solutions **as a pdf in one file**. Check their readability and that you have scanned all pages.
- All claims have to be proven (if they are not known from the lecture or exercises). In particular if you are asked for an example you have to **prove** that the example satisfies the requirements.
- Read through all problems before you start to work.
- If you are unable to solve the first part of a question, you can still use the result in the second part.

Exercise 16.25. [5 points] Consider the equivalence relation $\sim$ on $\mathbb{R}$, where $x \sim y$ iff x and y are both positive or both negative or both zero. Let $X = \mathbb{R}/\sim$ with the quotient topology. List (with proof) all closed subsets of X.

Exercise 16.26. [10 points] Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f\colon X \to Y$ a function. Let further $(C_i)_{i\in I}$ be a cover of X by closed subsets $C_i \subseteq X$, i.e. $\bigcup_{i\in I} C_i = X$. Assume that $f|_{C_i}\colon C_i \to Y$ is a continuous function for each $i \in I$, where we equip C_i with the subspace topology.

(a) Give an example of X, Y, f and $(C_i)_{i\in I}$ as above, where f is *not* continuous.
(b) Show that f is necessarily continuous if the indexing set I is finite.

Exercise 16.27. [6 points] Let $X = \mathscr{C}([0,1], \mathbb{R})$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. We consider on it the topology induced by the metric $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Show that X is path-connected.

Exercise 16.28. [6 points] Give examples $i, j\colon S^1 \to T$ of embeddings of the circle into the torus such that $T \setminus i(S^1)$ is not homeomorphic to $T \setminus j(S^1)$.

Exercise 16.29. [10 points]

(a) Consider the open cover $\mathscr{U} = \{(-2,1), (-1,2)\}$ of the interval $(-2,2)$ with the Euclidean topology. Give an example of a partition of unity subordinate to $\mathscr{U}$.
(b) Let $X = \{x, y\}$ be a set with two elements. Give an example of a topology on X such that every open cover[1] has a subordinate partition of unity, but the topology is not Hausdorff.

Exercise 16.30. [13 points] Let $p, q \in S^2$ be the points $(1,0,0)$ and $(-1,0,0)$. Let $X = S^2/\{p,q\}$ with the quotient topology.

(a) Show that X is not a 2-dimensional manifold.
(b) Show that $S^2 \setminus \{p,q\}$ is not homeomorphic to $\mathbb{R}^2$. (Hint: Use one-point compactifications.)

16.9 The 2021 retake exam (April 2021)

Exercise 16.31. [6 punten] Bekijk de verzamelingen

$$X = \left\{(x,y) \in \mathbb{R}^2 : xy = 1\right\}$$

en

$$Y = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1.\right\}$$

met de deelruimtetopologie van de Euclidische topologie op $\mathbb{R}^2$.

1. Schets X en Y. (1 pt)
2. Zijn X en Y homeomorf? Geef een bewijs. (5 pt)

Exercise 16.32. [7 punten] Zij $X = \{c,o\}$ een verzameling met twee elementen. Bekijk de topologie $\mathscr{T}_X = \{\varnothing, \{o\}, X\}$ op X. Bekijk $X \times X$ met de producttopologie.

1. Noem alle open deelverzamelingen van $X \times X$. (3pt)
2. Zij $A = \{(c,o)\} \subset X \times X$. Bepaal het inwendige en de afsluiting van A. (4pt)

Exercise 16.33. [12 punten] Geef een voorbeeld van een topologische ruimte die

1. tweedeaftelbaar, maar niet Hausdorff is (4pt);
2. Hausdorff, maar niet tweedeaftelbaar is (4pt);
3. compact, maar niet samenhangend is (4pt).

Geef in ieder voorbeeld een bewijs voor de eigenschappen (tenzij al bekend uit het hoorcollege).

Exercise 16.34. [7 punten] Zij $\mathbb{R}^+ = \mathbb{R} \cup \{\infty\}$ de één-punt-compactificatie van $\mathbb{R}$ met de topologie

$$\mathscr{T}^+ = \mathscr{T}_{\text{eucl}} \cup \left\{\mathbb{R}^+ \setminus K : K \subset \mathbb{R} \text{ compact}\right\},$$

waar $\mathscr{T}_{\text{eucl}}$ de Euclidische topologie op $\mathbb{R}$ is.
Bewijs dat er geen continue functie $f\colon \mathbb{R}^+ \to \mathbb{R}$ is met $f(x) = x$ voor alle $x \in \mathbb{R}$.

Exercise 16.35. [9 punten]

1. Geef een voorbeeld van een inbedding $i\colon S^1 \to T$ van een cirkel naar een torus zodat $T \setminus i(S^1)$ niet samenhangend is. (4pt)
2. Geef voorbeelden van inbeddingen $i, j\colon S^1 \to T$ zodat $T \setminus i(S^1)$ en $T \setminus j(S^1)$ niet homeomorf met elkaar zijn. (5pt)

Geef voor ieder voorbeeld een bewijs.

Exercise 16.36. [9 punten] Zij $N \in \mathbb{Z}_{>0}$ een positief getal en $(X, \mathscr{T})$ een compacte topologische ruimte met een inbedding $\Phi\colon X \to \mathbb{R}^N$, waar $\mathbb{R}^N$ de euclidische topologie heeft. Bewijs dat $\Phi(X) \subset \mathbb{R}^N$ niet open kan zijn als X niet-leeg is.

16.10 The 2021 exam (January 2021)

Schrijf leesbaar. Als je een deel van een opgave niet kunt oplossen, kun je het resultaat niettemin bij de volgende delen gebruiken. Veel succes!

Exercise 16.37. [4 punten] Zij $\mathscr{T}$ de verzameling

$$\{[a,b] : a < b \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$$

van deelverzamelingen van $\mathbb{R}$. Is $\mathscr{T}$ een topologie? Geef een bewijs.

Exercise 16.38. [11 punten] Bekijk de verzamelingen

$$X = (-1,0) \cup (0,1) \subset \mathbb{R}$$

en

$$Y = \left\{(x,y) \in \mathbb{R}^2 : d((0,0),(x,y)) \leq 1 \text{ of } d((2,0),(x,y)) \leq 1.\right\}$$

1. Schets X en Y. (1 pt)
2. Zijn X en Y homeomorf? Geef een bewijs. (4 pt)
3. Bereken voor X en Y het inwendige en de afsluiting. Geef voor al deze vier ruimtes aan of ze
   - samenhangend en/of
   - compact en/of
   - eerstaftelbaar

   zijn. Je hoeft bij deze deelopgave geen bewijzen te geven. (6 pt)

Exercise 16.39. [6 punten] Laat $(X, \mathscr{T}_X)$ en $(Y, \mathscr{T}_Y)$ topologische ruimtes zijn en $\mathscr{U} = \{U_i\}_{i \in I}$ een open dekking van X. Zij $f\colon X \to Y$ een afbeelding zodat de beperking $f|_{U_i}\colon U_i \to Y$ continu is voor iedere $i \in I$ (waar we U_i met de deelruimtetopologie bekijken). Bewijs dat f continu is.

Exercise 16.40. [7 punten] Geef een voorbeeld van een topologische ruimte $(X, \mathscr{T})$ zodat er voor iedere $x \in X$ een omgeving U bestaat zodat U en $\mathbb{R}^2$ homeomorf zijn, maar X niet Hausdorff is. Geef een bewijs. (Hint: Neem een quotient van twee kopieën van $\mathbb{R}^2$.)

Exercise 16.41. [10 punten] Zij K de kleinfles. Beantwoord iedere vraag met ja of nee en geef een bewijs.

1. Bestaat er een continue inbedding $g\colon S^1 \to K$ zodat $K \setminus g(S^1)$ samenhangend is? (5 pt)
2. Bestaat er een surjectieve continue afbeelding $f\colon [0,1] \to K$? (5 pt)

Exercise 16.42. [12 punten]

1. Bewijs dat er geen inbedding $S^1 \to \mathbb{R}$ bestaat. (6 pt)
2. Bewijs dat iedere injectieve continue afbeelding $S^1 \to S^1$ een homeomorfisme is. (6 pt)

16.11 The 2020 retake exam

Hertentamen Inleiding Topologie, WISB243 2020-04-14, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise! Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 40. The exam grade E will be obtained from your total score T by rounding off $\min(T/4, 10)$ to one decimal accuracy.
- You are free to write the solutions either in English, or in Dutch.

Succes !

10 pt total **Exercise 1.** Let $\mathscr{B}$ be the family of subsets of the form

$$\mathscr{B} = \{(a,b) : \ -\infty < a < b < \infty\} \bigcup \{\mathbb{Q} \cap (a,b) : \ -\infty < a < b < \infty\}.$$

2 pt (a) Show that $\mathscr{B}$ is a topology basis on $\mathbb{R}$.

2 pt (b) Let $\mathscr{T}$ be the topology generated by $\mathscr{B}$. Show that $(\mathbb{R}, \mathscr{T})$ is a Hausdorff space.

2 pt (c) Show that $\mathbb{R} \setminus \mathbb{Q}$ is closed in $(\mathbb{R}, \mathscr{T})$.

2 pt (d) Show that the topological space $(\mathbb{R}, \mathscr{T})$ is not normal.

2 pt (e) Let $F : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{eucl})$ be a continuous map with $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ and where $\mathscr{T}_{eucl}$ is the euclidean topology on $\mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

9 pt total **Exercise 2.** Let $X = \{(x,y) : \ y \geq 0, \ (x,y) \neq (0,0)\}$. We understand X as a topological space with the induced subspace topology from $X \subset \mathbb{R}^2$.

3 pt (a) Show that the subsets $E = \{(x,0) : \ x < 0\}$ and $F = \{(x,0) : \ x > 0\}$ are closed subsets in X.

3 pt (b) Construct explicitly a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ if $x \in E$ and $f(x) = 0$ if $x \in F$.

3 pt (c) Is X a normal topological space? (give a proof for your answer).

9 pt total **Exercise 3.** Let (X,d_1) and (X,d_2) be two metric spaces and assume that (X,d_1) is a compact topological space (with the topology indued by the metric). Let $f : X \to Y$ be a continuous map. Show that f is uniformly continuous, i.e. that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_1(x,y) < \delta \Rightarrow d_2(f(x),f(y)) < \varepsilon.$$

12 pt total **Exercise 4.** Let X,Y be topological spaces. We call a function $f : X \to Y$ proper if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$.

3 pt (a) Let X,Y be topological spaces and $f : X \to Y$ a continuous map. Show that $f(C) \subset Y$ is compact for every $C \subset X$ compact

3 pt (b) Let X,Y be locally compact Hausdorff spaces with one-point compactifications X_∞ and Y_∞. For a map $f : X \to Y$ we define $\hat{f} : X_\infty \to Y_\infty$ by setting $\hat{f}(x) = f(x)$ for $x \in X$ and $\hat{f}(\infty) = \infty$. Show that $\hat{f} : X_\infty \to Y_\infty$ is continuous if and only if the map $f : X \to Y$ is continuous and proper.

3 pt (c) Let X be a locally compact Hausdorff space and $C \subset X$ a closed subset. Show that $C \cup \{\infty\} \subset X_\infty$ is closed. Moreover show that a subset $C' \subset X_\infty$ is closed if and only if C' is compact.

3 pt (d) Let X,Y be locally compact Hausdorff spaces and $f : X \to Y$ a continuous and proper map. Use the previous parts of this exercise to show that if $C \subset X$ is closed, then $f(C) \subset Y$ is closed.

16.12 The 2020 exam (January 28, 2020)

Tentamen Inleiding Topologie, WISB243 2020-01-28, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise! Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 48.
- You are free to write the solutions either in English, or in Dutch.

Succes !

10 pt total **Exercise 1.** Let $\mathbb{N} = \{1,2,3,\dots,\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathscr{B}$ be the family of subsets of $\mathbb{N}$ of the form $U_{a,b} = \{a+nb : n \in \mathbb{N}_0\}$ where $a,b \in \mathbb{N}$ are coprime, i.e. $\gcd(a,b) = 1$.

5 pt (a) Show that $\mathscr{B}$ is a topology basis on $\mathbb{N}$.

5 pt (b) Let $\mathscr{T}$ be the topology on $\mathbb{N}$ generated by the topology basis $\mathscr{B}$. Show that the topological space $(\mathbb{N}, \mathscr{T})$ is Hausdorff.

9 pt total **Exercise 2.** Let $(X, \mathscr{T})$ be a locally compact Hausdorff space.

3 pt (a) Let $U \subset X$ be an open subset. Show that U is locally compact (with the induced subspace topology).

3 pt (b) Let $C \subset X$ be closed. Show that C is locally compact (with the induced subspace topology).

3 pt (c) Let $U \subset X$ open and $C \subset X$ closed. Show that $U \cap C$ is locally compact.

11 pt total **Exercise 3.** Let $J = [-1,1]$, let $S := \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $X := S \times J$ be equipped with the product topology. Let $\Gamma = \{1,-1\}$ be the multiplicative group of two elements. We consider the action φ of Γ by homeomorphisms on the space X given by

$$\varphi : \Gamma \times X \to X, \quad (\gamma, (z_1,z_2,t)) \mapsto (\gamma z_1, \gamma z_2, \gamma t).$$

2 pt (a) Show that every orbit for this action consists of precisely two points.

We equip the quotient X/Γ with the quotient topology, and denote the canonical projection by $p : X \to X/\Gamma$. We consider the map $\sigma : [0,1]^2 = [0,1] \times [0,1] \to X/\Gamma$ given by

$$\sigma(s,t) = p(\cos \pi s, \sin \pi s, 2t-1), \quad (0 \leq s,t \leq 1).$$

3 pt (b) Show that σ is surjective.

We define the equivalence relation $\sim$ on $[0,1]^2$ by $(s,t) \sim (s',t') \iff \sigma(s,t) = \sigma(s',t')$. The quotient space $[0,1]^2/\sim$ is equipped with the quotient topology. We denote the associated natural projection by $q : [0,1]^2 \to [0,1]^2/\sim$.

5 pt (c) Prove that there exists a homeomorphism

$$\bar{\sigma} : [0,1]^2/\sim \ \longrightarrow \ X/\Gamma$$

such that $\bar{\sigma} \circ q = \sigma$.

1 pt (d) Which of the following assertions is correct: (1) X/Γ is homeomorphic to the cylinder; (2) X/Γ is homeomorphic to the Möbius band. (Here you are not required to give any justification.)

10 pt total **Exercise 4.** We call a topological space extremally disconnected space if for every two disjoint open subsets U and V one has $\overline{U} \cap \overline{V} = \emptyset$. Here $\overline{U}$ and $\overline{V}$ are the closures of the sets U and V.

4 pt (a) Prove the following statement: A topological space is extremally disconnected if and only if the closure of every open subset U is open.

3 pt (b) Show that an extremally disconnected Hausdorff space with at least two elements is not connected.

3 pt (c) Give an example of a topological space X that has the following properties

- X has at least two points
- X is connected
- X is extremally disconnected
- for every pair of distinct points, each has a neighborhood not containing the other point.

8 pt total **Exercise 5.** Let X be a compact Hausdorff space and $x_0 \in X$.

3 pt (a) Show that $Y = X \setminus \{x_0\}$ is a locally compact Hausdorff space.

5 pt (b) Let Y_∞ be the one-point compactification of Y. Show that Y is homeomorphic to X.

16.13 The 2019 retake exam (April 15, 2019)

Herkansing Inleiding Topologie, WISB243 2019-04-15, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Aldo and Francesco, 2: Maarten) and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise!
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The grade will be obtained from your total score T by rounding off $\min(T/4, 10)$ to a half integer above 6 or an integer below 6.5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** For $\mathbb{R}$ we consider the collection $\mathscr{B}$ of all subsets of the form

$$[p,q) := \{x \in \mathbb{R} \mid p \leq x < q\}, \qquad \text{with } p,q \in \mathbb{Q}, p < q.$$

3 pt (a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. Denote by $\mathscr{T}$ the topology generated by $\mathscr{B}$.

2 pt (b) Show that $\mathscr{T}$ contains the Euclidean topology.

2 pt (c) Is $(\mathbb{R}, \mathscr{T})$ first countable, second countable? Is it Hausdorff?

3 pt (d) Show that $[0,1)$ is open and closed in $(\mathbb{R}, \mathscr{T})$. Is $(\mathbb{R}, \mathscr{T})$ connected?

3 pt (e) Show that $[0,1)$ and $[0,1]$ are not compact in $(\mathbb{R}, \mathscr{T})$.

9 pt total **Exercise 2.** Let $X := [0,1] \times [-2,2]$ and the subset $B := [0,1] \times [-1,1] \subset X$ both be equipped with the topology induced by the Euclidean topology on $\mathbb{R}^2$.

We equip X with the equivalence relation R whose equivalence classes are given by $\{(s,t)\}$ for $0 < s < 1$ and $-2 < t < 2$, $\{(s,-2),(s,2)\}$ for $0 < s < 1$; $\{(0,t),(1,-t)\}$ for $-2 < t < 2$ and, finally, $\{(0,\pm 2),(1,\pm 2)\}$. Accordingly, X/R equipped with the quotient topology, is the Klein bottle (which is known to be a Hausdorff space). The associated quotient map is denoted by $\pi : X \to X/R$.

3 pt (a) The restriction of R to B is the relation R_B defined by $b_1 R_B b_2 \iff \pi(b_1) = \pi(b_2)$, for $b_1, b_2 \in B$. Show that R_B is an equivalence relation on B and explicitly determine the associated equivalence classes in B.

1 pt (b) The quotient B/R_B is equipped with the quotient topology. To which well known space is this quotient homeomorphic? (You need not justify your answer.)

Let $\pi_B : B \to B/R_B$ be the associated quotient map.

2 pt (c) Show that there exists a unique map $f : B/R_B \to X/R$ such that for all $b \in B$ we have $f(\pi_B(b)) = \pi(b)$.

3 pt (d) Prove that the map f is an embedding.

11 pt total **Exercise 3.** Let M be a topological space, and assume that $\gamma \mapsto \varphi_\gamma$ is an action of the group $\mathbb{Z}_2 = \{-1, 1\}$ on M by homeomorphisms. Let $M/\mathbb{Z}_2$ be the associated quotient (equipped with the quotient topology), and $\pi : M \to M/\mathbb{Z}_2$ the quotient map.

2 pt (a) Given a subset $V \subset M$, show that $\pi^{-1}(\pi(V)) = V \cup \varphi_{-1}(V)$.

2 pt (b) Show that for $V \subset M$ open, the set $\pi(V)$ is open in $M/\mathbb{Z}_2$.

3 pt (c) Let $\{U_i\}_{i\in I}$ be an open cover of M. For every $m \in M$ let $i_m, j_m \in I$ be indices such that $m \in U_{i_m}$ and $\varphi(m) \in U_{j_m}$. Show that there exists an open neighborhood V_m of m such that $V_m \subset U_{i_m}$ and $\varphi(V_m) \subset U_{j_m}$.

4 pt (d) Show that M is compact if and only if $\pi(M)$ is compact. Hint: for one of the implications consider the collection $\{\pi(V_m)\}$ resulting from (c).

11 pt total **Exercise 4.** For M a locally compact Hausdorff space we denote by $C_c(M)$ the real linear space of continuous functions $M \to \mathbb{R}$ with compact support. If U is an open subset of M, we put $C_c(U) := \{f \in C_c(M) \mid \mathrm{supp} f \subset U\}$.

2 pt (a) If U is open in M, $f \in C_c(M)$, $\psi \in C(M)$ and $\mathrm{supp}\,\psi \subset U$, show that $\psi f \in C_c(U)$.

By a positive integral on an open subset U of M we mean a linear map $I : C_c(U) \to \mathbb{R}$ such that for all $f \in C_c(U)$ we have:

$$f \geq 0 \;\Rightarrow\; I(f) \geq 0.$$

A positive integral I on U is said to be strictly positive if for all $f \in C_c(U)$ we have

$$f \geq 0,\, I(f) = 0 \;\Rightarrow\; f = 0.$$

1 pt (b) Prove the following result. If I is a positive integral on an open subset U of M and $\psi : M \to \mathbb{R}$ is continuous function with $\psi \geq 0$ and $\mathrm{supp}\psi \subset U$ then $I_\psi : f \mapsto I(\psi f)$ is a positive integral on M.

5 pt (c) Assume that M is second countable, and that for every point $m \in M$ there exists an open neighborhood $U_m \ni m$ and a strictly positive integral $I_m : C_c(U_m) \to \mathbb{R}$ on U_m. Show that there exists a strictly positive integral on M.

In the next item you may use without proof that the map $J : C_c((0,1)^n) \to \mathbb{R}$ defined by the n-fold repeated Riemann integral $J(f) = \int_0^1 \cdots \int_0^1 f(x)\, dx_1 \cdots dx_n$, is a strictly positive integral on the open subset $(0,1)^n$ of $\mathbb{R}^n$.

3 pt (d) If M is a topological manifold of dimension $n \geq 1$, show that there exists a strictly positive integral on M.

16.14 The 2019 exam (January 29, 2019)

Tentamen Inleiding Topologie, WISB243 2019-01-29, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Aldo and Francesco, 2: Maarten) and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise!
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The exam grade E will be obtained from your total score T by rounding off $\min(T/4, 10)$ to one decimal accuracy.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** For $\mathbb{R}$ we consider the collection $\mathscr{B}$ of all subsets of the form

$$(a,n] := \{x \in \mathbb{R} \mid a < x \leq n\},$$

with $n \in \mathbb{Z}$ and $a \in \mathbb{R}$ such that $a < n$.

2 pt (a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. Denote by $\mathscr{T}$ the topology generated by $\mathscr{B}$.

4 pt (b) Is $(\mathbb{R}, \mathscr{T})$ second countable? Is it Hausdorff? Is it metrizable?

Consider the set $A := [\frac{1}{2}, 1]$.

3 pt (c) Determine the closure of A.

4 pt (d) Show that A is compact, and that $[0,1]$ is not compact.

10 pt total **Exercise 2.**

1 pt (a) Show that $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected for the topology induced by the Euclidean topology.

4 pt (b) Show that there exists no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

5 pt (c) If M is a two-dimensional topological manifold, show that there exists no continuous injective map $g : M \to S^1$. Hint: use (b).

11 pt total **Exercise 3.** Let X be topological space, and $A \subset X$ a subset. The space X/A obtained by collapsing A to a point is equipped with the quotient topology. The quotient map is denoted by $\pi : X \to X/A$.

1 pt (a) Show that $\pi^{-1}(\pi(A)) = A$.

2 pt (b) If $S \subset X - A$ show that $\pi^{-1}(\pi(S)) = S$.

2 pt (c) If $T \subset X$ satisfies $T \supset A$, show that $\pi^{-1}(\pi(T)) = T$.

From now on we assume that A is connected for the induced topology.

4 pt (d) If X is the union of two disjoint non-empty open subsets U and V, show that $\pi(U)$ and $\pi(V)$ are disjoint open subsets of X/A.

2 pt (e) Show that X is connected if and only if X/A is connected.

10 pt total **Exercise 4.** Let X be a paracompact topological space and A a closed subset of X. The goal of this exercise is to show that A is paracompact for the induced topology. For this, we assume that $\mathscr{V} = \{V_i\}_{i \in I}$ is an open covering of A.

1 pt (a) Show that there exists a family $\mathscr{U} = \{U_i\}_{i \in I}$ of open subsets of X such that $V_i = U_i \cap A$ for all $i \in I$.

2 pt (b) Show that $\mathscr{U}^* := \mathscr{U} \cup \{X - A\}$ is an open covering of X.

1 pt (c) Show that there exists a locally finite refinement $\mathscr{W} = \{W_j \mid j \in J\}$ of $\mathscr{U}^*$.

3 pt (d) Let $J_0 = \{j \in J \mid W_j \subset X - A\}$ and put $J_1 = J - J_0$. Show that $\mathscr{W}' = \{W_j \cap A\}_{j \in J_1}$ is an open covering of A which is subordinate to $\mathscr{V}$.

3 pt (e) Show that A is paracompact.

16.15 The 2018 retake exam (April 16, 2018)

Retake Inleiding Topologie, WISB243 2018-04-16, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Lauran, 2: Luca) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score T by rounding off $\min(T/4, 10)$ in the usual way.
- You are free to write the solutions either in English, or in Dutch.

Succes !

11 pt total **Exercise 1.** Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ such that

(1) $[0,1) \in \mathscr{T}$;

(2) the map $V : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$ is continuous with respect to $\mathscr{T}$;

(3) for each $c \in \mathbb{R}$ the map $T_c : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + c$ is continuous with respect to $\mathscr{T}$.

2 pt (a) Show that for every $U \in \mathscr{T}$ and $a \in \mathbb{R}$ the set $U + a = \{x + a \mid x \in U\}$ belongs to $\mathscr{T}$.

3 pt (b) Show that the collection $\mathscr{B} := \{[a, a + 2^{-n}) \mid a \in \mathbb{R},\ n \in \mathbb{N}\}$ is a subset of $\mathscr{T}$.

3 pt (c) Show that $\mathscr{B}$ is a topology basis.

3 pt (d) Show that the topology generated by $\mathscr{B}$ equals $\mathscr{T}$.

11 pt total **Exercise 2.** Let $J = [-1,1]$, let $S := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $X := S \times J$ be equipped with the product topology. Let $\Gamma = \{1, -1\}$ be the multiplicative group of two elements. We consider the action φ of Γ by homeomorphisms on the space X given by

$$\varphi : \Gamma \times X \to X, \quad (\gamma, (z_1, z_2, t)) \mapsto (\gamma z_1, \gamma z_2, \gamma t).$$

2 pt (a) Show that every orbit for this action consists of precisely two points.

We equip the quotient X/Γ with the quotient topology, and denote the canonical projection by $p : X \to X/\Gamma$. We consider the map $\sigma : [0,1]^2 = [0,1] \times [0,1] \to X/\Gamma$ given by

$$\sigma(s,t) = p(\cos \pi s, \sin \pi s, 2t-1), \quad (0 \leq s,t \leq 1).$$

3 pt (b) Show that σ is surjective.

We define the equivalence relation $\sim$ on $[0,1]^2$ by $(s,t) \sim (s',t') \iff \sigma(s,t) = \sigma(s',t')$. The quotient space $[0,1]^2/\sim$ is equipped with the quotient topology. We denote the associated natural projection by $q : [0,1]^2 \to [0,1]^2/\sim$.

5 pt (c) Prove that there exists a homeomorphism

$$\bar{\sigma} : [0,1]^2/\sim \ \longrightarrow \ X/\Gamma$$

such that $\bar{\sigma} \circ q = \sigma$.

1 pt (d) Which of the following assertions is correct: (1) X/Γ is homeomorphic to the cilinder; (2) X/Γ is homeomorphic to the Möbius band. (Here you are not required to give any motivation.)

11 pt total **Exercise 3.** We consider a (non-empty) topological space Y, a set Z and a map $f : Y \to Z$. The map f is said to be locally constant if for every $y \in Y$ there exists an open neighborhood U of y in Y such that the restriction $f|_U$ is constant. Equivalently, this means that $f(U)$ consists of a single element.

3 pt (a) Show that f is locally constant if and only if every fiber $f^{-1}(\{z\})$, for $z \in Z$, is open in Y.

3 pt (b) Show that $f : Y \to Z$ is locally constant if and only if f is continuous for the discrete topology on Z.

3 pt (c) If Y is connected and $f : Y \to Z$ locally constant, show that f is constant.

2 pt (d) If Y is not connected, show that there exists a set Z and a locally constant function $f : Y \to Z$ which is not constant.

11 pt total **Exercise 4.** Let X be a Hausdorff topological space and $f : X \to (0,\infty)$ a function such that that for every $a \in X$ there exists an open neighborhood U of a and a constant $m > 0$ such that $f(x) \geq m$ for all $x \in U$.

3 pt (a) If X is compact, show that there exists a constant $m_X > 0$ such that $f(x) \geq m_X$ for all $x \in X$.

We now assume that X is paracompact.

3 pt (b) Show that there exists a locally finite covering $\{V_i \mid i \in I\}$ of X such that for every $i \in I$ there exists a constant $m_i > 0$ such that $f(x) \geq m_i$ for all $x \in V_i$.

5 pt (c) Show that there exists a continuous function $\mu : X \to (0,\infty)$ such that $f(x) \geq \mu(x)$ for all $x \in X$.

16.16 The 2018 exam (January 29, 2018)

Exam Inleiding Topologie, WISB243 2018-01-29, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Lauran, 2: Luca) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 55. The final grade will be obtained from your total score through division by 5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

15 pt total **Exercise 1.** On $\mathbb{R}$ we consider the collection $\mathscr{B} \subset \mathscr{P}(\mathbb{R})$ consisting of $\mathbb{R}$ and all intervals of the form $[m,a)$, with $m \in \mathbb{Z}$ and $a \in \mathbb{R}$ (for convenience we agree that $[m,a) = \emptyset$ for $m \geq a$).

3 pt (a) Show that $\mathscr{B}$ is a topology basis, but not a topology.

Let $\mathscr{T}$ be the topology on $\mathbb{R}$ generated by $\mathscr{B}$.

2 pt (b) Is $\mathscr{T}$ Hausdorff?

2 pt (c) Is $\mathscr{T}$ second countable?

5 pt (d) Determine the interior and the closure of the set $A := [-\frac{1}{2}, \frac{1}{2}]$ with respect to $\mathscr{T}$.

3 pt (e) Determine for which $r > 0$ the subset $[0,r]$ is connected for the subspace topology (induced by $\mathscr{T}$). Prove the validity of your answer.

10 pt total **Exercise 2.** Let X and Y be non-empty topological spaces. We equip $X \times Y$ with the product topology.

5 pt (a) Show: if X and Y are Hausdorff spaces, then $X \times Y$ is a Hausdorff space.

5 pt (b) Show that the converse is also true: if $X \times Y$ is Hausdorff, then both X and Y are Hausdorff.

11 pt total **Exercise 3.** We consider a second countable locally compact Hausdorff space X. A function $\varphi : X \to [0,\infty)$ is said to be locally bounded if and only if for every $a \in X$ there exists a neighborhood V of a and a constant $M > 0$ such that $\varphi \leq M$ on V.

Let $f : X \to [0,\infty)$ be a function. Prove that the following assertions are equivalent.

(1) The function f is locally bounded.

(2) There exists a continuous function $g : X \to [0,\infty)$ such that $f \leq g$ on X.

4 pt for the proof that '(2) $\Rightarrow$ (1)'.

7 pt for the proof that '(1) $\Rightarrow$ (2)'.

19 pt total **Exercise 4.** Let S^1 be the unit circle in $\mathbb{R}^2$. Let Γ be the group $\{1, g\}$ of two elements, with group law given by $11 = g^2 = 1$ and $1g = g1 = g$.

We consider two actions α, β of Γ by homeomorpisms on S^1. These actions are given by $\alpha_1 = \beta_1 = \mathrm{id}_{S^1}$ and

$$\alpha_g(x) = (-x_1, -x_2), \quad \beta_g(x) = (x_1, -x_2), \qquad (x = (x_1, x_2) \in S^1).$$

In the following you may use without proof that α and β are indeed actions by homeomorphisms. For $\gamma \in \Gamma$ we define $\rho_\gamma : S^1 \times S^1 \to S^1 \times S^1$ by

$$\rho_\gamma((x,y)) = (\alpha_\gamma(x), \beta_\gamma(y)), \quad ((x,y) \in S^1 \times S^1).$$

2 pt (a) Show that ρ is an action of Γ on $S^1 \times S^1$ by homeomorphisms.

1 pt (b) For each point $p = (x,y)$ of the torus $S^1 \times S^1$, determine the orbit Γp.

In the following, q denotes the natural quotient map $S^1 \times S^1 \to S^1 \times S^1/\Gamma$ associated with the action ρ. We define the map $f : [0,1] \times S^1 \to S^1 \times S^1$ by $f(s,y) = (\cos \pi s, \sin \pi s, y)$.

3 pt (c) Show that f is a topological embedding.

3 pt (d) Show that $F := q \circ f$ is surjective from $[0,1] \times S^1$ onto the quotient $S^1 \times S^1/\Gamma$.

Let $\sim$ be the equivalence relation on $[0,1] \times S^1$ determined by $z \sim z' \iff F(z) = F(z')$.

4 pt (e) Prove that $S^1 \times S^1/\Gamma$ is homeomorphic to $[0,1] \times S^1/\sim$.

3 pt (f) Show that F is bijective from $[0,1) \times S^1$ onto $S^1 \times S^1/\Gamma$.

3 pt (g) Calculate the equivalence classes of $\sim$ in $[0,1] \times S^1$. Use a picture to indicate why $[0,1] \times S^1/\sim$ is homeomorphic to the Klein bottle.

16.17 The 2017 retake exam (April 18, 2017)

Retake Inleiding Topologie, WISB243 2017-04-18, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 55. The final grade will be obtained from your total score through division by 5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

14 pt total **Exercise 1.** Let $\mathscr{T}$ be the collection of sets $U \subset \mathbb{R}$ such that $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite.

2 pt (a) Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.

2 pt (b) Is $(\mathbb{R}, \mathscr{T})$ Hausdorff?

1 pt (c) Determine the closure of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathscr{T}$.

2 pt (d) Determine the interior of $[0,1]$ with respect to $\mathscr{T}$.

3 pt (e) Show that every subset S of $\mathbb{R}$ is compact for the topology induced by $\mathscr{T}$.

4 pt (f) Let $A \subset \mathbb{R}$ be a subset of at least two elements. Show that A is not connected with respect to the topology induced by $\mathscr{T}$ if and only if A is finite.

9 pt total **Exercise 2.** Let X and Y be topological spaces, and $f : X \to Y$ a continuous map. We assume that X is compact and that for every $x \in X$ there exists an open neighborhood U of x such that the restriction $f|_U : U \to Y$ is injective.

3 pt (a) Show that there exists an open covering $\{U_i \mid i \in I\}$ of X such that for every $y \in Y$ and $i \in I$ the intersection $U_i \cap f^{-1}(\{y\})$ consists of at most one point.

6 pt (b) Show that there exists an $N > 0$ such that for every $y \in Y$ the fiber $f^{-1}(\{y\})$ consists of at most N elements.

11 pt total **Exercise 3.** Let X be a topological space, and let $\{U_i \mid i \in I\}$ be an open cover of X such for each $i \in I$ the set U_i is connected for the induced topology. Let $A \subset X$ be open and closed.

2 pt (a) Show that for each $i \in I$ we have $A \cap U_i = \emptyset$ or $A \cap U_i = U_i$.

2 pt (b) Let $i, j \in I$ be such that $U_i \cap U_j \neq \emptyset$. Show that either A contains both U_i and U_j or is disjoint from both.

If $i, j \in I$, then we will write $i \sim j$ to indicate that there exists a sequence $i_0, \ldots i_n$ in I such that $i_0 = i$ and $i_n = j$ and $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ for all $k \in \{1, \ldots, n\}$. It is readily seen that $\sim$ defines an equivalence relation on I. You may use this without proof.

2 pt (c) If $i, j \in I$ are $\sim$-equivalent, show that either A contains both U_i and U_j or is disjoint from both.

2 pt (d) Show that if all elements of I are $\sim$-equivalent then X is connected.

3 pt (e) Conversely, if X is connected, show that all elements of I are $\sim$-equivalent.

11 pt total **Exercise 4.** We consider the closed unit disk in $\mathbb{C}$ given by

$$\bar{D} = \{Z \in \mathbb{C} \mid |z| \leq 1\}.$$

We consider the map $\varphi : \bar{D} \to \mathbb{C}^2$ given by

$$\varphi(z) = ((1 - \|z\|)z, z^2).$$

We define the equivalence relation R on $\bar{D}$ by $zRw :\iff \varphi(z) = \varphi(w)$.

4 pt (a) Show that $\bar{D}/R$ is homeomorphic to $\mathbb{P}^2(\mathbb{R})$.

4 pt (b) Show that there exists a topological embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

3 pt (c) We consider the algebra A of continuous functions $f : \bar{D} \to \mathbb{C}$ such that $f(z) = f(-z)$ for all $z \in \partial D$. Show that the topological spectrum $\mathbf{X}_A$ of A is homeomorphic to $\mathbb{P}^2(\mathbb{R})$.

10 pt total **Exercise 5.** Let X and Y be locally compact Hausdorff spaces. Their one-point compactifications are denoted by $\widehat{X} = X \cup \{\infty_X\}$ and $\widehat{Y} = Y \cup \{\infty_Y\}$. Let $f : X \to Y$ be a continuous map. We define $\widehat{f} : \widehat{X} \to \widehat{Y}$ by $\widehat{f} = f$ on X and $\widehat{f}(\infty_X) = \infty_Y$.

4 pt (a) Show: if $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous then for every compact set $K \subset Y$ the set $f^{-1}(K)$ is compact in X.

6 pt (b) Show that the converse implication is also valid: if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is compact in X, then $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous.

16.18 The 2017 exam (January 30, 2017)

Exam Inleiding Topologie, WISB243 2017-01-30, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Francesco, 2: Panagiotis) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score through division by 4.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** We consider the Euclidean topology $\mathscr{T}_{\text{eucl}}$ on $\mathbb{R}$ and also the topology $\mathscr{T}_l$ generated by the intervals $[a,b)$, for $a,b \in \mathbb{R},\ a < b$.

1 pt (a) Show that the intervals $[a,b)$ form a basis for the topology $\mathscr{T}_l$.

2 pt (b) Let $p,q \in \mathbb{R}\ p > 0$. Show that the map $T : \mathbb{R} \to \mathbb{R},\ x \mapsto px+q$ is a homeomorphism of the topological space $(\mathbb{R}, \mathscr{T}_l)$ to itself.

2 pt (c) Show that $\mathscr{T}_{\text{eucl}} \subset \mathscr{T}_l$.

1 pt (d) Show that $(\mathbb{R}, \mathscr{T}_l)$ is Hausdorff.

1 pt (e) Suppose $S \subset \mathbb{R}$ is compact for the topology induced by $\mathscr{T}_l$. Show that S is also compact for the topology induced by $\mathscr{T}_{eucl}$.

2 pt (f) Suppose $S \subset \mathbb{R}$ is compact for (the topology induced by) $\mathscr{T}_l$. Show that for every $a \in \mathbb{R}$ the set $(-\infty, a) \cap S$ is also compact in $(\mathbb{R}, \mathscr{T}_l)$.

2 pt (g) Show that $[0,1]$ is not compact in $(\mathbb{R}, \mathscr{T}_l)$.

2 pt (h) Show that $(\mathbb{R}, \mathscr{T}_l)$ is not locally compact.

9 pt total **Exercise 2.** Let Γ be the group of strictly positive real numbers, with the ordinary multiplication. We equip $\mathbb{R}$ with the Euclidean topology and consider that action of Γ on $\mathbb{R}$ given by $(\gamma, x) \mapsto \gamma x$. The quotient space $Y := \mathbb{R}/\Gamma$ is equipped with the quotient topology. Let $\pi : \mathbb{R} \to Y$ be the natural map.

2 pt (a) Show that Y consists of the three points $a := \pi(-1)$, $b := \pi(0)$ and $c := \pi(1)$.

4 pt (b) Determine the quotient topology $\mathscr{T}_Y$ of $Y = \{a,b,c\}$ explicitly, i.e., give a list of all open subsets of Y.

3 pt (c) Is Y Hausdorff, connected or compact? Prove the validity of each answer.

12 pt total **Exercise 3.** Let X be a locally compact Hausdorff topological space which is the disjoint union of two non-empty open subsets $X_1, X_2 \subset X$. Let $X^+ = X \cup \{\infty\}$ be the one point compactification of X.

3 pt (a) Show that the implication '$(1) \Rightarrow (2)$' holds for the following statements:

(1) At least one of the sets X_1 and X_2 is compact.

(2) X^+ is not connected.

In the rest of the exercise, assume that both X_1 and X_2 are connected.

3 pt (b) If U is non-empty, open and closed in X show that U contains one of the sets X_1 and X_2.

4 pt (c) Show that the implication '$(2) \Rightarrow (1)$' holds as well.

2 pt (d) Give an example of a non-connected locally compact Hausdorff space X such that X^+ is connected.

10 pt total **Exercise 4.** Let D be the closed unit disk in $\mathbb{R}^2$ and let $X := D \times \{-1, 1\} \subset \mathbb{R}^3$ be equipped with the restriction of the Euclidean topology on $\mathbb{R}^3$.

1 pt (a) Show that X is compact Hausdorff.

Let $\varphi : X \to \mathbb{R}^3$ be the map given by $\varphi(x, \pm 1) = (x, \pm\sqrt{1 - \|x\|^2})$, where $\|\cdot\|$ is the Euclidean norm. It is clear that φ is continuous.

2 pt (b) Show that the image of φ is the unit sphere S^2 in $\mathbb{R}^3$.

Let A be the set of continuous functions $f : X \to \mathbb{R}$ such that $f(x,-1) = f(x,+1)$ for all $x \in \partial D$. Here ∂D denotes the boundary of D in $\mathbb{R}^2$.

1 pt (c) Show that A is a subalgebra of $C(X)$.

6 pt (d) Show that the topological spectrum $\mathbf{X}_A$ of A is homeomorphic to S^2.

16.19 The 2016 exam (January 24, 2016)

Note: All the questions are worth 0.5 points (adding up to a total of 11 points). Please motivate all your answers. For instance, in Exercise 1, point c) do not just answer with "yes" or "no", but provide a proof. Also, in d) of the same exercise, please do not just write down the final result, but also explain how you found it. The most difficult questions are 4-d) and especially 3-c). You may write the answers in English or Dutch.

Exercise 16.43. For $\mathbb{R}$ we consider the family $\mathscr{S}$ of subsets consisting of all the intervals of type $(a,b]$ with $a<b\leq 0$ and all intervals of type $[c,d)$ with $0\leq c<d$. Let also $\mathscr{B}$ be the family which is $\mathscr{S}$ together with the subset $\{0\}$ of $\mathbb{R}$. We endow $\mathbb{R}$ with the smallest topology containing $\mathscr{S}$- denote it by $\mathscr{T}$.

a) Show that $\mathscr{S}$ is not a topology basis and that $\mathscr{B}$ is the smallest topology basis containing $\mathscr{S}$.
b) Show that $(\mathbb{R},\mathscr{T})$ is Hausdorff.
c) Is the sequence $x_n=\frac{1}{n}$ convergent in $(\mathbb{R},\mathscr{T})$?
d) Compute the closure and the interior of

$$A=(-3,-2]\cup(-1,0)\cup(0,1)\cup(2,3)$$

in the topological space $(\mathbb{R},\mathscr{T})$
e) Show that the identity map $\mathrm{Id}:(\mathbb{R},\mathscr{T})\to(\mathbb{R},\mathscr{T}_{\mathrm{Eucl}})$ is continuous.
f) Show that $[0,1]$, with the topology induced from $(\mathbb{R},\mathscr{T})$, is not compact.
g) Show that $(\mathbb{R},\mathscr{T})$ is not connected.
h) Describe all the connected subsets of $(\mathbb{R},\mathscr{T})$.
i) Show that any continuous function $f:(\mathbb{R},\mathscr{T}_{\mathrm{Eucl}})\to(\mathbb{R},\mathscr{T})$ is constant.

Exercise 16.44. Consider

$$X=\left\{(x,y)\in\mathbb{R}^2:x^2+y^2=1,\ \text{or } x=0\right\},$$
$$Y=\left\{(x,y)\in\mathbb{R}^2:xy=0\right\},$$

both endowed with the Euclidean topology. Show that:

a) Y can be embedded in X,
b) X cannot be embedded in Y,
c) that the one-point compactifications of X and Y are homeomorphic.

In this exercise you do not have to provide explicit formulas for the embedding and the homeomorphism from a) and c), respectively. But please use pictures to describe those functions and please provide all the arguments to support your claims (e.g., if you draw a picture for the one-point compactification of X, please explain why the picture you draw is indeed the one-point compactification).

Exercise 16.45. Show that:

a) If X is a compact Hausdorff space and A is nonempty closed in X, then the space X/A obtained from X by collapsing A to a point is compact and Hausdorff.
b) If $U\subset\mathbb{R}^n$ is a bounded nonempty open subset of $\mathbb{R}^n$ (with the Euclidean topology), considering $X=\overline{U}$ (the closure of U in $\mathbb{R}^n$) and $A=\partial(U)$ (the boundary of U in $\mathbb{R}^n$) show:

b1) X is compact and A is closed in X.
b2) the one-point compactification of U is homeomorphic to X/A.

c) If $U\subset\mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ then its one-point compactification can be embedded in $\mathbb{R}^{n+1}$. (a *complete* solution here is worth 1 point).

Exercise 16.46. Here we work over the field $\mathbb{R}$ of real numbers. Let A be an arbitrary algebra, assume that A is commutative (i.e. $a_0a_1 = a_1a_0$ for any $a_0, a_1 \in A$) and denote by $\mathbf{0_A}$ and $\mathbf{1_A}$ the zero and the unit element of A, respectively. Fix an element $\alpha \in A$. Consider

$$B = A \times A = \{(a_0, a_1) : a_0, a_1 \in A\}$$

with the operations

$$(a_0, a_1) + (a_0', a_1') = (a_0 + a_0', a_1 + a_1'),$$
$$\lambda \cdot (a_0, a_1) = (\lambda a_0, \lambda a_1),$$
$$(a_0, a_1) \bullet (a_0', a_1') = (a_0 a_0' + \alpha a_1 a_1', a_0 a_1' + a_0' a_1).$$

a) Show that B, together with the previous operations (written in red), is an algebra (don't forget the unit element of B!).
b) Assume that $\chi : B \to \mathbb{R}$ is a character and consider

$$\overline{\chi} : A \to \mathbb{R}, \quad \overline{\chi}(a) = \chi(a, \mathbf{0_A}),$$
$$r := \chi(\mathbf{0_A}, \mathbf{1_A}) \in \mathbb{R}.$$

Show that:

b1) $\overline{\chi}$ is a character on A.
b2) $\chi(a_0, a_1) = \overline{\chi}(a_0) + r\,\overline{\chi}(a_1)$ for any $(a_0, a_1) \in B$.
b3) $\overline{\chi}(\alpha) = r^2$.

c) Conversely, for any character $\overline{\chi}$ on A and any $r \in \mathbb{R}$ satisfying the equation from b3), show that χ defined by the formula from b2) is a character on B.
d) When $A := \mathbb{R}[X_0, X_1]$ is the algebra of polynomials in two variables and $\alpha := 1 - (X_0)^2 - (X_1)^2$, show that the topological spectrum of B is *homeomorphic* to S^2.

16.20 The 2015 exam (January 28, 2015)

Note: In all the questions below, **please explain/prove your answers** (e.g., in Exercise 1, part a., b., etc, please **do not just say "yes" or "no"**. Also, in part e. or g. of the same exercise, **do not just write down the final result**, but also explain how you found it- e.g. by explaining your reasoning using pictures). There are three questions marked with a "*"; they are probably more difficult than the rest. The final mark for the exam is the number of points that you collect, except for the case in which you collect more than 10 points, when the final mark will be 10 (**in total, the exercises below are worth 11.5 points**).

Exercise 16.47. Prove that, for any Hausdorff space $(X,\mathscr{T})$, any finite subset $F\subset X$ is closed in X.
(1 **point)**

Exercise 16.48. Consider the family $\mathscr{B}$ of subsets of $\mathbb{R}^2$ consisting of all the subsets of type $(a,b)\times(a,b)$ with $a<b$ real numbers:

$$\mathscr{B}=\{(a,b)\times(a,b):a,b\in\mathbb{R},a<b\}.$$

Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}^2$ containing $\mathscr{B}$. We also consider

$$A=[0,1]\times[0,2]\subset\mathbb{R}^2.$$

a. Is $(\mathbb{R}^2,\mathscr{T})$ second countable? *(0.5 points)*
b. Is $(\mathbb{R}^2,\mathscr{T})$ Hausdorff? *(0.5 points)*
c. Is the identity map $\mathrm{Id}:\mathbb{R}^2\to\mathbb{R}^2$ continuous as a map from $(\mathbb{R}^2,\mathscr{T})$ to $(\mathbb{R}^2,\mathscr{T}_{\mathrm{Eucl}})$? But as a map from $(\mathbb{R}^2,\mathscr{T}_{\mathrm{Eucl}})$ to $(\mathbb{R}^2,\mathscr{T})$? *(0.5 points)*
d. Is A, with the topology induced from $\mathscr{T}$, connected? Is it compact? *(0.5 points)*
e. Let $x=(0,2)\in\mathbb{R}^2$ (the point of coordinates 0 and 2). Compute the closure of $\{x\}$ in $(\mathbb{R}^2,\mathscr{T})$. *(0.5 points)*
f. Show that the sequence $(x_n)_{n\geq1}$ given by

$$x_n=(\sin^2(n),\cos^4(n+2015))\in\mathbb{R}^2$$

is convergent in $(\mathbb{R}^2,\mathscr{T})$ and has more than one limit. *(0.5 points)*
g. Compute the interior and the closure of A in $(\mathbb{R}^2,\mathscr{T})$. *(1 point)*
h* Show that any continuous map $f:(\mathbb{R}^2,\mathscr{T})\to\mathbb{R}$ must be constant. *(1 point)*

(total: $6\times0.5+2\times1=$ **5 points**)

Exercise 16.49. Consider the group of integers modulo 2, $\mathbb{Z}_2=\{\hat{0},\hat{1}\}$. Define the following action of $\mathbb{Z}_2$ on the closed unit disk $D^2=\{z\in\mathbb{C}:|z|\leq1\}$:

$$\hat{0}\cdot z=z,\ \hat{1}\cdot z=-z\quad(\text{for all } z\in D^2).$$

Prove that the resulting quotient $D^2/\mathbb{Z}_2$ is homeomorphic to D^2 (provide a complete argument; a "proof" based only on pictures is not enough to get the entire 1 point for this exercise).
(**1 point**)

Exercise 16.50. Let X be compact Hausdorff space and let $C(X)$ be the space of real-valued continuous functions on X, endowed with the topology induced by the *sup*-metric.

Prove that if $\mathscr{A} \subset C(X)$ is a dense subset of $C(X)$, then $\mathscr{A}$ must be point-separating.
(1 **point)**

Exercise 16.51. For any algebra A over $\mathbb{R}$ and any ideal I of A, we define

$$X(A,I) := \{\chi \in X_A : \chi(x) = 0 \ \ \forall\, x \in I\} \subset X_A$$

and we endow it with the topology induced from the topology on the spectrum X_A of A. Show that:

a. For any algebra A and any ideal I of A, $X(A,I)$ is closed in X_A. *(0.5 points)*
b. Applied to $A = \mathbb{R}[x,y]$ (polynomials in two variables) and I the ideal consisting of polynomials that are divisible by x^2+y^2-1, $X(A,I)$ is homeomorphic to S^1. *(0.5 points)*
c* Assume now that X is a compact Hausdorff space, $Y \subset X$ and set

$$A = C(X), \ \ I := \{f \in C(X) : f(y) = 0 \ \ \forall\, y \in Y\}.$$

Show that $X(A,I)$ is homeomorphic to the closure $\overline{Y}$ of Y in X (where the last space is endowed with the topology induced from X). *(1 point)*

(total: $2 \times 0.5 + 1 =$ **2 points**)

Exercise 16.52. For each natural number n we consider a space X_n that is obtained by removing n distinct points from $\mathbb{R}^2$. We consider the 1-point compactification X_n^+ and we denote by $\infty_n \in X_n^+$ the point at infinity (so that $X_n^+ = X_n \cup \{\infty_n\}$). Show that

a. X_n^+ can be embedded in $\mathbb{R}^3$ (here you **do not** have to write down explicit formulas for the embedding, but please explain your reasoning using pictures and mention what result(s) you use in order to reach the final conclusion). *(0.5 points)*
b* If X_n and X_m are homeomorphic, then $n = m$. *(1 point)*

(total: $0.5 + 1 =$ **1.5 points**)

16.21 The 2015 retake exam (March 11, 2015)

Note: Please explain/prove your answers. E.g., in Exercise 1, parts c., d., f. please **do not just say "yes" or "no"** but also prove your claims. Also, in part e. or h., **do not just write down the final result**, but also explain how you found it. Also, in Exercise 2, do not forget to prove that the examples you give are not homeomorphic; etc. etc.

Exercise 16.53. Let k be a natural number ($k \in \{0,1,2,\ldots\}$). Let $\mathscr{B}_k$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and all the open intervals (a,b) which contain at most k integers (when $k=0$, that means: intervals (a,b) that contain no integers). Show that:

a. $\mathscr{B}_k$ is a topology basis on $\mathbb{R}$. *(0.5 points)*
b. for $k \geq 1$, the topology induced by $\mathscr{B}_k$ on $\mathbb{R}$ is the Euclidean topology. *(0.5 points)*

Next, we assume that $k=0$ and we look at the topology $\mathscr{T}_0 := \mathscr{T}(\mathscr{B}_0)$ induced by $\mathscr{B}_0$.

c. Is $(\mathbb{R}, \mathscr{T}_0)$ Hausdorff? *(0.5 points)*
d. Is $(\mathbb{R}, \mathscr{T}_0)$ compact? *(0.5 points)*
e. Compute the interior and the closure of $A = [0,2)$ in $(\mathbb{R}, \mathscr{T}_0)$. *(0.5 points)*
f. Is the sequence $(x_n)_{n\geq 1}$ given by
$$x_n = \frac{n^{2015} + \sin(n)}{n^2}$$
convergent in $(\mathbb{R}, \mathscr{T}_0)$? *(0.5 points)*
g. Show that any continuous function $f : (\mathbb{R}, \mathscr{T}_0) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}})$ is constant. *(0.5 points)*
h. Compute the interior and the closure of
$$D := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\} \subset \mathbb{R}^2$$
in $\mathbb{R}^2$ endowed with the product topology $\mathscr{T}_0 \times \mathscr{T}_{\text{Eucl}}$. *(1 point)*

Exercise 16.54. Find two locally compact Hausdorff spaces X and Y which are not homeomorphic but which have homeomorphic one-point compactifications. *(0.75 pts)* Can you find such X and Y such that all the spaces X, Y, X^+ and Y^+ are connected? *(0.75 pts)*

Exercise 16.55. Let $\Gamma = \mathbb{R}_{>0}$ be the group of strictly positive real numbers, endowed with the usual multiplication. Let $X = \mathbb{R}^n \setminus \{0\}$. Show that
$$\Gamma \times X \to X, \quad (r,x) \mapsto rx$$
(the usual multiplication of vectors x by scalars r) defines an action of Γ on X and **prove** that X/Γ is homeomorphic to S^{n-1} (the $n-1$-dimensional sphere).

Exercise 16.56. Let X be a topological space.

a. If $\gamma_1, \gamma_2 : [0,1] \to X$ are continuous and $\gamma_1(1) = \gamma_2(0)$, show that
$$\gamma : [0,1] \to X, \quad \gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is continuous. *(0.5 points)*
b. If there exist $A, B \subset X$ path connected such that $X = A \cup B$ and $A \cap B \neq \emptyset$, then X is connected. *(0.5 points)*

Exercise 16.57. Let A be an algebra over $\mathbb{R}$, assume that $\chi : A \to \mathbb{R}$ is a character and let

$$I := \mathrm{Ker}(\chi) \ \ (= \{a \in A : \chi(a) = 0\}).$$

Show that:

a. I is an ideal of A. *(0.5 points)*
b. $a - \chi(a) \cdot \mathbf{1}_A \in I$ for all $a \in A$ (where $\mathbf{1}_A$ is the unit of A). *(0.5 points)*
c. I is a maximal ideal. *(0.5 points)*

Exercise 16.58. Let X be a compact Hausdorff space,

$$F = (F_1, \ldots, F_n) : X \to \mathbb{R}^n$$

a continuous function and consider

$$\mathscr{A}_F = \{f \in C(X) : f = P \circ F \ \text{ for some polynomial function } P : \mathbb{R}^n \to \mathbb{R}\}.$$

Show that F is an embedding if and only if $\mathscr{A}_F$ is dense in $C(X)$. *(1.5 points)*
(Recall: $C(X)$ denotes the space of all continuous functions from X to $\mathbb{R}$, and it is a topological space with the topology induced by the sup-norm).

16.22 The 2014 exam (January 29, 2014)

Note: The questions marked with "+" are worth 1 point, the rest 0.5 points. As agreed, you are allowed to use during the exam the three sheets of A4 paper (= six pages) containing definitions, theorems, etc from the course- that you prepared at home. **For some explanations/hints, please see the end of the exam!!!!!!!!!!!!**.

Exercise 16.59. For $\mathbb{R}$ we consider the family $\mathscr{S}$ of subsets consisting of all the intervals of type (m,M) with $m < M < 0$, all intervals of type (m,M) with $0 < m < M$ (m and M real numbers) and the interval $[-1,1)$. Denote by $\mathscr{T}$ the smallest topology on $\mathbb{R}$ containing $\mathscr{S}$.

a. show that $\mathscr{S}$ is not a topology basis and describe a basis of $(\mathbb{R},\mathscr{T})$.
b. is $(\mathbb{R},\mathscr{T})$ Hausdorff? Is it second countable?
c. find an interval of type $[a,b]$ whose closure inside $(\mathbb{R},\mathscr{T})$ is not an interval.
d. find an interval of type (a,b) whose interior inside $(\mathbb{R},\mathscr{T})$ is not an interval.
e. find an interval of type $[a,b]$ with the property that, together with the topology induced from $(\mathbb{R},\mathscr{T})$, is not compact.
f. find an interval of type (a,b) with the property that, together with the topology induced from $(\mathbb{R},\mathscr{T})$, is not connected.
g. (+) consider

$$f:(\mathbb{R},\mathscr{T})\to(\mathbb{R},\mathscr{T}_{\text{Eucl}}),\ f(x)=\begin{cases}0 \text{ if } x<-1\\ 1 \text{ if } x\geq -1\end{cases}$$

Is f continuous? Is f sequentially continuous?

Exercise 16.60. Let X be the open cylinder $(-1,1)\times S^1$ and let Y be the open Moebius band (i.e. the Moebius band discussed in the lectures, from which the boundary circle was removed).

a. Describe the 1-point compactification X^+ as a subspace of $\mathbb{R}^3$.
b. (+) Describe $X^+\subset\mathbb{R}^3$ by explicit formulas and write down an explicit embedding

$$f:X\to\mathbb{R}^3$$

so that X^+ is the image of f together with the extra-point $(0,0,0)$.
c. Show that the 1-point compactification of Y is homeomorphic to the projective plane $\mathbb{P}^2$.
d. Show that X and Y are not homeomorphic.

Exercise 16.61. Let $X=[-1,1]\times\mathbb{R}$.

a. (+) Find all the numbers $\lambda,a,b\in\mathbb{R}$ with the property that

$$n\cdot(x,y):=(\lambda^n x, a+by+\lambda n)$$

defines an action of the additive group $(\mathbb{Z},+)$ on X.
b. Show that, in all cases, $X/\mathbb{Z}$ is compact.
c. For the values of λ,a,b that you found compute the resulting quotients $X/\Gamma/$

Exercise 16.62. In this exercise we work over $\mathbb{R}$ (hence we consider real-valued functions and real algebras). Let X and Y be compact Hausdorff spaces. For $u \in \mathscr{C}(X)$, $v \in \mathscr{C}(Y)$, define $u \otimes v \in \mathscr{C}(X \times Y)$ given by

$$u \otimes v : X \times Y \to \mathbb{R}, \ \ (u \otimes v)(x,y) := u(x)v(y),$$

and we denote by $\mathscr{A} \subset \mathscr{C}(X \times Y)$ the set of functions of type

$$\sum_{i=1}^{k} u_i \otimes v_i \quad \text{with } k \in \mathbb{N}, u_1, \ldots, u_k \in \mathscr{C}(X), v_1, \ldots, v_k \in \mathscr{C}(Y)$$

Show that:

a. (+) $\mathscr{A}$ is a dense subalgebra of $\mathscr{C}(X \times Y)$.
b. For any $\chi \in X_{\mathscr{A}}$, $\chi_1 : \mathscr{C}(X) \to \mathbb{R}$ and $\chi_2 : \mathscr{C}(Y) \to \mathbb{R}$ given by

$$\chi_1(u) := \chi(u \otimes 1), \ \ \chi_2(v) := \chi(1 \otimes v)$$

are characters.
c. (+) $X_{\mathscr{A}}$ is homeomorphic to $X \times Y$.

Notes/hints:

1. Please motivate all your answers. For instance, in Exercise 1, for b. do not just give an yes/no answer, for, c.-f. prove why the intervals that you found do satisfy the required conditions, at point g. explain/prove why f is, or isn't, continuous or sequentially continuous, and similarly for the other exercises.
2. For items a. and c. of Exercise 2, and item c. of Exercise 3, you do not have to give explicit formulas; pictures are enough, provided they are properly explained.
3. Exercise 2: you may want to remember the models $T_{R,r}$ of the torus:

$$\begin{aligned} T_{R,r} &= \left\{ (x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2 \right\} = \\ &= \left\{ \big((R + r\cos(a))\cos(b), (R + r\cos(a))\sin(b), r\sin(a)\big) : a, b \in [-\pi, \pi] \right\} \end{aligned}$$

(where, to obtain a torus, one has to assume $R > r > 0$). For d.: you may want to remember that $\mathbb{P}^2$ is a 2-dimensional topological manifold (in particular, each point has a neighborhood homeomorphic to $\mathbb{R}^2$).
4. In exercise 3, if you encounter 0^0, take it to be 1 (say by convention).

16.23 The 2014 retake exam (March 12, 2014)

Exercise 16.63. Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ with the property that

$$f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \;\; f(x) = x^2$$

is continuous.

a. Describe a basis of $(\mathbb{R}, \mathscr{T})$ and show that any $U \in \mathscr{T}$ has the property that

$$-x \in U \quad \forall \; x \in U.$$

b. Find the closure and the interior of $(-1,2)$ in $(\mathbb{R}, \mathscr{T})$.
c. Is $(\mathbb{R}, \mathscr{T})$ Hausdorff? Can you find a sequence with two (distinct) limits?
d. Is $[-1,1]$ (with the topology induced from $\mathscr{T}$) compact? But connected?
e. Is $[-1,1)$ (with the topology induced from $\mathscr{T}$) compact?
f. Is $[-3,1) \cup (1,3]$ (with the topology induced from $\mathscr{T}$) connected?
g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R}, \mathscr{T})$?

Exercise 16.64. Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a continuous function between two *Hausdorff* topological spaces. Define $f^* \mathscr{T}_Y$ as the smallest topology on X with the property that

$$f : (X, f^* \mathscr{T}_Y) \to (Y, \mathscr{T}_Y)$$

is continuous, and define $f_* \mathscr{T}_X$ as the largest topology on Y with the property that

$$f : (X, \mathscr{T}_X) \to (Y, f_* \mathscr{T}_X)$$

is continuous. Show that

a. $f^* \mathscr{T}_Y = \left\{ f^{-1}(V) : V \in \mathscr{T}_Y \right\}$ and $f_* \mathscr{T}_X = \left\{ V \subset Y : f^{-1}(V) \in \mathscr{T}_X \right\}$.
b. If f is a homeomorphism then $f_* \mathscr{T}_X = \mathscr{T}_Y$ and $f^* \mathscr{T}_Y = \mathscr{T}_X$.
c. If $f_* \mathscr{T}_X = \mathscr{T}_Y$ and $(Y, \mathscr{T}_Y)$ is connected, then f is surjective.
d. If $f^* \mathscr{T}_Y = \mathscr{T}_X$ then f is injective.
e. $f^* \mathscr{T}_Y = \mathscr{T}_X$ holds if and only if f is an embedding.

Exercise 16.65. Assume that X is a sphere S^2 minus n points and Y is a sphere minus m points, where $m, n \geq 0$ are integers. Show that if X is homeomorphic to Y, then $n = m$.

Exercise 16.66. Assume that $(X, \mathscr{T}_X)$ is a compact Hausdorff space and A is a nonempty closed subset of X. We consider the complement of A in X,

$$Y := X - A = \{x \in X : x \notin A\},$$

we denote by X/A the space obtained from X by collapsing A to a point and we consider the canonical projection:

$$\pi : X \to X/A$$

(recall that X/A is endowed with the topology $\pi_* \mathscr{T}_X$). Show that:

a. Y is locally compact and Hausdorff.

b. For any open U in X with the property that $U \cap A = \emptyset$ or $A \subset U$, one has that $\pi(U)$ is open in X/A.
c. X/A is a compact Hausdorff space.
d. The one-point compactification of Y is homeomorphic to X/A.

Exercise 16.67. Let $\mathscr{C}(D^2)$ be the algebra of real-valued continuous functions on the unit disk D^2 and let $\mathscr{A}$ be the subset consisting of those $f \in \mathscr{C}(D^2)$ with the property that they are constant on the boundary circle S^1.

a. is $\mathscr{A}$ a sub-algebra of $\mathscr{C}(D^2)$?
b. is $\mathscr{A}$ dense in $\mathscr{C}(D^2)$?
c. show that the spectrum of $\mathscr{A}$ is homeomorphic to S^2.

Notes:

1. You may give your answers in Dutch or English.
2. All the questions a., b. etc are worth 0.5 points. Exercise 3 is worth 1.5 points (note also that the sum of all the points is 11 ...).
3. As before, you are allowed to use during the exam the three sheets of A4 paper (= six pages) containing definitions, theorems, etc from the course- that you prepared at home.
4. PLEASE MOTIVATE ALL YOUR ANSWERS!!!!!!!!!! (give details, explain your reasoning, use pictures whenever appropriate, etc etc).

16.24 The 2013 exam A (April 17, 2013)

Exercise 16.68. (*1 pt*) Let X and Y be two topological spaces. For $A \subset X$, $B \subset Y$, we consider $A \times B$ as a subset of $X \times Y$. Show that:

$$\mathrm{Int}(A \times B) = \mathrm{Int}(A) \times \mathrm{Int}(B)$$

(the interior of $A \times B$ inside $X \times Y$ (with respect to the product topology)= the product of the interior of A in X with the interior of B in Y).

Exercise 16.69. (*1 pt*) Give an example of a connected, bounded, open subset of $\mathbb{R}^2$ which cannot be written as a finite union of balls (here we use the Euclidean metric and topology on $\mathbb{R}^2$).

Exercise 16.70. Let $X = (-1, \infty)$.

(a) (*1 pt*) Find all the numbers $a, b \in \mathbb{R}$ with the property that

$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group $(\mathbb{Z}, +)$ on X.

(b) (*1 pt*) For the a and b that you found, show that the resulting quotient space $X/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 16.71. On $X = \mathbb{Z}$ we consider the family $\mathscr{B}$ of subsets of X consisting of the empty set and the subsets of type

$$N_{a,b} := a + b\mathbb{Z} = \{a + bn : n \in \mathbb{Z}\} \subset \mathbb{Z},$$

with $a, b \in \mathbb{Z}$, $b > 0$.

(a) (*0.25 pts*) Show that, for $a, a', b, b_1, b_2 \in \mathbb{Z}$ with $b, b_1, b_2 > 0$:

$$N_{a,1} = N_{a,-1} = \mathbb{Z},$$

$$N_{a,b_1 b_2} \subset N_{a,b_1} \cap N_{a,b_2}$$

and one has the following equivalences:

$$a' \in N_{a,b} \Longleftrightarrow a \in N_{a',b} \Longleftrightarrow N_{a,b} = N_{a',b}.$$

(b) (*0.5 pts*) Show that $\mathscr{B}$ is not a topology on X.

(c) (*0.5 pts*) Show that $\mathscr{B}$ is a topology basis on X. Let $\mathscr{T}$ be the induced topology.

(d) (*1 pt*) Compute the interior and the closure of $A := \{-1, 1\}$ in $(X, \mathscr{T})$.

(e) (*0.5 pts*) Show that $(X, \mathscr{T})$ is Hausdorff.

(f) (*0.25 pts*) Show that, for any $b \in \mathbb{Z}$, $b > 0$, $\mathbb{Z}$ can be written as a union of b nonempty subsets that belong to $\mathscr{B}$, each two of them being disjoint.

(g) (*0.5 pts*) Show that any subset of type $N_{a,b}$ is both open and closed in $(X, \mathscr{T})$.

(h) (*1 pt*) Show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p-\text{prime number}} N_{0,p}$$

and then, using (g) and (e), deduce that the set of prime numbers is infinite.

Exercise 16.72. Consider the 3-sphere S^3 viewed as a subspace of $\mathbb{C}^2$:

$$S^3 = \left\{(u,v) : u,v \in \mathbb{C}, |u|^2 + |v|^2 = 1\right\}.$$

Inside the sphere we consider

$$A := \left\{(u,v) \in S^3 : |v| = \frac{\sqrt{2}}{2}\right\}.$$

(a) (*1 pt*) Show that $S^3 \setminus A$ has two connected components.
(b) (*0.5 pts*) Show that the two connected components, denoted X_1 and X_2, satisfy:

$$\overline{X}_1 \cap \overline{X}_2 = \partial(X_1) = \partial(X_2) = A.$$

(where the closures and boundaries are inside the space S^3).
(c) (*1 pt*) Consider the unit circle and the closed unit disk

$$S^1 = \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\right\}, \quad D^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

By a solid torus we mean any space homeomorphic to $S^1 \times D^2$. Show that

$$f : S^1 \times D^2 \to \mathbb{R}^3, \; f((\alpha,\beta),(x,y)) = ((2-x)\alpha, (2-x)\beta, y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name "solid torus").
(d) (*1 pt*) Show that $\overline{X}_i$ is a solid torus for $i \in \{1,2\}$.
(e) (*1 pt*) Deduce that the 3-sphere can be obtained from two disjoint copies of $S^1 \times D^2$ (i.e. two solid tori) by gluing any point $(z_1,z_2) \in S^1 \times S^1$ in the boundary of the first copy with the point (z_2,z_1) in the boundary of the second.

Note 1: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 13 points in the game!).

Note 2: please MOTIVATE ALL YOUR ANSWERS (e.g., in Exercise 2, do not just give the example, but also explain/prove why it has the required properties).

16.25 The 2013 exam B (June 26, 2013)

Exercise 16.73. Prove that there is no continuous injective map $f : S^2 \to S^1$. (*1p*)
(warning: there are injective maps from S^2 to S^1!).

Exercise 16.74. Let $X = (0,\infty)$ and consider the open cover of X

$$\mathscr{U} = \{U_1, U_2, U_3, \ldots\} \quad \text{with } U_n = (0,n).$$

(a) Show that $\mathscr{U}$ does not admit a subcover which is locally finite. (*0.5p*)
(b) Describe a locally finite refinement of $\mathscr{U}$. (*0.5p*)

Exercise 16.75. Consider

$$X = \left\{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 = 1\right\} \setminus \{(0,0,1),(0,0,-1)\},$$

$$Y = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 2\sqrt{x^2+y^2}\right\} \setminus \{(0,0,0)\}.$$

$$\pi : X \to \mathbb{R}^3,\ \pi(u,v,w) = (2u\sqrt{u^2+v^2}, 2v\sqrt{u^2+v^2}, 2w\sqrt{u^2+v^2}).$$

(a) Compute the closure $\overline{Y}$ of Y in $\mathbb{R}^3$ and prove that it is compact. (*1p*)
(b) Prove that the one-point compactification of X is homeomorphic to $\overline{Y}$. (*1p*)
(c) Draw a picture of Y and explain the map π on the picture. (*1p*)

Exercise 16.76. Let X be a Hausdorff, locally compact, 2nd countable topological space and assume that U is an open in $X \times \mathbb{R}$ containing $X \times \{0\}$:

$$X \times \{0\} \subset U \subset X \times \mathbb{R}.$$

The aim of this exercise is to prove that there exists a continuous function $f : X \to (0,\infty)$ such that U contains

$$U_f := \{(x,t) \in X \times \mathbb{R} : |t| < f(x)\}.$$

Consider

$$r : X \to \mathbb{R},\ \ r(x) = \sup\{r \in (0,1] : \{x\} \times (-r,r) \subset U\}.$$

(a) Show that one can find an open cover $\{V_i : i \in I\}$ of X and a family $\{r_i : i \in I\}$ of strictly positive real numbers (for some indexing set I) such that

$$r(y) > r_i \quad \forall\, y \in V_i,\ \forall\, i \in I. \quad (0.5p)$$

(note: depending on the argument that you find, I that you construct may be countable, but it may also be "very large"- e.g. "as large as X").
(b) For $\{V_i : i \in I\}$, $\{r_i : i \in I\}$ as above, use a partition of unity argument to build a continuous function $f : X \to (0,\infty)$ such that $U_f \subset U$. (*1.5p*)
(c) Deduce that if X is actually compact, then f may be chosen to be constant. (*0.5p*)

Exercise 16.77. In this exercise we work over $\mathbb{R}$. Let X be a compact, Hausdorff topological space, $C(X)$ the space of real-valued continuous functions on X and let

$$\mathscr{A} \subset C(X).$$

be a point-separating subalgebra. The aim of this exercise is to understand whether the spectrum $X_{\mathscr{A}}$ is homeomorphic to X. The homeomorphism will be, in principle, provided by the map:

$$F : X \to X_{\mathscr{A}},\ F(x) = \chi_x|_{\mathscr{A}}$$

(the restriction of $\chi_x : C(X) \to \mathbb{R}$ to $\mathscr{A}$, where we recall that χ_x sends f to $f(x)$).

(a) Show that F is continuous. (*0.5p*)
(b) Show that F is injective. (*0.5p*)
(c) Give an example of X and A for which F is not surjective. (*0.5p*)

The next five steps are to prove that the image of F consists of those characters $\chi \in X_{\mathscr{A}}$ with the property that χ is continuous, where $\mathscr{A}$ is endowed with the topology induced by the sup-metric. Let χ be such a character.

(d) Show that, if $f, g \in \mathscr{A}$ and $f \geq g$, then $\chi(f) \geq \chi(g)$. (*0.5p*)
(Hint: recall that, in the proof of the Stone-Weierstrass theorem we showed that, for $f \in \mathscr{A}$ with $f \geq 0$, one has $\sqrt{f} \in \overline{\mathscr{A}}$).
(e) Show that for all $f \in \mathscr{A}$ one has $|\chi(f)| \leq ||f||_{\sup}$. (*0.5p*)
(f) Deduce that for any sequence $(f_n)_{n\geq 1}$ of elements in $\mathscr{A}$, convergent to some $f \in C(X)$, the sequence $(\chi(f_n))_{n\geq 1}$ is convergent. (*0.5p*)
(g) Deduce that there exists an extension of $\chi : \mathscr{A} \to \mathbb{R}$ to a continuous map

$$\tilde{\chi} : C(X) \to \mathbb{R}. \quad (0.5p)$$

(h) And then show that $\tilde{\chi}$ is a character. (*0.5p*)

Finally:

(i) Conclude that F is a homeomorphism between X and the space of continuous characters, endowed with the topology induced from X_A. (*0.5p*)

Notes:

- English or Dutch (or both)- doesn't matter, but please write clearly (thanks!).
- Please motivate your answers. For instance, in exercise 4, part (b), do not forget, after you write the function f, to check that f is well defined, continuous and that it has the desired property.
- In this exam, the sub-points of any exercise do not fully depend on each other. For instance, in the last exercise, you may do (e) without doing (d) (... but using it).

16.26 The 2013 retake exam (August, 2013)

Exercise 16.78. On $X=[0,\infty)$ we consider the family $\mathscr{B}$ of subsets consisting of the empty set and all intervals of type $[r,\infty)$ with $r\in\mathbb{R}_+$ and we denote by $\mathscr{T}$ the smallest topology on X containing $\mathscr{B}$.

(a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. (*0.5p*)
(b) Is $(X,\mathscr{T})$ Hausdorff? Is it compact? Is it connected? (*0.75p*)
(c) Is the sequence $(x_n)_{n\geq 1}$ given by

$$x_n=1+\frac{(-1)^n}{n}$$

convergent in $(X,\mathscr{T})$? To what? (*0.5p*)
(d) Compute the interior and the closure of $[1,2)\times(1,2]$ in $X\times X$ endowed with the product topology. (*0.5p*).

Exercise 16.79. Decide (and explain) which of the following statements hold true:

(a) $S^1\times\mathbb{R}$ can be embedded in $\mathbb{R}^2$. (*0.5p*)
(b) $S^1\times S^1\times S^1$ can be embedded in $\mathbb{R}^4$. (*0.5p*)
(c) S^2 can be embedded in S^1. (*0.5p*)
(d) the Moebius band can be embedded into the projective space P^2. (*0.5p*)
(e) the projective space P^3 can be embedded in $\mathbb{R}^6$. (*0.5p*)

Exercise 16.80. By a solid torus we mean any space homeomorphic to $S^1\times D^2$.

(a) Show that

$$f:S^1\times D^2\to\mathbb{R}^3,\ f((\alpha,\beta),(x,y))=((2-x)\alpha,(2-x)\beta,y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name solid torus). (*1p*)

Next, consider the 3-sphere S^3 viewed as a subspace $\mathbb{C}^2$:

$$S^3=\left\{(u,v):u,v\in\mathbb{C},|u|^2+|v|^2=1\right\}.$$

Inside the sphere we consider

$$A:=\left\{(u,v)\in S^3:|v|=\frac{\sqrt{2}}{2}\right\}.$$

(b) Show that $S^3\setminus A$ has two connected components; denote them by X_1 and X_2. (*0.5p*)
(c) Show that $\overline{X}_i$ is a solid torus for $i\in\{1,2\}$. (*0.5p*)
(d) Show that the 3-sphere S^3 can be obtained from two disjoint copies of $S^1\times D^2$ (i.e. two solid tori) by gluing any point $(z_1,z_2)\in S^1\times S^1$ in the boundary of the first copy with the point (z_2,z_1) in the boundary of the second.(*1p*)

Exercise 16.81. Give an example of two connected, Hausdorff, locally compact spaces X and Y which are not homeomorphic but have the property that their one-point compactifications are homeomorphic. (*1p*)

Exercise 16.82. If A and B are two commutative algebras over $\mathbb{R}$, a map

$$F:A\to B$$

is called an algebra homomorphism if it is $\mathbb{R}$-linear and satisfies

$$F(a_1a_2) = F(a_1)F(a_2)$$

for all $a_1, a_2 \in A$, and $F(1_A) = 1_B$, where $1_A \in A$ and $1_B \in B$ are the units. It is called an isomorphism of algebras if it is also bijective.

1. Show that for any algebra homomorphism $F : A \to B$, one has an induced continuous map

$$F^* : X_B \to X_A,\ F^*(\chi) = \chi \circ F.$$

   Moreover, if F is an isomorphism, show that F^* is a homeomorphism. (*0.5p*)
2. If X and Y are two compact Hausdorff spaces, show that for any algebra homomorphism

$$F : C(Y) \to C(X)$$

   one finds a continuous function $f : X \to Y$ such that

$$F(\phi) = \phi \circ f \quad \forall\ \phi \in C(Y). \quad (1p)$$

3. Compute the spectrum of

$$A := \left\{ f : \mathbb{R}^2 \to \mathbb{R} \text{ continuous} : f(x+1, y) = f(x, y+1) = f(x, y),\ \forall\ x, y \in \mathbb{R} \right\}. \quad (0.5p)$$

Note: As usual, please motivate/explain/prove all your statements!!!

16.27 The 2012 exam A (April 18, 2012)

Exercise 16.83. Let $\mathscr{B}$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets

$$[n,a) := \{r \in \mathbb{R} : n \leq r < a\} \quad \text{with } n \in \mathbb{Z}, a \in \mathbb{R}.$$

1. Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $\mathscr{T}$ the associated topology. (*1p*)
2. Is $(\mathbb{R}, \mathscr{T})$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)? (*1p*)
3. compute the closure, the interior and the boundary of $A = [-\frac{1}{2}, \frac{1}{2}]$ in $(\mathbb{R}, \mathscr{T})$. (*1.5p*)

Exercise 16.84. Prove directly that the abstract torus T_{abs} is homeomorphic to $S^1 \times S^1$. More precisely, define an explicit map

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^4$$

whose image is

$$S^1 \times S^1 = \left\{(x,y,z,t) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + t^2 = 1\right\}$$

and which induces a homeomorphism $f : T_{\text{abs}} \to \mathbb{S}^1 \times S^1$. Provide all the arguments. (*1.5p*).

Exercise 16.85. Let X be the space obtained from the sphere S^2 by gluing the north and the south pole (with the quotient topology). Show that X can be obtained from a square $[0,1] \times [0,1]$ by glueing some of the points on the *boundary* (note: you are not allowed to glue a point in the *interior* of the square to any other point). More precisely:

1. Describe the equivalence relation R_0 on S^2 encoding the glueing that defines X. (*0.25p*)
2. Make a picture of X in $\mathbb{R}^3$. (*0.25p*)
3. Describe an equivalence relation R on $[0,1] \times [0,1]$ encoding a glueing with the required properties. (*1p*)
4. Show that, indeed, X is homeomorphic to $[0,1] \times [0,1]/R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture). (*0.5p*)

Exercise 16.86. Show that:

1. There exist continuous surjective maps $f : S^1 \to S^1$ which are not injective. (*0.5p*)
2. Any continuous injective map $f : S^1 \to S^1$ is surjective. (*1p*)

Exercise 16.87. Show that any continuous map

$$f : (\mathbb{R}, \mathscr{T}_{\text{Eucl}}) \to (\mathbb{R}, \mathscr{T}_l)$$

must be constant (recall that $\mathscr{T}_l$ is the lower limit topology- i.e. the one generated by intervals of type $[a,b)$). (*1.5p*)

Note 1: Motivate all your answers. Whenever you use a Theorem or Proposition, please make that clear (e.g. by stating it). Please write clearly (English or Dutch).

Note 2: The final mark is $\min\{10, 1+p\}$ where p is the number of points you collect from the exercises.

16.28 The 2012 exam B (June 27, 2012)

Exercise 16.88. (*1p*) Show that

$$K := \left\{ (x,y) \in \mathbb{R}^2 : x^{2012} + y^{2012} \leq 10 \sin(e^x + e^y + 1000) + e^{\cos(x^2+y^2)} \right\}.$$

is compact.

Exercise 16.89. (*1.5p*) Let X be a bouquet of two circles:

$$X = \left\{ (x,y) \in \mathbb{R}^2 : ((x-1)^2 + y^2 - 1)((x+1)^2 + y^2 - 1) = 0 \right\}.$$

We say that a space Y is an *exam space* if there exist three distinct points $p,q,r \in X$ such that Y is homeomorphic to the one-point compactification of $X - \{p,q,r\}$.

Find the largest number l with the property that there exist exam spaces $Y_1, \ldots, Y_l$ with the property that any two of them are not homeomorphic (prove all the statements that you make!).

Exercise 16.90. (*1p*) Let X be a topological space and let $\gamma : [0,1] \to X$ be a continuous function. Assume that γ is locally injective, i.e. that, for any $t \in [0,1]$, there exists a neighborhood V of t in $[0,1]$ such that

$$\gamma|_V : V \to X$$

is injective. Show that, for any $x \in X$, the set

$$\gamma^{-1}(x) := \{ t \in [0,1] : \gamma(t) = x \}$$

is finite.

Exercise 16.91. (*1p*) Let X be a Hausdorff normal space and let $A \subset X$ be a subspace with the property that any two continuous functions $f,g : X \to \mathbb{R}$ which coincide on A must coincide everywhere on X. Show that A is dense in X (i.e. the closure of A in X coincides with X).

Exercise 16.92. (*1p*) Consider the following open cover of $\mathbb{R}$:

$$\mathscr{U} := \left\{ (r,s) : r,s \in \mathbb{R}, |r-s| < \frac{1}{3} \right\}.$$

Describe a locally finite subcover of $\mathscr{U}$.

Exercise 16.93. (each of the sub-questions is worth *0.5 p*) Let A be a commutative algebra over $\mathbb{R}$. Assume that it is finitely generated, i.e. there exist $a_1, \ldots, a_n \in A$ (called generators) such that any $a \in A$ can be written as

$$a = P(a_1, \ldots, a_n),$$

for some polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$. Recall that X_A denotes the topological spectrum of A; consider the functions

$$f_i : X_A \to \mathbb{R}, \;\; f_i(\chi) = \chi(a_i) \quad 1 \le i \le n,$$
$$f = (f_1, \ldots, f_n) : X_A \to \mathbb{R}^n.$$

Show that

(a) f is continuous.
(b) For any character $\chi \in X_A$ and any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$,

$$\chi(P(a_1, \ldots, a_n)) = P(\chi(a_1), \ldots, \chi(a_n)).$$

(c) f is injective.
(d) the topology of X_A is the smallest topology on X_A with the property that all the functions f_i are continuous.
(e) f is an embedding.

Next, for a subspace $K \subset \mathbb{R}^n$, we denote by $\mathrm{Pol}(K)$ the algebra of real-valued polynomial functions on K and let $a_1, \ldots, a_n \in \mathrm{Pol}(K)$ be given by

$$a_i : K \to \mathbb{R}, \;\; a_i(x_1, \ldots, x_n) = x_i.$$

Show that

(f) $\mathrm{Pol}(K)$ is finitely generated with generators $a_1, \ldots, a_n$.
(g) Show that the image of f (from the previous part) contains K.

Finally:

(h) For the $(n-1)$ sphere $K = S^{n-1} \subset \mathbb{R}^n$, deduce that f induces a homeomorphism between the spectrum of the algebra $\mathrm{Pol}(K)$ and K.
(i) For which subspaces $K \subset \mathbb{R}^n$ can one use a similar argument to deduce that the spectrum of $\mathrm{Pol}(K)$ is homeomorphic to K?

Note: Motivate all your answers; give all details; please write clearly (English or Dutch). The mark is given by the formula:

$$\min\{10, 1 + p\},$$

where p is the number of points you collect from the exercises.

16.29 The 2012 retake exam (August 22, 2012)

Exercise 16.94. On $X = \mathbb{R}$ consider the topology:

$$\mathscr{T} = \{(-a,a) : 0 \leq a \leq \infty\}.$$

(a) Is $(X,\mathscr{T})$ metrizable? (*0.5 p*)
(b) Is $(X,\mathscr{T})$ 1st countable? (*0.5 p*)
(c) Is $(X,\mathscr{T})$ connected? (*0.5 p*)
(d) Is the sequence $x_n = (-1)^n + \frac{1}{n}$ convergent in $(X,\mathscr{T})$? To what? (*0.5 p*)
(e) Find the interior and the closure of $A = (-1,2)$ in $(X,\mathscr{T})$. (*0.5 p*)
(f) Show that any continuous function $f : X \to \mathbb{R}$ is constant. (*0.5 p*)

Exercise 16.95. Let M be the Moebius band. For a continuous map $f : S^1 \to M$ we denote

$$X_f := M - f(S^1).$$

(a) Show that for any f, X_f is open in M, it is locally compact but, if nonempty, it is not compact. (*0.5 p*)
(b) Is there a function f such that X_f^+ is homeomorphic to the projective space $\mathbb{P}^2$? (*0.5 p*)

Exercise 16.96.

(a) Let A be a commutative algebra over $\mathbb{R}$ and assume that $a_0, a_1, \ldots, a_n \in A$ generate A, i.e. that any $a \in A$ can be written as

$$a = P(a_0, \ldots, a_n),$$

for some $P \in \mathbb{R}[X_0, \ldots, X_n]$. Let X_A be the topological spectrum of A. Show that

$$f : X_A \to \mathbb{R}^{n+1}, \;\; f(\chi) = (\chi(a_0), \ldots, \chi(a_n))$$

is an embedding. (*1.5 p*)
(b) If $A = \mathrm{Pol}(K)$ is the algebra of real-valued polynomial functions on a subset $K \subset \mathbb{R}^{n+1}$ and a_i are the polynomial functions

$$a_i : K \to \mathbb{R}, \;\; a_i(x_0, \ldots, x_n) = x_i, \quad (0 \leq i \leq n),$$

show that the image of the resulting f contains K. (*0.5 p*)
(c) For the sphere $S^n \subset \mathbb{R}^{n+1}$, deduce that the spectrum of the algebra $\mathrm{Pol}(S^n)$ is homeomorphic to S^n. (*1 p*)

Exercise 16.97. (*1 p*) Prove that there is no continuous map $g : \mathbb{C} \to \mathbb{C}$ satisfying $g(z)^2 = z$ for all $z \in \mathbb{C}$.

Exercise 16.98. (*1 p*) On $(0,\infty)$ we define the action of the group $(\mathbb{Z},+)$ by:

$$\mathbb{Z} \times (0,\infty) \to (0,\infty), \;\; (n,r) \mapsto 2^n r.$$

Show that the quotient $(0,\infty)/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 16.99. (*1 p*) Let X be a Hausdorff normal space. Show that, for $A, B \subset X$, A and B have the same closure in X if and only if, for any continuous function $f : X \to \mathbb{R}$, one has the equivalence

$$f|_A = 0 \iff f|_B = 0.$$

16.30 The 2011 exam A (April 20, 2011)

Notes:

- justify all your answers!!!
- the marking starts from 1 point. By solving the exercises, you can earn 9.5 more points. Your mark for this exam will be the minimum between your total number of points and 10.

Exercise 16.100. On $X = \mathbb{R}$ consider the family of subsets:

$$\mathscr{B} := \{(-p,p) : p \in \mathbb{Q}, p > 0\}, \ \mathscr{T} = \{(-a,a) : 0 \leq a \leq \infty\}.$$

- Show that $\mathscr{B}$ is a topology basis (*0.5 pt*).
- Show that $\mathscr{T}$ is the topology associated to $\mathscr{B}$. (*0.5 pt*).
- Is the sequence $x_n = (-1)^n + \frac{1}{n}$ convergent in $(X, \mathscr{T})$? To what? (*0.5 pt*).
- Find the interior and the closure of $A = (-1,2)$ in $(X, \mathscr{T})$ (*0.5 pt*).
- Show that any continuous function $f : X \to \mathbb{R}$ is constant (*0.5 pt*).
- For the topological space $(X, \mathscr{T})$, decide whether it is:

  1. Hausdorff (*0.5 pt*).
  2. 1st countable (*0.5 pt*).
  3. Metrizable (*0.5 pt*).
  4. Connected (*0.5 pt*).

Exercise 16.101. Which of the following spaces are homeomorphic and which are not:

- $(1,\infty)$ and $(0,\infty)$ (*0.5 pt*).
- $\mathbb{R}^2 - D^2$ and $\mathbb{R}^2 - \{0\}$ (*0.5 pt*).
- $(0,1)$ and $[0,1)$ (*0.5 pt*).
- $S^1 \times (\mathbb{R}^2 - \{0\})$ and $T^2 \times \mathbb{R}^*$ (*0.5 pt*).
- $S^1 \times (\mathbb{R}^2 - \{0\})$ and $T^2 \times \mathbb{R}^*_+$ (*0.5 pt*).

(here D^2 denotes the closed unit disk, T^2 denotes the torus, $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^*_+ = (0,\infty)$).

Exercise 16.102. Consider the map

$$\pi \colon \mathbb{R}^2 \to S^1 \times \mathbb{R}, \qquad (x,y) \mapsto \Big(\big(\cos(x+y), \sin(x+y)\big), x-y\Big) \in S^1 \times \mathbb{R}.$$

- Describe an equivalence relation R on $\mathbb{R}^2$ such that $(S^1 \times \mathbb{R}, \pi)$ is a quotient of $\mathbb{R}^2$ modulo R (*0.5 pt*).
- Find a group Γ and an action of Γ on $\mathbb{R}^2$ such that R is the equivalence relation induced by this action (*0.5 pt*).
- Show that, indeed, $\mathbb{R}^2/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}$ (*0.5 pt*).

Exercise 16.103. Show that, if a topological space X is Hausdorff, then the cone $\mathrm{Cone}(X)$ of X is Hausdorff (*0.5 pt*).

Exercise 16.104. Show that any continuous function $f : [0,1] \to [0,1]$ admits a fixed-point, i.e. there exists $t_0 \in [0,1]$ such that $f(t_0) = t_0$. (*0.5 pt*).

16.31 The 2011 exam B (June 29, 2011)

Exercise 16.105. Consider

$$X_1 = \left\{(x,y,z) \in \mathbb{R}^3 : (z=0) \text{ or } (x=y=0, z \geq 0)\right\},$$

$$X_2 = \left\{(x,y,z) \in \mathbb{R}^3 : (z=0) \text{ or } (x=0, y^2+z^2=1, z \geq 0)\right\}.$$

$$X_3 = \left\{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2+z^2=1) \text{ or } (y=0, z=0, \frac{1}{2} < |x| < 1)\right\},$$

(a) Show that X_1, X_2, X_3 are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures). (*0.5 p*)
(b) Show that the one-point compactifications of X_1, X_2 and X_3 are homeomorphic to each other. (*1 p*)

Exercise 16.106. Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathscr{R}_p$ the set of remainders modulo p. In other words,

$$\mathscr{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) : \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

We also denote by $\pi_p : \mathbb{R}[X_0, X_1, \ldots, X_n] \to \mathscr{R}_p$ the resulting quotient map. Show that:

(a) There is a unique algebra structure on $\mathscr{R}_p$ (i.e. unique operations $+$, $\cdot$ and multiplications by scalars, defined on $\mathscr{R}_p$) with the property that π_p is a morphism of algebras, i.e.

$$\pi_p(q_1+q_2) = \pi_p(q_1) + \pi_p(q_2),\ \pi_p(q_1 \cdot q_2) = \pi_p(q_1) \cdot \pi_p(q_2),\ \lambda\, \pi_p(q) = \pi_p(\lambda q)$$

for all $q_1, q_2 \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, $\lambda \in \mathbb{R}$. (*0.5 p*)
(b) For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathscr{R}_p$ has only one point. (*1 p*)
(c) For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathscr{R}_p$ is homeomorphic to S^n (*1 p*) .
(d) What is the spectrum for $p = x_0 x_1 \ldots x_n$? (*0.5 p*)

Exercise 16.107. Let X be the space of continuous maps $f : [0,1] \to [0,1]$ with the property that $f(0) = f(1)$. We endow it with the sup-metric $d_{\sup}$ and the induced topology (recall that $d_{\sup}(f,g) = sup\{|f(t) - g(t)| : t \in [0,1]\}$). Prove that:

(a) X is bounded and complete. (*1 p*)
(b) X is not compact. (*0.5 p*)

Exercise 16.108. Show that:

(a) The product of two sequentially compact spaces is sequentially compact. (*1 p*)
(b) Deduce that the product of two compact metric spaces is a compact space. (*1 p*)

Exercise 16.109. Show that the family of open intervals

$$\mathscr{U} := \{(q, q+1) : q \in \mathbb{R}\}$$

forms an open cover of $\mathbb{R}$ which admits no finite sub-cover, but which admits a locally finite sub-cover. (*1.5 p*)

Exercise 16.110. Prove that there is no continuous injective map $f : S^1 \vee S^1 \to S^1$, where $S^1 \vee S^1$ is a bouquet of two circles (two copies of S^1, tangent to each other). (*1.5 p*)

Note: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 11 points in the game!).

16.32 The 2011 retake exam (August 24, 2011)

Exercise 16.111. Show that the equation

$$x^5 + 7x^2 - 30x + 1 = 0$$

has at least two solutions $x_0, x_1 \in (0,2)$. (*1 p*)

Exercise 16.112. Consider the space $\mathscr{C}([0,1])$ of all continuous maps $f : [0,1] \to \mathbb{R}$, endowed with the sup-metric. Show that

$$A := \left\{ f \in \mathscr{C}([0,1]) : x^2 \leq e^{f(x)} + \sin(f(x)) \leq x \ \ \forall\, x \in [0,1] \right\}$$

is a closed subset of $\mathscr{C}([0,1])$, and that it is not bounded. (*1 p*)

Exercise 16.113. Describe a subspace $X \subset \mathbb{R}^2$ which is connected, whose closure (in $\mathbb{R}^2$) is compact, but with the property that X is not locally compact. (*1 p*)

Exercise 16.114. Let $G = (0,\infty)$ be the group of strictly positive reals, endowed with the usual product. Find an action of G on $\mathbb{R}^4 \setminus \{0\}$ with the property that $(\mathbb{R}^4 \setminus \{0\})/G$ is homeomorphic to S^3. (*1 p*)

Exercise 16.115. Let $X = \mathbb{R}^2$ endowed with the product topology $\mathscr{T}_l \times \mathscr{T}_l$, where $\mathscr{T}_l$ is the lower limit topology on $\mathbb{R}$.

a. Describe a topology basis for the topological space X. (*0.5 p*)
b. Find a sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^2$ which converges to $(0,0)$ with respect to the Euclidean topology, but which has no convergent subsequence in the topological space X. (*0.5 p*)
c. Compute the interior, the closure and the boundary (in X) of

$$A = [0,1) \times (0,1]. \quad (1p)$$

(please use pictures!).

Exercise 16.116. Decide (and explain) which of the following statements hold true:

a. $S^1 \times S^1 \times S^1$ can be embedded in $\mathbb{R}^4$. (*0.5 p*)
b. S^1 can be embedded in $(0,\infty)$. (*0.5 p*)
c. the cylinder $S^1 \times [0,1]$ can be embedded in the Klein bottle. (*0.5 p*)
d. The Moebius band can be embedded into the projective space $\mathbb{P}^2$. (*0.5 p*)
e. the projective space $\mathbb{P}^3$ can be embedded in $\mathbb{R}^6$. (*0.5 p*)

Exercise 16.117. Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathscr{R}_p$ the set of remainders modulo p. In other words,

$$\mathscr{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) : \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

For $q \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denoted by $[q] \in \mathscr{R}_p$ the induced equivalence class. Show that:

a. The operations (on $\mathscr{R}_p$) $+$, $\cdot$ and multiplications by scalars given by

$$[q_1] + [q_2] := [q_1 + q_2],\ [q_1] \cdot [q_2] := [q_1 \cdot q_2],\ \lambda[q] := [\lambda q]$$

are well-defined and make $\mathscr{R}_p$ into an algebra. (*0.5 p*)

b. For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathscr{R}_p$ has only one point. (*0.5 p*)

c. For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathscr{R}_p$ is homeomorphic to S^n (*1 p*) .

16.33 The 2010 exam (April 21, 2010)

Important: motivate/prove your answers to the questions from the exercises. When making pictures, try to make them as clear as possible. When using a result from the lecture notes, please give a clear reference.

Exercise 16.118. Let X be the (interior of an) open triangle, as drawn in the picture (the edges are not part of X!), viewed as a topological space with the topology induced from $\mathbb{R}^2$. Let $A \subset X$ be the open disk drawn in the picture (tangent to the edges of the closed triangle). Compute the closure and the boundary of A in X. (*1p*)

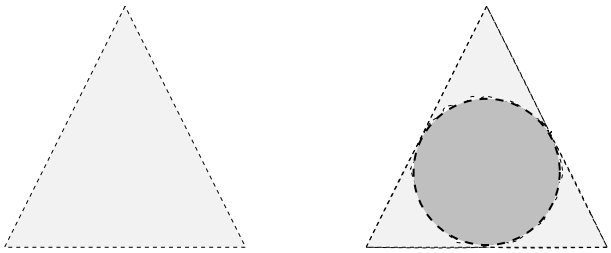


Fig. 16.1

Exercise 16.119. Let X be obtained by taking two disjoint copies of the interval $[0,2]$ (with the Euclidean topology) and gluing each t in the first copy with the corresponding t in the second copy, for all $t \in [0,2]$ different from the middle point. Explicitly, one may take the space

$$Y = [0,2] \times \{0\} \cup [0,2] \times \{1\} \subset \mathbb{R}^2$$

with the topology induced from the Euclidean topology, and X is the space obtained from Y by gluing $(t,0)$ to $(t,1)$ for all $t \in [0,2]$, $t \neq 1$. We endow X with the quotient topology.

(a) Is X Hausdorff? But connected? But compact? (*1.5p*)
(b) Can you find $A, B \subset X$ which, with the topology induced from X, are compact, but such that $A \cap B$ is not compact? (*1p*)
(c) Show that X can also be obtained as a quotient of the circle S^1. (*0.5p*)

Exercise 16.120. Let X, Y and Z be the spaces drawn in Figure 16.2.

(a) Show that any two of them are not homeomorphic. (*1.5p*)
(b) Compute their one-point compactifications X^+, Y^+ and Z^+. (*1p*)
(c) Which two of the spaces X^+, Y^+ and Z^+ are homeomorphic and which are not? (*1p*)

Exercise 16.121. Let M be the Moebius band. For any continuous function $f : S^1 \to M$ we denote by M_f the complement of its image:

$$M_f := M - f(S^1)$$

and we denote by M_f^+ the one-point compactification of M_f.

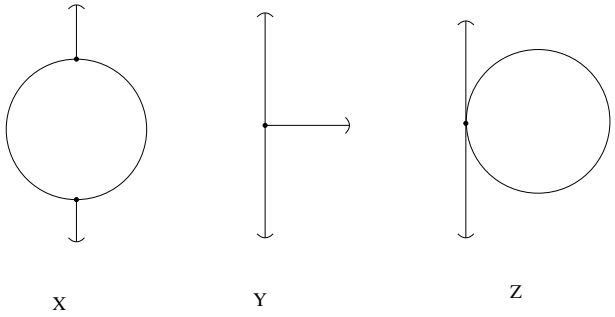


Fig. 16.2

(a) Show that for any f, M_f is open in M, it is locally compact but, if nonempty, it is not compact. (*1p*)
(b) Find an example of f such that M_f^+ is homeomorphic to D^2. Then one for which it is homeomorphic to S^2. And then one for $\mathbb{P}^2$. (*1.5p*)

16.34 The 2010 retake exam (August 27, 2010)

Exercise 16.122. Let $\mathbb{N} = \{0,1,2,\ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathscr{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0,1,\ldots,n\}$ with $n \in \mathbb{N}$.
- $\mathscr{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n,n+1,\ldots\}$ with $n \in \mathbb{N}$.

Questions:

1. Show that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on $\mathbb{N}$. (*1p*)
2. Show that the spaces $(\mathbb{N},\mathscr{T}_1)$ and $(\mathbb{N},\mathscr{T}_2)$ are not homeomorphic. (*0.5p*)
3. for each of the spaces $(\mathbb{N},\mathscr{T}_1)$ and $(\mathbb{N},\mathscr{T}_2)$ decide whether the space is Hausdorff, connected or compact. (*1.5p*)

Exercise 16.123. Let $\mathscr{T}_r$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathscr{B}_r := \{(a,b] : a,b \in \mathbb{R}, a < b\}.$$

Compute the interior, the closure and the boundary of

$$A := \left(\left(0,\frac{1}{3}\right) \cup \left[\frac{1}{2},1\right]\right) \times [0,1)$$

inside the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathscr{T}_r \times \mathscr{T}_r$. (*2p*).

Exercise 16.124. Let X be the connected sum of a Moebius band and a torus. Show how one can obtain X from a disk by gluing some of the points on the boundary of the disk (*2p*). Then describe on the picture a continuous map $f : S^1 \to X$ such that the one-point compactification of $X \setminus f(S^1)$ is homeomorphic to a sphere. (*0.5p*)

Exercise 16.125. Let X be the one-point compactification of the space obtained by removing two points from the torus. Show that:

1. X can be embedded in $\mathbb{R}^3$.(*1p*)
2. X is not homeomorphic to S^2. (*1p*)

16.35 The 2009 exam part A (April, 2009)

Exercise 16.126. (*3.25p*) Let $\mathscr{T}_l$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathscr{B}_l := \{[a,b) : a,b \in \mathbb{R}, a < b\}.$$

(a) With the topology induced from $(\mathbb{R}, \mathscr{T}_l)$, is $[0,1)$ connected? But compact? (*1p*)
(b) The same questions for $(0,1]$. (*1p*)
(c) Are $[0,1)$ and $(0,1]$ (with the induced topologies) homeomorphic? (*0.25p*)
(d) Compute the interior, the closure and the boundary of

$$A = (0,1] \times [0,1)$$

in the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathscr{T}_l \times \mathscr{T}_l$. (*1p*)

Exercise 16.127. (*2.5p*) Describe an embedding of the cylinder $S^1 \times [0,1]$ into the space X, show that its complement is locally compact, and find the one-point compactification of the complement, in each of the cases:

(a) X is a torus. (*1.5p*)
(b) X is the plane $\mathbb{R}^2$. (*1p*)

Exercise 16.128. (*2.5p*)

(a) Write the Moebius band as a union of two subspaces M and C where M is itself a Moebius band, C is a cylinder (i.e. homeomorphic to $S^1 \times [0,1]$) and $M \cap C$ is a circle. (*1p*)
(b) Similarly, decompose $\mathbb{P}^2$ as the union of a Moebius band M and another subspace Q, such that Q is a quotient of the cylinder and $M \cap Q$ is a circle. (*1p*)
(c) Deduce that $\mathbb{P}^2$ can be obtained from a Moebius band and a disk D^2 by gluing them along their boundary circles. (*0.5p*)

Exercise 16.129. (*1.75p*) Let X be a topological space and let $\mathrm{Cone}(X)$ be its cone. Show that:

(a) If X is compact then so is $\mathrm{Cone}(X)$. (*1p*)
(b) If X is path connected then so is $\mathrm{Cone}(X)$. (*0.5p*)
(c) If X is connected then so is $\mathrm{Cone}(X)$. (*0.25p*)

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

Overall hint: At the end of each question, you can see its "point value" (10 points in total). The more difficult questions of each exercise are the ones for which you get less points! So, try not to spend too much time with the more difficult questions before you answered the others. Finally: use pictures!

16.36 The 2009 retake exam (August, 2009)

Exercise 16.130. (*2.5p*) Let X be the (interior of an) open triangle, as drawn in the picture (the edges are not part of X!), viewed as a topological space with the topology induced from $\mathbb{R}^2$. Let $A \subset X$ be the open disk drawn in the picture (tangent to the edges of the closed triangle).

(a) Compute the closure and the boundary of A in X. (*1p*)
(b) Show that both of them are locally compact but not compact. (*0.5p*)
(c) For each one of them, compute their one-point compactification. (*1p*)

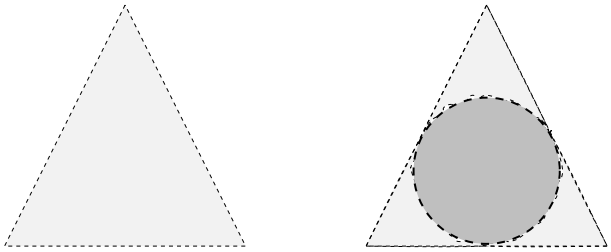


Fig. 16.3

Exercise 16.131. (*3.5p*) In each of the following examples, decide (and prove) whether X can be embedded in Y:

(a) $X = [0,1]$, $Y = [0,1)$. (*0.5p*)
(b) $X = \mathbb{R}$, $Y = S^1$. (*0.5p*)
(c) $X = S^1 \times \mathbb{R}$, $Y = \mathbb{R}^2$. (*0.5p*)
(d) $X = S^1 \times S^1$, $Y = \mathbb{R}^3$. (*0.5p*)
(e) $X =$ Moebius band, $Y =$ Klein bottle. (*0.5p*)
(f) $X =$ a bouquet of two circles, $Y = S^1$. (*1p*)

Exercise 16.132. (*4p*) Given two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on a set X, show that:

(a) $\mathscr{T}_1 \cap \mathscr{T}_2 := \{U \subset X : U \in \mathscr{T}_1, U \in \mathscr{T}_2\}$ is a topology on X. (*0.5p*)
(b) If $(X, \mathscr{T}_1)$ or $(X, \mathscr{T}_2)$ is connected, then so is $(X, \mathscr{T}_1 \cap \mathscr{T}_2)$. (*0.5p*)
(c) If $(X, \mathscr{T}_1)$ or $(X, \mathscr{T}_2)$ is compact, then so is $(X, \mathscr{T}_1 \cap \mathscr{T}_2)$. (*0.5p*)

Next, let $X = \mathbb{R}$, let $\mathscr{T}_l$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathscr{B}_l := \{[a,b) : a,b \in \mathbb{R}, a < b\}$$

and let $\mathscr{T}_u$ be defined similarly, using intervals of type $(a,b]$.

(d) What is $\mathscr{T}_l \cap \mathscr{T}_u$? (*1p*)
(e) Show that the discrete topology is the only topology on $\mathbb{R}$ containing $\mathscr{T}_l \cup \mathscr{T}_u$. (*0.5p*)
(f) Finally, show that the converses of (b) and (c) above do not hold in general. (*1p*)

16.37 The 2008 exam A (April 15, 2008)

Exercise 16.133. We consider the collection of subsets of $\mathbb{R}$:

$$\mathscr{B} := \{[a,b) : a,b \in \mathbb{R}, a \leq b\} \cup \{\mathbb{R}\}.$$

(a) Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. (*0.5p*)
(b) Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ containing $\mathscr{B}$. Show that $\mathscr{T}$ is larger than the Euclidean topology $\mathscr{T}_{\text{eucl}}$. (*1p*)
(c) In the topological space $(\mathbb{R}, \mathscr{T})$, find the closure, the interior and the boundary of

$$A = (0,1) \cup [2,3]. \qquad (1.5p)$$

(d) Show that $(\mathbb{R}, \mathscr{T})$ and $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$ are not homeomorphic.(*0.5p*)

Exercise 16.134. (*1.5p*) Show that the torus T contains a subspace C homeomorphic to a bouquet of two circles such that $T - C$ is homeomorphic to the open 2-disk. Similarly for the double torus and a bouquet of four circles.

Exercise 16.135. (*1.5p*) Consider the group $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$ of remainders modulo n (with the addition modulo n), and the action of Z_n on the circle S^1 given by

$$k \bullet (\cos(t), \sin(t)) = \left(\cos\left(t + \frac{2k\pi}{n}\right), \sin\left(t + \frac{2k\pi}{n}\right)\right)$$

(for $k \in \mathbb{Z}_n$, $(\cos(t), \sin(t)) \in S^1$). Show that the resulting quotient $S^1/\mathbb{Z}_n$ is homeomorphic to S^1. What can you say when $n = 2$?

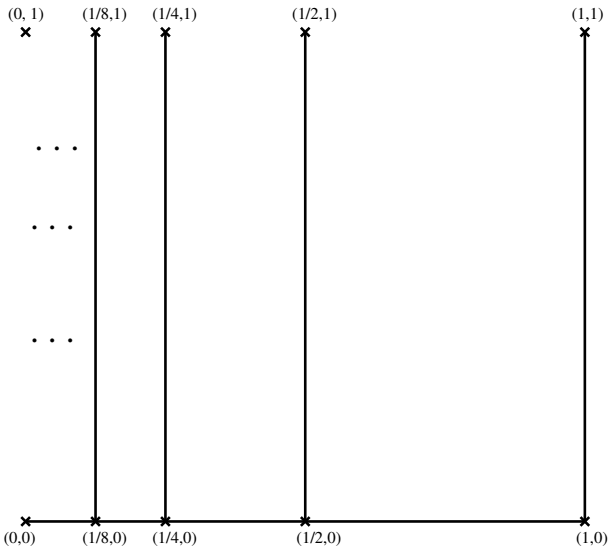


Fig. 16.4

Exercise 16.136. Consider the following subset of $\mathbb{R}^2$:

$$X = \bigcup_{n\geq 1 \text{ integer}} \left\{\frac{1}{2^n}\right\} \times [0,1] \quad \cup \quad [0,1]\times\{0\} \quad \cup \quad \{(0,1)\}$$

(see the picture), with the induced topology. Explain which of the following properties are true for X.

(a) it is Hausdorff. (*0.5p*)
(b) it is compact. (*1p*)
(c) it is locally compact. (*0.5p*)
(d) it is connected. (*0.5p*)
(e) it is path connected. (*0.5p*)

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

16.38 The 2007 exam A (April 16, 2007)

Exercise 16.137. Show that, in any topological space X, for any subspace $A \subset X$, one has

$$\partial(A) = \partial(X - A).$$

(1.5p)

Exercise 16.138.

(a) Is it true that any continuous surjective map $f : S^1 \to S^1$ is a homeomorphism? (0.75p)
(b) Show that S^1 cannot be embedded in $\mathbb{R}$. (1p)
(c) Show that any continuous injective map $f : S^1 \to S^1$ is a homeomorphism. (0.75p)

Exercise 16.139. Let (X,d) be a metric space and A,B closed subsets of X such that

$$d(A,B) = 0.$$

(Recall that $d(A,B) =: \inf\{d(a,b) : a \in A, b \in B\}$).

(a) Is it true that A and B must have a common point (i.e. $A \cap B \neq \emptyset$)? (1p)
(b) What if we assume that both A and B are compact? (0.75p)
(c) What if we assume that either A or B is compact? (0.75p)

Exercise 16.140. For any continuous map $f : S^1 \to T^2$ we define

$$X_f := T^2 - f(S^1).$$

(a) Is it true that, for any continuous function f, X_f is compact? But locally compact? But metrizable? But normal? But connected? (0.5p)
(b) Describe two embeddings $f_1, f_2 : S^1 \to T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic. (.75p)
(c) Describe the one-point compactifications $X_{f_1}^+$ and $X_{f_2}^+$. (0.75p)
(d) Describe $f : S^1 \to T^2$ continuous such that X_f^+ is homeomorphic to S^2. (0.5p)

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

16.39 The 2006 exam A (April 16, 2006)

Exercise 16.141. Let $(X, \mathscr{T})$ be a topological space and assume that $A, B \subset X$ are compact subsets. Prove that $A \cup B$ is compact. Prove that the same holds for $A \cap B$ if X is Hausdorff.

Exercise 16.142. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathscr{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$.
- $\mathscr{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n, n+1, \ldots\}$ with $n \in \mathbb{N}$.

Questions:

1. Show that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on $\mathbb{N}$.
2. Show that the spaces $(\mathbb{N}, \mathscr{T}_1)$ and $(\mathbb{N}, \mathscr{T}_2)$ are not homeomorphic.
3. for each of the spaces $(\mathbb{N}, \mathscr{T}_1)$ and $(\mathbb{N}, \mathscr{T}_2)$ decide whether the space is Hausdorff, connected or compact.

Exercise 16.143. Questions:

1. Is $[0, 1]$ homeomorphic to $[0, 1)$?
2. Does there exist a continuous surjective map $f : [0, 1] \to [0, 1)$? But a continuous injective one?
3. Does there exist a continuous surjective map $f : S^1 \to [0, 1)$? But an injective one?

Exercise 16.144. Consider

$$X = [0, 1] \times [0, 1)$$

with the topology induced from $\mathbb{R}^2$. Prove that X is a locally compact Hausdorff space and describe the one-point compactification (use a picture). What happens if we replace X by $Y = X \cup \{(1, 1)\}$?

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

Index

abstract quotient, 23
abstract torus, 53
action, of a group, 57
algebra, $*$-, 100
algebra, over a field, 98
Arzela-Ascoli theorem, 109

Baire space, 84
Banach algebra, 99
Banach space, 99
basis of neighborhoods, of a point, 35
basis, of a topological space, 40
boundary, of a set, 37
bounded set, 89
bounded set of maps, 107

Cauchy sequence, 88
Cauchy sequence, in metric space, 26
character, of an algebra, 103
cilinder on a space, 54
closed in a subset, 30
closed set, in topological space, 28
closed under finite sums, 112
closed under quotients, 112
closure, of a set, 37
co-countable, 28
co-finite topology, 28
coarser topology, 28
collapsing a set to a point, 54
compact convergence, 66
compact convergence, topology of, 105
compact topological space, 74
compactness in Euclidean space, 25
compactness, of metric space, 90
complete metric space, 26, 88
cone on a space, 54
connected, 70
connected component, 72
continuity at a point, 34
continuous function, 1
continuous function, between topological spaces, 31
convergence, 65
convergence, of a sequence, 34
converges, 3
coordinate chart, 41
cover, 74

dense subset, 38, 84
dimension of a topological space, 121
discrete topology, 28
disk, 5
distance, 3

embedding, 32
embedding theorem, 117
equicontinuous, family of functions, 106
equivalence relation, 21, 48
exhaustion of a space, 81
extreme topologies, 28

finer topology, 28
finite shrinking lemma, 113
first countability axiom, 35
first countable, 35

Gelfand-Naimark theorem, full version, 104
generated topology, 64
gluing, operation of, 6

Hausdorff, 39
hemisphere, 8
homeomorphic, 32
homeomorphism, 1
homeomorphism, between topological spaces, 32

ideal of an algebra, 103
induced topology, 30
initial topology, 64
interior, of a set, 37
invariant, topological, 1
involution, 100

Klein bottle, 17

larger topology, 28
Lebesgue lemma, 91
Lebesgues number, 91
locally compact, 81
locally finite, family of functions, 114
locally finite, family of sets, 85, 133
locally path connected, 73

maximal ideal, 103
metric, 3

metric space, 3
metrizable, 1
metrizable topology, 29
metrization theorem of Smirnov, 95
metrization theorem, of Urysohn, 94
modulo, equivalence relation, 22, 49
Moebius band, 11

neighborhood, of a point, 34
norm, on a vector space, 99
normal space, 42
normality, of compact Hausdorff space, 76

one-point compactification, 82
open ball, 25
open cover, 74
open in a subset, 30
open map, 32
open set, 25
open set, in topological space, 28

paracompact, 85
paracompactness, of metric space, 92
partition of unity, 113, 115
path connected, 70
point- separating set of functions, 101
pointwise bounded set of maps, 107
pointwise convergence, 65
pointwise topology, 105
pointwise totally bounded, 107
product topology, 61
projective plane, 19
projective space, 19
projective space, as quotient, 47

quotient, 22, 49
quotient map, 22, 49
quotient topology, 47
quotient, by a group action, 57

refinement of a cover, 85
relation, 21
relatively compact subset, 91
removing a point trick, 131

second countability axiom, 40
second countable, 40
separable metric space, 91
separated, 42
separated, by continuous functions, 42
sequentially compact, 80
sequentially continuous, 35
shrinking lemma, 115
smaller topology, 28
Smirnov metrization theorem, 95
smooth manifold, 170
spectrum, of an algebra, 103
sphere, n-dimensional, 8
square metric, 3
stereographic projection, 8
Stone-Weierstrass theorem, 101
subalgebra, 101
subcover, 74
subspace topology, 30
support of a function, 113
suspension of a space, 55

the flea and the comb, 72
topological manifold, 40
topological property, 32
topological space, 1, 28
topological spectrum, of an algebra, 104
topology, 1
topology basis, 63
topology generated, 64
topology induced by a metric, 25
topology of uniform convergence, 65
topology, on a set, 28
torus, 13
torus, abstract, 53
totally bounded set, 89
totally bounded set of maps, 107
triangle inequality, 3
trivial topology, 28
tube lemma, 77

uniform convergence, 65
uniform convergence on compacts, 65
uniform topology, 99, 105
uniformly continuous map, 106
uniformly equicontinuous, 106
Urysohn metrization theorem, 94