Inleiding Topologie

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- Utrecht Lecture Notes 2026-2027 -

PART I: THEORY

Utrecht University

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Chapter 1
Introduction: some standard spaces

1.1 Keywords for this course

In this course we study *topological spaces*. One may remember that in group theory one studies groups- and a group is a set G together with some extra-structure (the group operation) which allows us to multiply the elements of G. Similarly, a topological space is a set X together with some "extra-structure" which allows us to make sense of "two points getting close to each other" or, even better, it allows us to make sense of statements like: a sequence $(x_n)_{n\geq 1}$ of elements of X converges to $x \in X$. Of course, if X is endowed with a "metric" (i.e. a way to measure, or to give sense to, "the distance between two points of X"), then such statements have a clear intuitive meaning and can easily be made precise. However, the correct extra-structure that is needed is a bit more subtle- it is the notion of *topology on X* which will be explained in the next chapter.

The interesting functions in topology are the *continuous functions*. One may remember that, in group theory, the interesting functions between two groups G_1 and G_2 are not all arbitrary functions $f : G_1 \longrightarrow G_2$, but just those which "respect the group structure" (group homomorphisms). Similarly, in topology, the interesting maps between two topological spaces X and Y are those functions $f : X \longrightarrow Y$ which are continuous. Continuous means that "it respects the topological structures"- and this will be made precise later. But roughly speaking, f being continuous means that it maps convergent sequences to convergent sequences: if $(x_n)_{n\geq 1}$ is a sequence in X converging to $x \in X$, then the sequence $(f(x_n))_{n\geq 1}$ of elements of Y converges to $f(x) \in Y$.

The correct notion of isomorphism in topology is that of *homeomorphism*. In particular, we do not really distinguish between spaces which are *homeomorphic*. Thinking again back at group theory, there we do not really distinguish between groups which are isomorphic- and there the notion of isomorphism was: a bijection which preserves the group structure. Similarly, in topology, a homeomorphism between two topological spaces X and Y is a bijection $f : X \to Y$ so that f and f^{-1} are both continuous (note the apparent difference with group theory: there, a group isomorphism was a bijection $f : G_1 \to G_2$ such that f is a group homomorphism. The reason that, in group theory, we do not require that f^{-1} is itself a group homomorphism, is simple: it follows from the rest!).

Some of the *main questions* in topology are:

1. how to decide whether two spaces are homeomorphic (= the same topologically) or not?
2. how to decide whether a space is metrizable (i.e. the topology comes from a metric)?
3. when can a space be embedded ("pictured") in the plane, in the space, or in a higher $\mathbb{R}^n$?

These questions played the role of a driving force in Topology. Most of what we do in this course is motivated by these questions; in particular, we will see several results that give answers to them. There are several ways to tackle these questions. The first one - and this will keep us busy for a while- is that of finding special properties of topological spaces, called *topological properties* (such as Hausdorffness, connectedness, compactness, etc). For instance, a space which is compact (or connected, or etc) can never be homeomorphic to one which is not. Another way is that of associating *topological invariants* to topological spaces, so that, if two spaces have distinct topolog-

ical invariant, they cannot be homeomorphic. The topological invariants could be numbers (such as “the number of distinct connected components”, or “the number of holes”, or ”the Euler characteristic”), but they can also be more complicated algebraic objects such as groups. The study of such topological invariants is another field on its own (and is part of the course ”Topologie en Meetkunde”); what we will do here is to indicate from time to time the existence of such invariants.

In this course we will also devote quite some time to *topological constructions*- i.e. methods that allow us to construct new topological spaces out of ones that we already know (such as taking the product of two topological spaces, the cone of a space, quotients).

Finally, I would like to mention that these lecture notes are based on the book ”Topology” by James Munkres. But please be aware that the lecture notes should be self contained (however, you can have a look at the book if you want to find out more). The reason for writing lecture notes is that the book itself requires a larger number of lectures in order to achieve some of the main theorems of topology. In particular, in these lecture notes we present more direct approaches/proofs to such theorems. Sometimes, the price to pay is that the theorems we prove are not in full generality. Our principle is that: choose the version of the theorem that is most interesting for examples (as opposed to “most general”) and then find the shortest proof.

1.2 Spaces

In this chapter we present several examples of "topological spaces" before introducing the formal definition of "topological space" (but trying to point out the need for one). Hence please be aware: some of the statements made in this chapter are rather loose (un-precise)- and I try to make that clear by using quotes; the spaces that we mention here are rather explicit and intuitive, and when saying "space" (as opposed to "set"), we have in mind the underlying set (of elements, also called points) as well as the fact that we can talk (at least intuitively) about its points "getting closer to each other " (or, even better, about convergence of sequences of points in the set). For those who insist of being precise, let us mention that, in this chapter, all our spaces are metric spaces (so that convergence has a precise meaning); even better, although in some examples this is not entirely obvious, all the examples from this chapter are just subspaces of some Euclidean space $\mathbb{R}^n$. Recall here:

Definition 1.1. Let X be a set. A **metric** on X is a function

$$d : X \times X \to \mathbb{R}$$

which associates to a pair (x,y) of points x and y of X, a real number $d(x,y)$, called the distance between x and y, such that the following conditions hold:

(M1) $d(x,y) \geq 0$ for all $x,y \in X$.
(M2) $d(x,y) = 0$ if and only if $x = y$.
(M3) $d(x,y) = d(y,x)$.
(M4) (triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x,y,z \in X$.

A **metric space** is a pair (X,d) consisting of a set X together with a metric d.

Metric spaces are particular cases of "topological spaces"- since they allow us to talk about convergence and continuity. More precisely, given a metric space (X,d), and a sequence $(x_n)_{n\geq 1}$ of points of X, we say that $(x_n)_{n\geq 1}$ **converges** to $x \in X$ (in (X,d), or with respect to d) if $\lim_{n\to\infty} d(x_n,x) = 0$. When there is no danger of confusion (i.e. most of the times), we will just say that X is a metric space without specifying d. Given two metric spaces X and Y, a function $f : X \to Y$ is called **continuous** if for any convergent sequence $(x_n)_{n\geq 1}$ in X, converging to some $x \in X$, the sequence $(f(x_n))_{n\geq 1}$ converges (in Y) to $f(x)$. A continuous map f is called a **homeomorphism** if it is bijective and its inverse f^{-1} is continuous as well. Two spaces are called **homeomorphic** if there exists a homeomorphism between them.

The most intuitive examples of spaces are the real line $\mathbb{R}$, the plane $\mathbb{R}^2$, the space $\mathbb{R}^3$ or, more generally, the Euclidean space, one for each integer $k \geq 1$:

$$\mathbb{R}^k = \{(x^1,\ldots,x^k) : x^1,\ldots,x^k \in \mathbb{R}\}$$

For them we use the Euclidean metric and the notion of convergence and continuity with respect to this metric:

$$d(x,y) = \sqrt{(x^1-y^1)^2 + \ldots + (x^k-y^k)^2}$$

(for $x = (x^1,\ldots,x^k)$, $y = (y^1,\ldots,y^k) \in \mathbb{R}^k$). Another interesting metric on $\mathbb{R}^k$ is the **square metric** ρ, defined by:

$$\rho(x,y) = max\left\{|x^1-y^1|,\ldots,|x^k-y^k|\right\}.$$

The next exercise shows that, although the notion of metric allows us to talk about convergence, metrics do not encode convergence faithfully (two very different looking metrics can induce the same convergent sequences, hence the same "space"). The key of understanding the "topological content" of metrics (i.e. the one that allows us to talk about convergent sequences) is the notion of open subsets- to which we will come back later.

Exercise 1.2. Show that a sequence of points of $\mathbb{R}^n$ is convergent with respect to the Euclidean metric if and only if it is convergent with respect to the square metric.

Inside the Euclidean spaces one finds other interesting topological spaces such as intervals, circles, spheres, etc. In general, any subset

$$X \subset \mathbb{R}^k$$

can naturally be viewed as a "space", and as a metric space with the Euclidean metric.

Exercise 1.3. Show that, for any two numbers $a < b$

a) the interval $[a,b]$ is homeomorphic to $[0,1]$.
b) the interval $[a,b)$ is homeomorphic to $[0,1)$ and also to $[0,\infty)$.
c) the interval (a,b) is homeomorphic to $(0,1)$ and also to $(0,\infty)$ and to $\mathbb{R}$.

Exercise 1.4. Explain why the three subset of the plane drawn in Figure 1.1 are homeomorphic.

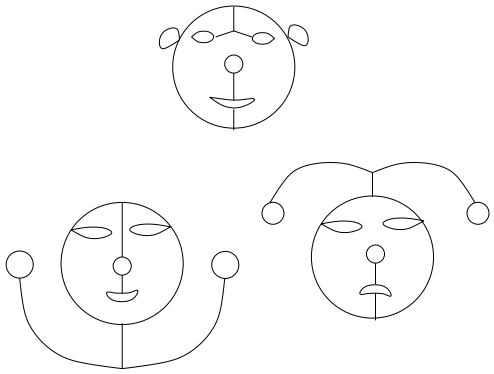


Fig. 1.1

Exercise 1.5. Which of the subset of the plane drawn in Figure 1.2 do you think are homeomorphic? (be aware that, at this point, we do not have the tools to prove which two are not!).

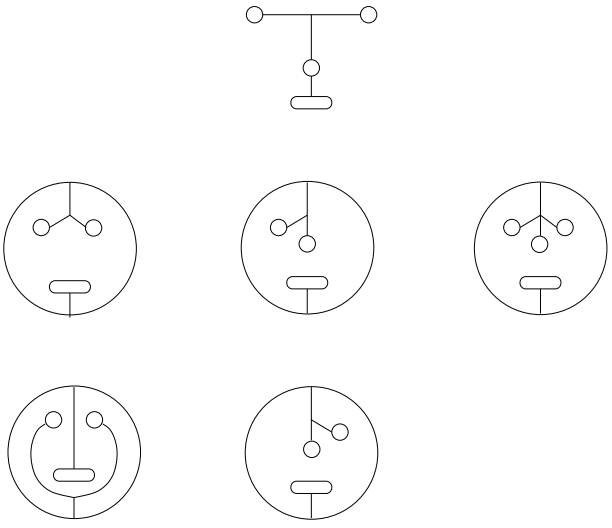


Fig. 1.2

1.3 The circle

In $\mathbb{R}^2$ one has the **unit circle**

$$S^1 = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\right\},$$

the **open disk**

$$\overset{\circ}{D}{}^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\right\},$$

the **closed disk**

$$D^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\right\}.$$

Next, we mention the standard parametrization of the unit circle: any point on the circle can be written as

$$e^{it} := (\cos(t), \sin(t))$$

for some $t \in \mathbb{R}$. This gives rise to a function

$$e : \mathbb{R} \to S^1, \;\; e(t) = e^{it}$$

which is continuous (explain why!). A nice picture of this function is obtained by first spiraling $\mathbb{R}$ above the circle and then projecting it down, as in Figure 1.3.

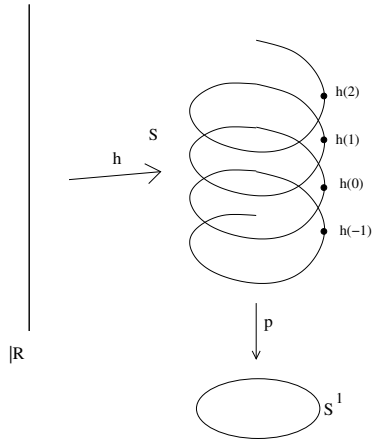


Fig. 1.3

Exercise 1.6. Make Figure 1.3 more precise. More precisely, find an explicit subspace $S \subset \mathbb{R}^3$ which looks like the spiral, and a homeomorphism h between $\mathbb{R}$ and S, so that the map e above is obtained by first applying h and then applying the projection p ($p(x,y,z) = (x,y)$). (Hint: $\left\{(x,y,z) \in \mathbb{R}^3 : x = \cos(z), y = \sin(z)\right\}$).

Note also that, if one restricts to $t \in [0,2\pi)$, we obtain a continuous bijection

$$f : [0,2\pi) \to S^1.$$

However, $[0,2\pi)$ and S^1 behave quite differently as topological spaces, or, more precisely, they are not homeomorphic. Note that this does not only mean that f is not a homeomorphism; it means that neither f nor any other bijection between $[0,2\pi)$ and S^1 is a homeomorphism. It will be only later, after some study of topological properties (e.g. compactness), that we will be able to prove this statement. At this point however, one can solve the following

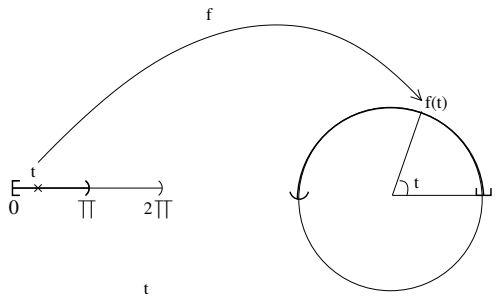


Fig. 1.4

Exercise 1.7. Show that the map $f : [0, 2\pi) \to S^1$ is not a homeomorphism.

Next, there is yet another way one can look at the unit circle: as obtained from the unit interval $[0,1]$ by "bending it" and "gluing" its end points, as pictured in Figure 1.5. This "gluing process" will be made more precise later and will give yet another general method for constructing interesting topological spaces.

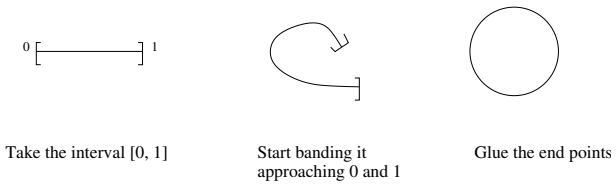


Fig. 1.5

In general, by a (topological) circle we mean any space which is homeomorphic to S^1. In general, they may be placed in the space in a rather non-trivial way. Some examples of circles are:

- circles S^1_r with a radius $r > 0$ different from 1, or other circles placed somewhere else in the plane.
- pictures obtained by twisting a circle in the space, such as in Figure 1.6.
- even pictures which, in the space, are obtained by breaking apart a circle, knotting it, and then gluing it back (see Figure 1.7).

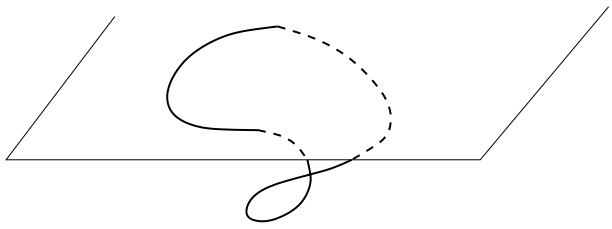


Fig. 1.6

Exercise 1.8. Explain on pictures that all the spaces enumerated above are homeomorphic to S^1. If you find it strange, try to explain to yourself what makes it look strange (is it really the circles, or is it more about the ambient spaces in which you realize the circles?).

Similarly, by a (topological) disk we mean any space which is homeomorphic to D^2. For instance, the unit square

$$[0,1] \times [0,1] = \big\{(x,y) \in \mathbb{R}^2 : x, y \in [0,1]\big\}$$

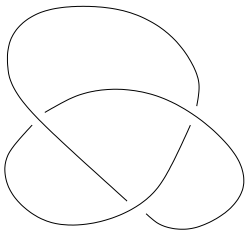
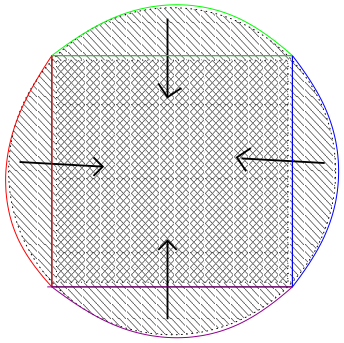


Fig. 1.7

is an important example of topological disk. More precisely, one has the following:

Exercise 1.9. Show that the unit disk D^2 is homeomorphic to the unit square, by a homeomorphism which restricts to a homeomorphism between the unit circle S^1 and the boundary of the unit square.



The unit disk and the square are homeomorphic

Fig. 1.8

1.4 The sphere and its higher dimensional versions

For each n, we have the **n-sphere**

$$S^n = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} : (x_0)^2 + \dots + (x_n)^2 = 1\right\} \subset \mathbb{R}^{n+1},$$

the **open** $(n+1)$**-disk**

$$\overset{\circ}{D}{}^{n+1} = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} : (x_0)^2 + \dots + (x_n)^2 < 1\right\} \subset \mathbb{R}^{n+1},$$

and similarly the **closed** $(n+1)$**-disk** D^{n+1}.

The points

$$p_N = (0,\dots,0,1),\ \ p_S = (0,\dots,0,-1) \in S^n$$

are usually called the north and the south pole, respectively, and

$$S^n_+ = \{(x_0,\dots,x_n) \in S^n : x_n \geq 0\}, \quad S^n_- = \{(x_0,\dots,x_n) \in S^n : x_n \leq 0\}$$

are called the north and the south hemisphere, respectively. See figure 1.9.

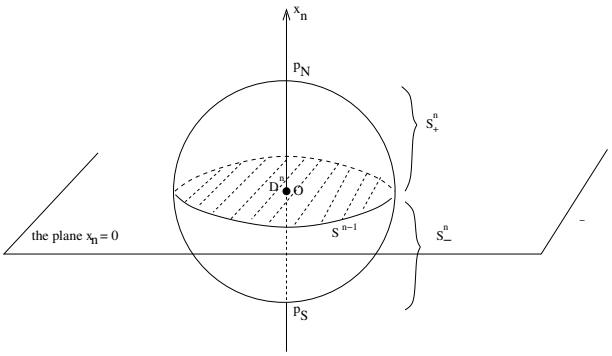


Fig. 1.9

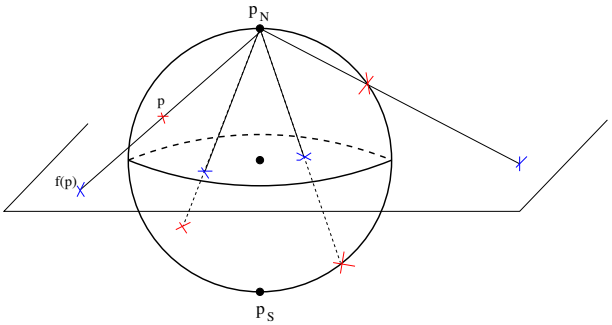
Exercise 1.10. Explain on a picture that $S^n_+ \cap S^n_-$ is homeomorphic to S^{n-1}, and S^n_+ and S^n_- are both homeomorphic to D^n.

As with the circle, we will call a sphere (or an n-sphere) any space which is homeomorphic to S^2 (or S^n). Also, we call a disk (or n-disk, or open n-disk) any space which is homeomorphic to D^2 (or D^n, or $\overset{\circ}{D}{}^n$). For instance, the previous exercise shows that the two hemispheres S^n_+ and S^n_- are n-disks, and S^n is the union of two n-disks whose intersection is an $n-1$-sphere. This also indicates that S^n can actually be obtained by gluing two copies of D^n along their boundaries. And, still as for circles, there are many subspaces of Euclidean spaces which are spheres (or disks, or etc), but look quite different from the actual unit sphere (or unit disk, or etc). An important example has already been seen in Exercise 1.9 .

Another interpretation of S^n is as adding to $\mathbb{R}^n$ "a point at infinity". (to be made precise later on). This can be explained using the stereographic projection

$$f : S^n - \{p_N\} \to \mathbb{R}^n$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 1.10).

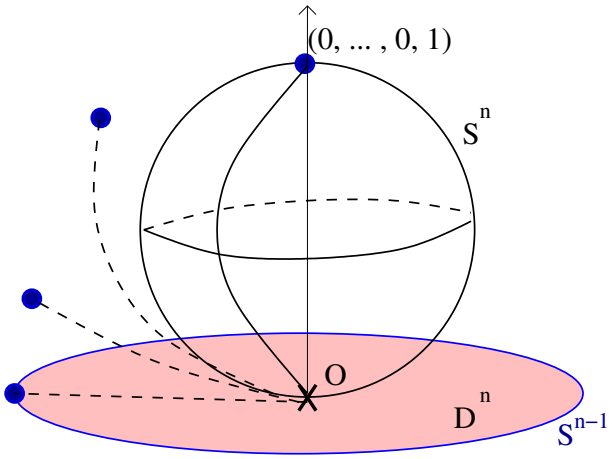


The stereographic projection (sending the red points to the blue ones)

Fig. 1.10

Exercise 1.11. Explain on the picture that the stereographic projection is a homeomorphism between $S^n - \{p_N\}$ and $\mathbb{R}^n$, and that it cannot be extended to a continuous function defined on the entire S^n. Then try to give a meaning to: "S^n can be obtained from $\mathbb{R}^n$ by adding a point at infinity". Also, find the explicit formula for f.

Here is another construction of the n-sphere. Take a copy of D^n, grab its boundary $S^{n-1} \subset D^n$ and glue it together (so that it becomes a point). You then get S^n (see Figure 1.11).



glue the boundary of D^n to a point to obtain S^n

Fig. 1.11

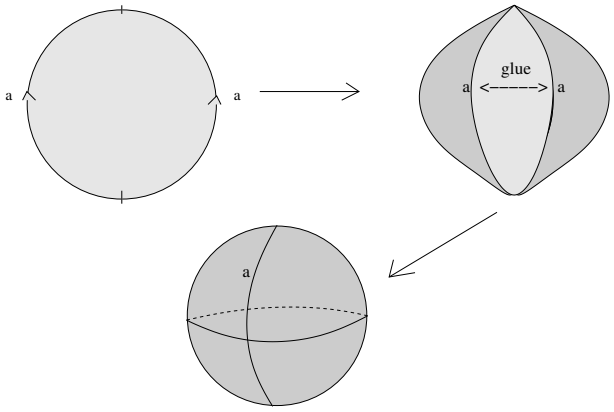
Exercise 1.12. Find explicitly the function

$$f : D^2 \to S^2$$

from Figure 1.11, check that $f^{-1}(p_N)$ is precisely $S^1 \subset D^2$, then generalize to arbitrary dimensions.

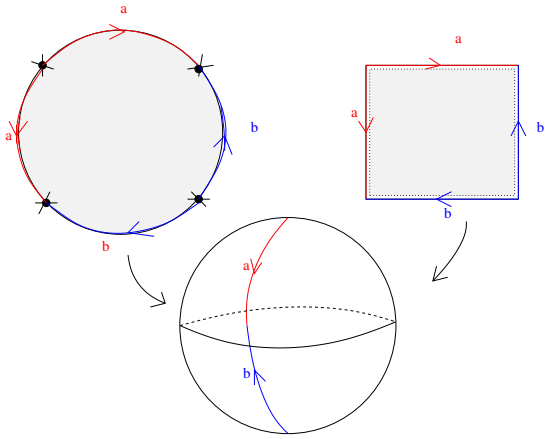
Another interesting way of obtaining the sphere S^2 is by taking the unit disk D^2, dividing its boundary circle S^1 into two equal sides and gluing the two half circles as indicated in the Figure 1.12.

A related construction of the sphere, which is quite important, is the following: take the disk D^2 and divide now its boundary circle into four equal sides, or take the unit square and its sides, and label them as in Figure 1.13. Glue now the two arcs denoted by a and the two arcs denoted by b.



The sphere obtained from the disk by glueing the two half circles denoted by "a" on the picture

Fig. 1.12



The sphere obtained from a disk or a square glueing as indicated in the pictu

Fig. 1.13

1.5 The Moebius band

"The Moebius band" is a standard name for subspaces of $\mathbb{R}^3$ which are obtained from the unit square $[0,1]\times[0,1]$ by "gluing" two opposite sides after twisting the square one time, as shown in Figure 1.14.

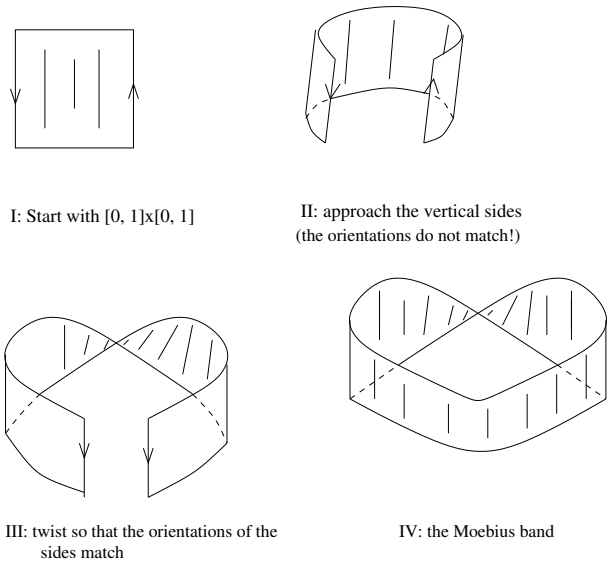


Fig. 1.14

As in the discussion about the unit circle (obtained from the unit interval by "gluing" its end points), this "gluing process" should be understood intuitively, and the precise meaning in topology will be explained later. The following exercise provides a possible parametrization of the Moebius band (inside $\mathbb{R}^3$).

Exercise 1.13. Consider

$$f:[0,1]\times[0,1]\to\mathbb{R}^3,$$

$$f(t,s)=\Big(\big(2+(2s-1)\sin(\pi t)\big)\cos(2\pi t),\ \big(2+(2s-1)\sin(\pi t)\big)\sin(2\pi t),\ (2s-1)\cos(\pi t)\Big).$$

You may want to check that $f(t,s)=f(t',s')$ holds only in the following cases:

- $(t,s)=(t',s')$.
- $t=0$, $t'=1$ and $s'=1-s$.
- $t=1$, $t'=0$ and $s'=1-s$.

(but this also follows from the discussion below). Based on this, explain why the image of f can be considered as the result of gluing the opposite sides of a square with the reverse orientation.

To understand where these formulas come from, and to describe explicit models of the Moebius band in $\mathbb{R}^3$, we can imagine the Moebius band as obtained by starting with a segment in $\mathbb{R}^3$ and rotating it around its middle point, while its middle point is being rotated on a circle. See Figure 1.15.

The rotations take place at the same time (uniformly), and while the segment rotates by 180°, the middle point makes a full rotation (360°). To write down explicit formulas, assume that

- the circle is situated in the XOY plane, is centered at the origin, and has radius R.
- the length of the segment is $2r$ and the starting position A_0B_0 of the segment is perpendicular on XOY with middle point $P_0=(R,0,0)$.
- at any moment, the segment stays in the plane through the origin and its middle point, which is perpendicular on the XOY plane.

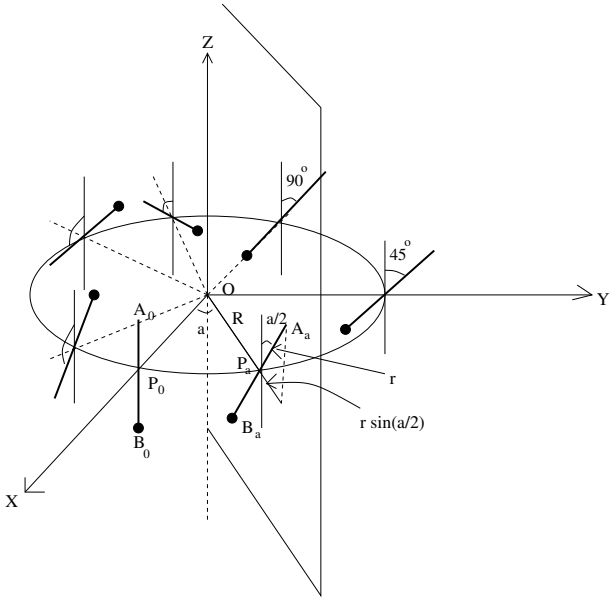


Fig. 1.15

We denote by $M_{R,r}$ the resulting subspace of $\mathbb{R}^3$ (note that we need to impose the condition $R > r$). To parametrize $M_{R,r}$, we parametrize the movement by the angle a which determines the middle point on the circle:

$$P_a = (R\cos(a), R\sin(a), 0).$$

At this point, the precise position of the segment, denoted A_aB_a, is determined by the angle that it makes with the perpendicular on the plane XOY through P_a; call it b. This angle depends on a. Due to the assumptions (namely that while a goes from 0 to 2π, b only goes from 0 to π, and that the rotations are uniform), we have $b = a/2$ (see 1.15). We deduce

$$A_a = \Big((R + r\sin(a/2))\cos(a),\ (R + r\sin(a/2))\sin(a),\ r\cos(a/2)\Big),$$

and a similar formula for B_a (obtained by replacing r by $-r$). Then, the Moebius band $M_{R,r}$ is:

$$M_{R,r} = \left\{ \Big((R + t\sin(a/2))\cos(a),\ (R + t\sin(a/2))\sin(a),\ t\cos(a/2)\Big) : a \in [0, 2\pi], t \in [-r, r] \right\} \tag{1.5.1}$$

Note that, although this depends on R and r, different choices of R and r produce homeomorphic spaces. To fix one example, one usually takes $R = 2$ and $r = 1$.

Exercise 1.14. Do the following:

a) Make a model of the Moebius band and cut it through the middle circle. You get a new connected object. Do you think it is a new Moebius band? Then cut it again through the middle circle and see what you get.
b) Prove (without using a paper model) that if you cut the Moebius band through the middle circle, you obtain a (space homeomorphic to a) cylinder. What happens if you cut it again?

1.6 The torus

"The torus" is a standard name for subspaces of $\mathbb{R}^3$ which look like a doughnut.

The simplest construction of the torus is by a gluing process: one starts with the unit square and then one glues each pair of opposite sides, as shown in Figure 1.16.

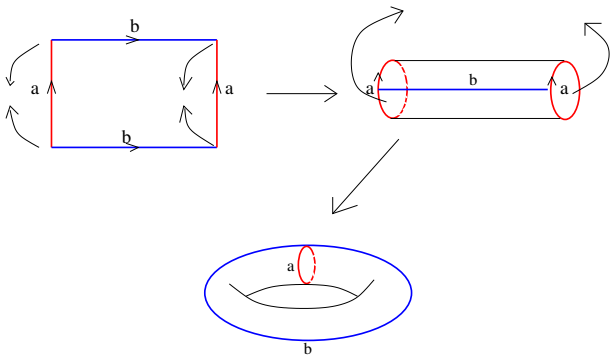


Fig. 1.16

As in the case of circles, spheres, disks, etc, by a torus we mean any space which is homeomorphic to the doughnut. Let's find explicit models (in $\mathbb{R}^3$) for the torus. To achieve that, we will build it by placing our hand in the origin in the space, and use it to rotate a rope which at the other end has attached a non-flexible circle. The surface that the rotating circle describes is clearly a torus (see Figure 1.17).

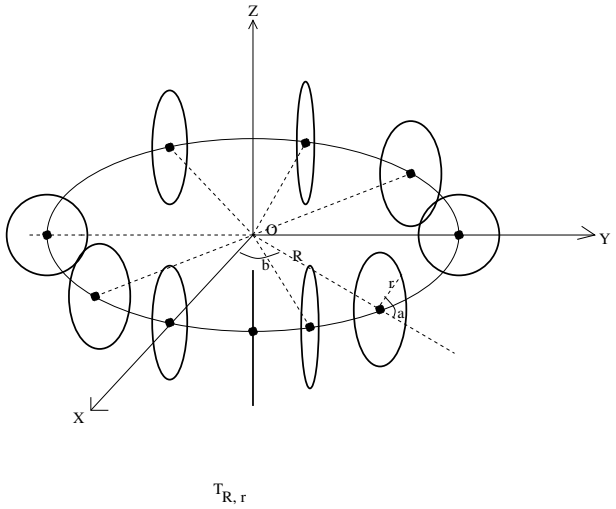


Fig. 1.17

To describe the resulting space explicitly, we assume that the rope rotates inside the XOY plane (i.e. the circle rotates around the OZ axis). Also, we assume that the initial position of the circle is in the XOZ plane, with center of coordinates $(R,0,0)$, and let r be the radius of the circle ($R > r$ because the length of the rope is $R-r$). We denote by $T^2_{R,r}$ the resulting subspace of $\mathbb{R}^3$. A point on $T^2_{R,r}$ is uniquely determined by the angles a and b indicated on the picture (Figure 1.17), and we find the parametric description:

$$T^2_{R,r} = \left\{ \Big((R+r\cos(a))\cos(b),\ (R+r\cos(a))\sin(b),\ r\sin(a) \Big) : a,b \in [0,2\pi] \right\} \subset \mathbb{R}^3. \qquad (1.6.1)$$

Exercise 1.15. Show that

$$T^2_{R,r} = \left\{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\right\}. \tag{1.6.2}$$

Although $T^2_{R,r}$ depends on R and r, different choices of R and r produce homeomorphic spaces. There is yet another interpretation of the torus, as the Cartesian product of two circles:

$$S^1 \times S^1 = \left\{(z,z') : z,z' \in S^1\right\} \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4.$$

Note that, a priori, this product is in the 4-dimensional space; the torus can be viewed as a homeomorphic copy inside $\mathbb{R}^3$.

Exercise 1.16. Show that

$$f : S^1 \times S^1 \to T^2_{R,r}, \quad f(e^{ia}, e^{ib}) = \Big((R + r\cos(a))\cos(b),\ (R + r\cos(a))\sin(b),\ r\sin(a)\Big)$$

is a bijection. Explain the map in the picture, and convince yourself that it is a homeomorphism.

Proving directly that f is a homeomorphism is not really pleasant, but the simplest way of proving that it is actually a homeomorphism will require the notion of compactness (and we will come back to this at the appropriate time).

The previous exercise shows that, topologically, a torus is the cartesian product of the circle with itself. More generally, for any integer $n \geq 1$, an n-torus is any space homeomorphic to

$$\underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}.$$

Exercise 1.17. Show that an n-torus can be embedded in $\mathbb{R}^{n+1}$.

Related to the torus is the double torus, pictured in Figure 1.18. Similarly, for each $g \geq 1$ integer, one can talk about the torus with g-holes.

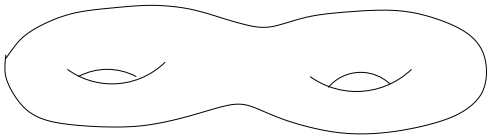


Fig. 1.18

Note that the double torus can be realized from two disjoint copies of the torus, by removing a small ball from each one of them, and then gluing them along the resulting circles (Figure 1.19).

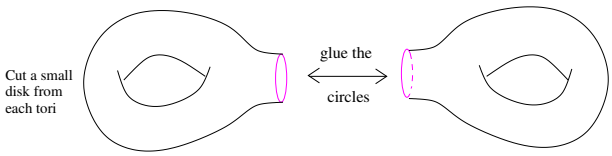


Fig. 1.19

Exercise 1.18. How should one glue the sides of a pentagon so that the result is a cut torus? (Hint: see Figure 1.20 and try to understand it. Try to make a paper model).

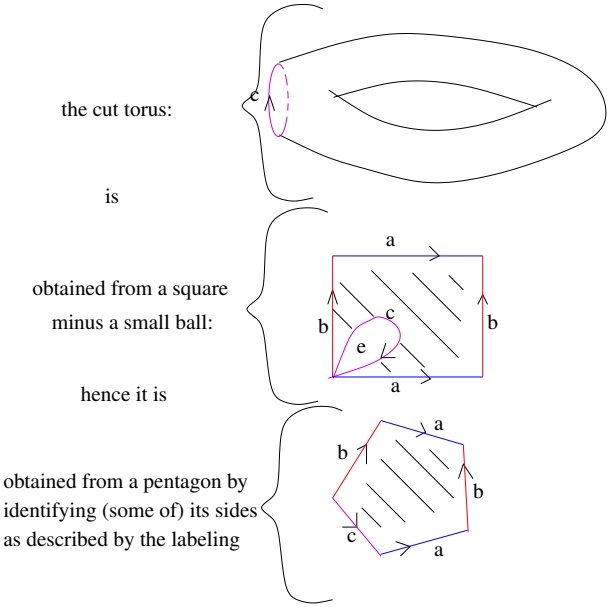


Fig. 1.20

Exercise 1.19. Show that the double torus can be obtained from an octagon by gluing some of its sides. (Hint: see Figure 1.21 and try to understand it. Try to make a paper model).

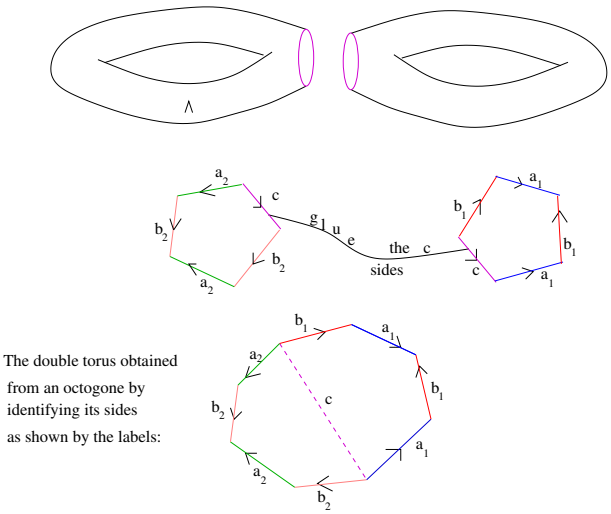


Fig. 1.21

Exercise 1.20. Show that the surface of a cup with a handle is homeomorphic to the torus (what about a cup with no handle?).

Exercise 1.21. Inside the sphere S^3 exhibit a subspace that is a torus.

Exercise 1.22. By a full torus we mean any space homeomorphic to $S^1 \times D^2$.

- Can you explain the terminology?
- Describe a full torus inside the 3-sphere S^3.

Exercise 1.23. As an analogue of Exercise 1.18 but for the Moebius band, show that if one glues the sides of a pentagon according to Figure 1.22 then one obtains a Moebius band with a hole inside.

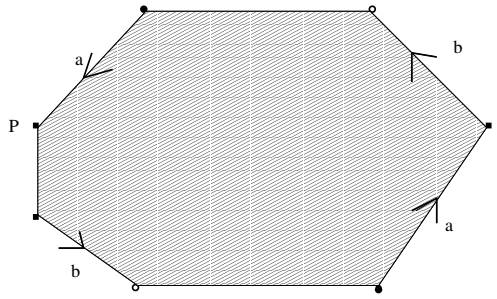


Fig. 1.22

Exercise 1.24. And, as the analogue of Exercise 1.19 show how the double Moebius band (Figure 1.23) can be obtained from a (full) polygon by gluing some of its sides.

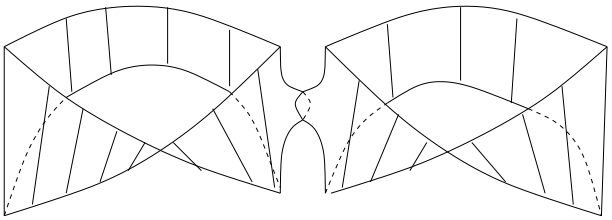


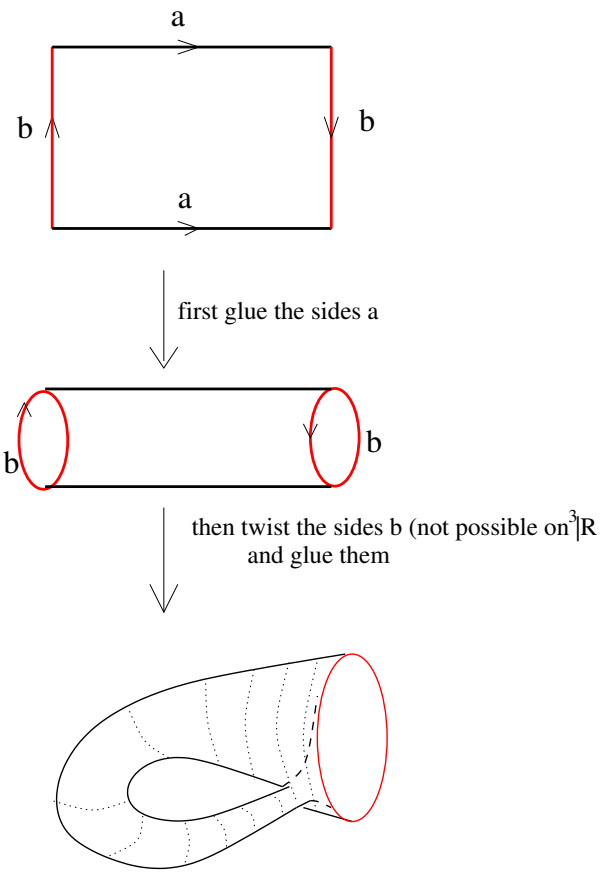
Fig. 1.23

1.7 The Klein bottle

We have seen that, starting from an unit square and gluing some of its sides, we can produce the sphere (Figure 1.13), the Moebius band (Figure 1.14) or the torus (Figure 1.16). What if we try to glue the sides differently? The next in this list of example would be the space obtained by gluing the opposite sides of the square but reversing the orientation for one of them, as indicated in Figure 1.24. The resulting space is called the Klein bottle, denoted here by K. Trying to repeat what we have done in the previous examples, we have trouble when "twisting the cylinder". Is that really a problem? It is now worth having a look back at what we have already seen:

- starting with an interval and gluing its end points, although the interval sits on the real line, we did not require the gluing to be performed without leaving the real line (it would not have been possible). Instead, we used one extra-dimension to have more freedom and we obtained the circle.
- similarly, when we constructed the Moebius band or the torus, although we started in the plane with a square, we did not require the gluing to take place inside the plane (it wouldn't even have been possible). Instead, we used an extra-dimension to have more freedom for "twisting", and the result was sitting in the space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

Something very similar happens in the case of the Klein bottle: what seems to be a problem only indicates the fact that the gluing cannot be performed in $\mathbb{R}^3$; instead, it indicates that K cannot be pictured in $\mathbb{R}^3$, or, more precisely, that K cannot be embedded in $\mathbb{R}^3$. Instead, K can be embedded in $\mathbb{R}^4$. The following exercise is an indication of that.



The Klein botlle: fake picture in $|R^3$

Fig. 1.24

Exercise 1.25. Consider the map

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^4,$$

$$\tilde{f}(t,s) = \Big((2-\cos(2\pi s))\cos(2\pi t),\ (2-\cos(2\pi s))\sin(2\pi t),\ \sin(2\pi s)\cos(\pi t),\ \sin(2\pi s)\sin(\pi t) \Big).$$

Explain why the image of this map in $\mathbb{R}^4$ can be interpreted as the space obtained from the square by gluing its opposite sides as indicated in the picture (hence can serve as a model for the Klein bottle).

Exercise 1.26. Explain how can one obtain a Klein bottle by starting with two Moebius bands and gluing them along their boundaries.

(Hint: look at Figure 1.25).

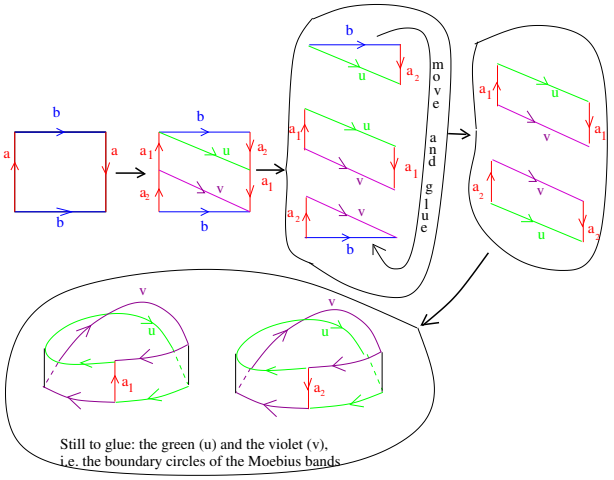


Fig. 1.25

1.8 The projective plane $\mathbb{P}^2$

In the same spirit as that of the Klein bottle, let's now try to glue the sides of the square as indicated on the left hand side of Figure 1.26.

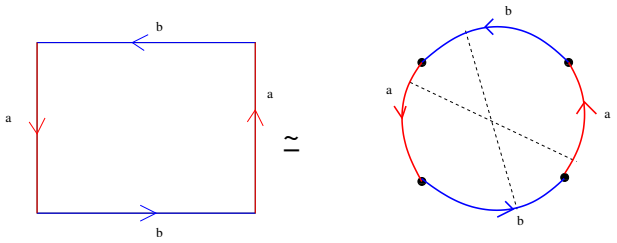


Fig. 1.26

Still as in the case of the Klein bottle, it is difficult to picture the result because it cannot be embedded in $\mathbb{R}^3$ (but it can be embedded in $\mathbb{R}^4$!). However, the result is very interesting: it can be interpreted as the set of all lines in $\mathbb{R}^3$ passing through the origin- denoted $\mathbb{P}^2$. Let us adopt here the standard definition of the projective space:

$$\mathbb{P}^2 := \left\{ l : l \subset \mathbb{R}^3 \text{ is a line through the origin} \right\},$$

(i.e. $l \subset \mathbb{R}^3$ is a one-dimensional vector subspace). Note that this is a "space" in the sense that there is a clear intuitive meaning for "two lines getting close to each other". We will explain how $\mathbb{P}^2$ can be interpreted as the result of the gluing that appears in Figure 1.26.

Step 1: First of all, there is a simple map:

$$f : S^2 \to \mathbb{P}^2$$

which associates to a point on the sphere, the line through it and the origin. Since every line intersects the sphere exactly in two (antipodal) points, this map is surjective and has the special property:

$$f(z) = f(z') \Longleftrightarrow z = z' \text{ or } z = -z' \ (z \text{ and } z' \text{ are antipodal}).$$

In other words, $\mathbb{P}^2$ can be seen as the result of gluing the antipodal points of the sphere.

Step 2: In this gluing process, the lower hemisphere is glued over the upper one. We see that, the result of this gluing can also be seen as follows: start with the upper hemisphere S^2_+ and then glue the antipodal points which are on its boundary circle.

Step 3: Next, the upper hemisphere is homeomorphic to the horizontal unit disk (by the projection on the horizontal plane). Hence we could just start with the unit disk D^2 and glue the opposite points on its boundary circle.

Step 4: Finally, recall that the unit ball is homeomorphic to the square (by a homeomorphism that sends the unit circle to the contour of the square). We conclude that our space can be obtained by the gluing indicated in the initial picture (Figure 1.26).

Note that, since $\mathbb{P}^2$ can be interpreted as the result of gluing the antipodal points of S^2, the following exercise indicates why $\mathbb{P}^2$ can be seen inside $\mathbb{R}^4$:

Exercise 1.27. Show that

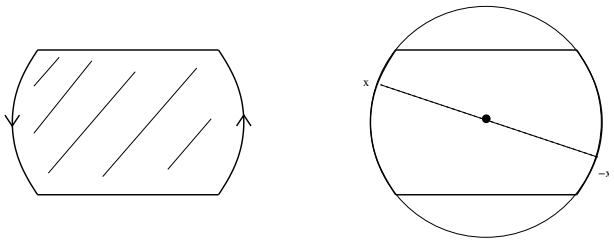
$$\tilde{f} : S^2 \to \mathbb{R}^4 \ , \tilde{f}(x,y,z) = (x^2 - y^2, xy, xz, yz)$$

has the property that, for $p, p' \in S^2$, $\tilde{f}(p) = \tilde{f}(p')$ holds if and only if p and p' are either equal or antipodal.

Note also that (a model of) the Moebius band can be seen as sitting inside (a model of) the projective plane $M \hookrightarrow \mathbb{P}^2$. To see this, recall that $\mathbb{P}^2$ can be seen as obtained from D^2 by gluing the antipodal points on its boundary. Consider inside D^2 the "band"

$$B = \left\{ (x,y) \in D^2 : -\frac{1}{2} \leq y \leq \frac{1}{2} \right\} \subset D^2.$$

The gluing process that produces $\mathbb{P}^2$ affects B in the following way: it glues the "opposite curved sides" of B as in the picture (Figure 1.27), and gives us the Moebius band.



The Moebius band inside the projective plane

Fig. 1.27

Paying attention to what happens to $D^2 - B$ in the gluing process, you can now try the following.

Exercise 1.28. Indicate how $\mathbb{P}^2$ can be obtained by starting from a Moebius band and a disk, and gluing them along the boundary circle.

1.9 Gluing (or quotients): a first look

We have already seen some examples of spaces obtained by gluing some of their points. When the gluing becomes less intuitive or more complicated, we start asking ourselves:

1. What does gluing really mean?
2. What is the result of such a gluing?

Here we address these questions. In examples such as the circle, torus or Moebius band, the answer was clear intuitively:

1. Gluing had the intuitive meaning- done effectively by using paper models.
2. the result was a new object, or rather a shape (it depends on how much we twist and pull the piece of paper).

Emphasize again that there was no preferred torus or Moebius band, but rather models for it (each two models being homeomorphic). Moving to the Klein bottle, things started to become less intuitive, since the result of the gluing cannot be pictured in $\mathbb{R}^3$ (and things become probably even worse in the case of the projective plane). But, as we explained above, if we use an extra-dimension, the Klein bottle exists in $\mathbb{R}^4$- and Exercise 1.25 produces a subset of $\mathbb{R}^4$ which is an explicit *model* for it.

And things become much worse if we now start performing more complicated gluing of more complicated objects (one can even get "spaces" which cannot be "embedded" in any of the spaces $\mathbb{R}^n$!). It is then useful to have a more conceptual (but abstract) understanding of what "gluing" and "the result of a gluing" means.

Start with a set X and assume that we want to glue some of its points. Which points we want to glue form the initial "gluing data", which can be regarded as a subset

$$R \subset X \times X$$

consisting of all pairs (x,y) with the property that we want x and y to be glued. This subset must have some special properties (e.g., if we want to glue x to y, y to z, then we also have to glue x to z). This brings us to the notions of equivalence relation:

Definition 1.29. An **equivalence relation** on a set X is a subset $R \subset X \times X$ satisfying the following:

1. If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.
2. If $(x,y) \in R$ then also $(y,x) \in R$.
3. $(x,x) \in R$ for all $x \in X$.

Hence, as a gluing data we start with any equivalence relation on X.

Remark 1.30 (using $\sim$ instead of $R \subset X \times X$). First of all, recall that a (binary) **relation** on a set X is nothing but a subset

$$R \subset X \times X.$$

However, depending on the properties of R, one uses notations of type

$$x \sim_R y, \quad x \cong_R y, \quad x <_R y, \quad x \leq_R y, \quad \text{etc},$$

or generically $x \rho_R y$, to say that the pair $(x,y) \in X \times X$ is in R. Even more, relations are often introduced by specifying what the comparison $x \sim_R y$ (or $x \cong_R y$ etc) means, without any reference to R, and it is self-understood that

$$R = \{(x,y) \in X \times X : x \sim y\} \subset X \times X.$$

In that situation R will be called the **graph of** $\sim$. This is completely similar to the fact that functions $f : X \to Y$ can also be described by using their graphs $\text{Graph}(f) \subset X \times Y$.

The choice between using $\sim$ or R is mainly a notational one. It is often made according to what the author finds more aesthetic or to what the author finds as conveying better the nature and aspects of the equivalence relations that are discussed. With that in mind, we choose to use R because we find the notations nicer, but we will occasionally refer to $\sim$ as well, when it conveys better the message.

Exercise 1.31. Describe explicitly the equivalence relation R on the square $X = [0,1] \times [0,1]$ which describe the gluing that we performed to construct the Moebius band. Similarly for the equivalence relation on the disk that described the gluing from Figure 1.11.

What should the result of the gluing be? First of all, it is going to be a new set Y. Secondly, any point of X should give a point in Y, i.e. there should be a function $\pi : X \to Y$ which should be surjective (the gluing should not introduce new points). Finally, π should really reflect the gluing, in the sense that $\pi(x) = \pi(y)$ should only happen when $(x,y) \in R$. Here is the formal definition.

Definition 1.32. Given an equivalence relation R on a set X, a **quotient of X modulo R** is a pair (Y, π) consisting of a set Y and a surjection $\pi : X \to Y$ (called the quotient map) with the property that

$$\pi(x) = \pi(y) \Longleftrightarrow (x,y) \in R.$$

Hence, a quotient of X modulo R can be viewed as a model for the set obtained from X by gluing its points according to R.

Example 1.33. Consider $X = [0,1]$. The equivalence relation R that corresponds to the gluing of 0 and 1 is:

$$(t,s) \in R \Longleftrightarrow (t = s) \text{ or } (t = 0, s = 1) \text{ or } (t = 1, s = 0).$$

A quotient of $[0,1]$ modulo R is the circle S^1 together with

$$\pi := e : [0,1] \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$

Indeed, $e(t) = e(s)$ happens if and only if $(t,s) \in R$.

Example 1.34. Consider on $X = \mathbb{R}^3 \setminus \{0\}$ the equivalence relation R defined by

$$(x,y) \in R \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}^*.$$

Then the projective space $\mathbb{P}^2$ (the set of all lines in $\mathbb{R}^3$, passing through the origin), appears as a quotient of X modulo R, when endowed with the map

$$\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2,\ \pi(x) := \text{the (unique) line through the origin and } x.$$

Example 1.35. Consider on $X = \mathbb{R}^3 \setminus \{0\}$ the equivalence relation R defined by

$$(x,y) \in R \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}_{>0}.$$

Then the 2-sphere S^2 together with

$$\pi : \mathbb{R}^3 \setminus \{0\} \to S^2,\ \pi(x) = \frac{1}{||x||} x$$

is a quotient of X modulo R (think of a picture!).

Exercise 1.36. Describe the equivalence relation R on $[0,1]\times[0,1]$ that encodes the gluing that we performed when constructing the torus. Then prove that the explicit model $T^2_{R,r}$ of the torus given by formula (1.6.2) is a quotient of $[0,1]\times[0,1]$ modulo R, in the sense of the previous definition (of course, you also have to describe the map π).

One can wonder "how many" abstract quotients can one build? Well, only one – up to isomorphism. More precisely:

Exercise 1.37. Show that if (Y_1,π_1) and (Y_2,π_2) are two quotients of X modulo R, then there exists and is unique a bijection $f: Y_1 \to Y_2$ such that $f\circ\pi_1 = \pi_2$.

One can also wonder: can one always build quotients? The answer is yes but, in full generality (for arbitrary X and R), one has to construct the model abstractly. Namely, for each $x\in X$ we define the R-equivalence class of x as

$$R(x) := \{y\in X : (x,y)\in R\}$$

(a subset of X) and define

$$X/R = \{R(x) : x\in X\}$$

(a new set whose elements are subsets of X). There is a simple function

$$\pi_R : X \to X/R,\ \ \pi_R(x) = R(x),$$

called the canonical projection. This is the abstract quotient of X modulo R. When using $\sim$ to refer to R, the R-orbit through x is often called **the equivalence class of** x w.r.t. to $\sim$ and denoted $[x]_\sim$, $[x]$, $\widehat{x}$ or $\overline{x}$. Hence, with the first notation, the abstract quotient and the abstract quotient map read:

$$X/\sim = \{[x]_\sim : x\in X\},\qquad \pi : X\to X/\sim,\quad \pi(x) = [x]_\sim,$$

and one has:

$$[x]_\sim = [y]_\sim \quad\Longleftrightarrow\quad x\sim y.$$

Example 1.38. Let $n\geq 1$ be an integer. Consider $X=\mathbb{Z}$ with the equivalence relation R_n defined by:

$$(x,y)\in R_n \Longleftrightarrow x\equiv y \pmod n.$$

Then the equivalence class of an arbitrary element $k\in\mathbb{Z}$ is

$$R_n(k) = \{\ldots, k-2n, k, k-n, k+n, k+2n, \ldots\}\ \ (\text{usually denoted } (k \bmod n) \text{ or just } \hat{k}).$$

Of course, $\mathbb{Z}/R_n$ is the usual set $\mathbb{Z}_n$ of integers modulo n.

Example 1.39. Let us return to Example 1.35. Then, for $x\in X=\mathbb{R}^3\setminus\{0\}$, its equivalence class is

$$R(x) = \{\lambda\cdot x : \lambda\in\mathbb{R}_{>0}\},$$

which even has a geometric interpretation in this case: it is the half line from the origin passing through x. Hence X/R is the set of such half lines. The bijection between X/R and S^2 is clear because a half line is uniquely determined by its intersection with the sphere!

Example 1.40. Let us return to Example 1.34. Then, for $x\in X=\mathbb{R}^3\setminus\{0\}$, its equivalence class is

$$R(x) = \{\lambda\cdot x : \lambda\in\mathbb{R}^*\}$$

which, again, can be interpreted as the line through the origin and x (but with the origin removed). This is in clear bijection with $\mathbb{P}^2$.

Example 1.41. Let us return to the gluing of the end-points of $[0,1]$ (Example 1.33). The example is simpler, but X/R will look a bit more abstract: it is the collection of subsets of $[0,1]$ consisting of

- the subset $\{0,1\}$.
- for each $t \in (0,1)$ the subset $\{t\}$.

Also, the projection π_R is given by

$$\pi_R(0) = \{0,1\}\,,\ \pi_R(1) = \{0,1\}\,,\ \text{and } \pi_R(t) = \{t\}\ \text{ for } t \in (0,1).$$

We see that, by its very construction, $(X/R, \pi_R)$ is a quotient of X modulo R. Of course, there is nothing particular about this example, and the same statement (and for the same tautological reasons) holds in general:

Exercise 1.42. Show that:

a) For any equivalence relation R on a set X, $(X/R, \pi_R)$ is a quotient of X modulo R.
b) For any surjective $f : X \to Y$ there is a unique relation R on X such that (Y, f) is a quotient of X mod R.

Remark 1.43. This discussion has been set-theoretical, so let's now go back to the case that X is a subset of some $\mathbb{R}^n$, and R is some equivalence relation on X. It is clear that, in such a "topological setting", we do not look for arbitrary models (quotients), but only for those which are in agreement with our intuition. In other words we are looking for "topological quotients" ("topological models"). What that really means will be made precise later on (since it requires the precise notion of topology). As a first attempt one could look for quotients (Y, π) of X modulo R with the property that

(1) Y is itself is a subspace of some $\mathbb{R}^k$.
(2) π is continuous.

These requirements pose two problems:

- Insisting that Y is a subspace of some $\mathbb{R}^k$ is too strong- see e.g. the exercise below. (Instead, Y will be just "topological space").
- The list of requirements is not complete. This can already be seen when $R = \{(x,x) : x \in X\}$ (i.e. when no gluing is required). Clearly, in this case, a "topological model" is X itself, and any other model should be homeomorphic to X. However, the requirements above only say that $\pi : X \to Y$ is a continuous bijection which, as we have already seen, does not imply that π is a homeomorphism.

As we already said, the precise list of requirements will be made precise later on. (There is a good news however: if X is "compact", then any quotient (Y, π) of X mod R which satisfies (1) and (2) is a good topological model!).

Exercise 1.44. Let $X = S^1$, and we want to glue any two points $e^{ia}, e^{ib} \in S^1$ with the property that $b = a + 2\pi\sqrt{2}$. Show that there is no model (Y, π) with Y-a subset of some space $\mathbb{R}^n$ and $\pi : X \to Y$ continuous.

Exercise 1.45. Let

$$\mathbb{P}^n := \left\{ l : l \subset \mathbb{R}^{n+1} \text{ is a line through the origin} \right\},$$

(i.e. $l \subset \mathbb{R}^{n+1}$ is a one-dimensional vector subspace). Explain how $\mathbb{P}^n$ (with the appropriate quotient maps) can be seen as:

a) a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ modulo an equivalence relation that you have to specify (see second part of Exercise 1.42).
b) obtained from S^n by gluing every pairs of its antipodal points.
c) obtained from D^n by gluing every pair of antipodal points situated on the boundary sphere S^{n-1}.

1.10 Metric aspects versus topological ones

Of course, most of what we discussed so far can be done within the world of metric spaces- a notion that did allow us to talk about convergence, continuity, homeomorphisms. There are however several reasons to allow for more flexibility and leave this world.

One was already indicated in Exercise 1.2 which shows that metric spaces do not encode convergence faithfully. Very different looking metrics on $\mathbb{R}^n$ (e.g. the Euclidean d and the square one ρ- in that exercise) can induce the same notion of convergence, i.e. they induce the same "topology" on $\mathbb{R}^n$- the one that we sense with our intuition. Of course, that exercise just tells us that $\mathrm{Id} : (\mathbb{R}^n, d) \to (\mathbb{R}^n, \rho)$ is a homeomorphism, or that d and ρ induce the same "topology" on $\mathbb{R}^n$ (we will make this precise a few line below).

The key of understanding the "topological content" of metrics (i.e. the one that allows us to talk about convergent sequences) is the notion of opens with respect to a metric. This is the first step toward the abstract notion of topological space.

Definition 1.46. Let (X,d) be a metric space. For $x \in X$, $\varepsilon > 0$ one defines the **open ball** with center x and radius ε (with respect to d):

$$B_d(x,\varepsilon) = \{y \in X : d(y,x) < \varepsilon\}.$$

A set $U \subset X$ is called **open with respect to** d if

$$\forall\ x \in U\ \ \exists\ \ \varepsilon > 0\ \text{ such that }\ B(x,\varepsilon) \subset U. \tag{1.10.1}$$

The topology induced by d, denoted $\mathscr{T}_d$, is the collection of all such opens $U \subset X$.

With this we have:

Exercise 1.47. Let d and d' be two metrics on the set X. Show that convergence in X with respect to d coincides with convergence in X with respect to d' if and only if $\mathscr{T}_d = \mathscr{T}_{d'}$.

It is not a surprise that the notion of convergence and continuity can be rephrased using opens only .

Exercise 1.48. Let (X,d) be a metric space, $(x_n)_{n\geq 1}$ a sequence in X, $x \in X$. Then $(x_n)_{n\geq 1}$ converges to x (in (X,d)) if and only if: for any open $U \in \mathscr{T}_d$ containing x, there exists an integer n_U such that

$$x_n \in U \quad \forall \quad n \geq n_U.$$

Exercise 1.49. Let (X,d) and (Y,d') be two metric spaces, and $f : X \to Y$ a function. Then f is continuous if and only if

$$f^{-1}(U) \in \mathscr{T}_d \ \ \forall \ \ U \in \mathscr{T}_{d'}.$$

The main conclusion is that the topological content of a metric space (X,d) is retained by the family $\mathscr{T}_d$ of opens in the metric space. This is the first example of a topology. I would like to emphasize here that our previous discussion does NOT mean that we should not use metrics and that we should not talk about metric spaces. Not at all! When metrics are around, we should take advantage of them and use them! However, one should be aware that some of the simple operations that we make with metric spaces (e.g. gluing) may take us out of the world of metric spaces. But, even when staying within the world of metric spaces, it is extremely useful to be aware of what depends on the metric itself and what just on the topology that the metric induces. We give two examples here.

Compactness The first example is the notion of compactness. You have probably seen this notion for subspaces of $\mathbb{R}^n$: a subset $\mathbb{R}^n$ is called compact if it is bounded and closed in $\mathbb{R}^n$ (see Dictaat Inleiding Analyse, Stelling

4.20, page 78). With this definiton it is very easy to work with compactness (... of subspaces of $\mathbb{R}^n$). E.g., the torus, the Moebius band, etc, they are all compact. However, one should be careful here: what we can say is that all the models of the torus, etc that we built are compact. What about the other ones? In other words, is compactness a topological condition? I.e., if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are homeomorphic, is it true that the compactness of A implies the compactness of B? With the previous definition of compactness, the answer may be no, as both the condition "bounded" and "closed" make reference to the way that A sits inside $\mathbb{R}^n$, and even to the Euclidean metric on $\mathbb{R}^n$ (for boundedness). See also Exercise 1.52 below. However, as we shall see, the answer is: yes, compactness is a topological property (and this is extremely useful).

Completeness Another notion that is extremely important when we talk about metric spaces is that of completeness. Recall (see Dictaat Inleiding Analyse, page 74):

Definition 1.50. Given a metric space (X,d) and a sequence $(x_n)_{n\geq 1}$ in X, we say that $(x_n)_{n\geq 1}$ is a **Cauchy sequence** if

$$\lim_{n,m\to\infty} d(x_n,x_m) = 0,$$

i.e., for each $\varepsilon > 0$, there exists an integer n_ε such that

$$d(x_n,x_m) < \varepsilon$$

for all $n,m \geq n_\varepsilon$. One says that (X,d) is **complete** if any Cauchy sequence is convergent. We say that $A \subset X$ is complete if A, together with the restriction of d to A, is complete.

For instance, $\mathbb{R}^n$ with the Euclidean metric is complete, as is any closed subspace of $\mathbb{R}^n$. Now, is completeness a topological property? This time, the answer is no, as the following exercise shows. But, again, this does not mean that we should ignore completeness in this course (and we will not). We should be aware that it is not a topological property, but use it whenever possible!

Exercise 1.51. On $\mathbb{R}$ we consider the metric $d'(x,y) = |e^x - e^y|$. Show that

a) d' induces the same topology on $\mathbb{R}$ as the Euclidean metric d.
b) although $(\mathbb{R},d)$ is complete, $(\mathbb{R},d')$ is not.

(Hint: $log(\frac{1}{n})$).

Exercise 1.52. For a metric space (X,d) we define $\hat{d} : X \times X \to \mathbb{R}$ by

$$\hat{d}(x,y) = \min\{d(x,y),1\}.$$

Show that:

a) $\hat{d}$ is a metric inducing the same topology on X as d.
b) (X,d) is complete if and only if $(X,\hat{d})$ is.

Chapter 2
Topological spaces

1. **Topological spaces**

2. **Continuous functions; homeomorphisms**

3. **Neighborhoods and convergent sequences**

4. **Inside a topological space: closure, interior and boundary**

5. **Hausdorffness; 2nd countability; topological manifolds**

6. **More on separation**

7. **More exercises**

2.1 Topological spaces

We start with the abstract definition of topological spaces.

Definition 2.1. A **topology** on a set X is a collection $\mathscr{T}$ of subsets of X, satisfying the following axioms:

(T1) $\emptyset$ and X belong to $\mathscr{T}$.
(T2) The intersection of any two sets from $\mathscr{T}$ is again in $\mathscr{T}$.
(T3) The union of any collection of sets of $\mathscr{T}$ is again in $\mathscr{T}$.

A **topological space** is a pair $(X,\mathscr{T})$ consisting of a set X and a topology $\mathscr{T}$ on X. Given a topological space $(X,\mathscr{T})$, we will use the following terminology:

- a subset $U \subset X$ is called **open** in the topological space $(X,\mathscr{T})$ if it belongs to $\mathscr{T}$.
- A subset $A \subset X$ is called **closed** in the topological space $(X,\mathscr{T})$ if $X - A$ is open.

Given two topologies $\mathscr{T}$ and $\mathscr{T}'$ on X, we say that $\mathscr{T}'$ is **larger** (or finer) than $\mathscr{T}$, or that $\mathscr{T}$ is **smaller** (or coarser) than $\mathscr{T}'$, if $\mathscr{T} \subset \mathscr{T}'$.

Exercise 2.2. Show that, in a topological space $(X,\mathscr{T})$, any finite intersection of open sets is open: for each $k \geq 1$ integer, $U_1,\ldots,U_k \in \mathscr{T}$, one must have $U_1 \cap \ldots \cap U_k \in \mathscr{T}$. Would it be reasonable to require that arbitrary intersections of open sets is open? What can you say about intersections or union of closed subsets of $(X,\mathscr{T})$?

Conventions 2.3. When referring to a topological space $(X,\mathscr{T})$, when no confusion may arise, we will simply say that "X is a topological space". Also, the opens in $(X,\mathscr{T})$ will simply be called "opens in X" (and similarly for "closed"). In other words, we will not mention $\mathscr{T}$ all the time; its presence is implicit in the statement "X is a topological space", which allows us to talk about "opens in X".

Example 2.4. (*Extreme topologies*) On any set X we can define the following:

1. The **trivial topology** on X, $\mathscr{T}_{\text{triv}}$: the topology whose open sets are only $\emptyset$ and X.
2. The **discrete topology** on X, $\mathscr{T}_{\text{dis}}$: the topology whose open sets are all subsets of X.
3. The **co-finite topology** on X, $\mathscr{T}_{\text{cf}}$: the topology whose open sets are the empty set and complements of finite subsets of X.
4. The **co-countable** topology on X, $\mathscr{T}_{\text{cc}}$: the topology whose open sets are the empty set and complements of subsets of X which are at most countable.

An important class of examples comes from metrics.

Proposition 2.5. *For any metric space (X,d), the family $\mathscr{T}_d$ of opens in X with respect to d is a topology on X. Moreover, this is the smallest topology on X with the property that it contains all the balls*

$$B_d(x;r) = \{y \in X : d(x,y) < r\} \quad (x \in X, r > 0).$$

Proof. Axiom (T1) is immediate. To prove (T2), let $U,V \in \mathscr{T}_{\text{d}}$ and we want to prove that $U \cap V \in \mathscr{T}_{\text{d}}$. We have to show that, for any point $x \in U \cap V$, there exists $r > 0$ such that $B_d(x,r) \subset U \cap V$. So, let $x \in U \cap V$. That means that $x \in U$ and $x \in V$. Since $U,V \in \mathscr{T}_{\text{d}}$, we find $r_1 > 0$ and $r_2 > 0$ such that

$$B_d(x,r_1) \subset U, B_d(x,r_2) \subset V.$$

Then $r = \min\{r_1,r_2\}$, has the desired property: $B_d(x,r) \subset U \cap V$.

To prove axiom (T3), let $\{U_i : i \in I\}$ be a family of elements $U_i \in \mathscr{T}_{\text{d}}$ (indexed by a set I) and we want to prove that $U := \cup_{i\in I} U_i \in \mathscr{T}_{\text{d}}$. We have to show that, for any point $x \in U$, there exists $r > 0$ such that $B_d(x,r) \subset U$. So, let

$x \in U$. Then $x \in U_i$ for some $i \in I$; since $U_i \in \mathscr{T}_{\mathrm{d}}$, we find $r > 0$ such that $B_d(x,r) \subset U_i$. Since $U_i \subset U$, r has the desired property $B_d(x,r) \subset U$.

Assume now that $\mathscr{T}$ is a topology on X which contains all the balls and we prove that $\mathscr{T}_d \subset \mathscr{T}$. Let $U \in \mathscr{T}_d$ and we prove $U \in \mathscr{T}$. From the definition of $\mathscr{T}_d$, for each $x \in U$ we find $r_x > 0$ with

$$\{x\} \subset B(x; r_x) \subset U.$$

Taking the union over all $x \in U$ we deduce that

$$U \subset \bigcup_{x \in U} B(x; r_x) \subset U.$$

Hence $U = \cup_x B(x, r_x)$ and then, since all the balls belong to $\mathscr{T}$, U belongs itself to $\mathscr{T}$.

Definition 2.6. A topological space $(X, \mathscr{T})$ is called **metrizable** if there exists a metric d on X such that $\mathscr{T} = \mathscr{T}_d$.

Remark 2.7. And here is one of the important problems in topology:

The metrizability problem: which topological spaces are metrizable?

More exactly, one would like to find the special properties that a topology must have so that it is induced by a metric. Such properties will be discussed throughout the entire course.

The most basic metric is the Euclidean metric on $\mathbb{R}^n$ which was behind the entire discussion of Chapter 1. Also, the Euclidean metric can be (and was) used as a metric on any subset $A \subset \mathbb{R}^n$.

Conventions 2.8. The topology on $\mathbb{R}^n$ induced by the Euclidean metric is called the **Euclidean topology** on $\mathbb{R}^n$ and is denoted $\mathscr{T}_{\mathrm{Eucl}}^{\mathbb{R}^n}$. Whenever we talk about "the space $\mathbb{R}^n$" without specifying the topology, we always mean the Euclidean topology. Similarly for subsets $A \subset \mathbb{R}^n$, with the notation $\mathscr{T}_{\mathrm{Eucl}}^{A}$

Example 2.9. (interesting topologies on $\mathbb{R}$) On $\mathbb{R}$ we have seen several interesting topologies: the $\mathscr{T}_{\mathrm{triv}}$, $\mathscr{T}_{\mathrm{dis}}$, $\mathscr{T}_{\mathrm{cf}}$ and $\mathscr{T}_{\mathrm{cc}}$ from Example 2.4 as well as the Euclidean topology $\mathscr{T}_{\mathrm{Eucl}}^{\mathbb{R}}$. For the Euclidean one notice that, for $U \subset \mathbb{R}$:

$$\begin{aligned} U \in \mathscr{T}_{\mathrm{Eucl}}^{\mathbb{R}} &\iff \quad \forall\ x \in U \quad \exists\ (a,b) \subset \mathbb{R} \quad \text{s.t.} \quad x \in (a,b) \subset U \\ &\iff \quad U \text{ is a union of open intervals} \end{aligned} \tag{2.1.1}$$

and, furthermore, $\mathscr{T}_{\mathrm{Eucl}}^{\mathbb{R}}$ is the smallest topology on $\mathbb{R}$ which contains all the open intervals.

Another interesting topology is **the lower limit topology** on $\mathbb{R}$, denoted $\mathscr{T}_l$, which has exactly the same properties as $\mathscr{T}_{\mathrm{Eucl}}^{\mathbb{R}}$ but with respect to intervals of type $[a,b)$ instead of (a,b): for $U \subset \mathbb{R}$:

$$\begin{aligned} U \in \mathscr{T}_l &\iff \quad \forall\ x \in U \quad \exists\ [a,b) \subset \mathbb{R} \quad \text{s.t.} \quad x \in [a,b) \subset U \\ &\iff \quad U \text{ is a union of intervals of type } [a,b). \end{aligned} \tag{2.1.2}$$

and, furthermore, $\mathscr{T}_l$ is the smallest topology on $\mathbb{R}$ which contains all intervals of type $[a,b)$ with $a,b \in \mathbb{R}$. To see this one takes the first line above as definition of $\mathscr{T}_l$ and then notices that the arguments used for metric topologies still apply to prove the rest. Since any open interval can be written as a union of intervals of type $[a,b)$ it follows that

$$\mathscr{T}_{\mathrm{Eucl}}^{\mathbb{R}} \subset \mathscr{T}_l.$$

Exercise 2.10. Provide the complete proofs for the statements we have made about the lower limit topology.

General methods to construct topologies will be discussed in the next chapter. Here we mention:

Example 2.11. (subspace topology) In general, given a topological space $(X, \mathscr{T})$, any subset $A \subset X$ inherits a topology on its own. More precisely, one defines the restriction of $\mathscr{T}$ to A, or the topology induced by $\mathscr{T}$ on A (or simply the **induced topology** on A) as:

$$\mathscr{T}|_A := \{B \subset A : B = U \cap A \text{ for some } U \in \mathscr{T}\}.$$

Exercise 2.12. Show that $\mathscr{T}|_A$ is indeed a topology.

Conventions 2.13. Given a topological space $(X, \mathscr{T})$, whenever we deal with a subset $A \subset X$ without specifying the topology on it, we always consider A endowed with $\mathscr{T}|_A$.

The following shows that, for subsets of $\mathbb{R}^n$, there is no conflict between our last two conventions.

Exercise 2.14. Show that for any $A \subset \mathbb{R}^n$, the restriction to A of the Euclidean topology of $\mathbb{R}^n$ coincides with the Euclidean topology of A: $\mathscr{T}^{\mathbb{R}^n}_{\text{Eucl}}|_A = \mathscr{T}^A_{\text{Eucl}}$.

Definition 2.15. Given a topological space $(X, \mathscr{T})$ and $A, B \subset X$, we say that B is **open in** A if $B \subset A$ and B is an open in the topological space $(A, \mathscr{T}|_A)$. Similarly, we say that B is **closed in** A if $B \subset A$ and B is closed in the topological space $(A, \mathscr{T}|_A)$.

Exercise 2.16. The interval $[0,1) \subset \mathbb{R}$:

a) is neither open nor closed in $(-1,2)$.
b) is open in $[0,\infty)$ but it is not closed in $[0,\infty)$.
c) is closed in $(-1,1)$ but it is not open in $(-1,1)$.
d) it is both open and closed in $(-\infty,-1) \cup [0,1) \cup (2,\infty)$.

2.2 Continuous functions

Definition 2.17. Given two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, and a function $f : X \to Y$, we say that f is **continuous** (with respect to the topologies $\mathscr{T}_X$ and $\mathscr{T}_Y$) if:

$$f^{-1}(U) \in \mathscr{T}_X \quad \forall \quad U \in \mathscr{T}_Y.$$

Exercise 2.18. Show that a map $f : X \to Y$ between two topological spaces (X with some topology $\mathscr{T}_X$, and Y with some topology $\mathscr{T}_Y$) is continuous if and only if $f^{-1}(A)$ is closed in X for any closed subspace A of Y.

Example 2.19. Some extreme examples first:

1. If Y is given the trivial topology then, for any other topological space $(X, \mathscr{T}_X)$, any function $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_{\text{triv}})$ is automatically continuous.
2. If X is given the discrete topology then, for any other topological space $(Y, \mathscr{T}_Y)$, any function $f : (X, \mathscr{T}_{\text{dis}}) \to (Y, \mathscr{T}_Y)$ is automatically continuous.
3. The composition of two continuous functions is continuous: If $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ and $g : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$ are continuous, then so is $g \circ f : (X, \mathscr{T}_X) \to (Z, \mathscr{T}_Z)$. Indeed, for any $U \in \mathscr{T}_Z$, $V := g^{-1}(U) \in \mathscr{T}_Y$, hence
$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$$
must be in $\mathscr{T}_X$.
4. For any topological space $(X, \mathscr{T})$, the identity map $\text{Id}_X : (X, \mathscr{T}) \to (X, \mathscr{T})$ is continuous. More generally, if $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on X, then the identity map $\text{Id}_X : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous if and only if $\mathscr{T}_2$ is smaller than $\mathscr{T}_1$.

Example 2.20. Given $f : X \to Y$ a map between two metric spaces (X, d) and (Y, d'), Exercise 1.49 says that $f : X \to Y$ is continuous as a map between metric spaces (in the sense discussed in the previous chapter) if and only if $f : (X, \mathscr{T}_d) \to (Y, \mathscr{T}_{d'})$ is continuous as a map between topological spaces.

That is good news: all functions $f : \mathbb{R}^n \to \mathbb{R}^m$ between Euclidean spaces that we knew (e.g. from the Analysis course) to be continuous, are continuous in the sense of the previous definition as well. This applies in particular to all the elementary functions such as polynomial ones, $\exp, \sin, \cos$, etc.

Even more, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ such that $f(A) \subset B$, then the restriction of f to A, viewed as a function from A to B, is automatically continuous (check that!). Finally, the usual operations of continuous functions are continuous:

Exercise 2.21. Let X be a topological space, $f_1, \ldots, f_n : X \to \mathbb{R}$ and consider

$$f := (f_1, \ldots, f_n) : X \to \mathbb{R}^n.$$

Show that f is continuous if and only if $f_1, \ldots, f_n$ are. Deduce that the sum and the product of two continuous functions $f, g : X \to \mathbb{R}$ are themselves continuous.

2.3 Homeomorphisms

Definition 2.22. Given two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, a **homeomorphism** between them is a bijective function $f : X \to Y$ with the property that f and f^{-1} are continuous. We say that X and Y are **homeomorphic** if there exists a homeomorphism between them.

Remark 2.23. In the definition of the notion of homeomorphism (and as we have seen already in the previous chapter), it is not enough to require that $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous and bijective (it may happen that f^{-1} is not continuous!). For f^{-1} to be continuous, one would need that for each $U \subset X$ open, $f(U) \subset Y$ is open. Functions with this property (that send opens to opens) are called **open maps**.

For instance, the function

$$f : [0, 2\pi) \to S^1, f(t) = (\cos(t), \sin(t)),$$

that we also discussed in the previous chapter, is continuous and bijective, but it is not a homeomorphism. More precisely, it is not open: $[0, \pi)$ is open in X, while $f([0, \pi))$ is a half circle closed at one end and open at the other- hence not open (Figure 1.4 in the previous chapter).

Remark 2.24. We would like to emphasize that the notion of "homeomorphism" is the correct notion "isomorphism in the topological world". A homeomorphism $f : X \to Y$ allows us to move from X to Y and backwards carrying along any topological argument (i.e. any argument which is based on the notion of opens) and without losing any topological information. For this reason, in topology, homeomorphic spaces are not viewed as being different from each other.

Another important question in Topology is:

Central question of topology: how do we decide if two spaces are homeomorphic or not?

Actually, all the topological properties that we will discuss in this course (the countability axioms, Hausdorffness, connectedness, compactness, etc) could be motivated by this problem. For instance, try to prove now that $(0, 2)$ and $(0, 1) \cup (1, 2)$ are not homeomorphic (if you managed, you have probably discovered the notion of connectedness). Try to prove that the open disk and the closed disk are not homeomorphic (if you managed, you have probably discovered the notion of compactness). Let us be slightly more precise about the meaning of "topological property".

Conventions 2.25. We call **topological property** any property $\mathscr{P}$ of topological spaces (that a space may or may not satisfy) such that, if X and Y are homeomorphic, then X has the property $\mathscr{P}$ if and only if Y has it.

For instance, the property of being metrizable (see Definition 2.6) is a topological property:

Exercise 2.26. Let X and Y be two homeomorphic topological spaces. Show that X is metrizable if and only if Y is.

Definition 2.27. A continuous function $f : X \to Y$ (between two topological spaces) is called an **embedding** if f is injective and, as a function from X to its image $f(X)$, it is a homeomorphism (where $f(X) \subset Y$ is endowed with the induced topology).

Example 2.28. There are injective continuous maps that are not embeddings. This is the case already with the function $f(\alpha) = (\cos(\alpha), \sin(\alpha))$ already discussed, viewed as a function $f : [0, 2\pi) \to \mathbb{R}^2$.

Remark 2.29. Again, one of the important questions in Topology is:

The embedding problem: understand when a space X can be embedded in another given space Y

When $Y = \mathbb{R}^2$, that means intuitively that Xcan be pictured topologically on a piece of paper. When $Y = \mathbb{R}^3$, it is about being able to make models of X in space. Of course, one of the most interesting versions of this question is whether X can be embedded in some $\mathbb{R}^N$ for some N. As we have seen, the torus and the Moebius band can be embedded in $\mathbb{R}^3$; one can *prove* that they cannot be embedded in $\mathbb{R}^2$; also, one can *prove* that the Klein bottle cannot be embedded in $\mathbb{R}^3$. However, all these proofs are far from trivial.

2.4 Neighborhoods and convergent sequences

Definition 2.30. Given a topological space $(X, \mathscr{T}_X)$, $x \in X$, **a neighborhood of** x in the topological space $(X, \mathscr{T}_X)$ is any subset $V \subset X$ with the property that there exists $U \in \mathscr{T}_X$ such that

$$x \in U \subset V.$$

When V is itself open, we call it an **open neighborhood** of x. We denote:

$$\mathscr{T}_X(x) := \{U \in \mathscr{T} : x \in U\}, \quad \mathscr{N}_{\mathscr{T}_X}(x) := \{V \subset X : \ \exists\, U \in \mathscr{T}_X(x) \text{ such that } U \subset V\}$$

where the notation $\mathscr{N}_{\mathscr{T}_X}(x)$ is further simplified to $\mathscr{N}_X(x)$ when there is no danger of confusion.

Example 2.31. In a metric space (X,d), from the definition of $\mathscr{T}_d$ we deduce:

$$\mathscr{N}_{\mathscr{T}_d}(x) = \{V \subset X : \ \exists\, \varepsilon > 0 \text{ such that } B(x,\varepsilon) \subset V\}. \tag{2.4.1}$$

Remark 2.32. What are neighborhoods good for? They are the " topological pieces" which are relevant when looking at properties which are "local", in the sense that they depend only on what happens "near points". For instance, we can talk about continuity at a point.

Definition 2.33. We say that a function $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is **continuous at** x if

$$f^{-1}(V) \in \mathscr{N}_X(x) \quad \forall\ V \in \mathscr{N}_Y(f(x)). \tag{2.4.2}$$

Proposition 2.34. *A function is continuous if and only if it is continuous at all points.*

Proof. Assume first that f is continuous, $x \in X$. For $V \in \mathscr{N}_Y(f(x))$, there exists $U \in \mathscr{T}_Y(f(x))$ with $U \subset V$; then $f^{-1}(U)$ is open, contains x and is contained in $f^{-1}(V)$; hence $f^{-1}(V) \in \mathscr{N}_X(x)$. For the converse, assume that f is continuous at all points. Let $U \subset Y$ open; we prove that $f^{-1}(U)$ is open. For each $x \in f^{-1}(U)$, continuity at x implies that $f^{-1}(U)$ is a neighborhood of x, hence we find $U_x \subset f^{-1}(U)$, with U_x-open containing x. It follows that $f^{-1}(U)$ is the union of all U_x with $x \in f^{-1}(U)$, hence it must be open.

Neighborhoods also allow us to talk about convergence.

Definition 2.35. Given a sequence $(x_n)_{n\geq 1}$ of elements in a topological space $(X, \mathscr{T}_X)$, $x \in X$, we say that $(x_n)_{n\geq 1}$ **converges to** x in $(X, \mathscr{T})$, and we write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) if for each $V \in \mathscr{N}_X(x)$, there exists an integer n_V such that

$$x_n \in V \quad \forall\, n \geq n_V. \tag{2.4.3}$$

Example 2.36. Let X be a set. Then, in $(X, \mathscr{T}_{\text{triv}})$, any sequence $(x_n)_{n\geq 1}$ of points in X converges to any $x \in X$. In contrast, in $(X, \mathscr{T}_{\text{dis}})$, a sequence $(x_n)_{n\geq 1}$ converges to an $x \in X$ if and only if $(x_n)_{n\geq 1}$ is stationary equal to x, i.e. there exists n_0 such that $x_n = x$ for all $n \geq n_0$.

To clarify the relationship between convergence and continuity, we introduce:

Definition 2.37. Let $(X,\mathscr{T}_X)$, $(Y,\mathscr{T}_Y)$ be topological spaces, $f:X\to Y$. We say that f is **sequentially continuous** if, for any sequence $(x_n)_{n\geq 1}$ in X, $x\in X$, we have:

$$x_n\to x \text{ in } (X,\mathscr{T}_X)\Longrightarrow f(x_n)\to f(x) \text{ in } (Y,\mathscr{T}_Y).$$

Theorem 2.38. *Any continuous function is sequentially continuous.*

Proof. Assume that $x_n\to x$ (in $(X,\mathscr{T}_X)$). To show that $f(x_n)\to f(x)$ (in $(Y,\mathscr{T}_Y)$, let $V\in\mathscr{N}_Y(f(x))$ arbitrary and we have to find n_V such that $f(x_n)\in V$ for all $n\geq n_V$. Since f is continuous, we must have $f^{-1}(V)\in\mathscr{N}_X(x)$; since $x_n\to x$, we find n_V such that $x_n\in f^{-1}(V)$ for all $n\geq n_V$. Clearly, this n_V has the desired properties.

Definition 2.39. Let $(X,\mathscr{T})$ be a topological space and $x\in X$. A **basis of neighborhoods** of x (in the topological space $(X,\mathscr{T})$) is a collection $\mathscr{B}_x$ of neighborhoods of x with the property that

$$\forall\, V\in\mathscr{T}(x)\quad \exists\, B\in\mathscr{B}_x:\quad B\subset V.$$

Example 2.40. If (X,d) is a metric space, $x\in X$, the family of all balls centered at x,

$$\mathscr{B}_d(x):=\{B(x;\varepsilon):\varepsilon>0\},\tag{2.4.4}$$

is a basis of neighborhoods of x.

Remark 2.41. What are bases of neighborhoods good for? They are collections of neighborhoods which are "rich enough" to encode the local topology around the point. I.e., instead of proving conditions for all $V\in\mathscr{N}_X(x)$, it is enough to do it only for the elements of a basis. For instance, in the definition of convergence $x_n\to x$ (Definition 2.35), if we have a basis $\mathscr{B}_x$ of neighborhoods of x, it suffices to check the condition from the definition only for neighborhoods $V\in\mathscr{B}_x$ (why?). In the case of a metric space (X,d), we recover the more familiar description of convergence: using the basis (2.4.4) we find that $x_n\to x$ if and only if:

$$\forall\,\varepsilon>0,\ \exists\, n_\varepsilon\in\mathbb{N}:d(x_n,x)<\varepsilon\ \forall\, n\geq n_\varepsilon.$$

A similar discussion applies to the notion of continuity at a point- Definition 2.33: if we have a basis $\mathscr{B}_{f(x)}$ of neighborhoods of $f(x)$, then it suffices to check (2.4.2) for all $V\in\mathscr{B}_{f(x)}$. As before, if f is a map between two metric spaces (X,d_X) and (Y,d_Y), we find the more familiar description of continuity: using the basis (2.4.4) (with x replaced by $f(x)$), and using (2.4.1), we find that f is continuous at x if and only if, for all $\varepsilon>0$, there exists $\delta>0$ such that

$$d_Y(f(y),f(x))<\varepsilon\quad\forall\, y\in X \text{ satisfying } d_X(y,x)<\delta.$$

Definition 2.42. We say that $(X,\mathscr{T})$ satisfies the first countability axiom, or that it is **1st-countable**, if for each point $x\in X$ there exists a countable basis of neighborhoods of x.

Exercise 2.43. Show that the first-countability is a topological property.

Example 2.44. Any metric space (X,d) is 1st countable: for $x \in X$,

$$\mathscr{B}'_d(x) := \left\{ B\left(x; \frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

is a countable basis of neighborhoods of x. Hence, in relation with the metrizability problem, we deduce: if a topological space is metrizable, then it must be 1st countable.

Exercise 2.45. Let $(X, \mathscr{T})$ be a topological space and $x \in X$. Show that if x admits a countable basis of neighborhoods, then one can also find a decreasing one, i.e. one of type

$$\mathscr{B}_x = \{B_1, B_2, B_3, \ldots\}, \text{with } \ldots \subset B_3 \subset B_2 \subset B_1.$$

(Hint: $B_n = V_1 \cap V_2 \cap \ldots \cap V_n$).

The role of the first countability axiom is a theoretical one: "it is the axiom under which the notion of sequence can be used in its full power". For instance, Theorem 2.38 can be improved:

Theorem 2.46. *If X is 1st countable (in particular, if X is a metric space) then a map $f : X \to Y$ is continuous if and only if it is sequentially continuous.*

Proof. We are left with the converse implication. Assume f is sequentially continuous but not continuous. By Prop. 2.34, we find $x \in X$ such that f is not continuous at x. Hence we find $V \in \mathscr{N}_Y(f(x))$ such that $f^{-1}(V) \notin \mathscr{N}_X(x)$. Let $\{B_n : n \in \mathbb{N}\}$ be a countable basis of neighborhoods of x; by the previous exercise, we may assume it is decreasing. Since $f^{-1}(V) \notin \mathscr{N}_X(x)$, for each n we find $x_n \in B_n - f^{-1}(V)$. Since $x_n \in B_n$, it follows that $(x_n)_{n\geq 1}$ converges to x (see Remark 2.41). But note that $(f(x_n))$ cannot converge to $f(x)$ since $f(x_n) \notin V$ for all n. This contradicts the hypothesis.

Another good illustration of the fact that, under the first-countability axiom, "convergent sequences contain all the information about the topology", is given in Exercise 10.38. Another illustration is the characterisation of Hausdorffness (Theorem 2.58 below).

2.5 Inside a topological space: closure, interior and boundary

Definition 2.47. Let $(X,\mathscr{T})$ be a topological space. Given $A\subset X$, define:

- **the interior of** A:
$$\overset{\circ}{A}=\bigcup_{U-\text{open contained in } A} U.$$
(The union is over all the subsets U of A which are open in $(X,\mathscr{T})$). It is sometimes denoted by $\mathrm{Int}(A)$. Note that $\overset{\circ}{A}$ is open, is contained in A, and it is the largest set with these properties.
- **the closure of** A:
$$\overline{A}=\bigcap_{F-\text{closed containing } A} F.$$
(The intersection is over all the subsets F of X which contain A and are closed in $(X,\mathscr{T})$). It is sometimes denoted by $\mathrm{Cl}(A)$. Note that $\overline{A}$ is closed, contains A, and it is the smallest set with these properties.
- **the boundary of** A:
$$\partial(A)=\overline{A}-\overset{\circ}{A}.$$

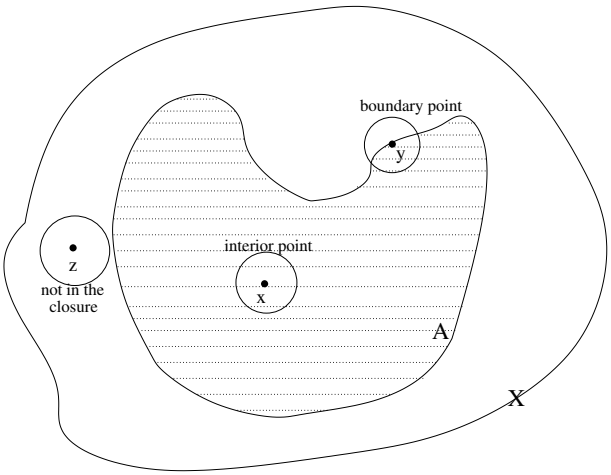


Fig. 2.1

Lemma 2.48. *Let $(X,\mathscr{T})$ be a topological space, $x\in X$. Then:*

(a) $x\in\overset{\circ}{A}$ if and only if there exists $U\in\mathscr{T}(x)$ such that $U\subset A$.
(b) $x\in\overline{A}$ if and only if, for all $U\in\mathscr{T}(x)$, $U\cap A\neq\emptyset$.
(c) If $(X,\mathscr{T})$ is metrizable (or just 1st countable) then $x\in\overline{A}$ if and only if there exists a sequence $(a_n)_{n\geq 1}$ of elements of A such that $a_n\to x$.

Furthermore, in the first two items one may replace $\mathscr{T}(x)$ by any basis $\mathscr{B}_x$ of neighborhoods around x.

See Figure 2.1.

Proof. (of the lemma) You should first convince yourself that (a) is easy; we prove here (b) and (c). To prove the equivalence in (b), is sufficient to prove the equivalence of the negations, i.e.

$$[x \notin \overline{A}] \Longleftrightarrow [\exists\, U \in \mathscr{B}_x : U \cap A = \emptyset].$$

From the definition of $\overline{A}$, the left hand side is equivalent to:

$$\exists\, F - \text{closed} : A \subset F, x \notin F.$$

Since closed sets are those of type $F = X - U$ with U-open, this is equivalent to

$$\exists\, U - \text{open} : A \cap U = \emptyset, x \in U,$$

i.e.: there exists $U \in \mathscr{T}(x)$ such that $U \cap A = \emptyset$. On the other hand, any $U \in \mathscr{T}(x)$ contains at least one $B \in \mathscr{B}_x$, and the condition $U \cap A = \emptyset$ will not be destroyed if we replace U by B. This concludes the proof of (b). For (c), let us assume we are in the metric case (the more general one is left as an exercise). First assume that $x = \lim a_n$ for some sequence of elements of A. Then, for any $U \in \mathscr{T}(x)$, we find n_U such that $a_n \in U$ for all $n \geq n_U$, which shows that $U \cap A \neq \emptyset$. By (b), $x \in \overline{A}$. For the converse, one uses that fact that $B(x, \frac{1}{n}) \cap A \neq \emptyset$ hence, for each n, we find $a_n \in A$ with $d(a_n, x) < \frac{1}{n}$. Clearly $a_n \to a$.

Example 2.49. Take the "open disk" in the plane

$$A = \overset{\circ}{D}{}^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}.$$

Then the interior of A is A itself (it is open!), the closure is the "closed disk"

$$\overline{A} = D^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\},$$

while the boundary is the unit circle

$$\partial(A) = S^1 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}.$$

Example 2.50. Take $A = [0,1) \cup \{2\} \cup [3,4)$ in $X = \mathbb{R}$. Using the lemma (and the basis given by open intervals) we find

$$\overset{\circ}{A} = (0,1) \cup (3,4),\ \overline{A} = [0,1] \cup \{2\} \cup [3,4],\ \partial(A) = \{0,1,2,3,4\}.$$

However, considering A inside $X' = [0,4)$ (with the topology induced from $\mathbb{R}$),

$$\overset{\circ}{A} = [0,1) \cup (3,4),\ \overline{A} = [0,1] \cup \{2\} \cup [3,4), \partial(A) = \{1,2,3\}.$$

For the case of metric spaces, let us point out the following corollary. To state it, recall that given a metric space (X,d), $A \subset X$, and $x \in X$, one defines the distance between x and A as

$$d(x,A) = \inf\{d(x,a) : a \in A\}.$$

Corollary 2.51. *If A is a subspace of a metric space (X,d), $x \in X$, then the following are equivalent:*
(1) $x \in \overline{A}$.
(2) there exists a sequence (a_n) of elements of A such that $a_n \to x$.
(3) $d(x,A) = 0$.

Proof. The equivalence of (1) and (2) follows directly from (c) of the lemma. Next, the condition (3) means that, for all $\varepsilon > 0$, there exists $a \in A$ such that $d(x,a) < \varepsilon$. In other words, $A \cap B(x,\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Using (b) of the lemma (with $\mathscr{B}_x$ being the collection of all balls centered at x), we find that (3) is equivalent to (1).

Definition 2.52. Given a topological space X, a subset $A \subset X$ is called dense in X if $\overline{A} = X$.

Example 2.53. $\overset{\circ}{D}{}^n$ is dense in D^n; $\mathbb{Q}$ is dense in $\mathbb{R}$.

2.6 Hausdorffness; 2nd countability; topological manifolds

One of the powers of the notion of topological space comes from its generality, which gives it a great flexibility when it comes to examples and general constructions. However, in many respects the definition is "too general". For instance, for proving interesting results one often has to impose extra-axioms. Sometimes these axioms are rather strong (e.g. compactness), but sometimes they are rather weak (in the sense that most of the interesting examples satisfy them anyway). The most important such (weak) axiom is "Hausdorffness". This axiom is also important for the metrizability problem, for which we have to understand the special topological properties that a topology must satisfy in order to be induced by a metric. And Hausdorffness is the most basic one.

Definition 2.54. We say that a topological space $(X,\mathscr{T})$ is **Hausdorff** if for any $x,y\in X$ with $x\neq y$, there exist $V\in\mathscr{N}_X(x)$ and $W\in\mathscr{N}_X(y)$ such that $V\cap W=\emptyset$.

Example 2.55. Looking at the extreme topologies: $\mathscr{T}_{\text{triv}}$ is not Hausdorff (unless X is empty or consists of one point only), while $\mathscr{T}_{\text{dis}}$ is Hausdorff. In the light of the Hausdorffness property, the cofinite topology $\mathscr{T}_{\text{cf}}$ becomes more interesting (see Exercise 10.68).

Exercise 2.56. Show that Hausdorffness is a topological property.

As promised, one has:

Proposition 2.57. *Any metric space is Hausdorff.*

Proof. Given (X,d), $x,y\in X$ distinct, we must have $r:=d(x,y)>0$. We then choose $V=B(x;\frac{r}{2})$, $W=B(y;\frac{r}{2})$. We claim these are disjoint. If not, we find z in their intersection, i.e. $z\in X$ such that $d(x,z)$ and $d(y,z)$ are both less than $\frac{r}{2}$. From the triangle inequality for d we obtain the following contradiction

$$r=d(x,y)\leq d(x,z)+d(z,y)<\frac{r}{2}+\frac{r}{2}=r.$$

However, one of the main reasons that Hausdorffness is often imposed comes from the fact that, under it, sequences behave "as expected".

Theorem 2.58. *Let $(X,\mathscr{T})$ be a topological space. If X is Hausdorff, then every sequence $(x_n)_{n\geq 1}$ has at most one limit in X. The converse holds if we assume that $(X,\mathscr{T})$ is 1st countable.*

Proof. Assume that X is Hausdorff. Assume that there exists a sequence $(x_n)_{n\geq 1}$ in X converging both to $x\in X$ and $y\in X$, with $x\neq y$; the aim is to reach a contradiction. Choose $V\in\mathscr{N}_X(x)$ and $W\in\mathscr{N}_X(y)$ such that $V\cap W=\emptyset$. Then we find n_V and n_W such that $x_n\in V$ for all $n\geq n_V$, and similarly for W. Choosing $n>\max\{n_V,n_W\}$, this will contradict the fact that V and W are disjoint.

Let's now assume that X is 1st countable and each sequence in X has at most one limit, and we prove that X is Hausdorff. Assume it is not. We find then $x\neq y$ two elements of X such that $V\cap W\neq\emptyset$ for all $V\in\mathscr{N}_X(x)$ and $W\in\mathscr{N}_X(y)$. Choose $\{V_n:n\geq 1\}$ and $\{W_n:n\geq 1\}$ bases of neighborhoods of x and y, which we may assume to be decreasing (cf. Exercise 2.45). For each n, we find an element $x_n\in V_n\cap W_n$. As in the previous proofs, this implies that x_n converges both to x and to y- which contradicts the hypothesis.

Besides Hausdorffness, there is another important axiom that one often imposes on the spaces one deals with (especially on the spaces that arise in Geometry).

Definition 2.59. Let $(X,\mathscr{T})$ be a topological space. A **basis of the topological space** $(X,\mathscr{T})$ is a family $\mathscr{B}$ of opens of X with the property that any open $U\subset X$ can be written as a union of opens that belong to $\mathscr{B}$.

We say that $(X,\mathscr{T})$ satisfies the second countability axiom, or that it is **second-countable** (also written **2nd countable**) if it admits a countable basis.

Exercise 2.60. Given a topological space $(X,\mathscr{T})$ and a family $\mathscr{B}$ of opens of X, show that $\mathscr{B}$ is a basis of $(X,\mathscr{T})$ if and only if, for each $x\in X$,

$$\mathscr{B}_x := \{B\in\mathscr{B} : x\in B\}$$

is a basis of neighborhoods of x. Deduce that any 2nd countable space is also 1st countable.

Example 2.61. For $(\mathbb{R},\mathscr{T}_{\text{Eucl}})$,

$$\mathscr{B} := \{(a,b) : a,b\in\mathbb{R}\},\quad \mathscr{B}^{\mathbb{Q}} := \{(p,q) : p,q\in\mathbb{Q}\}$$

are both bases, the second one being countable. In particular, $(\mathbb{R},\mathscr{T}_{\text{Eucl}})$ is 2nd countable.

Example 2.62. For $(\mathbb{R},\mathscr{T}_l)$, by the very construction of the lower limit topology (see (2.1.2)), we see that

$$\mathscr{B}_l := \{[a,b) : a,b\in\mathbb{R}\}$$

is a basis. However,

$$\mathscr{B}_l^{\mathbb{Q}} := \{[p,q) : p,q\in\mathbb{Q}\}$$

is not a basis! One can actually prove that $(\mathbb{R},\mathscr{T}_l)$ is not even 2nd countable, but that is a bit more difficult.

Example 2.63. In a metric space (X,d), the collection of all balls

$$\mathscr{B}_d := \{B(x,r) : x\in X, r>0\}$$

is a basis for the topology $\mathscr{T}_d$ (see the end of the proof of Proposition 2.5). Although metric spaces are always 1st countable (cf. Example 2.44), not all are 2nd countable. However:

Example 2.64. For $\mathbb{R}^n$, one can restrict to balls centered at points with rational coordinates:

$$\mathscr{B}_{\text{Eucl}}^{\mathbb{Q}} := \left\{B\left(x,\frac{1}{k}\right) : x\in\mathbb{Q}^n, k\in\mathbb{Q}_+\right\}.$$

This is a countable family since $\mathbb{Q}$ is countable and products of countable sets are countable.

Exercise 2.65. Show that $\mathscr{B}_{\text{Eucl}}^{\mathbb{Q}}$ is a basis of $\mathbb{R}^n$. Deduce that any $A\subset\mathbb{R}^n$ is 2nd countable.

Finally, we arrive at the notion of topological manifold.

Definition 2.66. An n-dimensional **topological manifold** is any Hausdorff, 2nd countable topological space X which has the following property: any point $x\in X$ admits an open neighborhood U which is homeomorphic to $\mathbb{R}^n$.

Remark 2.67. Of course, the most important condition is the one requiring X to be locally homeomorphic to $\mathbb{R}^n$. A pair (U,χ) consisting of an open $U \subset X$ and a homeomorphism

$$\chi : U \to \mathbb{R}^n,\ x \mapsto \chi(x) = (\chi_1(x),\dots,\chi_n(x))$$

is called a (local) coordinate chart for X; U is called the domain of the chart; $\chi_1(x),\dots,\chi_n(x)$ are called the coordinates of x in the chart (U,χ). Given another chart $\psi : V \to \mathbb{R}^n$,

$$c := \psi \circ \chi^{-1} : \chi(U \cap V) \to \psi(U \cap V)$$

(a homeomorphism between two opens in $\mathbb{R}^n$) is called the change of coordinates from χ to ψ (it satisfies $\psi_i(x) = c_i(\chi_1(x),\dots,\chi_n(x))$ for all $x \in U \cap V$). By definition, topological manifolds can be covered by (domains of) coordinate charts; hence they can be thought of as obtained by "patching together" several copies of $\mathbb{R}^n$, glued according to the change of coordinates.

Remark 2.68. One may wonder why the "2nd countability" condition is imposed. Well, there are many reasons. The simplest one: we do hope that a topological manifold can be embedded in some $\mathbb{R}^N$ for N large enough. However, as the previous exercise shows, this would imply that X must be 2nd countable anyway. Also, the 2nd countability condition implies that X can be covered by a countable family of coordinate charts (see Exercise 10.61).

Example 2.69. Of course, $\mathbb{R}^n$ is itself a topological manifold. Using the stereographic projection (see the previous chapter), we see that the spheres S^n are topological n-manifolds. But note that, while the open disks are topological manifolds, the closed disks are not.

Exercise 2.70. Show that the torus is a 2-dimensional topological manifold. What about the Klein bottle? What about the Moebius band? Try to define "manifolds with boundary".

2.7 More on separation; normality and Urysohn's lemma

The Hausdorffness is just one of the possible "separation axioms" that one may impose (the most important one!). Such separation axioms are relevant to the metrizability problem, as they are automatically satisfied by metric spaces. Here is the precise definition.

Definition 2.71. We say that two subspaces A and B of a topological space $(X,\mathscr{T})$ can be **separated topologically** (or simply separated) if there are open sets U and V such that

$$A \subset U, B \subset V, \text{ and } U \cap V = \emptyset.$$

A topological space is called **normal** if any two disjoint *closed* subsets can be separated topologically.

Example 2.72. When A and B are closed disjoint intervals inside $\mathbb{R}$ (with the Euclidean topology), then no matter how close they are to each other they can always be separated. For instance, for $A = (-\infty, 0]$, $B = [\varepsilon, 10]$ with $\varepsilon > 0$ very small, one can always choose $U = (-\infty, \frac{\varepsilon}{2})$, $V = (\frac{\varepsilon}{2}, 11)$. However, if we do not require A *and* B to be closed, one allows for examples like:

$$A = (-\infty, 0], \quad B = (0, 10]. \tag{2.7.1}$$

Exercise 2.73. Please show that (2.7.1) above, seen as subspace of the Euclidean line, cannot be separated? Then show that in *any* topological space X, if $A, B \subset X$ can be separated then the closure $\overline{A}$ of A must be disjoint from B.

Example 2.74. In any metric space (X,d), any two disjoint closed subsets A and B can be separated: indeed,

$$U = \{x \in X : d(x,A) < d(x,B)\}, \; V = \{x \in X : d(x,A) > d(x,B)\} \tag{2.7.2}$$

are disjoint opens containing A and B. To check that these are indeed opens one can either proceed directly or make use of the continuity of the function $d_A : X \to \mathbb{R}$, $x \mapsto d(x,A)$ (see Exercise 10.18) and the similar function d_B. Hence any metric space is normal. Together with Proposition 2.57, we obtain necessary conditions for metrizability:

Corollary 2.75. *If a space X is metrizable then it must be both Hausdorff as well as normal.*

Remark 2.76. The previous corollary can also serve as motivation for the notion of "normality": it is a topological condition that is necessary for metrizability. Of course, one may wonder: are we maybe done? Is it true that any Hausdorff and normal space X is metrizable? Then answer is "no", but we are actually not very far from a "yes". More precisely, as we shall see later on, if the space X is also assumed to be 2nd countable, then metrizability follows (Urysohn metrization theorem).

Since the ultimate goal for such metrizability results is to produce a metric that induces the given topology on X, being able to produce $\mathbb{R}$-valued functions that interact nicely with the topology can be seen as an intermediate step. That is precisely what the Urysohn lemma presented below does (lemma which will be the main ingredient for the promised Urysohn metrization theorem).

Definition 2.77. We say that two subspaces A and B of a topological space $(X,\mathscr{T})$ can be **separated by continuous functions** if there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$, $f|_B = 1$ (and we say that f separates A and B).

Example 2.78. In any metric space X any two disjoint closed subsets $A, B \subset X$ can be separated by continuous functions. Indeed, one can use again the distance functions d_A and d_B to build a function that separates A and B:

$$f : X \to [0,1], \quad f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}. \tag{2.7.3}$$

Definition 2.77 (separation by continuous functions) is closely related to Definition 2.71 (topological separation and normality). Going in one direction is rather easy:

Lemma 2.79. *In any topological space X, if $A, B \subset X$ can be separated by continuous functions then they can also be separated topologically.*

Proof. If f separates A and B, then we can take

$$U = f^{-1}\left(\left(-\infty, \frac{1}{2}\right)\right), V = f^{-1}\left(\left(\frac{1}{2}, \infty\right)\right).$$

These are clearly disjoint, containing A and B, respectively. They are opens in X because they are pre-images of opens by a continuous map defined on X.

Example 2.80. If X is a metric space and we apply the argument from the proof to the separating function (2.7.3), then one obtains precisely the separating opens U and V from (2.7.2).

Going the other way around, from topological separation and normality to separation by continuous functions. is far from clear. After all, one starts from a topology and topological properties and one has to produce actual (continuous) functions, which are far from the obvious (constant) ones. Trying a bit using what we have learned so far really looks like trying to extract water from a stone. With all that in mind, the following is rather surprising and truly remarkable.

Theorem 2.81 (Urysohn's lemma). *If a topological space X is Hausdorff and normal then any two closed disjoint subsets $A, B \subset X$ can be separated by continuous functions.*

Before we start with the proof, we provide an equivalent reformulation of the notion of normality. The reformulation may look a bit more technical/nontransparent, but it is sometimes more useful.

Lemma 2.82. *A topological space X is normal if and only if for any*

$$A \subset O \subset X$$

with A-closed and O-open in X, there exists an open O' in X such that

$$A \subset O' \subset \overline{O'} \subset O.$$

Proof. Since $A \subset O$, A and $X - O$ are disjoint. They are both closed, hence we know that we can find disjoint opens O' and O'' such that $A \subset O'$, $X - O \subset O''$. The condition $O' \cap O'' = \emptyset$ is equivalent to $O' \subset X - O''$. Since $X - O''$ is a closed containing O', this implies $\overline{O'} \subset X - O''$. On the other hand, $X - O \subset O''$ can be re-written as $X - O'' \subset O$. Hence $\overline{O'} \subset X - O'' \subset O$.

Assume now that the condition holds, and let $A, B \subset X$ be disjoint closed subspaces. Then $O := X \setminus B$ is an open that contains A. By the hypothesis, we find an open O' such that $A \subset O' \subset \overline{O'} \subset O$. It follows that $U := O'$ and $V := X \setminus \overline{O'}$ separate A and B.

Proof (Proof of Theorem 2.81:). Fix A and B disjoint closed subsets. From now on, when saying that "A is closed" or "D is open", we mean that they are closed (open) in the given topological space $(X, \mathscr{T})$. The proof is organised as a sequence of claims. In some sense, Lemma 2.82 is "Claim 0" and will be used repeatedly.

Claim 1: Then there is a family of open sets $\{U_q : q \in \mathbb{Q}\}$ such that

(C1) $U_q = \emptyset$ for $q < 0$, U_0 contains A, $U_1 = X - B$, $U_q = X$ for $q > 1$.
(C2) $\overline{U}_q \subset U_{q'}$ for all $q < q'$.

Proof. The condition (C1) forces the definition of U_q for $q < 0$ and for $q \geq 1$. For $q = 0$, we choose U_0 to be any open set such that

$$A \subset U_0 \subset \overline{U}_0 \subset U_1.$$

This is possible since $A \cap B = \emptyset$ means that $A \subset X - B = U_1$ hence we can apply Lemma 2.82.

We are left with the construction of U_q for $q \in \mathbb{Q} \cap (0,1)$. Writing

$$\mathbb{Q} \cap [0,1] = \{q_0, q_1, q_2, \ldots\},$$

with $q_0 = 0$, $q_1 = 1$, we will define U_{q_n} by induction on n such that (C2) holds for all $q = q_i$, $q' = q_j$ with $0 \leq i, j \leq n$. Assume that U_q is constructed for $q \in \{q_0, \ldots, q_n\}$ and we construct it for $q = q_{n+1}$. Looking at all intervals of type (q_i, q_j) with $0 \leq i, j \leq n$, there is a smallest one containing q_{n+1}. Call it (q_a, q_b). Since $q_a < q_b$, by the induction hypothesis we have

$$\overline{U}_{q_a} \subset U_{q_b}$$

hence, by Lemma 2.82, we find an open U such that

$$\overline{U}_{q_a} \subset U \subset \overline{U} \subset U_{q_b}.$$

Define $U_{q_{n+1}} = U$. We have to check that (C2) holds for $q, q' \in \{q_0, \ldots, q_{n+1}\}$. Fix q, q'. If $q \neq q_{n+1}$ and $q' \neq q_{n+1}$, $\overline{U}_q \subset U_{q'}$ holds by the induction hypothesis. Hence we may assume that $q = q_{n+1}$ or $q' = q_{n+1}$. We treat the case $q = q_{n+1}$, the other one being similar. Write $q' = q_j$ with $j \in \{0, 1, \ldots, n\}$. The assumption is that $q_{n+1} < q_j$ and we want to show that

$$\overline{U}_{q_{n+1}} \subset U_{q_j}.$$

But, since $q_{n+1} < q_j$ and (q_a, q_b) is the smallest interval of this type containing q_{n+1}, we must have $q_j \geq q_b$. But then

$$\overline{U}_{q_{n+1}} = \overline{U} \subset U_{q_b} \subset U_{q_j}.$$

Claim 2: The function $f : X \to [0,1]$, $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$ satisfies:

(1) $f(x) > q \Longrightarrow x \notin \overline{U}_q$.
(2) $f(x) < q \Longrightarrow x \in U_q$.

(in particular, $f(x) = q$ for $x \in \partial U_q$).

Proof. For (1), we prove its contrapositive, i.e. that $x \in \overline{U}_q$ implies $f(x) \leq q$. Hence assume that $x \in \overline{U}_q$. From (C2) we deduce that $x \in U_{q'}$ for all $q' > q$. Hence $f(x) \leq q'$ for all $q' > q$. This implies $f(x) \leq q$. For (2), we assume that $f(x) < q$. By the definition of $f(x)$ (as an infimum), there exists $q' < q$ such that $x \in U_{q'}$. But $q' < q$ implies $U_{q'} \subset U_q$, hence $x \in U_q$.

Claim 3: $f|_A = 0$, $f|_B = 1$, and f is continuous.

Proof. The first two conditions are immediate from the definition of f and properties (C1) of the first claim. We now prove that f is continuous. We have to prove that for any open interval (a,b) in $\mathbb{R}$, and any $x \in f^{-1}((a,b))$, there exists an open U containing x such that $f(U) \subset (a,b)$. Fix (a,b) and x such that $f(x) \in (a,b)$ and look for U satisfying the desired condition. Choosing $p, q \in \mathbb{Q}$ such that

$$a < p < f(x) < q < b,$$

then $U := U_q - \overline{U}_p$ will do the job. Indeed:

1. using Claim 2, $f(x) > p$ implies $x \notin \overline{U}_p$, while $f(x) < q$ implies $x \in U_q$. Hence $x \in U$.
2. for $y \in U$ arbitrary, we have $f(y) \in (a,b)$ because:
   - $y \in U_q \subset \overline{U}_q$ which, by the previous claim, implies $f(y) \leq q < b$.
   - $y \notin \overline{U}_p$, hence $y \notin U_p$ which, by the previous claim, implies $f(y) \geq p > a$.

Chapter 3
Constructions of topological spaces

1. **Quotient topologies**

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3. **Special classes of quotients I: quotients modulo group actions**

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9. **Example: some spaces of functions**

10. **More exercises**

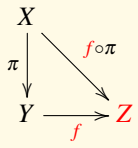
3.1 Quotient topologies

Definition 3.1. A map $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between two topological spaces is called a **topological quotient map** if it is surjective and satisfies the condition that, for $V \subset Y$, one has

$$\pi^{-1}(V) \in \mathscr{T}_X \iff V \in \mathscr{T}_Y.$$

Intuitively, one may think of this situation as saying that "$(X, \mathscr{T}_X)$ sits nicely above $(Y, \mathscr{T}_Y)$"; actually, in many examples, X is a "simpler' space which unravels Y", and then π allows one to study Y via studying the simpler X. One such example would be the exponential map $e : \mathbb{R} \to S^1$, pictured in Figure 1.3/Example 1.6. Here is a precise statement that supports this intuition.

Proposition 3.2. *Given a topological quotient map $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ then, for any other topological space Z, a function $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is.*



(3.1.1)

Proof. Recall that f being continuous means that for any $W \subset Z$ open, $f^{-1}(W)$ is open in Y. By the quotient map condition, $V := f^{-1}(W)$ is open in Y if and only if $\pi^{-1}(V) = \pi^{-1}(f^{-1}(W))$ is open in X. But $\pi^{-1}(f^{-1}(W)) = (f \circ \pi)^{-1}(W)$, hence f is continuous if and only if for any $W \subset Z$ open $(f \circ \pi)^{-1}(W)$ is open in X, i.e., if and only if $f \circ \pi$ is continuous.

Here is a simple criterion (sufficient, but not necessary).

Lemma 3.3. *Any continuous map $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ which admits a continuous right inverse, i.e., a continuous function $i : Y \to X$ satisfying $\pi \circ i = Id_Y$, is automatically a topological quotient map.*

Proof. Notice first that $\pi \circ i = \mathrm{Id}_Y$ implies that $\pi(i(Y)) = Y$, hence π is surjective. We now assume that $V \subset Y$ has the property that $U := \pi^{-1}(V) \in \mathscr{T}_X$ and we want to prove that $V \in \mathscr{T}_Y$. To that end we use the continuity of i to say that $i^{-1}(U) \in \mathscr{T}_Y$ and we notice that $i^{-1}\left(\pi^{-1}(V)\right) \in \mathscr{T}_Y$. But

$$i^{-1}(U) = i^{-1}\left(\pi^{-1}(V)\right) = (\pi \circ i)^{-1}(V) = \mathrm{Id}_Y^{-1}(V) = V.$$

Therefore $V \in \mathscr{T}_Y$ as desired.

Example 3.4. The map $\pi : \mathbb{R}^3 \setminus \{0\} \to S^2$, $\pi(x) = \frac{1}{||x||}x$ is a topological quotient because the obvious inclusion $i : S^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$ is a right inverse.

Here is an even more useful criterion; its proof and more insight will be provided after we discuss compactness.

Proposition 3.5. *(to be proven later on) Assume that $\pi : X \to Y$ is a continuous surjective map between two subspaces of Euclidean spaces (endowed with the induced (Euclidean) topology)*

$$X \subset \mathbb{R}^k, \quad Y \subset R^n.$$

If X is closed and bounded in $\mathbb{R}^k$ then π is automatically a topological quotient map.

The notion of "topological quotient map" can also be looked from a different perspective- that of construction of new topologies. To that end one starts with

$$\text{a surjective map} \quad \pi : \underbrace{(X, \mathscr{T}_X)}_{\text{topological space}} \longrightarrow \underbrace{Y}_{\text{set}}$$

and the conclusion is that Y will carry a natural topology, called **the quotient topology** on Y induced by π, denoted $\pi_*(\mathscr{T}_X)$, which is the unique topology on Y that makes π into a topological quotient map, i.e.,

$$\pi_*(\mathscr{T}_X) := \left\{V \subset Y : \pi^{-1}(V) \in \mathscr{T}_X\right\}.$$

Theorem 3.6. *$\pi_*(\mathscr{T}_X)$ is indeed a topology on Y. Moreover, it is the largest topology on Y with the property that $\pi : X \to Y$ becomes continuous.*

Proof. The axioms (T1)-(T3) for $\pi_*(\mathscr{T}_X)$ follow right away from the same axioms for $\mathscr{T}_X$ and the following properties of taking pre-images of π:

$$\pi^{-1}(Y) = X,\ \pi^{-1}(\emptyset) = \emptyset,\ \pi^{-1}(V_1 \cap V_2) = \pi^{-1}(V_1)) \cap \pi^{-1}(V_2),\ \pi^{-1}(\cup_i V_i) = \cup_i \pi^{-1}(V_i)$$

The last part follows from the definition of continuity and of $\pi_*(\mathscr{T})$.

Example 3.7. [The projective *space*] A very good illustration of the use of quotient topologies is the construction of the projective space $\mathbb{P}^n$, as a topological space. Set theoretically, $\mathbb{P}^n$ is

$$\mathbb{P}^n = \left\{l \subset \mathbb{R}^{n+1} : l \text{ is a line through the origin}\right\},$$

the set of all lines in $\mathbb{R}^{n+1}$ through the origin. Can one relate it to a topological space that we already know? The answer is immediate once one realises one can use the basic property of lines: lines are uniquely determined by any two distinct points lying on it. In particular, to specify a line $l \in \mathbb{P}^n$ it suffices to specify a point $x \in \mathbb{R}^{n+1} \setminus \{0\}$- a property that can be encoded in the surjective map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n \tag{3.1.2}$$

that sends $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ to the line through the origin and x:

$$l_x := \overline{Ox} = \{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) : \lambda \in \mathbb{R}\} \in \mathbb{P}^n.$$

Using the Euclidean topology on $\mathbb{R}^{n+1} \setminus \{0\}$, the **projective space** $\mathbb{P}^n$ is now defined as the set $\mathbb{P}^n$ endowed with the resulting quotient topology .

Example 3.8 (Gluings). Many of the cool examples of "spaces Y" can be illustrated by starting with a strip of paper and performing certain gluings to it. Mathematically the strip of paper is represented by the unit square $X = [0,1] \times [0,1]$, and the process of gluing comes with a map

$$\pi : [0,1] \times [0,1] \to Y$$

onto the outcome Y of the gluing process. The map π is the map that keeps track of the points before and after the gluing. As the square has a natural topology on it, the Euclidean topology, we are now precisely in the scenario described above, hence: the outcome Y of the gluing carries a natural topology: the resulting quotient topology.

For a more concrete example, one can look at "the Moebius band". The quotes are used to remind you that, so far, we have been talking only about models of the Moebius band, rather than one single one. For instance, one can use the mathematically concrete model $M_{R,r}$ from (1.5.1), conveniently re-parametrised by the square by writing $u = (2s-1)r$ and $a = 2\pi t$. In other words, $M_{R,r}$ is the collection of points in $\mathbb{R}^3$ of type

$$\Big(\big(R+(2s-1)r\sin(\pi t)\big)\cos(2\pi t),\ \big(R+(2s-1)r\sin(\pi t)\big)\sin(2\pi t),\ (2s-1)r\cos(\pi t)\Big) \tag{3.1.3}$$

with $(t,s)\in[0,1]\times[0,1]$, Of course, the map π sends (t,s) to this expression. Therefore, one can now endow $M_{R,r}$ with the resulting quotient topology $\pi_*(\mathscr{T}_{\text{Eucl}})$.

Exercise 3.9. On the other hand, since we have used a model that already sits inside $\mathbb{R}^3$, $M_{R,r}$ also carries the Euclidean topology. Do the two topologies on $M_{R,r}$ coincide? Use Proposition 3.5 to prove that they do.

3.2 How to handle "gluings": equivalence relations and quotients

To make the last class of examples mathematically precise in full generality, one still has to formalise the process of "gluing". In principle, we will be repeating section 1.9 but taking advantage of the precise notions of Topology that we have developed so far.

Definition 3.10. An **equivalence relation** on a set X is a subset $R\subset X\times X$ satisfying the following:

1. If $(x,y)\in R$ and $(y,z)\in R$, then $(x,z)\in R$.
2. If $(x,y)\in R$ then also $(y,x)\in R$.
3. $(x,x)\in R$ for all $x\in X$.

Given R, two elements $x,y\in X$ are said to be R**-equivalent**, and we write $x\sim_R y$, if $(x,y)\in R$.

The key point of our discussion is that one can think of an equivalence relation R on X as encoding a "gluing rule" for the elements of X, or:

$$x\sim_R y \quad\Longleftrightarrow\quad x \text{ and } y \text{ become the same point after the gluing.} \tag{3.2.1}$$

In practice one often starts with describing the most relevant gluing information, omitting the obvious ones such as "any x is glued to itself" (reflexivity). By adding all the consequences that follow from reflexivity, symmetry and transitivity, one generates the equivalence relation.

Example 3.11. Let us make this precise for the gluing underlying the Moebius band, and describe the corresponding equivalence relation R_{Moebius}. First of all, $X=[0,1]\times[0,1]$. Then the key information about the gluing is:

1. for each $t\in[0,1]$, $x=(0,t)$ is glued to $y=(1,1-t)$. Hence for such pairs (x,y) must be in R_{Moebius}

Note that this does not describe all the situations in which "x and y become the same point after the gluing" (see (3.2.1)). Equivalently, and as predicted, this is not yet an equivalence relation. But it generates one by adding

2. for each $t\in[0,1]$, $x=(1,1-t)$ is glued to $y=(0,t)$
3. for each $x\in X$, x is glued to itself

In other words, the relevant equivalence relation is $\sim_{\text{Moebius}}$ defined by:

$$\Big(x\sim_{\text{Moebius}} y\Big)\Longleftrightarrow\Big(x=y \text{ or } \{x,y\}=\{(0,t),(1,1-t)\} \text{ for some } t\in[0,1]\Big).$$

Exercise 3.12. Check that this $\sim_{\text{Moebius}}$ is, indeed, an equivalence relation.

Example 3.13. In general, any surjective function $f:X\to Y$ gives rise to an equivalence relation, namely

$$R_f:=\{(x,y):f(x)=f(y)\}.$$

Thinking of R_f as encoding a "gluing rule", it is clear what the result of the gluing should be: Y itself.

Definition 3.14. Given an equivalence relation R on a set X, a **quotient of** X **modulo** R is a pair (Y,π) consisting of a set Y together with a surjection $\pi : X \to Y$ (called the quotient map) with the property that

$$\pi(x) = \pi(y) \Longleftrightarrow (x,y) \in R. \tag{3.2.2}$$

When $(X,\mathscr{T}_X)$ is a topological space, a **topological quotient of** $(X,\mathscr{T}_X)$ **modulo** R is a topological space $(Y,\mathscr{T}_Y)$ together with a topological quotient map $\pi : X \to Y$ satisfying (3.2.2).

Example 3.15. Assume that we want to glue the end-points of the interval $[0,1]$ ... with the rather intuitively "obvious" outcome: a circle. This is now made precise by:

- taking $X = [0,1]$,
- introducing the equivalence relation R on X given by

$$t \sim_R s \Longleftrightarrow (t = s) \text{ or } (t = 0, s = 1) \text{ or } (t = 1, s = 0),$$

- noticing that the circle S^1 together with

$$\pi = e : [0,1] \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t))$$

  is a quotient of $[0,1]$ modulo R in the sense of the previous definition. This claim is immediate (when seeing this for the first time, the non-triviality is to digest the definitions).
- upgrading the previous item to the topological setting: S^1 together with π is a topological quotient of $[0,1]$ modulo R (where $[0,1]$ as well as S^1 are endowed with their Euclidean topologies).

The difference between the last two items is that in the last one one still has to prove that π is a topological quotient map, and that is not completely obvious. Checking it directly is a rather tedious exercise, though possibly instructive. The most elegant way is however by making use of compactness (to be discussed), and that is the reason we already stated Proposition 3.17.

Example 3.16. It should be clear now that starting with the equivalence relation underlying the Moebius band (Example 3.11 above), the concrete model $M_{R,r}$ discussed before (as well as any other "paper models") are quotients of $X = [0,1] \times [0,1]$ modulo R_{Moebius}. As in the previous example, the same is true also topologically, but please notice that one still has to check that the map $[0,1] \times [0,1] \to M_{R,r}$ is a topological quotient map (where both the domain as well as the codomain are endowed with the Euclidean topology). Again, Proposition 3.17 can be invoked.

Notice that the use of Proposition 3.17 as in the previous examples can be implemented in the following general situation:

Proposition 3.17. *(reformulation of Proposition 3.5) Start with an equivalence relation R on a subspace X of some Euclidean space,*

$$X \subset \mathbb{R}^k$$

endowed with the (induced) Euclidean topology. We assume that X is closed and bounded in $\mathbb{R}^k$ and that we find a continuous map into some other Euclidean space $\mathbb{R}^n$,

$$\pi : X \to \mathbb{R}^n,$$

satisfying $\pi(x) = \pi(y) \Longleftrightarrow (x,y) \in R$. Let Y be the image of π endowed with the Euclidean topology:

$$Y := \pi(X) \subset \mathbb{R}^n.$$

Then (Y,π) is a topological quotient of X modulo R.

Example 3.18. Consider on $X = \mathbb{R}^3 \setminus \{0\}$ the equivalence relation R defined by

$$x \sim_R y \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}_{>0}.$$

Then the 2-sphere S^2 together with

$$\pi : \mathbb{R}^3 \setminus \{0\} \to S^2,\ \pi(x) = \frac{1}{||x||} x$$

is a quotient of X modulo R (picture this!). Notice that the fact that π is actually a topological quotient map cannot be obtained by applying Proposition 3.17; instead, as already pointed out in Example 3.4, one can apply the criterion from Lemma 3.3.

Example 3.19. Another interesting example is when $X = \mathbb{R}^{n+1} \setminus \{0\}$, with the equivalence relation R for which

$$x \sim_R y \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}.$$

Equivalently, R consists of those pairs (x,y) with the property that $0 \in \mathbb{R}^{n+1}$, x and y are collinear. It should be clear now that $\mathbb{P}^n$ together with π from (3.1.2) is a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ modulo R. Notice that, by the construction of the $\mathbb{P}^n$ as a topological space, $\mathbb{P}^n$ is a topological quotient of X modulo R.

The last example indicates how to proceed to build up a "canonical" quotient in general. More precisely, for any equivalence relation R on a set X:

- for $x \in X$ we define the **R-orbit through** x (also called the equivalence class of x w.r.t. to R) as the subset $R(x)$ of X consisting of elements that are R-equivalent to x:

$$R(x) := \{y \in X : (x,y) \in R\}.$$

- we define the **abstract quotient of X modulo R** as the collection of all such R-orbits:

$$X/R := \{R(x) : x \in X\},$$

  and the abstract quotient map

$$\pi_R : X \to X/R, \quad \pi_R(x) := R(x).$$

- when X is a topological space, X/R endowed with the quotient topology is called the **abstract topological quotient of X modulo R**.

We leave it as an exercise to show that, in general, for $x, y \in X$, one has:

$$R(x) = R(y) \quad \Longleftrightarrow \quad (x,y) \in R. \tag{3.2.3}$$

In other words, the abstract quotient $(X/R, \pi_R)$ is always a quotient of X modulo R in the sense of Definition 3.14.

Example 3.20. On $X = \mathbb{Z}$, any integer $n \geq 1$ gives rise to the equivalence relation R_n on $\mathbb{Z}$ of being "congruent modulo n":

$$(x,y) \in R_n \Longleftrightarrow x \equiv y \pmod{n}.$$

Then the R_n-orbit of an integer $k \in \mathbb{Z}$ is

$$R_n(k) = \{\ldots, k-2n, k, k-n, k+n, k+2n, \ldots\} \quad (\text{usually denoted } (k \bmod n) \text{ or just } \hat{k})$$

and it is usually denoted $\widehat{k}$. Of course, $\mathbb{Z}/R_n$ is the usual set $\mathbb{Z}_n$ of integers modulo n.

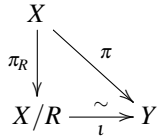
Example 3.21 (THE Moebius band). From the point of view of gluings the abstract quotient has the great advantage that it provides a model for the result of the gluing that does not depend on our influence on how we actually perform the gluing. E.g. we can now finally make sense of **THE Moebius band** independently on the various ways one can twist and stretch the strip before gluing the ends: it is defined as the (topological) abstract quotient of $X = [0,1] \times [0,1]$ modulo the equivalence relation R_{Moebius} described in Examples 3.11.

Of course, it is important to notice the "uniqueness up to isomorphisms" of the result of the gluing, and that is what the next proposition ensures.

Proposition 3.22. *Any quotient of X modulo R is isomorphic to the abstract one. More precisely, given an equivalence relation R on a set X then any quotient Y of X modulo R (cf. Definition 3.14),*

1. *for any quotient (Y,π) of X modulo R, the following defines a bijection between X/R and Y:*
$$\iota : X/R \xrightarrow{\sim} Y, \quad R(x) \mapsto \pi(x).$$
2. *similarly in the topological context: if $(X,\mathscr{T}_X)$ is a topological space and (Y,π) is any topological quotient of X modulo R, then ι is a homeomorphism.*

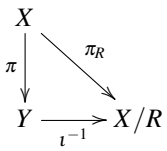
Notice also that ι can also be described as the unique map satisfying $\pi = \iota \circ \pi_R$, equality that we depict in a commutative diagram:



Proof. Notice that the commutativity of the previous diagram, i.e. the equality $\pi = \iota \circ \pi_R$, forces ι to be defined as $R(x) \mapsto \pi(x)$. The only thing one has to take care of is to ensure that this is well-defined: if $R(x) = R(x')$ we have to ensure that $\pi(x) = \pi(x')$, but this follows right away from the direct implication in (3.2.3) and the fact that π is a quotient map. The reverse implication in (3.2.3) shows that ι is injective, while the surjectivity of ι follows from the one of π.

In the topological context one still has to check that

- ι is a continuous map: this can be seen as an immediate consequence of Proposition 3.2 applied to π_R and $f = \iota$.
- ι^{-1} is continuous as well: this is a consequence of the same proposition but applied now to π and $f = \iota^{-1}$:



, $\quad \pi_R = \iota^{-1} \circ \pi.$

Example 3.23. Returning to the equivalence relation on $X = \mathbb{R}^3 \setminus \{0\}$ discussed in Example 3.18, we see that the R-orbit of an arbitrary $x \in X$ is
$$R(x) = \{\lambda \cdot x : \lambda \in \mathbb{R}_{>0}\},$$
i.e. precisely the half line from the origin passing through x. Therefore, the abstract quotient X/R is precisely the collection of all such half lines. The proposition ensures us that the model S^2 discussed in Example 3.18 can be identified with X/R; that identification has a simple geometric interpretation: a half line is uniquely determined by its intersection with the sphere.

Example 3.24. (the Moebius band again) Let us return to the Moebius band
$$M_{\text{abs}} := [0,1] \times [0,1]/R_{\text{Moebius}}$$

and the concrete model $M_{R,r}$ from Example 3.8. The previous proposition allows us to turn the parametrisation (3.1.3) into a homeomorphism $M_{\text{abs}} \xrightarrow{\sim} M_{R,r}$,

$$R_{Moebius}(t,s) \mapsto \Big(\big(R+(2s-1)r\sin(\pi t)\big)\cos(2\pi t),\ \big(R+(2s-1)r\sin(\pi t)\big)\sin(2\pi t),\ (2s-1)r\cos(\pi t) \Big). \quad (3.2.4)$$

Notice that the proposition also ensures that all the paper models of the Moebius that you produced yourself are homeomorphic to the (abstract) Moebius band. It is worth taking a minute and re-think why, when producing such paper models, even if one uses 11 twists before gluing, one still obtains "the same thing, topologically". And maybe try to identify why you have found that weird at first.

Let us point out that the previous proposition can be slightly improved by replacing Y by a more general space Z which is no longer assumed to be a quotient of X modulo R and trying to describe continuous functions from X/R to Z. One sees that any map $f : X/R \to Z$ can be obtained from a map $\tilde{f} : X \to Z$ by setting

$$f(R(x)) = \tilde{f}(x) \qquad (*)$$

The only issue is to make sure that f is well-defined: if $R(x) = R(x')$ then $\tilde{f}(x) = \tilde{f}(x')$. Proposition 3.2 allows us to handle continuity: f is continuous if and only if $\tilde{f}$ is. One obtains the following.

Corollary 3.25. *Consider the abstract topological quotient X/R of a space X modulo an equivalence relation R. Then, for any other topological space Z,*

$$\tilde{f} \mapsto f \text{ defined by } (*)$$

(see diagram (3.1.1)) defines a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{continuous maps} \\ \tilde{f} : X \to Z \\ \text{satisfying } \tilde{f}(x) = \tilde{f}(x') \text{ whenever } (x,x') \in R \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{continuous maps} \\ f : X/R \to Z \end{array} \right\} \quad (3.2.5)$$

Exercise 3.26. In (3.2.5), what is the condition on $\tilde{f}$ that ensures that f is injective? But surjective?

Example 3.27. (and the Moebius band again) The various concrete models for the Moebius band and the fact that they are all homeomorphic, see Example 3.24, are most naturally interpreted as providing concrete embeddings of the (abstract) Moebius band into $\mathbb{R}^3$. For instance the formula from (3.1.3) provides a map

$$\tilde{f}_{R,r} : [0,1] \times [0,1] \to \mathbb{R}^3,$$

which can then be turned into an (injective) continuous map

$$f_{R,r} : M_{\text{abs}} \to \mathbb{R}^3 \quad (3.2.6)$$

whose image is the concrete model $M_{R,r}$. From the discussion above it follows that $f_{R,r}$ is an embedding. Notice that, somewhere in the argument, we have used Proposition 3.17. Actually, notice that the previous corollary allows us to reformulate the conclusion of Proposition 3.17 so that it suits perfectly this point of view:

Addendum to Proposition 3.17: Equivalently, π induces an embedding of X/R into $\mathbb{R}^n$,

$$X/R \hookrightarrow \mathbb{R}^n, \quad R(x) \mapsto \pi(x). \quad (3.2.7)$$

Example 3.28 (the torus). While the Moebius band served as an example throughout the development of the theory, let us make some similar remarks in the case of "the torus", seen as the result of gluing the opposite edges

of a square. As for the Moebius band, there are many different "shapes" that can be obtained as the result of such a gluing, some of which are shown in Figure 3.1. When saying "torus" we would like to think about the intrinsic space itself, and make sense of "shapes" as different ways to embed the torus into $\mathbb{R}^3$. The way to achieve this can then be described as follows:

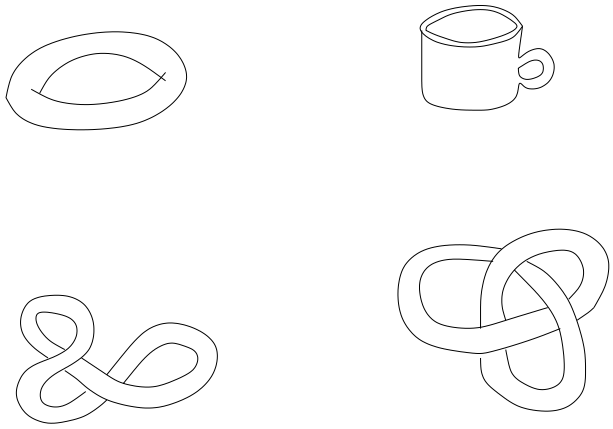


Fig. 3.1

1. Describe the relevant equivalence relation R_{torus}: it is the smallest equivalence relation on $X = [0,1] \times [0,1]$ for which, if $(t,s) = (t',s')$, or if $\{t,t'\} = \{0,1\}$ and $s' = s$, or if $\{s,s'\} = \{0,1\}$ and $t = t'$, then (t,s) is R_{torus}-equivalent to (t',s'). More explicitly, (t,s) is R_{torus}-equivalent to (t',s') if and only if both of the following conditions hold:

- $t = t'$ or $\{t,t'\} = \{0,1\}$,
- $s = s'$ or $\{s,s'\} = \{0,1\}$.

2. Consider the abstract torus, defined as the abstract quotient

$$T_{\text{abs}} := [0,1] \times [0,1]/R_{\text{torus}}.$$

This defines T_{abs} both as a set as well as a topological space.

3. Embed the abstract torus: realizing the torus more concretely in $\mathbb{R}^3$, i.e. finding explicit models of it, translates now into the question of describing embeddings

$$f : T_{\text{abs}} \to \mathbb{R}^3. \tag{3.2.8}$$

By the 1-1 correspondence (3.2.5), continuous f's correspond to continuous maps

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^3$$

with the property that $\tilde{f}(t,s) = \tilde{f}(t',s')$ whenever (t,s) and (t',s') are R_{torus}-equivalent. The injectivity of f is equivalent to the condition that the last equality holds iff (t,s) and (t',s') are R_{torus}-equivalent. Proposition 3.17 (or better: its addendum described at the end of Example 3.27) implies that f is an embedding. Such $\tilde{f}$'s arise, for instance, from the explicit realizations of the torus (see Chapter 1):

$$\tilde{f}(t,s) = \Big(\big(R + r\cos(2\pi t)\big)\cos(2\pi s),\ \big(R + r\cos(2\pi t)\big)\sin(2\pi s),\ r\sin(2\pi t) \Big). \tag{3.2.9}$$

Exercise 3.29. Do the same for the Klein bottle and $\mathbb{P}^2$.

3.3 Special classes of quotients I: collapsing a subspace, cones, suspensions

An interesting and rather large class of quotient spaces are quotients obtained by collapsing a subspace to a point.

Definition 3.30. Let X be a topological space and let $A \subset X$. We define X/A as the topological space obtained from X by collapsing A to a point (i.e. by identifying to each other all the points of A). Equivalently,

$$X/A = X/R_A,$$

where R_A is the equivalence relation on X defined by

$$R_A = \{(x,y) : x = y \text{ or } x,y \in A\}$$

or, in the $\sim$ notation,

$$x \sim_{R_A} y \quad \Longleftrightarrow \quad (x = y) \text{ or } (x,y \in A)$$

Remark 3.31. We always endow X/A with the quotient topology. To handle the topological space X/A, it is useful to summarize what the general theory, including Corollary 3.25, gives us. To that end, we start with a quotient X/A obtained by collapsing a subset $A \subset X$ to a point and let Z be another topological space. We then obtain:

(a) Given a map $\tilde{f} : X \to Z$, the following map

$$f : X/A \to Z, \quad f\Big(R_A(x)\Big) = \tilde{f}(x)$$

is well-defined if and only if $\tilde{f}$ is **constant on** A, i.e., $\tilde{f}|_A : A \to Z$ is constant.

(b) In the situation from (a), f is continuous if and only if $\tilde{f}$ is.

(c) In the situation from (a), f is a homeomorphism if and only if $\tilde{f}$ is a topological quotient map satisfying:

$$\tilde{f}(x) = \tilde{f}(x') \text{ with } x,x' \in X \quad \Longrightarrow \quad x = x' \text{ or } x,x' \in A.$$

(d) In general, the construction from (a) gives rise to a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{continuous maps } \tilde{f} : X \to Z \\ \text{which are constant on } A \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{continuous maps} \\ f : X/A \to Z \end{array} \right\} \tag{3.3.1}$$

Here are some more particular classes of quotients obtained by collapsing, classes that are interesting on their own. Starting with a topological space X, we can talk about:

- **The cylinder on** X is defined as

$$\boxed{\text{Cyl}(X) := X \times [0,1]}$$

endowed with the product topology, where the unit interval is endowed with the Euclidean topology. It contains two interesting copies of X: $X \times \{1\}$ and $X \times \{0\}$.

- **The cone on** X is defined as the quotient obtained from $\text{Cyl}(X)$ by collapsing $X \times \{1\}$ to a point:

$$\boxed{\text{Cone}(X) := X \times [0,1]/(X \times \{1\})}$$

endowed with the quotient topology. The cone contains the copy $X \times \{0\}$ of X and, intuitively, $\text{Cone}(X)$ looks like a cone with base X.

- **The suspension of** X is defined as the quotient obtained from $\mathrm{Cone}(X)$ by further collapsing the base $X \times \{0\}$:

$$\mathrm{S}(X) := \mathrm{Cone}(X)/(X \times \{0\}).$$

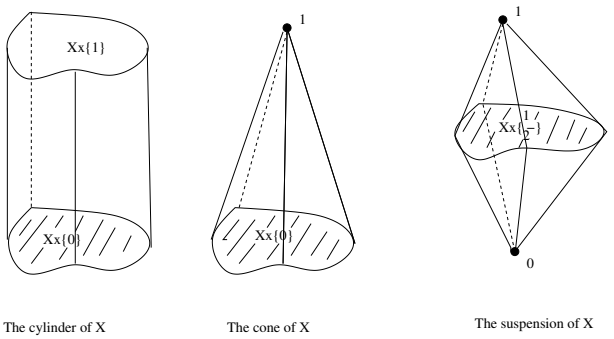


Fig. 3.2

Example 3.32. The general constructions of quotients, such as the quotient by collapsing a subspace to a point, the cone construction and the suspension construction, are nicely illustrated by the various relations between the closed unit balls $D^n \subset \mathbb{R}^n$ and the unit spheres $S^n \subset \mathbb{R}^{n+1}$. We mention here the following:

(a) D^n is homeomorphic to $\mathrm{Cone}(S^{n-1})$- the cone of S^{n-1}.
(b) S^n is homeomorphic to $\mathrm{S}(S^{n-1})$- the suspension of S^{n-1}.
(c) S^n is homeomorphic to D^n/S^{n-1}- the space obtained from D^n by collapsing its boundary to a point.

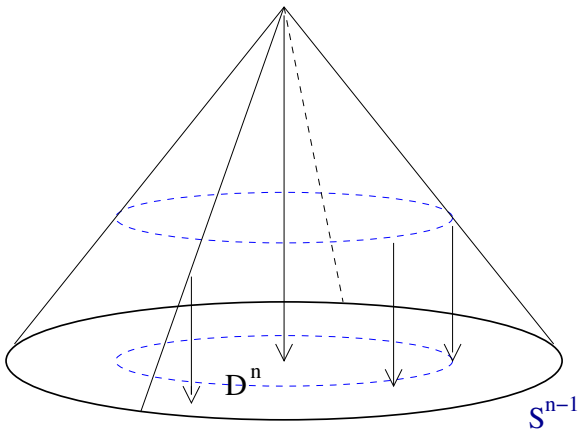


Fig. 3.3

Proof. The first homeomorphism is indicated in Figure 3.3 (project the cone down to the disk). It is not difficult to make this precise: we have a map

$$\tilde{f} : S^{n-1} \times [0,1] \to D^n, \tilde{f}(x,t) = (1-t)x.$$

This is clearly continuous and surjective, and it has the property that

$$\tilde{f}(x,t) = \tilde{f}(x',t') \Longleftrightarrow (x,t) = (x',t') \text{ or } t = t' = 1,$$

hence, by the previous discussion, induces a continuous bijective map

$$f : \text{Cone}(S^{n-1}) = S^{n-1} \times [0,1]/(S^{n-1} \times \{1\}) \to D^n.$$

To conclude that f is actually homeomorphism, one needs to make sure that $\tilde{f}$ is a topological quotient map. This follows from Proposition 3.17 but, again, one should keep in mind that this part of the argument will become clearer after we discuss compactness. Note that this f sends the image of $S^{n-1} \times \{0\}$ in the cone to the boundary of D^n, hence (b) will follow from (c). In turn, (c) is clear on the picture (see Figure 1.11 in the previous Chapter); the map from D^n to S^n indicated on the picture can be written explicitly as

$$\tilde{g} : D^n \to S^n, x \mapsto \left(\frac{x_1}{||x||} \sin(\pi||x||), \ldots, \frac{x_n}{||x||} \sin(\pi||x||), \cos(\pi||x||) \right)$$

and which sends 0 to the north pole $(0,\ldots,0,1)$.

Again, $\tilde{g}$ is a topological quotient map by Proposition 3.17 (...) and one can check directly that

$$\tilde{g}(x) = \tilde{g}(x') \Longleftrightarrow x = x' \text{ or } x,x' \in S^{n-1}.$$

Hence, by the remark above, we obtain a homeomorphism

$$g : D^n/S^{n-1} \to S^n$$

3.4 Special classes of quotients II: quotients modulo group actions

In this section we discuss quotients by group actions. Let X be a topological space. We denote by $\mathrm{Homeo}(X)$ the set of all homeomorphisms from X to X. Together with composition of maps, this is a group. Let Γ be another group, whose operation is denoted multiplicatively.

Definition 3.33. An **action** of the group Γ on the topological space X is a group homomorphism

$$\phi : \Gamma \to \mathrm{Homeo}(X),\ \gamma \mapsto \phi_\gamma.$$

Hence, for each $\gamma \in \Gamma$, one has a homeomorphism ϕ_γ of X ("the action of γ on X"), so that

$$\phi_{\gamma\gamma'} = \phi_\gamma \circ \phi_{\gamma'} \quad \forall\, \gamma, \gamma' \in \Gamma.$$

Sometimes $\phi_\gamma(x)$ is also denoted $\gamma(x)$, or simply $\gamma \cdot x$, and one looks at the action as a map

$$\Gamma \times X \to X,\ (\gamma, x) \to \gamma \cdot x$$

and the conditions for this to be an action translates into:

$$e \cdot x = x, \quad \gamma \cdot (\gamma' \cdot x) = (\gamma\gamma') \cdot x$$

for all $\gamma, \gamma' \in \Gamma$, $x \in X$, where e is the unit of Γ. The action induces an equivalence relation R_Γ on X defined by:

$$(x, y) \in R_\Gamma \iff \exists\, \gamma \in \Gamma \text{ s.t. } y = \gamma \cdot x.$$

Note that the R_Γ-orbit through an element $x \in X$ is precisely its Γ-orbit:

$$\Gamma \cdot x := \{\gamma \cdot x : \gamma \in \Gamma\}.$$

The resulting topological quotient is called **the quotient of X by the action of Γ**, and is denoted by X/Γ. Hence

$$X/\Gamma = \left\{\Gamma \cdot x : x \in X\right\}, \quad \pi : X \to X/\Gamma,\ x \mapsto \Gamma \cdot x,$$

and X/Γ is endowed with the quotient topology induced by π (the unique one s.t. π is a topological quotient map).

Example 3.34. The additive group $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ (n, r) \mapsto \phi_n(r) = n \cdot r := n + r.$$

The resulting quotient is (homeomorphic to) S^1. More precisely, one uses the 1-1 correspondence (3.2.5) again to see that the map $\tilde{f} : \mathbb{R} \to S^1$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ induces a continuous bijection $f : \mathbb{R}/\mathbb{Z} \to S^1$. Proving that f is a homeomorphism can now be done directly, e.g. using sequential continuity. Notice that this example is very much related to that from Example 3.15 just that, strictly speaking, Proposition 3.17 cannot be applied right away. Do you see a way to overcome that problem?

Remark 3.35. Again, it is useful to summarize what the general theory, including Corollary 3.25, gives us in this case of such quotients. To that end, we start with a quotient X/Γ induced by an action of a group Γ on a space X and let Z be another topological space. We then obtain:

(a) Given a map $\tilde{f}: X \to Z$, the following map

$$f: X/\Gamma \to Z, \quad f(\Gamma \cdot x) = \tilde{f}(x).$$

is well-defined if and only if $\tilde{f}$ is **Γ-invariant**, i.e., it satisfies:

$$\tilde{f}(\gamma \cdot x) = \tilde{f}(x) \quad \forall\, \gamma \in \Gamma,\ x \in X.$$

(b) In the situation from (a), f is continuous if and only if $\tilde{f}$ is.
(c) In the situation from (a), f is a homeomorphism if and only if $\tilde{f}$ is a topological quotient map satisfying:

$$\tilde{f}(x) = \tilde{f}(x') \text{ with } x, x' \in X \quad \Longrightarrow \quad \exists\, \gamma \in \Gamma \text{ such that } x' = \gamma \cdot x.$$

(d) In general, the construction from (a) gives rise to a 1-1 correspondence

$$\left\{ \begin{array}{c} \Gamma\text{-invariant continuous} \\ \text{maps } \tilde{f}: X \to Z \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{continuous maps} \\ f: X/\Gamma \to Z \end{array} \right\} \tag{3.4.1}$$

Here is a fortunate case in which Hausdorffness is preserved when passing to quotients.

Theorem 3.36. *If X is a Hausdorff space and Γ is a finite group acting on X, then X/Γ is Hausdorff.*

Proof. Let $\Gamma x, \Gamma y \in X/\Gamma$ be two distinct points ($x, y \in X$). That they are distinct means that, for each $\gamma \in \Gamma$, $x \neq \gamma \cdot y$. Hence, for each $\gamma \in \Gamma$, we find disjoint opens $U_\gamma, V_\gamma \subset X$ containing x, and γy, respectively. Note that

$$W_\gamma = \phi_\gamma^{-1}(V_\gamma)$$

is an open containing y, and what we know is that

$$U_\gamma \cap \phi_\gamma(W_\gamma) = \emptyset.$$

Since Γ is finite, $U := \cap_\gamma U_\gamma$, $V := \cap_\gamma W_\gamma$ will be open neighborhoods of x and y, respectively, with the property that

$$U \cap \phi_a(V) = \emptyset, \ \ \forall\, a \in \Gamma.$$

Using the quotient map $\pi: X \to X/\Gamma$, we consider $\pi(U), \pi(V)$, and we claim that they are disjoint opens in X/Γ separating Γx and Γy. That they are disjoint follows from the previous property of U and V. To see that $\pi(U)$ is open, we have to check that $\pi^{-1}(\pi(U))$ is open, but

$$\pi^{-1}(\pi(U)) = \cup_{\gamma \in \Gamma} \phi_\gamma(U)$$

(check this!) is a union of opens, hence open. Similarly, $\pi(V)$ is open. Clearly, $\Gamma \cdot x = \pi(x) \in \pi(U)$ and $\Gamma \cdot y = \pi(y) \in \pi(V)$, hence we separated $\Gamma \cdot x, \Gamma \cdot y \in X/\Gamma$ by two disjoint opens.

3.5 The projective space $\mathbb{P}^n$ again, from several perspectives

A very good illustration of the use of quotient topologies is the construction of the projective space, as a topological space. Recall that, set theoretically, $\mathbb{P}^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$:

$$\mathbb{P}^n = \left\{ l \subset \mathbb{R}^{n+1} : l - \text{one dimensional vector subspace} \right\}.$$

To realize it as a topological space, we relate it to spaces that we already know. There are several different ways.

As a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$:

The first approach is the one discussed already in Example 3.7, making use of the canonical map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n,\ x \mapsto l_x,$$

where l_x is the line through the origin and x:

$$l_x = \mathbb{R}x = \{\lambda x : \lambda \in \mathbb{R}\} \subset \mathbb{R}^{n+1}.$$

Furthermore, as we pointed out Example 3.19 (and served as inspiration for the definitions that followed), $\mathbb{P}^n$ can also be seen as the abstract quotient modulo the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ given by

$$x \sim_R y \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}.$$

What we can add here is the remark that this equivalence relation comes from a group action: the group $\Gamma = \mathbb{R}^*$ (with respect to the multiplication) acting on $X = \mathbb{R}^{n+1} \setminus \{0\}$ by scalar multiplication

$$\phi_\lambda(x) = \lambda x \ \text{ for } \lambda \in \mathbb{R}^*, x \in \mathbb{R}^{n+1} - \{0\}.$$

Therefore, the projective space becomes

$$\mathbb{P}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*.$$

As a quotient of S^n:

This is based on another simple remark: a line in $\mathbb{R}^{n+1}$ through the origin is uniquely determined by its intersection with the unit sphere $S^n \subset \mathbb{R}^{n+1}$- which is a set consisting of two antipodal points (the first picture in Figure 3.4). This indicates that $\mathbb{P}^n$ can be obtained from S^n by identifying (gluing) its antipodal points. Again, this is a quotient that arises from a group action: the group $\mathbb{Z}_2$ acting on S^n. Using the multiplicative description $\mathbb{Z}_2 = \{1, -1\}$, the action is: ϕ_1 is the identity map, while ϕ_{-1} is the map sending $x \in S^n$ to its antipodal point $-x$. We are arriving at:

Proposition 3.37. *$\mathbb{P}^n$ is homeomorphic to $S^n/\mathbb{Z}_2$.*

Proof. The conclusion of the previous discussion is that there is a set-theoretical bijection:

$$\phi : S^n/\mathbb{Z}_2 \to \mathbb{P}^n,$$

which sends the $\mathbb{Z}_2$-orbit of $x \in S^n$ to the line l_x through x, with the inverse

$$\psi : \mathbb{P}^n \to S^n/\mathbb{Z}_2$$

which sends the line l to $S^n \cap l$ (a $\mathbb{Z}_2$-orbit!). We have to check that they are continuous. We use Proposition 3.2 and its corollary. To see that ϕ is continuous, we have to check that the composition with the quotient map $S^n \to S^n/\mathbb{Z}_2$

is continuous. But this composition is precisely the restriction of the quotient map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n$ to S^n, hence is continuous. In conclusion, ϕ is continuous.

To see that ψ is continuous, we have to check that its composition with the quotient map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n$ is continuous. But this composition- which is a map from $\mathbb{R}^{n+1} - \{0\}$ to $S^n/\mathbb{Z}_2$ can be written as the composition of two continuous maps: the map $\mathbb{R}^{n+1} - \{0\} \to S^n$ sending x to $x/||x||$ and the quotient map $S^n \to S^n/\mathbb{Z}_2$.

Corollary 3.38. *The projective space $\mathbb{P}^n$ is Hausdorff.*

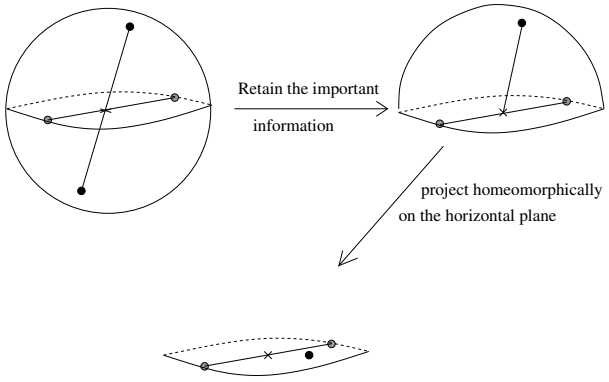
As a quotient of D^n:

Again, the starting remark is very simple: the orbits of the action of $\mathbb{Z}_2$ on S^n always intersect the upper hemisphere S^n_+ (for notations, see Section 1.4 in the first chapter). Moreover, such an orbit either lies entirely in the boundary of S^n_+, or intersects its interior in a unique point. See the second picture in Figure 3.4. This indicates that $\mathbb{P}^n$ can be obtained from S^n_+ by gluing the antipodal points that belong to its boundary. On the other hand, the orthogonal projection onto the horizontal hyperplane defines a homeomorphism between S^n_+ and D^n (see Figure 3.4). Passing to D^n, we obtain an equivalence relation R on D^n given by:

$$(x,y) \in R \iff (x = y) \text{ or } (x,y \in S^{n-1} \text{ and } x = -y).$$

Exercise 3.39. Show that, indeed, $\mathbb{P}^n$ is homeomorphic to D^n/R. What happens when $n = 1$?

Corollary 3.40. *$\mathbb{P}^n$ for $n = 2$ is homeomorphic to the projective plane as defined in Chapter 1 (Section 1.8), i.e. obtained from the square by gluing the opposite sides as indicated in Figure 3.5.*



Different ways to encode the lines in the space

Fig. 3.4

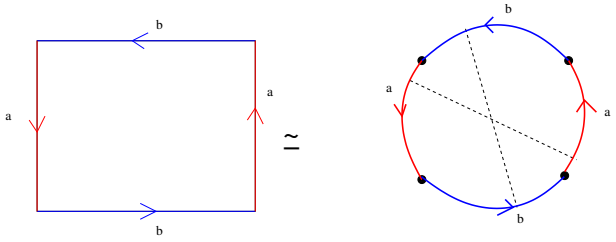


Fig. 3.5

3.6 Constructions of topologies: products

In this section we explain how the Cartesian product of two topological spaces is naturally a topological space itself. Given two sets X and Y we consider their Cartesian product

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

Given a topology $\mathscr{T}_X$ on X and a topology $\mathscr{T}_Y$ on Y, one defines a topology on $X \times Y$, the "product topology" $\mathscr{T}_X \widehat{\times} \mathscr{T}_Y$, as follows. We say that a subset $D \subset X \times Y$ is open if and only if

$$\forall\ (x,y) \in D\ \exists\ U \in \mathscr{T}_X, V \in \mathscr{T}_Y\ \text{ such that }\ x \in U, y \in V, U \times V \subset D. \tag{3.6.1}$$

We denote by $\mathscr{T}_X \widehat{\times} \mathscr{T}_Y$ the collection of all such D's and we call it **the product topology**.

One should notice the similarity with the description of the Euclidean topology in (2.1.1) (or, more generally, of metric topologies in (1.10.1)), as well as of the lower limit topology (2.1.2). As in those cases, the opens in $X \times Y$ can also be recognised as those subsets that are unions $\cup_i U_i \times V_i$ of such products of opens (see the proof below).

Proposition 3.41. *Given $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, $\mathscr{T}_X \widehat{\times} \mathscr{T}_Y$ is indeed a topology on $X \times Y$. Moreover, it is the smallest topology on $X \times Y$ with the property that the two projections*

$$pr_X : X \times Y \to X,\ pr_Y : X \times Y \to Y$$

(sending (x,y) to x, and y, respectively) are continuous.

Proof. Axiom (T1) is clear. For (T2), let D_1, D_2 be in the product topology, and we show that $D := D_1 \cap D_2$ is as well. To check (3.6.1), let $(x,y) \in D$. Since D_1 and D_2 satisfy (3.6.1), for each $i \in \{1,2\}$, we find $U_i \in \mathscr{T}_X$ and $V_i \in \mathscr{T}_Y$ such that

$$(x,y) \in U_i \times V_i \subset D_i.$$

Then $U := U_1 \cap U_2 \in \mathscr{T}_X$ (axiom (T2) for $\mathscr{T}_X$), and similarly $V := V_1 \cap V_2 \in \mathscr{T}_Y$, while clearly we have $x \in U$, $y \in V$, $U \times V \subset D$. The proof of the axiom (T3) is similar.

For the second part, note that a topology $\mathscr{T}$ on $X \times Y$ has the property that both projections are continuous if and only if $U \times Y \in \mathscr{T}$ and $X \times V \in \mathscr{T}$ for all $U \in \mathscr{T}_X$ and $V \in \mathscr{T}_Y$. Clearly $\mathscr{T}_X \widehat{\times} \mathscr{T}_Y$ has the property, hence the projections are continuous with respect to the product topology. For an arbitrary topology $\mathscr{T}$ on $X \times Y$ with the same property, since

$$U \times V = (U \times Y) \cap (X \times V),$$

we deduce that $U \times V \in \mathscr{T}$ for all $U \in \mathscr{T}_X$, $V \in \mathscr{T}_Y$. To show that $\mathscr{T}_X \widehat{\times} \mathscr{T}_Y \subset \mathscr{T}$, let D be an open in the product topology and we show that it must belong to $\mathscr{T}$. Since D satisfies (3.6.1), for each $z = (x,y) \in D$ we find $U_z \in \mathscr{T}_X$, $V_z \in \mathscr{T}_Y$ such that

$$\{z\} \subset U_z \times V_z \subset D.$$

Taking the union over all $z \in D$, we deduce that

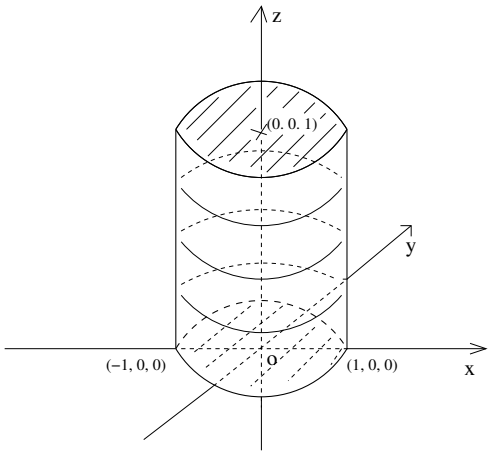
$$D = \bigcup_{z \in D} U_z \times V_z.$$

But, as we have already seen, all members $U_z \times V_z$ must be in $\mathscr{T}$ hence, using axiom (T3) for $\mathscr{T}$, we deduce that $D \in \mathscr{T}$.

Example 3.42. In $\mathbb{R}^3$ we have the cylinder

$$C = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \le z \le 1\right\},$$

which is pictured in Figure 3.6. According to our conventions, C is considered with the topology induced from $\mathbb{R}^3$. On the other hand, since



The cyclinder in $\mathbb{R}^3$

Fig. 3.6

$$C = S^1 \times [0,1],$$

where S^1 is the unit circle in $\mathbb{R}^2$, C carries yet another natural topology, namely the product topology. These two topologies are the same. This can be proven in a much greater generality, as described in Exercise 11.32.

Exercise 3.43. Products are handy when trying to express continuity of "operations in several variables". An instance of this principle is obtained when looking at group operations: a **topological group** is a group $(G,\cdot)$ endowed with a topology on G such that all the group operations, i.e.

(a) the inversion map $\tau : G \to G, g \mapsto g^{-1}$,
(b) the composition map $m : G \times G \to G$, $(g,h) \mapsto g \cdot h$

are continuous. Here, $G \times G$ is endowed with the product topology.

Note that the sets of matrices $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$ that appear in Exercise 10.72, together with multiplication of matrices, are groups. The same exercise describe natural topologies on them (induced from $M_n(\mathbb{R})$). Show that, with respect to these topologies, they are all topological groups.

3.7 Constructions of topologies: Bases for topologies

In the construction of metric topologies the balls were the building pieces. Similarly for the product topology, where the building pieces were the subsets $U \times V$ with $U \in \mathscr{T}_X$, $V \in \mathscr{T}_Y$. In both cases the collection of "building pieces" was not a topology, but "generated" one. Here is the abstract notion underlying these constructions:

Definition 3.44. Let X be a set and let $\mathscr{B}$ be a collection of subsets of X. We say that $\mathscr{B}$ is a **topology basis** if it satisfies the following two axioms:

(B1) for each $x \in X$ there exists $B \in \mathscr{B}$ such that $x \in B$.
(B2) for each $B_1, B_2 \in \mathscr{B}$, and $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$.

In this case, we define **the topology** induced by $\mathscr{B}$ as the collection

$$\mathscr{T}(\mathscr{B}) := \Big\{ U \subset X : \quad \forall\, x \in U \ \exists\, B \in \mathscr{B} \text{ s.t. } x \in B,\ B \subset U \Big\}.$$

Exercise 3.45. Show that, indeed, for any metric d on X, the collection $\mathscr{B}_d$ of all open balls is a topology basis and topology $\mathscr{T}(\mathscr{B}_d) = \mathscr{T}_d$. Prove a similar statement for the product topology.

We still have to prove that $\mathscr{T}(\mathscr{B})$ is, indeed, a topology. We first point out a different description of $\mathscr{T}(\mathscr{B})$ (which we have already seen in the case of metric and of product topologies- and this is a hint for the next exercise!).

Exercise 3.46. Let X be a set and let $\mathscr{B}$ be a collection of subsets of X. Then a subset $U \subset X$ is in $\mathscr{T}(\mathscr{B})$ if and only if there exist $B_i \in \mathscr{B}$ with $i \in I$ (I-an index set) such that $U = \cup_{i \in I} B_i$.

Proposition 3.47. *Given a collection $\mathscr{B}$ of subsets of a set X, the following are equivalent:*

1. *$\mathscr{B}$ is a topology basis.*
2. *$\mathscr{T}(\mathscr{B})$ is a topology on X.*

In this case $\mathscr{T}(\mathscr{B})$ is the smallest topology on X which contains $\mathscr{B}$; moreover, $\mathscr{B}$ is a basis for the topological space $(X, \mathscr{T}(\mathscr{B}))$, in the sense of Definition 2.59.

Proof. We prove that the axioms (T1), (T2) and (T3) of a topology (applied to $\mathscr{T}(\mathscr{B})$) are equivalent to axioms (B1) and (B2) of a topology basis (applied to $\mathscr{B}$). First of all, the previous exercise shows that (T3) is satisfied without any assumption on B. Next, due to the definition of $\mathscr{T}(\mathscr{B})$, (B1) is equivalent to $X \in \mathscr{T}(\mathscr{B})$. Since clearly $\emptyset \in \mathscr{T}(\mathscr{B})$, (B1) is equivalent to (T1). Hence it suffices to prove that (T2) (for $\mathscr{T}(\mathscr{B})$) is equivalent to (B2) (for $\mathscr{B}$). That (T2) implies (B2) is immediate: given $B_1, B_2 \in \mathscr{B}$, since they are in $\mathscr{T}(\mathscr{B})$ so is their intersection, i.e. for all $x \in B_1 \cap B_2$ there exists $B \in \mathscr{B}$ such that $x \in B$, $B \subset B_1 \cap B_2$. For the converse, assume that (B2) holds. To prove $(T2)$ for $\mathscr{T}(\mathscr{B})$, we start with $U, V \in \mathscr{T}(\mathscr{B})$ and we want to prove that $U \cap V \in \mathscr{T}(\mathscr{B})$. I.e., for an arbitrary $x \in U \cap V$, we have to find $B \in \mathscr{B}$ such that $x \in B \subset U \cap V$. Since $x \in U \in \mathscr{T}(\mathscr{B})$, we find $B_1 \in \mathscr{B}$ such that $x \in B_1 \subset U$. Similarly, we find $B_2 \in \mathscr{B}$ such that $x \in B_2 \subset V$. By (B2) we find $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$. We deduce that $x \in B \subset U \cap V$, proving (T2). Finally, the last part of the proposition follows from the previous exercise, as any topology which contains $\mathscr{B}$ must contain all unions of sets in $\mathscr{B}$.

Next, since many topologies are defined with the help of a basis, it is useful to know how to compare topologies by only looking at basis elements (see Exercises 11.34 and 11.35).

Lemma 3.48. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two topology bases on X. Then $\mathscr{T}_1$ is smaller than $\mathscr{T}_2$ if and only if: for each $B_1 \in \mathscr{B}_1$ and each $x \in B_1$, there exists $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subset B_1$.*

Proof. What we have to show is that $\mathscr{T}_1 \subset \mathscr{T}_2$ is equivalent to $\mathscr{B}_1 \subset \mathscr{T}_2$. The direct implication is clear since $\mathscr{B}_1 \subset \mathscr{T}_1$. For the converse, we use the fact that every element in $\mathscr{T}_1 = \mathscr{T}(\mathscr{B}_1)$ can be written as a union of elements of $\mathscr{B}_1$, hence it is itself in $\mathscr{T}_2$ if $\mathscr{B}_1 \subset \mathscr{T}_2$.

3.8 Constructions of topologies: Generating topologies

Generated Topologies

There is a slightly more general recipe for generating topologies. What it may happen is that we have a set X, and we are looking for a topology on X which contains certain (specified) subsets of X. In other words,

- we start with a set X and a collection $\mathscr{S}$ of subsets of X

and we are looking for a (interesting) topology $\mathscr{T}$ on X which contains $\mathscr{S}$. Of course, the discrete topology $\mathscr{T}_{\text{dis}}$ is always a choice, but it is not a very interesting one (it does not even depend on $\mathscr{S}$). Is there a "best" one? Or:

- is there a smallest possible topology on X which contains $\mathscr{S}$?

Example 3.49. If $\mathscr{S} = \mathscr{B}$ is a topology basis on the set X, Proposition 3.47 shows that the answer is positive, and the resulting topology is precisely $\mathscr{T}(\mathscr{B})$.

The answer to the question is always "yes", for any collection $\mathscr{S}$. Indeed, Exercise 10.10 of the previous chapter tells us that any intersection of topologies is a topology. Hence one can just proceed abstractly and define:

$$\langle \mathscr{S} \rangle := \bigcap_{\mathscr{T}-\text{topology on } X \text{ containing } \mathscr{S}} \mathscr{T}$$

This is called **the topology generated by** $\mathscr{S}$. By Exercise 10.10, it is a topology. By construction, it is the smallest one containing $\mathscr{S}$. Of course, this abstract description is not the most satisfactory one. However, using exactly the same type of arguments as in the proof of Proposition 3.47:

Proposition 3.50. *Let X be a set and let $\mathscr{S}$ be a collection of subsets of X. Define $\mathscr{B}(\mathscr{S})$ as the collection of subsets of X which can be written as finite intersections of subsets that belong to $\mathscr{S}$. Then $\mathscr{B}(\mathscr{S})$ is a topology basis and the associated topology is precisely $\langle \mathscr{S} \rangle$. In conclusion, a subset $U \subset X$ belongs to $\langle \mathscr{S} \rangle$ if and only if it is a union of finite intersections of members of $\mathscr{S}$ (where we recall that the empty intersection is, say by convention, the entire X).*

Initial topologies

Here is a general principle for constructing topologies. Many topological constructions are what we call "natural", or "canonical" (in any case, not arbitrary). Very often when one looks for a topology, one wants certain maps to be continuous. This happens e.g. with induced and product topologies. A general setting is as follows.

- start with a set X and maps $\{f_i : X \to X_i\}_{i \in I}$ (I–an index set) where each X_i is endowed with a topology $\mathscr{T}_i$

We are looking for (interesting) topologies $\mathscr{T}_X$ on X such that all the maps f_i become continuous. As before, this has an obvious but unsatisfactory answer: $\mathscr{T}_X = \mathscr{T}_{\text{dis}}$ (which does not see the functions f_i). One should also remark that the smaller $\mathscr{T}_X$ becomes, the smaller are the chances that f_i are continuous. Hence, the interesting question is:

- find the smallest topology on X such that all the functions f_i become continuous.

Now, by the definition of continuity, a topology on X makes the functions f_i continuous if and only if all subsets of type $f_i^{-1}(U_i)$ with $i \in I$, $U_i \in \mathscr{T}_i$, are open. Hence, denoting

$$\mathscr{S} := \left\{ U \subset X : \ \exists\, i \in I, \exists\, U_i \in \mathscr{T}_i \text{ such that } U = f_i^{-1}(U_i) \right\}$$

the answer to the previous question is: the topology $\langle \mathscr{S} \rangle$ generated by $\mathscr{S}$. This is called **the initial topology on** X associated to the starting data (the topological spaces X_i and the functions f_i).

Example 3.51. Given a subset A of a topological space $(X, \mathscr{T})$, the natural map here is the inclusion $i : A \to X$. The associated initial topology on A is the induced topology $\mathscr{T}|_A$.

Given two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, the Cartesian product $X \times Y$ comes with two natural maps: the projections $\text{pr}_X : X \times Y \to X$, $\text{pr}_Y : X \times Y \to Y$. The associated initial topology is the product topology on $X \times Y$.

3.9 Example: some spaces of functions (... and the Peano curve)

Given two sets X and Y we denote by $\mathscr{F}(X,Y)$ the set of all functions from X to Y. In many parts of mathematics, when interested in a certain problem, one deals with subsets of $\mathscr{F}(X,Y)$, endowed with a topology which is relevant to the problem; the topology is dictated by the type of convergence one has to deal with. The list of examples is huge; we will look at some topological examples, i.e. at the set of continuous functions $\mathscr{C}(X,Y) \subset \mathscr{F}(X,Y)$ between two spaces. The general setting will be discussed later. Here we treat the particular case

$$X = I \subset \mathbb{R} \text{ an interval, } Y = \mathbb{R}^n \text{ endowed with the Euclidean metric } d.$$

Here I could be any interval, open or not, closed or not, equal to $\mathbb{R}$ or not.

There are several notions of convergence on the set $\mathscr{F}(I,\mathbb{R}^n)$ of functions from I to $\mathbb{R}^n$.

Definition 3.52. Let $\{f_n\}_{n\geq 1}$ be a sequence in $\mathscr{F}(I,\mathbb{R}^n)$, $f \in \mathscr{F}(I,\mathbb{R}^n)$. We say that:

- f_n converges **pointwise** to f, and write $f_n \overset{pt}{\to} f$, if $f_n(x) \to f(x)$ for all $x \in I$.
- f_n converges **uniformly** to f, and write $f_n \rightrightarrows f$, if for any $\varepsilon > 0$, there exists n_ε s.t.

$$d(f_n(x), f(x)) < \varepsilon \quad \forall\, n \geq n_\varepsilon,\ \forall\, x \in I.$$

- f_n converges **uniformly on compacts** to f, and write $f_n \overset{cp}{\to} f$, if $f_n|_K \rightrightarrows f|_K$ for all $K \subset I$ compact interval.

We show that these convergences correspond to certain topologies on $\mathscr{F}(I,\mathbb{R}^n)$. First the pointwise convergence. For $x \in I$, $U \subset \mathbb{R}^n$ open, we define

$$S(x,U) := \{f \in \mathscr{F}(I,\mathbb{R}^n) : f(x) \in U\} \quad \subset \mathscr{F}(I,\mathbb{R}^n).$$

These form a family $\mathscr{S}$. The **topology of pointwise convergence**, denoted $\mathscr{T}_{pt}$, is the topology on $\mathscr{F}(I,\mathbb{R}^n)$ generated by $\mathscr{S}$. Hence $\mathscr{S}$ defines a topology basis, consisting of finite intersections of members of $\mathscr{S}$, and $\mathscr{T}_{pt}$ is the associated topology.

Proposition 3.53. *The pointwise convergence coincides with the convergence in* $(\mathscr{F}(I,\mathbb{R}^n), \mathscr{T}_{pt})$.

Proof. Rewrite the condition that $f_n \to f$ with respect to $\mathscr{T}_{pt}$. It means that, for any neighborhood of f of type $S(x,U)$ there exists an integer N such that $f_n \in S(x,U)$ for $n \geq N$. I.e., for any $x \in I$ and any open U containing $f(x)$, there exists an integer N such that $f_n(x) \in U$ for all $n \geq N$. I.e., for any $x \in I$, $f_n(x) \to f(x)$ in $\mathbb{R}^n$.

For uniform convergence, the situation is more fortunate: it is induced by a metric. Given two functions $f, g \in \mathscr{F}(I,\mathbb{R}^n)$, we define the sup-distance between f and g by

$$d_{\sup}(f,g) = \sup\{d(f(x),g(x)) : x \in I\}.$$

Since this supremum may be infinite for some f and g (and only for that reason!), we define $\hat{d}_{\sup}(f,g) = \min(d_{\sup}(f,g),1)$. Note that $d_{\sup}$ and $\hat{d}_{\sup}$ are morally the same when it comes to convergence: $d_{\sup}(f,g)$ is" small" if and only if $\hat{d}_{\sup}(f,g)$ is. The associated topology is called the **topology of uniform convergence**.

Exercise 3.54. Show that $\hat{d}_{\sup}$ is a metric on $\mathscr{F}(I,\mathbb{R}^n)$.

Proposition 3.55. *The uniform convergence coincides with the convergence in* $(\mathscr{F}(I,\mathbb{R}^n), \hat{d}_{sup})$.

Proof. According to the definition of uniform convergence, $f_n \rightrightarrows f$ if and only if for each $\varepsilon > 0$, we find n_ε such that $d_{\sup}(f_n, f) \leq \varepsilon$ for all $n \geq n_\varepsilon$. Since only small enough ε matter here, we recover the convergence wrt $\hat{d}_{\sup}$.

We now move to uniform convergence on compacts. Given $K \subset I$ compact interval, $\varepsilon > 0$, $f \in \mathscr{F}(I, \mathbb{R}^n)$, we define

$$B_K(f, \varepsilon) := \left\{ g \in \mathscr{F}(I, \mathbb{R}^n) : \sup_{x \in K} d(f(x), g(x)) < \varepsilon \right\}.$$

The **topology of compact convergence**, denoted $\mathscr{T}_{cp}$, is the topology on $\mathscr{F}(I, \mathbb{R}^n)$ generated by the family of all the subsets $B_K(f, \varepsilon)$. As above, the definitions immediately imply:

Proposition 3.56. *The uniform convergence on compacts coincides with the convergence in the topological space* $(\mathscr{F}(I, \mathbb{R}^n), \mathscr{T}_{cp})$.

In topology, we are interested in continuous functions. The situation is as follows:

Theorem 3.57. *For a sequence of continuous functions* $f_n \in \mathscr{C}(I, \mathbb{R}^n)$, *and* $f \in \mathscr{F}(I, \mathbb{R}^n)$:

$$(f_n \rightrightarrows f) \Longrightarrow (f_n \xrightarrow{cp} f) \Longrightarrow (f \in \mathscr{C}(I, \mathbb{R}^n)). \tag{3.9.1}$$

More precisely, one has an inclusion of topologies

$$(\textit{pointwise}) \subset (\textit{uniform on compacts}) \subset (\textit{uniform})$$

and $\mathscr{C}(I, \mathbb{R}^n)$ *is closed in* $(\mathscr{F}(I, \mathbb{R}^n), \mathscr{T}_{cp})$ *(hence also in* $(\mathscr{F}(I, \mathbb{R}^n), \hat{d}_{sup})$*).*

Proof. The comparison between the three topologies is again a matter of checking the definitions. Also, the first implication in (3.9.1) is trivial; the second one follows from the last part of the theorem, on which we concentrate next. We first show that $\mathscr{C}(I, \mathbb{R}^n)$ is closed in $(\mathscr{F}(I, \mathbb{R}^n), \hat{d}_{\sup})$. Assume that f is in the closure, i.e. $f : I \to \mathbb{R}^n$ is the uniform limit of a sequence of continuous functions f_n. We show that f is continuous. Let $x_0 \in I$ and we show that f is continuous at x_0. I.e., we fix $\varepsilon > 0$ and we look for a neighborhood V_ε of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for all $x \in V_\varepsilon$. Since $f_n \rightrightarrows f$, we find N such that $d(f_n(x), f(x)) < \varepsilon/3$ for all $n \geq N$ and all $x \in I$. Since f_N is continuous at x_0, we find a neighborhood V_ε such that $d(f_N(x), f_N(x_0)) < \varepsilon/3$ for all $x \in V_\varepsilon$. But then, for all $x \in V_\varepsilon$,

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < 3 \times \varepsilon/3 = \varepsilon.$$

Finally, we show that $\mathscr{C}(I, \mathbb{R}^n)$ is closed in $(\mathscr{F}(I, \mathbb{R}^n), \mathscr{T}_{cp})$. Assume that f is in the closure. Notice that for f to be continuous it suffices that $f|_K$ is continuous for any compact interval $K \subset I$. Fix such a K. For neighborhoods of type $B_K(f, 1/n)$, we find $f_n \in \mathscr{C}(I, \mathbb{R}^n)$ lying in this neighborhood. But then $f_n|_K \rightrightarrows f|_K$ hence $f|_K$ is continuous.

Here is the most important property of the uniform topology ($\mathscr{T}_{cp}$ will be discussed later).

Theorem 3.58. $(\mathscr{F}(I, \mathbb{R}^n), \hat{d}_{sup})$ *and* $(\mathscr{C}(I, \mathbb{R}^n), \hat{d}_{sup}))$ *are complete metric spaces.*

Proof. Using the previous theorem and the simple fact that closed subspaces of complete metric spaces are complete, we are left with showing that $(\mathscr{F}(I, \mathbb{R}^n), \hat{d}_{\sup})$ is complete. So, let $(f_n)_{n \geq 1}$ be a Cauchy sequence with respect to $\hat{d}_{\sup}$ (as mentioned above, for such arguments there is no difference between using d or $\hat{d}$). Since for all $x \in I$,

$$d(f_n(x), f_m(x)) \leq d_{\sup}(f_n, f_m),$$

it follows that $(f_n(x))_{n \geq 1}$ is a Cauchy sequence in $(\mathbb{R}^n, d)$, for all $x \in I$. Denoting by $f(x)$ the limit, we obtain a function $f \in \mathscr{F}(I, \mathbb{R}^n)$. To show that $f_n \rightrightarrows f$, let $\varepsilon > 0$ and we look for n_ε such that $d_{\sup}(f_n, f) < \varepsilon$ for all $n \geq n_\varepsilon$. For that, use that $(f_n)_{n \geq 1}$ is Cauchy and choose n_ε such that $d_{\sup}(f_n, f_m) < \varepsilon/2$ for all $n, m \geq n_\varepsilon$. Combining with the previous displayed inequality, we have $d(f_n(x), f_m(x)) < \varepsilon/2$ for all such n, m and all $x \in I$. Taking $m \to \infty$, we find that $d(f_n(x), f(x)) \leq \varepsilon/2 < \varepsilon$ for all $n \geq n_\varepsilon$ and $x \in I$, i.e. $d_{\sup}(f_n, f) < \varepsilon$ for all $n \geq n_\varepsilon$.

We now describe a cool application: the existence of a "space filling curve", by which we mean a continuous surjective map from the unit interval $[0,1]$ to the unit square. That curve will be obtained as the limit of a sequence of curves $(\gamma_n)_{n\geq 0}$ that fill the square better and better. The sequence is constructed by an iteration process, as indicated in the picture.

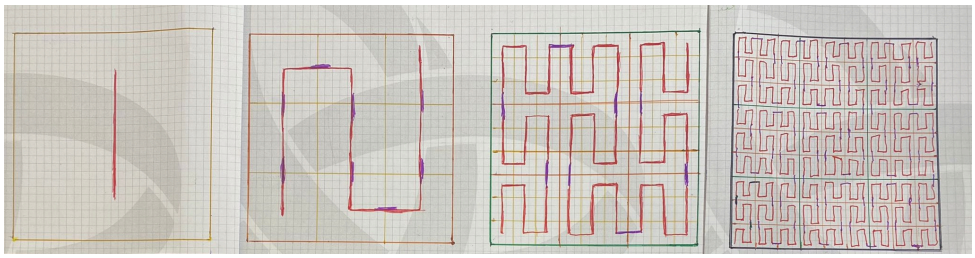
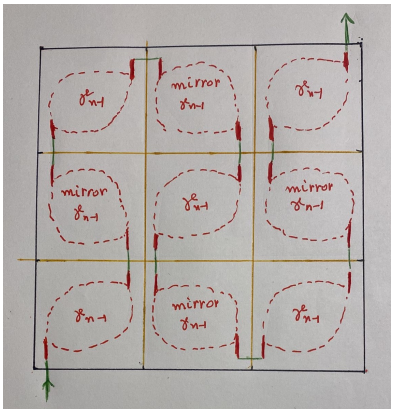


Fig. 3.7 γ_0 (left) and its first three iterations in the sequence whose limit is the Peano curve



To go from γ_{n-1} to γ_n one divides the square into 9 pieces and, inside each of those, one places a rescaled γ_{n-1} in a way that the 9 rescaled copies fit to each other as in the picture. In fact, when constructing γ_n, we are dividing the square into 9^n smaller squares; at the same time we also divide the domain $[0,1]$ into $9^n = 9\cdot 9^{n-1}$ smaller intervals and, while $t\in[0,1]$ travels through those smaller intervals, in the obvious order, $\gamma_n(t)$ travels through the 9^n smaller squares (but the order depends on the previous γ_{n-1}). While the picture should be clear now, each γ_n is defined to be linear on each straight segment (hence piecewise linear).

Theorem 3.59. *The sequence $(\gamma_n)_{n\geq 0}$ is convergent in the space $(\mathscr{C}(I,\mathbb{R}^2),\hat{d}_{sup})$ and its limit*

$$\gamma : [0,1] \to [0,1]\times[0,1]$$

*is a continuous surjective map. (γ is known under the name of **the Peano curve**).*

Proof. Given the iteration process, for any $n\geq m$ and any $t\in[0,1]$, $\gamma_n(t)$ and $\gamma_m(t)$ belong together to one of the 9^m little squares of side length $a_m = \left(\frac{1}{3}\right)^m$; hence the distance between them is less than the length of the diagonals:

$$d(\gamma_n(t),\gamma_m(t)) \leq a_m\sqrt{2}.$$

Since this holds for all $t\in[0,1]$, we have $d_{\sup}(\gamma_n,\gamma_m)\leq a_m\sqrt{2}$ as well. This implies that $(\gamma_n)_{n\geq 0}$ is a Cauchy sequence in $(\mathscr{C}(I,\mathbb{R}^2),\hat{d}_{\sup})$ and, therefore, convergent in that space. Hence we can talk about γ, and γ is continuous. To prove it is surjective, let $p=(x,y)\in[0,1]\times[0,1]$ be arbitrary. Then, for each $n\geq 1$, looking at the partition of the unit square into 9^n smaller squares, (x,y) will belong to at least one of them; hence we find $t_n\in[0,1]$ such that

$$d(p,\gamma_n(t_n)) < a_n\cdot\sqrt{2}.$$

We see that, if $(t_n)_{n\geq 1}$ was convergent to some t_∞, or at least if it had a convergent subsequence then, passing to the limit, we would obtain that $p=\gamma(t_\infty)$ and we would be done. Now, you may remember from Analysis that any sequence in $[0,1]$ does, indeed, admit a convergent subsequence. This will also be discussed in the next chapter, with the conclusion that this is actually a result that belongs 100% to Topology rather than Analysis. See Lemma 4.23 and Theorem 4.43.

Chapter 4
Topological properties

1. **Connectedness**

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   - Connected components
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2. **Compactness**

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3. **Local compactness and the one-point compactification**

   - Local compactness
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4. **Paracompactness**

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4.1 Connectedness

Definitions and examples

Definition 4.1. We say that a topological space $(X,\mathscr{T})$ is **connected** if X cannot be written as the union of two disjoint non-empty opens $U,V\subset X$.

We say that a topological space $(X,\mathscr{T})$ is **path connected** if for any $x,y\in X$, there exists a path γ connecting x and y, i.e. a continuous map $\gamma:[0,1]\to X$ such that $\gamma(0)=x$, $\gamma(1)=y$.

Given $(X,\mathscr{T})$, we say that a subset $A\subset X$ is connected (or path connected) if A, together with the induced topology, is connected (path connected).

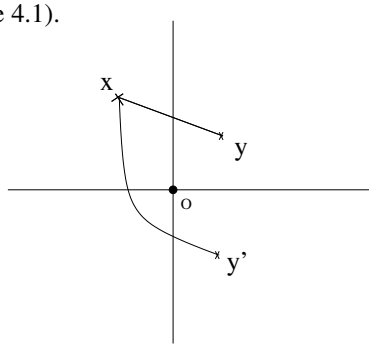
As we shall soon see, path connectedness implies connectedness. This is good news since, unlike connectedness, path connectedness can be checked more directly (see the examples below).

Example 4.2.

(1) $X=\{0,1\}$ with the discrete topology is not connected. Indeed, $U=\{0\}$, $V=\{1\}$ are disjoint non-empty opens (in X) whose union is X.
(2) Similarly, $X=[0,1)\cup[2,3]$ is not connected (take $U=[0,1)$, $V=[2,3]$). More generally, if $X\subset\mathbb{R}$ is connected, then X must be an interval. Indeed, if not, we find $r,s\in X$ and $t\in(r,s)$ such that $t\notin X$. But then $U=(-\infty,t)\cap X, V=(t,\infty)\cap X$ are opens in X, nonempty (as $r\in U$, $s\in V$), disjoint, with $U\cup V=X$ (as $t\notin X$).
(3) However, although true, the fact that any interval $I\subset\mathbb{R}$ is connected is not entirely obvious. In contrast, the path connectedness of intervals is clear: for any $x,y\in I$,

$$\gamma:[0,1]\to\mathbb{R},\gamma(t)=(1-t)x+ty$$

takes values in I (since I is an interval) and connects x and y.
(4) Similarly, any convex subset $X\subset\mathbb{R}^n$ is path connected (recall that X being convex means that for any $x,y\in X$, the whole segment $[x,y]$ is contained in X).
(5) $X=\mathbb{R}^2-\{0\}$, although not convex, is path connected: if $x,y\in X$ and the segment $[x,y]$ does not contain the origin, we use the linear path from x to y. But even if $[x,y]$ contains the origin, we can join them by a path going around the origin (see Figure 4.1).



$\mathbb{R}^2-\{0\}$ is path connected

Fig. 4.1

Lemma 4.3. *The unit interval* $[0,1]$ *is connected.*

Proof. We assume the contrary: $\exists$ disjoint non-empty U,V, opens in $[0,1]$ such that $U\cup V=[0,1]$. Since $U=[0,1]-V$, U must be closed in $[0,1]$. Hence, as a limit of points in U, $R:=\sup U$ must belong to U. We claim that $R=1$. If not, we find $(R-\varepsilon,R+\varepsilon)\cap[0,1]\subset U$ and then $R+\frac{1}{2}\varepsilon$ is an element in U strictly greater than its supremum- which is impossible. In conclusion, $1\in U$. But exactly the same argument shows that $1\in V$, and this contradicts the fact that $U\cap V=\emptyset$.

Basic properties

Proposition 4.4.

(a) If $f : X \to Y$ is a continuous map and X is connected, then $f(X)$ is connected.
(b) Given $(X, \mathscr{T})$, if for any two points $x, y \in X$, there exists $\Gamma \subset X$ connected such that $x, y \in \Gamma$, then X is connected.

Proof. For (a), replacing Y by $f(X)$, we may assume that f is surjective, and we want to prove that Y is connected. If it is not, we find $U, V \subset Y$ disjoint nonempty opens whose union is Y. But then $f^{-1}(U), f^{-1}(V) \subset X$ are disjoint (since U and V are), nonempty (since U and V are and f is surjective) opens (because f is continuous) whose union is X- and this contradicts the connectedness of X. For (b) we reason again by contradiction, and we assume that X is not connected, i.e. $X = U \cup V$ for some disjoint nonempty opens U and V. Since they are non-empty, we find $x \in U$, $y \in V$. By hypothesis, we find Γ connected such that $x, y \in \Gamma$. But then

$$U' = U \cap \Gamma,\ V' = V \cap \Gamma$$

are disjoint non-empty opens in Γ whose union is Γ- and this contradicts the connectedness of Γ.

Theorem 4.5. *Any path connected space X is connected.*

Proof. We use (b) of the proposition. Let $x, y \in X$. We know there exists $\gamma : [0,1] \to X$ joining x and y. But then $\Gamma = \gamma([0,1])$ is connected by using (a) of the proposition and the fact that $[0,1]$ is connected; also, $x, y \in \Gamma$.

Since a connected subset of $\mathbb{R}$ must be an interval, and that any interval is path connected, the theorem implies:

Corollary 4.6. *The only connected subsets of $\mathbb{R}$ are the intervals.*

Combining with part (a) of Proposition 4.4 we deduce the following:

Corollary 4.7. *If X is connected and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is an interval.*

Corollary 4.8. *If X is connected, then any quotient of X is connected.*

Example 4.9. There are a few more consequences that one can derive by combining connectedness properties with the "removing one point trick" (Exercise 10.33 in Chapter 2).

(1) $\mathbb{R}$ cannot be homeomorphic to S^1. Indeed, if we remove a point from $\mathbb{R}$ the result is disconnected, while if we remove a point from S^1, the result stays connected.
(2) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$. The argument is similar to the previous one (recall that $\mathbb{R}^2 - \{0\}$ is path connected, hence connected).
(3) The more general statement that $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic if $n \neq m$ is much more difficult to prove. One possible proof is a generalization of the argument given above (when $n = 1$, $m = 2$)- but that is based on "higher versions of connectedness", a notion which is at the core of algebraic topology.

Exercise 4.10. Show that $[0,1)$ and $(0,1)$ are not homeomorphic.

Connected components

Definition 4.11. Let $(X,\mathscr{T})$ be a topological space. A **connected component** of X is any maximal connected subset of X, i.e. any connected $C\subset X$ with the property that, if $C'\subset X$ is connected and contains C, then C' must coincide with C.

Proposition 4.12. *Let $(X,\mathscr{T})$ be a topological space. Then*

(a) Any point $x\in X$ belongs to a connected component of X.
(b) If C_1 and C_2 are connected components of X then either $C_1=C_2$ or $C_1\cap C_2=\emptyset$.
(c) Any connected component of X is closed in X.

Proof. To prove the proposition we will use the following

Exercise 4.13. If $A,B\subset X$ are connected and $A\cap B\neq\emptyset$, then $A\cup B$ is connected.

For (a) of the proposition, let $x\in X$ and define $C(x)$ as the union of all connected subsets of X containing x. We claim that $C(x)$ is a connected component. The only thing that is not clear is the connectedness of $C(x)$. To prove that, we will use the criterion given by (b) of Proposition 4.4. For $y,z\in C(x)$ we have to find $\Gamma\subset C(x)$ connected such that $y,z\in\Gamma$. Due to the definition of $C(x)$, we find C_y and C_z- connected subsets of X, both containing x, such that $y\in C_y$, $z\in C_z$. The previous exercise implies that $\Gamma:=C_y\cup C_z$ is connected containing both y and z.

To prove (b), we use the exercise applied to $A=C_1$ and $B=C_2$ and the maximality of connected components.

To prove (c), due to the maximality of connected components, it suffices to prove that if $C\subset X$ is connected, then $\overline{C}$ is connected. Assume that $Y:=\overline{C}$ is not connected. We find D_1,D_2 nonempty opens in Y, disjoint, such that $Y=D_1\cup D_2$. Take $U_i=C\cap D_i$. Clearly, U_1 and U_2 are disjoint opens in C, with $C=U_1\cup U_2$. To reach a contradiction (with the fact that C is connected) it suffices to show that U_1 and U_2 are nonempty. To show that U_i is non-empty ($i\in\{1,2\}$), we use the fact that D_i is non-empty. We find a point $x_i\in D_i$. Since x_i is in the closure of C in Y, we have $U\cap C\neq\emptyset$ for each neighborhood U of x_i in Y. Choosing $U=D_i$, we have $C\cap D_i\neq\emptyset$.

Remark 4.14. From the previous proposition we deduce that, for any topological space $(X,\mathscr{T})$, the family $\{C_i\}_{i\in I}$ of connected components of $(X,\mathscr{T})$ (where I is an index set) gives a partition of X:

$$X=\bigcup_{i\in I}C_i,\ \ C_i\cap C_j=\emptyset\ \forall\, i\neq j,$$

called the partition of X into connected components.

Exercise 4.15. Let X be a topological space and assume that $\{X_1,\dots,X_n\}$ is a finite partition of X. Show that all X_i's are closed if and only if all of them are open. Deduce from this that, if each X_i is closed and connected, then $\{X_1,\dots,X_n\}$ must coincide with the partition of X into connected components.

Connected versus path connected, again

Theorem 4.5 shows that path connectedness implies connectedness. However, the converse does not hold in general. The standard example is "the flea and comb", drawn in Figure 4.2. Explicitly:

$$X=C\cup\{f\}\quad\text{ where } C=[0,1]\times\{0\}\cup\{(1/n,y):y\in[0,1],\ n\in\mathbb{Z}_{>0}\}\text{ and } f=(0,1).$$

Exercise 4.16. Show that, indeed, X is connected but not path connected.



The flea and comb

Fig. 4.2

However, there is a partial converse to Theorem 4.5 if $(X,\mathscr{T})$ is **locally path connected** in the sense that for any $x \in X$, inside any neighborhood of x one finds a path connected one.

Theorem 4.17. *Let $(X,\mathscr{T})$ be a topological space. If it has the property that any point of X has a path connected neighborhood (in particular, if X is locally path connected), then $(X,\mathscr{T})$ is path connected if and only if it is connected.*

Proof. We still have to prove that if X is connected then it is also path connected. We define on X the relation: $x \sim y$ if and only if x and y can be joined by a (continuous) path. This is an equivalence relation. Indeed, for $x,y,z \in X$:

- $x \sim x$: consider the constant path.
- if $x \sim y$ then $y \sim x$: if γ is a path from x to y, then $\gamma^-(t) = \gamma(1-t)$ is a path from y to x.
- if $x \sim y$ and $y \sim z$, then $x \sim z$. Indeed, if γ_1 is a path from x to y, while γ_2 from y to z, then the following is a path from x to z (all continuous)

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0,\frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } t \in [\frac{1}{2},1] \end{cases}$$

For $x \in X$, we denote by $C(x)$ the equivalence class of x:

$$C(x) = \{y \in X : y \sim x\}.$$

Note that each $C(x)$ is path connected. We claim that $C(x) = X$ for any $x \in X$. Since $\sim$ is an equivalence,

$$\{C(x) : x \in X\}$$

is a partition of X: if $C(x) \cap C(x') \neq \emptyset$, then $C(x) = C(x')$. What we want to prove is that this partition consists of one set only. Since X is connected, it is enough to prove that $C(x)$ is open for each $x \in X$. Fixing x, we want to prove that for any $y \in C(x)$, there is an open U such that $y \in U \subset C(x)$. To see this, we use the hypothesis to find a path connected neighborhood V of y. Since $y \sim x$ and $z \sim y \; \forall \, z \in V$, we deduce that $z \sim x \; \forall \, z \in V$, hence $V \subset C(x)$. Since V is a neighborhood of y, we find U open such that $y \in U \subset V$, and this U clearly has the desired properties.

Remark 4.18. Inspecting the proof we see there are a few general conclusions to draw, for any space X (connected or not). First of all, one can talk about **path connected components**, defined as the equivalence classes $C(x)$ w.r.t. the equivalence relation described above. Secondly, the last part of the proof shows that, if X is locally path connected then each $C(x)$ is open in X. Finally, the proof of the equality $C(x) = X$ now works to prove that the partition by the path connected components coincides with the one by connected components. In particular:

Theorem 4.19. *If $(X,\mathscr{T})$ is locally path connected, then its connected components are both open and closed in X.*

4.2 Compactness

Definition and first examples

Probably many of you have seen the notion of compact space in the context of subsets X of $\mathbb{R}^n$, as sets which are "closed and bounded". Notice that both being "closed" as well as being "bounded" depends on the ambient space $\mathbb{R}^n$ and the Euclidean metric. Hence it is rather surprising (and not at all obvious) that this is a topological property of X, i.e. it can be defined using open sets only.

Definition 4.20. Given a topological space $(X,\mathscr{T})$ an **open cover of** X is a family $\mathscr{U}=\{U_i : i\in I\}$ (I-some index set) consisting of open sets $U_i\subset X$ such that

$$X=\bigcup_{i\in I} U_i.$$

A **subcover** is any cover $\mathscr{V}$ with the property that $\mathscr{V}\subset\mathscr{U}$.

We say that a topological space $(X,\mathscr{T})$ is **compact** if **from any** open cover $\mathscr{U}=\{U_i : i\in I\}$ of X one can extract a finite open subcover, i.e. there exist $i_1,\dots,i_k\in I$ such that

$$X=U_{i_1}\cup\ldots\cup U_{i_k}.$$

We say that a subset $A\subset X$ is compact if, seen as a topological space the with topology induced from the one of X (see Example 2.11), it is a compact topological space.

Remark 4.21. Given a topological space $(X,\mathscr{T})$ and $A\subset X$, the compactness of A (viewed as a topological space with the topology induced from X) can be expressed using "open coverings of A in X", i.e. families $\mathscr{U}=\{U_i : i\in I\}$ (I-some index set) consisting of open sets $U_i\subset X$ such that

$$A\subset\bigcup_{i\in I} U_i.$$

A subcover is any cover (of A in X) $\mathscr{V}$ with the property that $\mathscr{V}\subset\mathscr{U}$. With these, A is compact if and only if from any open cover of A in X one can extract a finite open subcover. This follows immediately from the fact that the opens for the induced topology on A are of type $A\cap U$ with $U\in\mathscr{T}$ and from the fact that, for a family $\{U_i : i\in I\}$, we have

$$\bigcup_i (A\cap U_i)=A\Longleftrightarrow A\subset\bigcup_i U_i,$$

Example 4.22.

(1) $(X,\mathscr{T}_{\text{discr}})$ is compact if and only if X is finite (use the cover of X by the open-point opens).

(2) $\mathbb{R}$ is not compact. Indeed,

$$\mathbb{R}=\bigcup_{k\in\mathbb{Z}}(-k,k)$$

is an open cover from which we cannot extract a finite open subcover. By the same argument, any compact $A\subset\mathbb{R}^n$ must be bounded (e.g., when $n=1$, write $A\subset\cup_k(-k,k)$).

(3) $[0,1)$ is not compact. Indeed,

$$[0,1)\subset\bigcup_k(-\infty,1-\frac{1}{k})$$

defines an open cover of $[0,1)$ in $\mathbb{R}$, from which we cannot extract a finite open subcover. By a similar argument, any compact $A \subset \mathbb{R}^n$ must be closed in $\mathbb{R}^n$. To show this, assume for simplicity that $n = 1$. We proceed by contradiction and assume that there exists $a \in \overline{A} - A$. Since $a \notin A$,

$$U_\varepsilon := \mathbb{R} - [a-\varepsilon, a+\varepsilon]$$

form an open cover of A in $\mathbb{R}$ indexed by $\varepsilon > 0$. Extracting a finite subcover, we find

$$A \subset U_{\varepsilon_1} \cup \ldots \cup U_{\varepsilon_k} = U_\varepsilon \text{ where } \varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}.$$

But this implies that $A \cap (a-\varepsilon, a+\varepsilon) = \emptyset$ which contradicts $a \in \overline{A}$.

(4) $[0,1]$ is compact, see the lemma below.

Lemma 4.23. *The set* $[0,1]$, *equipped with the topology induced by the Euclidean metric, is compact.*

Proof. Let $\mathscr{U}$ be an open cover for $[0,1]$. We consider the set A of $a \in [0,1]$ such that $[0,a]$ is covered by a finite collection of sets from $\mathscr{U}$. The point 0 is contained in a set U_* from $\mathscr{U}$ and by openness of U_* in $[0,1]$, there exists an element $m \in (0,1]$ such that $[0,m] \subset U_*$. Hence, $[0,m] \subset A$.

The set A contains $[0,m]$, and is bounded from above, hence has a supremum s. Clearly, $0 < m \leq s \leq 1$. Since $\mathscr{U}$ covers $[0,1]$, there exists a set $U_0 \in \mathscr{U}$ such that $s \in U_0$. By openness of the latter set, there exists a $\delta > 0$ such that $(s-\delta, s] \subset U_0$. Since $s - \delta < s$, there exists an element $a \in A$ such that $s - \delta < a < s$. Hence, there exists a finite number of sets $U_1, \ldots, U_n$ from $\mathscr{U}$ such that $[0,a]$ is contained in the union of $U_1, \ldots, U_n$. Now $[0,s] \subset [0,a] \cup U_0$ and we see that

$$[0,s] \subset U_0 \cup \cdots \cup U_n \qquad (*)$$

so that $s \in A$. It now suffices to show that $s = 1$. Assume this were not the case. Then there would exist $s' \in (s,1]$ such that $[s,s'] \subset U_0$; hence (*) would hold with s' in place of s, whence $s' \in A$. This contradicts that s is an upper bound of A.

Exercise 4.24. Show that the set

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

is compact in $\mathbb{R}$.

Exercise 4.25. Inside $X = \mathbb{R}$ endowed with the lower limit topology $\mathscr{T}_l$ we consider $A = [0,1]$. Is it compact?

Topological properties of compact spaces

In this section we point out some topological properties of compact spaces.

The first one says that "closed inside compact is compact".

Proposition 4.26. *If $(X,\mathscr{T})$ is a compact space, then any closed subset $A\subset X$ is compact.*

The second one says that "compact inside Hausdorff is closed".

Theorem 4.27. *In a Hausdorff space $(X,\mathscr{T})$, any compact set is closed.*

The third one says that "disjoint compacts inside a Hausdorff can be separated".

Proposition 4.28. *In a Hausdorff space $(X,\mathscr{T})$, any two disjoint compact sets A and B can be separated topologically, i.e. there exist opens $U,V\subset X$ such that*

$$A\subset U,\ B\subset V,\ U\cap V=\emptyset.$$

Corollary 4.29. *Any compact Hausdorff space is normal.*

Proof. (of Proposition 4.26) If $\mathscr{U}$ is an open cover of A in X, then adding $X-A$ to $\mathscr{U}$ (which is open since A is closed), we get an open cover of X. Extracting a finite subcover (which may or may not contain $X-A$), denoting by $U_1,\dots,U_n$ the elements of this finite subcover which are different from $X-A$, this will define a finite subcover of the original cover of A in X.

Proof. (of Proposition 4.28): We introduce the following notation: given $Y,Z\subset X$ we write $Y|Z$ if Y and Z can be separated, i.e. if there exist opens U and V (in X) such that $Y\subset U$, $Z\subset V$ and $U\cap V=\emptyset$. We claim that, if Z is compact and $Y|\{z\}$ for all $z\in Z$, then $Y|Z$.

Proof. (of the claim) We know that for each $z\in Z$ we find opens U_z and V_z such that $Y\subset U_z$, $z\in V_z$ and $U_z\cap V_z=\emptyset$. Note that $\{V_z : z\in Z\}$ is an open cover of Z in X: indeed, any $z\in Z$ belongs at least to one of the opens in the cover (namely V_z). By compactness of Z, we find a finite number of points $z_1,\dots,z_n\in Z$ such that

$$Z\subset V_{z_1}\cup\ldots\cup V_{z_n}.$$

Denoting by V the last union, and considering

$$U=U_{z_1}\cap\ldots\cap U_{z_n},$$

V is an open containing Z, U is an open containing Y. Moreover, $U\cap V=\emptyset$: indeed, if x is in the intersection, since $x\in V$ we find k such that $x\in V_{z_k}$; but $x\in U$ hence $x\in U_{z_k}$, which is impossible since $U_{z_k}\cap V_{z_k}=\emptyset$. In conclusion, U and V show that Y and Z can be separated.

Back to the proof of the proposition, let $a\in A$ arbitrary. Now, since X is Hausdorff, we have $\{a\}|\{b\}$ for all $b\in B$. Since B is compact, the claim above implies that $\{a\}|B$, or, equivalently, $B|\{a\}$. This holds for all $a\in A$, hence using again the claim (and the fact that A is compact) we deduce that $A|B$.

Proof. (of Theorem 4.27) For Theorem 4.27, assume that $A\subset X$ is compact, and we prove that $\overline{A}\subset A$: if $x\in\overline{A}$, then, for any neighborhood U of x, $U\cap A\neq\emptyset$, and this shows that x cannot be separated from A; using the proposition, we conclude that $x\in A$.

Exercise 4.30. Deduce that a subset $A\subset\mathbb{R}$ is compact if and only if it is closed and bounded. What is missing to prove the same for subsets of $\mathbb{R}^n$?

Compactness of products, and compactness in $\mathbb{R}^n$

Next, we are interested in the compactness of the product of two compact spaces. We will use the following:

Lemma 4.31. (*Tube Lemma*) Let X and Y be two topological spaces, $x_0 \in X$, and let $U \subset X \times Y$ be an open (in the product topology) such that

$$\{x_0\} \times Y \subset U.$$

If Y is compact, then there exists $W \subset X$ open containing x_0 such that

$$W \times Y \subset U.$$

Proof. For $y \in Y$, since $(x_0, y) \in U$ and U is open in the product topology, there exist opens $W_y \subset X$, $V_y \subset Y$ s.t.

$$W_y \times V_y \subset U.$$

Now, $\{V_y : y \in Y\}$ will be an open cover of Y, hence we find $y_1, \ldots, y_n \in Y$ such that $Y = V_{y_1} \cup \ldots \cup V_{y_n}$. Choose $W = W_{y_1} \cap \ldots \cap W_{y_n}$, which is an open containing x_0, as a finite intersection of such. To check $W \times Y \subset U$, let $(x,y) \in W \times Y$. Since the V_{y_i}'s cover Y, we find i s.t. $y \in V_{y_i}$. But $x \in W$ implies $x \in W_{y_i}$, hence $(x,y) \in W_{y_i} \times V_{y_i}$. But $W_z \times V_z \subset U$ for all $z \in Y$, hence $(x,y) \in U$.

Theorem 4.32. *If X and Y are compact spaces, then $X \times Y$ is compact.*

Proof. Let $\mathscr{U}$ be an open cover of $X \times Y$. For each $x \in X$,

$$\{x\} \times Y \subset X \times Y$$

is compact (why?), hence we find a $\mathscr{U}_x \subset \mathscr{U}$ finite such that

$$\{x\} \times Y \subset \bigcup_{U \in \mathscr{U}_x} U. \tag{4.2.1}$$

Using the previous lemma, we find W_x open containing x such that

$$W_x \times Y \subset \bigcup_{U \in \mathscr{U}_x} U. \tag{4.2.2}$$

Now, $\{W_x : x \in X\}$ is an open cover of X, hence we find a finite subcover

$$X = W_{x_1} \cup \ldots \cup W_{x_p}.$$

Then

$$\mathscr{V} = \mathscr{U}_{x_1} \cup \ldots \cup \mathscr{U}_{x_p}$$

is finite union of finite collections, hence finite. Moreover, $\mathscr{V}$ still covers $X \times Y$: given (x,y) arbitrary, using (4.2.2), we find i such that $x \in W_{x_i}$. Hence $(x,y) \in W_{x_i} \times Y$, and using (4.2.2) we find $U \in \mathscr{U}_{x_i}$ such that $(x,y) \in U$. Hence we found $U \in \mathscr{V}$ such that $(x,y) \in U$,

Corollary 4.33 (Heine-Borel theorem). *A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. The direct implication was already mentioned in Example 4.22 (and that A is closed follows also from Theorem 4.27). For the converse, since A is bounded, we find $R, r \in \mathbb{R}$ such that $A \subset [r,R]^n$. The intervals $[r,R]$ are homeomorphic to $[0,1]$, hence compact. The previous theorem implies that $[r,R]^n$ is compact, hence A must be compact as closed inside a compact.

Example 4.34. In particular, spaces like the spheres S^n, the closed disks D^n, the Moebius band, etc, are compact.

Compactness and continuous functions

Theorem 4.35. *If $f : X \to Y$ is a continuous function and $A \subset X$ is compact, then $f(A) \subset Y$ is compact.*

Proof. If $\mathscr{U}$ is an open cover of $f(A)$ in Y, then $f^{-1}(\mathscr{U}) := \left\{ f^{-1}(U) : U \in \mathscr{U} \right\}$ is an open cover of A in X, hence it has a finite subcover $\left\{ f^{-1}(U_i) : 1 \leq i \leq n \right\}$ with $U_i \in \mathscr{U}$. But then $\{U_i\}$ will be a finite subcover of $\mathscr{U}$.

Finally, we can state the property of compact spaces that we referred to several times when having to prove that certain continuous injections are homeomorphisms.

Theorem 4.36. *If $f : X \to Y$ is continuous and bijective, and if X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We have to show that the inverse g of f is continuous. For this we show that if U is open in X then the pre-image $g^{-1}(U)$ is open in Y. Since the open (or closed) sets are just the complements of closed (respectively open) subsets, and $g^{-1}(X \setminus U) = Y \setminus g^{-1}(U)$, we see that the continuity of g is equivalent to: if A is closed in X then the pre-image $g^{-1}(A)$ is closed in Y. To prove this, let A be a closed subset of X. Since g is the inverse of f, we have $g^{-1}(A) = f(A)$, hence we have to show that $f(A)$ is closed in Y. Since X is compact, Proposition 4.26 implies that A is compact. By the previous theorem, $f(A)$ must be compact. Since Y is Hausdorff, Theorem 4.27 implies that $f(A)$ is closed in Y.

Corollary 4.37. *Any continuous injection $f : X \to Y$ of a compact space into a Hausdorff one is an embedding.*

Example 4.38. Here is an example which shows the use of compactness. We will show that there is no injective continuous map $f : S^1 \to \mathbb{R}$.

Proof (sketch; see also the exercises at the end of the chapter). Assume there is such a map. Since S^1 is connected and compact, its image is a closed interval $[m,M]$; f becomes a continuous bijection $f : S^1 \to [m,M]$, hence a homeomorphism (cf. the previous theorem). Then use the "removing a point trick".

Example 4.39. (back to the torus; recap) The last theorem together its corollary, combined with the Heine-Borel theorem, imply Proposition 3.5, its reformulation from Proposition 3.17 and its addendum from (3.2.7). Recall that those propositions were used to conclude that the identification (3.2.4) of the Moebius band was a homeomorphism, or that its embedding is actually a topological embedding. Similarly for the torus, with the embedding (3.2.8) induced by (3.2.9), or other examples such as Example 3.18 or Example 3.32. Of course, one can also build examples in which one needs the general Theorem 4.36 instead of Proposition 3.5. And not only that: even in the examples in which the proposition suffices, the general discussion behind the theorem clearly gives more insight.

All together, in some sense, it is only now that the discussion on the Moebius band, the torus, and other similar examples are complete. For clarity, let's now give a final overview of our discussions on the torus. First, in Chapter 1, Section 1.6, we introduced the torus intuitively, by gluing the opposite sides of a square. The result was a subspace of $\mathbb{R}^3$ (or rather a shape). After the definition of topological spaces, we learned that these subspaces of $\mathbb{R}^3$ are topological spaces on their own- endowed with the induced topology. In the previous chapter, in section 3.2, we gave a precise meaning to the process of gluing and then in Example 3.28 we introduced the abstract torus T_{abs}. In the same example we also produced one (of the many possible) continuous injections

$$f : T_{\text{abs}} \to \mathbb{R}^3.$$

The previous corollary implies that f defines an embedding of the abstract T_{abs} into $\mathbb{R}^3$, which is a homeomorphism onto its image, image which is the explicit model $T_{R,r}$ described already in the first chapter, in (1.6.1) and (1.6.2).

Exercise 4.40. Prove that the torus is homeomorphic to $S^1 \times S^1$.

Embeddings of compact manifolds

Theorem 4.41. *Any n-dimensional compact topological manifold can be embedded in $\mathbb{R}^N$, for some $N \in \mathbb{N}$.*

Proof. We use the Euclidean distance in $\mathbb{R}^n$ and we denote by B_r and $\overline{B}_r$ the resulting open and closed balls of radius r centered at the origin. We choose a function

$$\eta : \mathbb{R}^n \to [0,1] \text{ such that } \eta|_{B_1} = 1, \;\; \eta|_{\mathbb{R}^n - B_2} = 0.$$

For instance, we could choose

$$\eta(x) = \frac{d(x, \mathbb{R}^n - B_2)}{d(x, \overline{B}_1) + d(x, \mathbb{R}^n - B_2)}.$$

For a coordinate chart

$$\chi : U \to \mathbb{R}^n$$

and any radius $r > 0$, we consider:

$$U(r) := \chi^{-1}(B_r), \;\; U[r] = \overline{U(r)} = \chi^{-1}(\overline{B}_r).$$

Since X is compact, we find a finite number of coordinate charts

$$\chi_i : U_i \to \mathbb{R}^n, \;\; 1 \leq i \leq k,$$

such that $\{U_i(1) : 1 \leq i \leq k\}$ cover X. For each i, consider $\eta \circ \chi_i : U_i \to [0,1]$; since it vanishes on $U_i - U_i(2)$, extending it to be 0 outside U_i will give us a continuous map

$$\eta_i : X \to [0,1].$$

Similarly, since the product $\eta_i \cdot \chi_i : U_i \to \mathbb{R}^n$ vanishes on $U_i - U_i(2)$, extending it by 0 gives us continuous maps

$$\tilde{\chi}_i : X \to \mathbb{R}^n.$$

Finally, we define

$$f = (\eta_1, \ldots, \eta_k, \tilde{\chi}_1, \ldots, \tilde{\chi}_k) : X \to \mathbb{R}^{k(n+1)}.$$

It is continuous by construction hence, by Theorem 4.36, it suffices to show that f is injective. Assume that $f(x) = f(y)$ with $x, y \in X$. From the choice of the charts, we find i such that $x \in U_i(1)$. Then $\eta_i(x) = 1$. But $f(x) = f(y)$ implies that $\eta_i(y) = \eta_i(x) = 1$. On one hand, this implies that $\eta_i(y) \neq 0$, hence y must be inside U_i (even inside $U_i(2)$). But these imply

$$\tilde{\chi}_i(x) = \eta_i(x)\chi_i(x) = \chi_i(x)$$

and similarly for y. Finally, $f(x) = f(y)$ also implies that $\tilde{\chi}_i(x) = \tilde{\chi}_i(y)$. Hence x and y are in the domain of χ_i and are sent by χ_i into the same point. Hence $x = y$.

Corollary 4.42. *Any n-dimensional compact topological manifold is metrizable.*

Sequential compactness

When one deals with sequences, one often sees statements of type "we now consider a subsequence with this property". Compactness is related to the existence of convergent subsequences. In general, given a topological space $(X,\mathscr{T})$, one says that X is **sequentially compact** if any sequence $(x_n)_{n\geq 1}$ of elements of X has a convergent subsequence. Recall that a subsequence of $(x_n)_{n\geq 1}$ is a sequence $(y_k)_{k\geq 1}$ of type

$$y_k = x_{n_k},\ \text{with } n_1 < n_2 < n_3 < \dots.$$

However, we have already mentioned (and seen in various other cases) that topological properties involving sequences usually require the axiom of first countability.

Theorem 4.43. *Any first countable compact space is sequentially compact.*

Proof. Let X be first countable and compact, and assume that $(x_n)_{n\geq 1}$ is an arbitrary sequence in X. For each integer $n \geq 1$ we put

$$U_n = X - \overline{\{x_n, x_{n+1}, \dots\}}.$$

Note that these define an increasing sequence of open subsets of X:

$$U_1 \subset U_2 \subset U_3 \subset \dots.$$

We now claim that $\cup_n U_n \neq X$. If this is not the case, $\{U_n : n \geq 1\}$ is an open cover of X hence we find a finite set F such that $\{U_i : i \in F\}$ covers X. Since our cover is increasing, we find that $U_p = X$ where $p = \max F$, and this is clearly impossible. In conclusion, $\cup_n U_n \neq X$. Hence there exists $x \in X$ such that, for all $n \geq 1$, $x \notin U_n$. Choose a countable basis of neighborhoods of x

$$V_1 \supset V_2 \supset V_3 \supset \dots.$$

Since $x \notin U_1$, we have $V \cap \{x_1, x_2, \dots\} \neq \emptyset$ for all neighborhoods V of x. Choosing $V = V_1$, we find n_1 such that

$$x_{n_1} \in V_1.$$

Next, we use the fact that $x \notin U_n$ for $n = n_1 + 1$. This means that $V \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset$ for all neighborhoods V of x. Choosing $V = V_2$, we find $n_2 > n_1$ such that

$$x_{n_2} \in V_2.$$

We continue this process inductively (e.g. the next step uses $x \notin U_n$ for $n = n_1 + n_2 + 1$) and we find $n_1 < n_2 < \dots$ such that

$$x_{n_k} \in V_k$$

for all $k \geq 1$. Then $(x_{n_k})_{k\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$ converging to x.

Corollary 4.44. *Any compact metrizable space is sequentially compact.*

Actually, as we shall see in the next heading, for metric spaces compactness is equivalent to sequential compactness.

4.3 Local compactness and the one-point compactification

Local compactness

Definition 4.45. A space X is called **locally compact** if any point of X admits a compact neighborhood.

We will be mainly interested in locally compact spaces which are Hausdorff.

Exercise 4.46. Prove that, in a locally compact Hausdorff space X, for each $x \in X$ the collection of all compact neighborhoods of x is a basis of neighborhoods of x, i.e.: $\forall\, U \in \mathscr{T}_X(x)$, $\exists\, N \in \mathscr{N}_{\mathscr{T}_X}(x)$ compact with $N \subset U$.

Example 4.47.

1. any compact Hausdorff space $(X, \mathscr{T})$ is locally compact Hausdorff.
2. $\mathbb{R}^n$ is locally compact (use closed balls as compact neighborhoods).
3. any open $U \subset \mathbb{R}^n$ is locally compact (use small enough closed balls). In general, any open subset of a locally compact Hausdorff space is locally compact (use the previous exercise).
4. any closed $A \subset \mathbb{R}^n$ is locally compact. Indeed, for any $a \in A$, $B[a,1] \cap A$ is a neighborhood of a (in A) that is compact (closed inside compact is compact). Similarly for closed subsets A of a locally compact Hausdorff X.
5. the interval $(0,1]$ is locally compact (combine the arguments from (2) and (3)).
6. $\mathbb{Q}$ is not locally compact. To prove it assume that 0 has a compact neighborhood N. Then $(-\varepsilon,\varepsilon) \cap \mathbb{Q} \subset N$ for some $\varepsilon > 0$. Passing to closures in $\mathbb{R}$, since N is closed (as compact), we find $[-\varepsilon,\varepsilon] \subset \overline{N} = N$. Contradiction!

Locally compact Hausdorff spaces which are 2nd countable deserve special attention: they include topological manifolds and, as we shall see later on, they are easier to handle. The most basic property (to be used several times) is that they can be "exhausted" by compact spaces.

Definition 4.48. Let $(X, \mathscr{T})$ be a topological space. An **exhaustion** of X is a family $\{K_n : n \in \mathbb{Z}_+\}$ of compact subsets of X such that $X = \cup_n K_n$ and $K_n \subset \overset{\circ}{K}_{n+1}$ for all n.

Theorem 4.49. *Any locally compact, Hausdorff, 2nd countable space admits an exhaustion.*

Proof. Let $\mathscr{B}$ be a countable basis and consider $\mathscr{V} = \left\{B \in \mathscr{B} : \overline{B} - \text{compact}\right\}$. Then $\mathscr{V}$ is a basis: for any open U and $x \in X$ we choose a compact neighborhood N inside U; since $\mathscr{B}$ is a basis, we find $B \in \mathscr{B}$ s.t. $x \in B \subset N$; this implies $\overline{B} \subset N$ and then $\overline{B}$ must be compact; hence we found $B \in \mathscr{V}$ s.t. $x \in B \subset U$. In conclusion, we may assume that we have a basis $\mathscr{V} = \{V_n : n \in \mathbb{Z}_+\}$ where $\overline{V}_n$ is compact for each n. We define the exhaustion $\{K_n\}$ inductively, as follows. We set $K_1 = \overline{V}_1$. Since $\mathscr{V}$ covers the compact K_1, we find i_1 such that

$$K_1 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_1}.$$

Denoting by D_1 the right hand side of the inclusion above, we set $K_2 = \overline{D}_1 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_1}$. This is compact because it is a finite union of compacts. Since $D_1 \subset K_2$ and D_1 is open, we must have $D_1 \subset \overset{\circ}{K}_2$; since $K_1 \subset D_1$, we have $K_1 \subset \overset{\circ}{K}_2$. Next, we choose $i_2 > i_1$ such that

$$K_2 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_2},$$

we denote by D_2 the right hand side of this inclusion, and we set $K_3 = \overline{D}_2 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_2}$. As before, K_3 is compact, its interior contains D_2, hence also K_2. Continuing inductively we obtain the desired exhaustion.

The one-point compactification

Intuitively, the idea of the one-point compactification is to "add a point at infinity" to achieve compactness.

Definition 4.50. Let $(X, \mathscr{T})$ be a topological space. A **one-point compactification** of X is a compact Hausdorff space $(\tilde{X}, \tilde{\mathscr{T}})$ together with an embedding $i : X \to \tilde{X}$, such that $\tilde{X} - i(X)$ has precisely one point.

From the remarks above it follows that, if X admits a one-point compactification, then it must be locally compact and Hausdorff. Conversely, we have:

Theorem 4.51. *If X is a locally compact Hausdorff space, then*

1. *It admits a one-point compactification X^+.*
2. *Any two one-point compactifications of X are homeomorphic.*

Moreover, if X is 2nd countable, then so is X^+.

Example 4.52.

1. If $X = (0,1]$, then X^+ is (homeomorphic to) $[0,1]$, with i the inclusion $i : (0,1] \to [0,1]$.
2. If $X = (0,1)$, then X^+ is (homeomorphic to) the circle S^1. Indeed, $i : (0,1) \to S^1$ as in Figure 4.3 (e.g. $i(t) = (\cos(2\pi t), \sin(2\pi t))$ has the properties from the previous proposition.

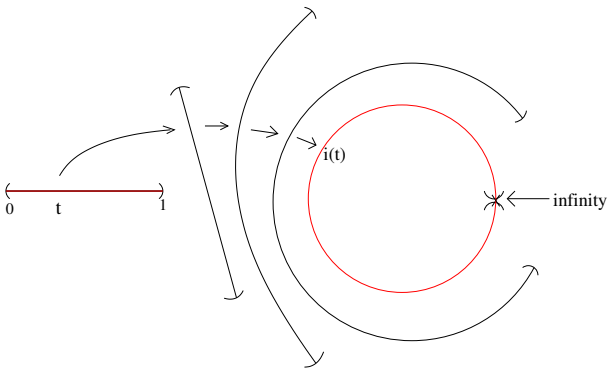


Fig. 4.3

3. If $X = [-1,0) \cup (1,2) \subset \mathbb{R}$, X^+ is shown in Figure 4.4.
4. If $X = \overset{\circ}{D}{}^n$ the one-point compactification is S^n.

Proof. (of Theorem 4.51) For the existence, choose a symbol $\infty \notin X$ and consider

$$X^+ = X \cup \{\infty\}.$$

Since $X \subset X^+$, any subset of X is a subset of X^+. We consider the family of subsets of X^+:

$$\mathscr{T}^+ = \mathscr{T} \cup \mathscr{T}(\infty), \quad \text{where} \quad \mathscr{T}(\infty) = \left\{X^+ - K : K \subset X, K - \text{compact}\right\}.$$

We claim that $\mathscr{T}^+$ is a topology on X^+. First, we show that $U \cap V \in \mathscr{T}^+$ whenever $U, V \in \mathscr{T}^+$. We have three cases. If $U, V \in \mathscr{T}$, we know that $U \cap V \in \mathscr{T}$. If U and V are both in $\mathscr{T}(\infty)$, then so is their intersection because union of

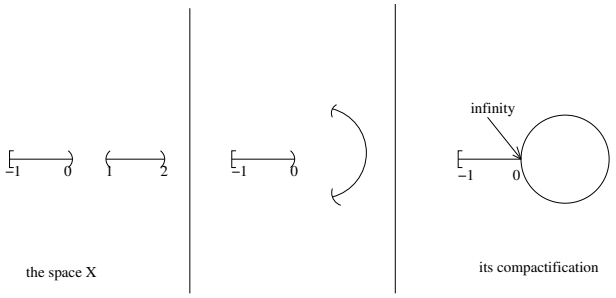


Fig. 4.4

two compacts is compact (show this!). Finally, if $U \in \mathscr{T}$ and $V = X^+ - K \in \mathscr{T}(\infty)$, then $U \cap V = U \cap (X - K)$ is open in X because U and $X - K$ are. Next, we show that arbitrary union of sets from $\mathscr{T}^+$ is in $\mathscr{T}^+$. This property holds for $\mathscr{T}$, and also for $\mathscr{T}(\infty)$ since intersection of compacts is compact (why?). Hence it suffices to show that $U \cup V \in \mathscr{T}^+$ whenever $U \in \mathscr{T}$, $V \in \mathscr{T}(\infty)$. Writing $V = X^+ - K$ with $K \subset X$ compact, we have $U \cup V = X^+ - K'$, where $K' = K \cap (X - U)$. Since $K - K' = K \cap U$, $K - K'$ is open in K, i.e. K' is closed in the compact K, hence K' is compact (Proposition 4.26 again!). Hence $U \cup V \in \mathscr{T}^+$.

We show that $(X^+, \mathscr{T}^+)$ is compact. Let $\mathscr{U}$ be an open cover of X^+. Choose $U = X^+ - K \in \mathscr{U}$ containing ∞. The sets $V \cap X$, with $V \in \mathscr{U}$, are open in X and cover the compact K. Hence there exist $V_1, \ldots, V_r \in \mathscr{U}$ such that $K \subset (V_1 \cap X) \cup \cdots \cup (V_r \cap X)$. Then $\{U, V_1, \ldots, V_r\}$ is a finite subcover of $\mathscr{U}$ covering X^+.

Next we show that X^+ is Hausdorff. Let $x, y \in X^+$ distinct; we are looking for $U, V \in \mathscr{T}^+$ such that $U \cap V = \emptyset$, $x \in U, y \in V$. When $x, y \in X$, just use the Hausdorffness of X. So, let's assume $y = \infty$. Then choose a compact neighborhood K of x and we consider $U \subset X$ open with $x \in U \subset K$. Then $x \in U$, $\infty \in X^+ - K$ and $U \cap (X^+ - K) = \emptyset$.

Next, we show that the inclusion $i : X \to X^+$ is an embedding, i.e. that $\mathscr{T}^+|_X = \mathscr{T}$. Now,

$$\mathscr{T}^+|_X = \mathscr{T} \cup \{U \cap X : U \in \mathscr{T}(\infty)\}$$

(just apply the definition!), and just remark that, for $U = X^+ - K \in \mathscr{T}(\infty)$, $U \cap X = X - K \in \mathscr{T}$. This concludes the proof of 1. For 2, let $\tilde{X}$ be another one-point compactification and we prove that it is homeomorphic to X^+. Choose $y_\infty \in \tilde{X}$ such that $\tilde{X} = i(X) \cup \{y_\infty\}$ and define

$$f : \tilde{X} \to X^+, \; f(y) = \begin{cases} x & \text{if } y = i(x) \in i(X) \\ \infty & \text{if } y = y_\infty \end{cases}$$

Since f is bijective, $\tilde{X}$ is compact and X^+ is Hausdorff, it suffices to show that f is continuous (Theorem 4.36). Let $U \in \mathscr{T}^+$; we prove $f^{-1}(U) \in \tilde{\mathscr{T}}$. If $U \in \mathscr{T}$, then $f^{-1}(U) \subset i(X)$ and then

$$f^{-1}(U) = \{y = i(x) : f(y) \in U\} = \{i(x) : x \in U\} = i(U)$$

is open in $i(X)$ since i is an embedding. But, since $\tilde{X}$ is Hausdorff, $i(X) = \tilde{X} - \{y_\infty\}$ is open in $\tilde{X}$, hence so is $f^{-1}(U)$. The other case is when $U = X^+ - K$ with $K \subset X$ compact. Then $f^{-1}(U) = \tilde{X} - i(K)$ is open in $\tilde{X}$ since $i(K)$ is compact as the image of a compact by a continuous function.

To prove the last part of the theorem we start with an exhaustion $\{K_n : n \in \mathbb{Z}_+\}$ and a countable basis $\mathscr{B}$ of X and we claim that the following is a basis of X^+:

$$\mathscr{B}^+ := \mathscr{B} \cup \mathscr{B}(\infty), \text{ where } \mathscr{B}(\infty) = \left\{X^+ - K_n : n \in \mathbb{Z}_+\right\}.$$

To prove: for any $U \in \mathscr{T}^+$ and any $x \in U$, there exists $B \in \mathscr{B}^+$ such that $x \in B \subset U$. If $U \in \mathscr{T}$, just use that $\mathscr{B}$ is a basis. Similarly, if $U = X^+ - K \in \mathscr{T}(\infty)$, the interesting case is when $x = \infty$. Then we look for $B = X^+ - K_n$ such that $B \subset U$ (i.e. $K \subset K_n$). But $\{\overset{\circ}{K_n}\}$ is an open cover of X, hence also of K; since K is compact and $K_n \subset K_{n+1}$, we find n such that $K \subset \overset{\circ}{K_n}$.

4.4 The Baire property

Here is another important topological property which is useful in applications of Topology to Analysis but not only (e.g. to "the uniform boundedness theorem", or to the size of the space of continuous functions that are nowhere differentiable).

Definition 4.53. A topological space X is called a **Baire space** if the following condition holds: *for any countable family* $\{U_n : n \in \mathbb{N}\}$ *consisting of dense open subsets of* X, $\cap_n U_n$ *is dense in* X.

Recall here $S \subset X$ dense in X means that $\overline{S} = X$. To refresh your memory, please do the following:

Exercise 4.54. Show that S is dense in X if and only if $S \cap V \neq \emptyset$ for any nonempty open $V \subset X$.

Example 4.55 (discrete and cofinite topologies). Sets endowed with the discrete topology are automatically Baire. We now look at X with the cofinite topology, $(X, \mathscr{T}_{\text{cf}})$. If X is finite then X is discrete, hence Baire. Assume now that X is infinite. Notice that a subset $S \subset X$ will be dense in X if and only if S is infinite or $S = X$. We see that any nonempty open of X, i.e. $U = X \setminus F$ with F finite, will automatically be dense (since we assume X to be infinite). Hence the Baire property becomes: for any collection of finite sets $\{F_n : n \in \mathbb{N}\}$, the set $\cap_n (X \setminus F_n) = X \setminus (\cup_n F_n)$ must be infinite (or X). This will fail precisely when X is countable. In conclusion, $(X, \mathscr{T}_{\text{cf}})$ is a Baire space if and only if X is finite or uncountable!

Theorem 4.56. *Any locally compact Hausdorff space* X *is Baire.*

Proof. We start with a family $\{U_n : n \in \mathbb{N}\}$ consisting of dense open subsets of X. We have to show that $(\cap_n U_n) \cap V \neq \emptyset$ for any nonempty open $V \subset X$. We proceed by contradiction assuming that there is a V with

$$(\cap_n U_n) \cap V = \emptyset, \quad V \neq \emptyset.$$

Set $V_0 = V$. Since U_1 is dense in X, we have $U_1 \cap V_0 \neq \emptyset$. By choosing a point in the intersection, and then using local compactness (see Exercise 4.46), we find a nonempty open V_1 such that $\overline{V}_1 \subset U_1 \cap V_0$, $\overline{V}_1$-compact. Next, since U_2 is dense in X, we have $U_2 \cap V_1 \neq \emptyset$ and, as before, we find an non-empty openV_2 such that $\overline{V}_2 \subset U_2 \cap V_1$, $\overline{V}_2$-compact. In this way we build a decreasing sequence

$$\overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \dots$$

with the property that $\cap_n \overline{V}_n \subset (\cap_n U_n) \cap V$. The last inclusions combined with the original assumption imply $\cap_n \overline{V}_n = \emptyset$. But then passing to complements in the previous sequence one obtains an increasing sequence of opens:

$$X \setminus \overline{V}_1 \subset X \setminus \overline{V}_2 \subset X \setminus \overline{V}_3 \subset \dots$$

whose union is $X \setminus \cap_n \overline{V}_n = X$. In particular, these provide a cover of the compact $\overline{V_1}$ by opens in X. By compactness, $\overline{V_1}$ will still be covered by a finite number of them but, since the sequence is increasing, that means that $\overline{V_1}$ must be cover by one single $X \setminus \overline{V}_N$ for some N. But that would give $\overline{V}_N \subset \overline{V}_1 \subset X \setminus \overline{V}_N$ which is impossible since each V_N was non-empty. This provides the desired contradiction.

Example 4.57 (the lower limit topology). The previous theorem implies $(\mathbb{R}, \mathscr{T}_{\text{Eucl}})$ is Bair but it cannot be used to draw the same conclusion about $(\mathbb{R}, \mathscr{T}_l)$ since it is not locally compact (exercise!). On the other hand, using the previous exercise, one deduces that $(\mathbb{R}, \mathscr{T}_l)$ and $(\mathbb{R}, \mathscr{T}_{\text{Eucl}})$ have the same dense subsets. This is an indication that $(\mathbb{R}, \mathscr{T}_l)$ may be Bair- which is actually the case and is left as an exercise (hint: prove first that, for $U \in \mathscr{T}_l$ dense, its Euclidean interior is Euclidean-dense).

4.5 Paracompactness

Paracompactness can be seen as being somewhat analogous to compactness, just that instead of "subcovers" one considers "refinements" and instead of "finite" one allows "locally finite". Therefore, let us start by introducing these concepts. First of all:

Definition 4.58. Let X be a topological space and let $\mathscr{A}$ be a cover of X. A **refinement** of $\mathscr{A}$ is any other cover $\mathscr{B}$ with the property that any $B \in \mathscr{B}$ is contained in some $A \in \mathscr{A}$.

Example 4.59. For $X = (0,1)$ and $\mathscr{A} = \{(0,\varepsilon) : \varepsilon \in (0,1]\}$, $\mathscr{B} = \{(0,1/n) : n \in \mathbb{Z}_+\}$, $\mathscr{B}$ is subcover (hence also a refinement) of $\mathscr{A}$ but, at the same time, $\mathscr{A}$ is a refinement of $\mathscr{B}$.

Definition 4.60. Let $(X,\mathscr{T})$ be a topological space and let $\mathscr{S} = \{S_i\}$ be a family of subsets of X. We say that $\mathscr{S}$ is **locally finite** (in the space X) if for any $x \in X$, there exists a neighborhood V_x of x such that V_x intersects only finitely many subsets that belong to $\mathscr{S}$.

Example 4.61. The collection $\mathscr{S} = \{(0,1/n) : n \in \mathbb{Z}_+\}$ is locally finite in $(0,1)$, but not in $\mathbb{R}$.

We can now introduce the notion of paracompactness:

Definition 4.62. A topological space X is called **paracompact** if every open cover admits a locally finite open refinement.

Example 4.63. Paracompactness is shared by large classes of spaces and, in particular, by almost all the spaces that arise in geometry. First of all compact spaces are paracompact, simply because any subcover of $\mathscr{U}$ is, in particular, a refinement. Later on we will show that all metric spaces are paracompact (Theorem 5.14). Hence paracompactness is shared by the most important classes of spaces. A related large class is that of locally compact, Hausdorff, 2nd countable space (hence also any topological manifold).

As we will prove later on, see Theorem 4.64, any locally compact, Hausdorff, 2nd countable space (hence also any topological manifold) is paracompact.

Theorem 4.64. *Any Hausdorff, locally compact and 2nd countable space is paracompact.*

Proof. (of Theorem 4.64) We use an exhaustion $\{K_n\}$ of X (Theorem 4.49), setting $K_{-1} = K_0 = \emptyset$. Let $\mathscr{U}$ be an open cover of X. For each $n \geq 1$, there is a finite family $\mathscr{V}_n$ covering $K_n - \mathrm{Int}(K_{n-1})$, consisting of opens V with $V \subset \mathrm{Int}(K_{n+1}) - K_{n-2}$ and $V \subset U$ for some $U \in \mathscr{U}$. Indeed, for each $x \in K_n - \mathrm{Int}(K_{n-1})$, intersect a member of $\mathscr{U}$ containing x with $\mathrm{Int}(K_{n+1}) - K_{n-2}$. This is an open neighborhood of x, since $K_n \subset \mathrm{Int}(K_{n+1})$ and $K_{n-2} \subset \mathrm{Int}(K_{n-1})$; by compactness of $K_n - \mathrm{Int}(K_{n-1})$, finitely many of these neighborhoods cover it. Set $\mathscr{V} = \bigcup_n \mathscr{V}_n$. This is an open refinement of $\mathscr{U}$ covering X, since each $K_n - K_{n-1}$ is covered by $\mathscr{V}_n$. Finally, it is locally finite: given $x \in X$, choose N with $x \in \mathrm{Int}(K_N)$. For $m \geq N+2$, we have $K_N \subset K_{m-2}$, while every member of $\mathscr{V}_m$ is disjoint from K_{m-2}. Hence $\mathrm{Int}(K_N)$ can meet only members of $\mathscr{V}_1 \cup \cdots \cup \mathscr{V}_{N+1}$, a finite family.

On the other hand, it is fair to say that paracompactness is best understood as the topological condition that is necessary for the existence of partitions of unity- to be discussed in Chapter 8. An important role in that discussion is played by the following lemma whose proof is very similar to the proof of the normality property of compact spaces (Proposition 4.27).

Proposition 4.65. *Any paracompact Hausdorff space X is normal.*

Proof. The proof is very similar to the compact case, i.e. the proof of Proposition 4.28. We use the same idea and the same notations. We see that it suffices to show that, for $Y,Z \subset X$, if Z is *closed* and $Y|\{z\}$ for all $z \in Z$, then $Y|Z$. To prove this, we first make a general remark: the condition $Y|Z$ implies (and it is actually equivalent to) the existence of an open neighborhood V of Z such that $Y \cap \overline{V} = \emptyset$. Indeed, if $U \cap V = \emptyset$ for some open neighborhoods U of Y and V of Z, then $V \subset X - U$ where the last set is closed, hence $\overline{V} \subset X - U$, hence $\overline{V} \cap U = \emptyset$; since $Y \subset U$, we must have $\overline{V} \cap Y = \emptyset$ (for the converse, just take $U = X - \overline{V}$).

Hence we assume now that $Y|\{z\}$ for all $z \in Z$ and we prove $Y|Z$. For each $z \in Z$ choose an open neighborhood V_z such that $Y \cap \overline{V}_z = \emptyset$. Then $\{V_z : z \in Z\} \cup \{X - Z\}$ is an open cover of X. Let $\mathscr{U}$ be a locally finite refinement and let $\mathscr{W} = \{W_i : i \in I\}$ consisting of those members of $\mathscr{U}$ which intersect Z. Define $V = \cup_i W_i$. This is an open neighborhood of Z. Note that $Y \cap \overline{W}_i = \emptyset$ for all i (since each W_i is inside some V_z and $Y \cap \overline{V}_z = \emptyset$ by construction). Also, due to local finiteness (and Exercise 10.46),

$$\overline{V} = \cup_i \overline{W}_i.$$

Hence $\overline{V} \cap Y = \emptyset$, proving that $Y|Z$. In conclusion X must be normal.

Chapter 5
Topological properties versus metric ones

1. **Completeness and the Baire property**

2. **Boundedness and total boundedness**

3. **Compactness**

4. **Paracompactness**

5. **More exercises**

5.1 Completeness and the Baire property

Probably the most important metric property is that of completeness which we now recall.

Definition 5.1. Given a metric space (X,d) and a sequence $(x_n)_{n\geq 1}$ in X, we say that $(x_n)_{n\geq 1}$ is a **Cauchy sequence** if

$$\lim_{n,m\to\infty} d(x_n,x_m)=0,$$

i.e., for each $\varepsilon>0$, there exists an integer n_ε such that

$$d(x_n,x_m)<\varepsilon$$

for all $n,m\geq n_\varepsilon$. One says that (X,d) is **complete** if any Cauchy sequence is convergent.

Very simple examples (see e.g Exercise 1.51 from the first chapter) show that completeness is not a topological property. However, it does have topological consequences. The first one is a relative topological property for complete spaces.

Proposition 5.2. *If (X,d) is a complete metric space then $A\subset X$ is complete (with respect to the restriction of d to A) if and only if A is closed in X.*

Proof. Assume first that A is complete and show that $\overline{A}=A$. Let $x\in\overline{A}$. Then we find a sequence (a_n) in A converging (in (X,d)) to x. In particular, (a_n) is Cauchy. But the completeness of A implies that the sequence is convergent (in A!) to some $a\in A$. Hence $x=a\in A$. This proves that A is closed. For the converse, assume A is closed and let (a_n) be a Cauchy sequence in A. Of course, the sequence is Cauchy also in X. Since X is complete, it will be convergent to some $x\in X$. Since A is closed, $x\in A$, i.e. (a_n) is convergent in A.

The next topological property that complete metric spaces automatically have is:

Theorem 5.3. *Any complete metric space (X,d) is a Baire space.*

Proof. Assume now that $\{U_n\}_{n\geq 1}$ consists of open dense subsets of X. We show that any $x\in X$ is in the closure of $\cap_n U_n$. Let U be an open containing x; we have to show that U intersects $\cap_n U_n$. First, since U_1 is dense in X, $U\cap U_1\neq\emptyset$; choosing x_1 in this intersection, we find $r_1>0$ such that $B[x_1,r_1]\subset U\cap U_1$. We may assume $r_1<1$. Next, since U_2 is dense in X, $B(x_1,r_1)\cap U_2\neq\emptyset$; choosing x_2 in this intersection, we find $r_2>0$ such that $B[x_2,r_2]\subset B(x_1,r_1)\cap U_2$. We may assume $r_2<1/2$. Similarly, we find x_3 and $r_3<1/3$ such that $B[x_3,r_3]\subset B(x_2,r_2)\cap U_3$ and we continue inductively. Then the resulting sequence (x_n) is Cauchy because $d(x_n,x_m)<r_n$ for $n\leq m$. This implies that (x_n) is convergent to some $y\in X$ and $d(x_n,y)\leq r_n$ for all n. Hence $y\in B[x_n,r_n]\subset U_n$, i.e. $y\in\cap_n U_n$. Also, since $B[x_1,r_1]\subset U$, $y\in U$. Hence $U\cap(\cap_n U_n)\neq\emptyset$, as we wanted.

Remark 5.4 (nowhere differentiable functions). This theorem applied to the space $(\mathscr{C}(I,\mathbb{R}),\hat{d}_{\sup})$ (see Theorem 3.58) can be used to prove that there exist continuous functions $f:[0,1]\to\mathbb{R}$ that are nowhere differentiable ... actually quite a bit more: that the subspace $\mathscr{C}\subset\mathscr{C}(I,\mathbb{R})$ of such nowhere differentiable continuous function is even dense in $\mathscr{C}(I,\mathbb{R})$!

5.2 Boundedness and total boundedness

Another notion that strongly depends on a metric is the notion of boundedness.

Definition 5.5. Given a metric space (X,d), we say that $A \subset X$ is

1. **bounded** in (X,d) (or with respect to d) if there exists $x \in X$ and $R > 0$ such that $A \subset B(x,R)$.
2. **totally bounded** in (X,d) if, for any $\varepsilon > 0$, there exist a finite number of balls in X of radius ε covering A.

When $A = X$, we say that (X,d) is bounded, or totally bounded, respectively.

You should convince yourself that, when $X = \mathbb{R}^n$ and d is the Euclidean metric, total boundedness with respect to d is equivalent to the usual notion of boundedness.

A few remarks are in order here. First of all, these properties are not really relative properties (i.e. they did not depend on the way that A sits inside X), but properties of the metric space (A,d_A) itself, where d_A is the induced metric on A.

Exercise 5.6. Given a metric space (X,d) and $A \subset X$, A is bounded in (X,d) if and only if (A,d_A) is bounded. Similarly for totally bounded.

Another remark is that the property of "totally bounded" is an improvement of that of "bounded". The following exercise shows that, by a simple trick, a metric d can always be made into a bounded metric $\hat{d}$ without changing the induced topology; although the notion of boundedness is changed, total boundedness with respect to d and $\hat{d}$ is the same.

Exercise 5.7. As in Exercise 1.52, for a metric space (X,d) we define $\hat{d} : X \times X \to \mathbb{R}$ by

$$\hat{d}(x,y) = \min\{d(x,y), 1\}.$$

We already know that $\hat{d}$ is a metric inducing the same topology on X as d, and that $(X,\hat{d})$ is complete if and only if (X,d) is. Also, it is clear that $(X,\hat{d})$ is always bounded. Show now that $(X,\hat{d})$ is totally bounded if and only if (X,d) is.

Finally, here is a lemma that we will use later on:

Lemma 5.8. *Given a metric space (X,d) and $A \subset X$, then A is totally bounded if and only if $\overline{A}$ is.*

Proof. Let $\varepsilon > 0$. Choose $x_1, \ldots, x_k$ such that A is covered by the balls $B(x_i, \varepsilon/2)$. Then $\overline{A}$ will be covered by the balls $B(x_i,\varepsilon)$. Indeed, if $y \in \overline{A}$, we find $x \in A$ such that $d(x,y) < \varepsilon/2$; also, we find x_i such that $x \in B(x_i,\varepsilon/2)$; from the triangle inequality, $y \in B(x_i,\varepsilon)$.

5.3 Compactness

The main criterion to recognize when a subspace $A \subset \mathbb{R}^n$ is compact is by checking whether it is closed and bounded in $\mathbb{R}^n$. For general metric spaces:

Theorem 5.9. *A subset A of a complete metric space (X,d) is compact if and only if it is closed (in X) and totally bounded (with respect to d).*

This theorem will actually be an immediate consequence of another theorem, which also clarifies the relationship between compactness and sequential compactness for metric spaces.

Theorem 5.10. *For a metric space (X,d), the following are equivalent:*

1. *X is compact.*
2. *X is sequentially compact.*
3. *X is complete and totally bounded.*

Proof. We first prove Theorem 5.10. The implication $1 \Longrightarrow 2$ is Corollary 4.44. For $2 \Longrightarrow 3$, assume that X is sequentially compact. We first prove that X is complete. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence. By hypothesis, we find a convergent subsequence $(x_{n_k})_{k\geq 1}$. Let x be its limit. We prove that the entire sequence (x_n) converges to x. Let $\varepsilon > 0$. We look for an integer N_ε such that $d(x_n,x) < \varepsilon$ for all $n > N_\varepsilon$. Since (x_n) is Cauchy we find N'_ε such that

$$d(x_n,x_m) < \varepsilon/2$$

for all $n,m \geq N'_\varepsilon$. Since $(x_{n_k})_{k\geq 1}$ converges to x, we find k_ε such that

$$d(x_{n_k},x) < \varepsilon/2$$

for all $k \geq k_\varepsilon$. Choose $N_\varepsilon = \max\{N'_\varepsilon, n_{k_\varepsilon}\}$. Then, for $n > N_\varepsilon$, choosing k such that $n_k > n$ (such a k exists since $n_1 < n_2 < \ldots$ is a sequence that tends to ∞), we must have $k > k_\varepsilon$ and $n_k > N'_\varepsilon$, hence

$$d(x_n,x_{n_k}) < \varepsilon/2,\ d(x_{n_k},x) < \varepsilon/2.$$

The triangle inequality implies $d(x_n,x) < \varepsilon$ for all $n \geq N_\varepsilon$, hence $x_n \to x$. We now prove that X is totally bounded. Assume it is not, i.e., $\exists\ r > 0$ such that X cannot be covered by a finite number of balls of radius r. Construct a sequence $(x_n)_{n\geq 1}$ as follows. Start with any $x_1 \in X$. Since $X \neq B(x_1,r)$, we find $x_2 \in X - B(x_1,r)$. Since $X \neq B(x_1,r) \cup B(x_2,r)$, we find $x_3 \in X - (B(x_1,r) \cup B(x_2,r))$. Continuing we find a sequence with $x_n \notin B(x_m,r)$ for $n > m$. Hence $d(x_n,x_m) \geq r$ for all $n \neq m$. But by hypothesis, (x_n) has a subsequence $(x_{n_k})_{k\geq 1}$ which converges to some $x \in X$. But then we find N such that $d(x_{n_k},x) < r/2$ for all $k \geq N$, hence $d(x_{n_k},x_{n_l}) \leq d(x_{n_k},x) + d(x,x_{n_l}) < r$ for all $k,l \geq N$, and this contradicts the condition "$d(x_n,x_m) \geq r$ for all $n \neq m$".

$3 \Longrightarrow 1$: Assume that (X,d) is complete and totally bounded. The last condition ensures that for each integer $n \geq 0$, there is a finite set $F_n \subset X$ such that

$$X = \bigcup_{x\in F_n} B\left(x, \frac{1}{2^n}\right).$$

Let $\mathscr{U} = \{U_i : i \in I\}$ be an open cover of X, and we want to prove that we can extract a finite subcover of $\mathscr{U}$. Assume this is not possible. We construct a sequence $(x_n)_{n\geq 1}$ inductively as follows. Since $\cup_{x\in F_1} B\left(x, \frac{1}{2}\right) = \cup_i U_i$, and F_1 is finite, we find $x_1 \in F_1$ such that $B(x_1, \frac{1}{2})$ cannot be covered by a finite number of opens from $\mathscr{U}$. Since

$$B\left(x_1, \frac{1}{2}\right) = \bigcup_{x\in F_2} (B\left(x_1, \frac{1}{2}\right) \cap B\left(x, \frac{1}{4}\right)$$

we find $x_2 \in F_2$ such that

$$B\left(x_1, \frac{1}{2}\right) \cap B\left(x_2, \frac{1}{4}\right) \neq \emptyset$$

and $B\left(x_2, \frac{1}{4}\right)$ cannot be covered by a finite numbers of opens from $\mathscr{U}$. Continuing inductively we find $x_n \in F_n$ s.t.

$$B\left(x_{n-1}, \frac{1}{2^{n-1}}\right) \cap B\left(x_n, \frac{1}{2^n}\right) \neq \emptyset$$

and $B\left(x_n, \frac{1}{2^n}\right)$ cannot be covered by a finite number of opens from $\mathscr{U}$. Note that, choosing an element y in the (non-empty) intersection above, the triangle inequality implies that

$$d(x_{n-1}, x_n) < \frac{1}{2^{n-1}} + \frac{1}{2^n} = \frac{3}{2^n},$$

from which we deduce that $(x_n)_{n\geq 1}$ is a Cauchy sequence (why?). By hypothesis, it will converge to an element $x \in X$. Choose $U \in \mathscr{U}$ such that $x \in U$. Since U is open, we find $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Since $x_n \to x$, we find n_ε such that $d(x_n, x) < \varepsilon/2$ for all $n > n_\varepsilon$. Using the triangle inequality, we deduce that $B(x_n, \varepsilon/2) \subset U$ for all $n \geq n_\varepsilon$. Choosing n so that also $1/2^n < \varepsilon/2$, we deduce that $B(x_n, 1/2^n) \subset U$, which contradicts the fact that $B(x_n, 1/2^n)$ cannot be covered by a finite number of opens from $\mathscr{U}$.

This ends the proof of Theorem 5.10. For Theorem 5.9, one uses the equivalence between 1 and 3 above, applied to the metric space (A, d_A), and Proposition 5.2.

We now derive some more properties of compactness in the metric case. In what follows, given $F \subset X$, we say that F is **relatively compact** in X if the closure $\overline{F}$ in X is compact.

Corollary 5.11. *For a subset F of a complete metric space (X, d), the following are equivalent*

1. *F is relatively compact in X.*
2. *any sequence in F admits a convergent subsequence (with some limit in X).*
3. *F is totally bounded.*

Proof. We apply Theorem 5.10 to $\overline{F}$. We know that 3 is equivalent to the same condition for $\overline{F}$ (Lemma 5.8). We prove the same for 2; the non-obvious part is to show that $\overline{F}$ satisfies 2 if F does. So, let (y_n) be a sequence in $\overline{F}$. For each n we find $x_n \in F$ such that $d(x_n, y_n) < 1/n$. After eventually passing to a subsequence, we may assume that (x_n) is convergent to some $x \in X$ and $d(x_n, y_n) \to 0$ as $n \to \infty$. This implies that (y_n) itself must converge to x.

Corollary 5.12. *Any compact metric space (X, d) is separable, i.e. there exists $A \subset X$ which is at most countable and which is dense in X.*

Proof. For each n choose a finite A_n s.t. X is covered by $B\left(a, \frac{1}{n}\right)$ with $a \in A_n$. Then $A := \cup A_n$ is dense in X: for $x \in X$, $\varepsilon > 0$ we have to show that $B(x, \varepsilon) \cap A \neq \emptyset$; but we find n with $\frac{1}{n} < \varepsilon$ and $a \in A_n$ such that $x \in B\left(a, \frac{1}{n}\right)$; then $a \in B(x, \varepsilon) \cap A$.

Proposition 5.13. *(the Lebesgue lemma) If (X, d) is a compact metric space then, for any open cover $\mathscr{U}$ of X, there exists $\delta > 0$ such that*

$$A \subset X,\ diam(A) < \delta \Longrightarrow\ \exists\, U \in \mathscr{U} \text{ such that } A \subset U.$$

(δ is called a Lebesgue number for the cover $\mathscr{U}$).

Proof. It suffices to show that there exists δ such that each ball $B(x, \delta)$ is contained in some $U \in \mathscr{U}$. If no such δ exists, we find $\delta_n \to 0$ such that $B(x_n, \delta_n)$ is not inside any $U \in \mathscr{U}$. Using (sequential) compactness we may assume that (x_n) is convergent, with some limit $x \in X$ (if not, pass to a convergent subsequence). Let $U \in \mathscr{U}$ with $x \in U$ and let $r > 0$ with $B(x, r) \subset U$. Since $\delta_n \to 0$, $x_n \to x$, we find n s.t. $\delta_n < r/2$, $d(x_n, x) < r/2$. From the triangle inequality, $B(x_n, \delta_n) \subset B(x, r)$ $(\subset U)$ which contradicts the choice of x_n and δ_n.

5.4 Paracompactness

Finally, we show that:

Theorem 5.14. *Any metric space is paracompact.*

Proof. Start with an arbitrary open cover $\mathscr{U} = \{U_i : i \in I\}$ of X. We consider an order relation "$\leq$" on I, which makes I into a well-ordered set (i.e. so that any nonempty subset of I has a smallest element). We will construct a locally finite refinement of type $\mathscr{V} = \cup_{n\in\mathbb{N}} \mathscr{V}_n$ where, for each n, the family $\mathscr{V}(n)$ will have one member for each $i \in I$; i.e. it is of type:

$$\mathscr{V}(n) = \{V_i(n) : i \in I\}.$$

We set $X(n) = \cup_i V_i(n)$. The definition of $\mathscr{V}(n)$ is by induction on n. For $n = 1$:

$$V_i(1) := \bigcup_{a\in U_i-(\cup_{j<i}U_j) \text{ with } B(a,\frac{3}{2})\subset U_i} B(a, \frac{1}{2}).$$

Assuming that $\mathscr{V}(1), \ldots, \mathscr{V}(n-1)$ have been constructed, we define, for each $i \in I$:

$$V_i(n) = \bigcup_{a\in U_i-(\cup_{j<i}U_j) \text{ with } B(a,\frac{3}{2^n})\subset U_i,\ a\notin X(1)\cup\ldots\cup X(n-1)} B(a, \frac{1}{2^n}).$$

It is clear that $\mathscr{V}$ is a refinement of $\mathscr{U}$. Next, we claim that $X = \cup_n X(n)$ (i.e. $\mathscr{V}$ is a cover): for $x \in X$, choose the smallest i such that $x \in U_i$ and choose n such that $B(x, 3/2^n) \subset U_i$; then either $x \in X(1) \cup \ldots \cup X(n-1)$ and we are done, or x can serve as an index in the definition of $V_i(n)$, hence $x \in X(n)$. Before showing local finiteness, we remark that, for each n:

$$d(V_i(n), V_j(n)) \geq \frac{1}{2^n} \quad \forall\, i \neq j. \tag{5.4.1}$$

To see this, assume that $i < j$ and let $x \in V_i(n)$, $y \in V_j(n)$. Then $x \in B(a, \frac{1}{2^n})$ for some $a \in X$ with $B(a, \frac{3}{2^n}) \subset U_i$ and $y \in B(b, \frac{1}{2^n})$ for some $b \in X$ with $b \notin U_i$. These imply that $b \notin B(a, \frac{3}{2^n})$, i.e. $d(a,b) \geq \frac{3}{2^n}$. From the triangle inequality:

$$d(x,y) \geq d(a,b) - d(a,x) - d(b,y) > \frac{3}{2^n} - \frac{1}{2^n} - \frac{1}{2^n} = \frac{1}{2^n}.$$

We now show local finiteness. Let $x \in X$. Fix $n_0 \geq 1$ integer, $i_0 \in I$ with $x \in V_{i_0}(n_0)$. Also, choose $n_1 \geq 1$ integer with

$$B(x, \frac{1}{2^{n_1}}) \subset V_{i_0}(n_0). \tag{5.4.2}$$

We claim that

$$V := B(x,r) \quad \text{where } r = \frac{1}{2^{n_0+n_1}}$$

intersects only a finite number of members of $\mathscr{V}$. This follows from the following two remarks

1. For $n < n_0 + n_1$, V intersects at most one member of the family $\mathscr{V}(n)$.
2. For $n \geq n_0 + n_1$, V intersects no member of the family $\mathscr{V}(n)$.

Part 1 follows from (5.4.1): if V intersects both $V_i(n)$ and $V_j(n)$ with $i \neq j$, we would find $a, b \in V$ with $d(a,b) \geq \frac{1}{2^n}$ but $d(a,b) \leq d(a,x) + d(x,b) < 2r \leq \frac{1}{2^n}$ for all $a, b \in V$.

For part 2, assume that $n \geq n_0 + n_1$. Assume, for a contradiction, that $V \cap V_i(n) \neq \emptyset$ for some $i \in I$. From the definition of $V_i(n)$, we then find $B(a, \frac{1}{2^n}) \cap V \neq \emptyset$ for some $a \in U_i - (\cup_{j<i} U_j)$, with $B(a, \frac{3}{2^n}) \subset U_i$, $a \notin X(1) \cup \ldots \cup X(n-1)$. Since $n > n_0$, we have $a \notin X(n_0)$, hence $a \notin V_{i_0}(n_0)$. From the choice of n_1 (see (5.4.2) above), $a \notin B(x, \frac{1}{2^{n_1}})$, hence $d(a,x) \geq \frac{1}{2^{n_1}}$. But, by the triangle inequality again, this implies that $B(a, \frac{1}{2^n}) \cap B(x,r) = \emptyset$. I.e., for any a which contributes to the definition of $V_i(n)$, its contribution $B(a, \frac{1}{2^n})$ does not intersect V. Hence $V \cap V_i(n) = \emptyset$.

Chapter 6
Metrizability theorems

1. **The Urysohn metrization theorem**

2. **The Smirnov metrization theorem**

3. **Consequences: the compact case, the locally compact case, manifolds**

4. **More exercises**

6.1 The Urysohn metrization theorem

The following is known as the Urysohn metrization theorem.

Theorem 6.1. *Any topological space which is Hausdorff, normal and second countable is metrizable.*

The rest of this section is devoted to the proof of this theorem.

Claim 1: $\exists$ a countable family $(f_n)_{n\geq 1}$ of continuous functions $f_n : X \to [0,1]$ satisfying:

$$(\forall\, U - \text{open},\ x \in U),\ \ (\exists\, N \in \mathbb{Z}_+) \text{ such that: } (f_N(x) = 1, f_N = 0 \text{ outside } U). \tag{6.1.1}$$

Proof. Let $\mathscr{B} = \{B_1, B_2, \ldots\}$ be a countable basis for the topology $\mathscr{T}$. Notice that $I = \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \overline{B}_n \subset B_m\right\}$ is countable (subset of countable is countable), hence there is an enumeration $I = \{(n_1,m_1),(n_2,m_2),\ldots\}$. For each i, using Urysohn's lemma, we find a continuous function $f_i : X \to [0,1]$ such that $f_i|_{\overline{B}_{n_i}} = 1$, $f_i|_{X - B_{m_i}} = 0$. Then $(f_i)_{i\geq 1}$ has the desired properties: for $U \in \mathscr{T}$ and $x \in U$, we can choose m such that $x \in B_m \subset U$. By Lemma 2.82, we find V-open containing x such that $x \in V \subset \overline{V} \subset B_m$. Since $\mathscr{B}$ is a basis, we find n such that $x \in B_n \subset V$. Then $\overline{B}_n \subset \overline{V} \subset B_m$, hence $(n,m) \in I$. Writing $(n,m) = (n_N, m_N)$ with $N \in \mathbb{N}$, since $x \in B_n$ we have $f_N(x) = 1$, and since $B_m \subset U$, we have $f_N = 0$ outside B_m hence also outside U.

Claim 2: The following is a metric on X inducing the topology of X:

$$d : X \times X \to \mathbb{R},\ \ d(x,y) = \sup\left\{\frac{|f_n(x) - f_n(y)|}{n} : n \geq 1 \text{ integer}\right\}.$$

Proof. Note that $d(x,y)$ is finite since $0 \leq f_n \leq 1$. The triangle inequality is implied by the fact that, for all n:

$$\frac{|f_n(x) - f_n(y)|}{n} \leq \frac{|f_n(x) - f_n(z)|}{n} + \frac{|f_n(z) - f_n(y)|}{n} \leq d(x,z) + d(z,y).$$

To see that $d(x,y) \neq 0$ when $x \neq y$ choose $U \in \mathscr{T}$ containing x and not containing y, choose N as in (6.1.1) and remark that $|f_N(x) - f_N(y)| = 1$, hence $d(x,y) \geq 1/N > 0$. To show that $\mathscr{T} \subset \mathscr{T}_d$, let $U \in \mathscr{T}$ and we prove that:

$$\forall\, x \in U,\ \exists\, \varepsilon > 0 : B_d(x,\varepsilon) \subset U.$$

Let $x \in U$ and choose N as in (6.1.1). Then $\varepsilon := \frac{1}{N}$ does the job. Indeed, if $y \in B_d(x,\varepsilon)$, then

$$\frac{|1 - f_N(y)|}{N} = \frac{|f_N(x) - f_N(y)|}{N} \leq d(x,y) < \frac{1}{N},$$

hence $f_N(y) \neq 0$ and this can only happen if $y \in U$.

Finally, we show that $\mathscr{T}_d \subset \mathscr{T}$. It suffices to prove that, for each ball $B(x,\varepsilon)$, there exists $U = U_{x,\varepsilon} \in \mathscr{T}$ such that $x \in U \subset B(x,\varepsilon)$. This will imply that $B(x,\varepsilon)$ is open in X: indeed, for any $y \in B(x,\varepsilon)$ we can choose $r > 0$ such that $B(y,r) \subset B(x,\varepsilon)$ (e.g. take $r = \varepsilon - d(x,y)$ and use the triangle inequality), and then $U_{y,r}$ will be an open in X contained in $B(x,\varepsilon)$. So, let us fix $x \in X$, $\varepsilon > 0$ and look for $U \in \mathscr{T}$ with $x \in U \subset B(x,\varepsilon)$. Choose $n_0 > 2/\varepsilon$ and set

$$U := \bigcap_{n=1}^{n_0} \left\{y \in X : \frac{|f_n(y) - f_n(x)|}{n} < \varepsilon\right\}.$$

Since this is a finite intersection and each f_n is continuous, we have $U \in \mathscr{T}$. Clearly, $x \in U$. Note also that, from the choice of n_0 and the fact that $0 \leq |f_n| \leq 1$,

$$\frac{|f_n(x) - f_n(y)|}{n} \leq \frac{2}{n_0} < \varepsilon \quad \forall\, n \geq n_0.$$

We deduce that $d(x,y) < \varepsilon$ for all $y \in U$, i.e. $U \subset B(x,\varepsilon)$.

6.2 The Smirnov Metrization Theorem

The following is known as the Smirnov Metrization Theorem.

Theorem 6.2. *A space X is metrizable iff it is Hausdorff, paracompact and locally metrizable.*

Theorem 5.14 takes care of the direct implication. Here we prove the converse. The proof is very similar to the proof of the Urysohn metrization theorem.

Claim 1: There exists a basis $\mathscr{B}$ for the topology of X, of type $\mathscr{B} = \cup_{n\in\mathbb{N}^*}\mathscr{B}_n$, where each $\mathscr{B}_n$ is a locally finite family. Moreover, for each $B \in \mathscr{B}$, there is a continuous function

$$f_B : X \to [0,1] \quad \text{such that} \quad B = \{x \in X : f_B(x) \neq 0.\}\,.$$

Proof. From the hypothesis it follows that there is a cover $\mathscr{U} = \{U_i : i \in I\}$ of X by opens in X, on which the topology is induced by a metric d_i; we may assume that $d_i \leq 1$ (cf. e.g. Exercise 1.52). For each $i \in I$, we denote by $B_i(x,r)$ the balls induced by d_i. They are open subsets of U_i, hence also open in X. By the shrinking lemma (Lemma 8.13), we can find another locally finite cover $\{V_i : i \in I\}$ with $\overline{V}_i \subset U_i$. For each integer n, we consider the open cover of X

$$\left\{B_i(x,\frac{1}{n}) \cap V_i : i \in I,\ x \in U_i\right\}.$$

Let $\mathscr{B}_n$ be a locally finite refinement of it and $\mathscr{B} = \cup_n \mathscr{B}_n$. For each $B \in \mathscr{B}$, choose i such that $B \subset V_i$. If $B \neq U_i$, define $f_B(x) := d_i(x, U_i - B)$ on U_i. This is continuous, takes values in $[0,1]$, and is nonzero precisely on B. Since $\overline{B} \subset \overline{V}_i \subset U_i$ (all closures being taken in X), its extension by zero outside U_i is continuous. If $B = U_i$, then $U_i = B \subset V_i \subset \overline{V}_i \subset U_i$, so B is both open and closed in X; in this case, take f_B to be 1 on B and 0 outside B. Finally, we show that $\mathscr{B}$ is a basis. Consider $U \subset X$ open, $x \in U$; we show that $x \in B \subset U$ for some $B \in \mathscr{B}$. Since $\{V_i\}$ is locally finite, the set $F_x := \{i \in I : x \in V_i\}$ is finite. For each $i \in F_x$, $U \cap U_i$ is open in (U_i, d_i) and contains x, hence we find $\varepsilon_i > 0$ such that $B_i(x,\varepsilon_i) \subset U \cap U_i$. Choose $m \geq 1$ with $2/m < \varepsilon_i$ for all $i \in F_x$, and choose $B \in \mathscr{B}_m$ containing x. By the definition of $\mathscr{B}_m$, we have $B \subset B_i(y,1/m) \cap V_i$ for some $i \in I$, $y \in U_i$. In particular, $x \in V_i$, hence $i \in F_x$. Since $x \in B_i(y,1/m)$, the triangle inequality gives $B_i(y,1/m) \subset B_i(x,2/m) \subset B_i(x,\varepsilon_i) \subset U$. Thus $x \in B \subset U$, as required.

Claim 2: The following is a metric on X inducing the topology $\mathscr{T}$ of X.

$$d : X \times X \to \mathbb{R},\ d(x,y) = \sup\left\{\frac{1}{n}|f_B(x) - f_B(y)| : n \geq 1 \text{ integer}, B \in \mathscr{B}_n\right\}.$$

Proof. By the same argument as in the Urysohn metrization theorem, d is a metric. Next, we show that $\mathscr{T} \subset \mathscr{T}_d$. Let $U \subset X$ open, $x \in U$. We have to find $r > 0$ such that $B_d(x,r) \subset U$. Since $\mathscr{B}$ is a basis, we find $B \in \mathscr{B}_n$ for some n, with $x \in B \subset U$. We claim that $r = \frac{1}{n}|f_B(x)|$ does the job. Indeed, if $y \in B_d(x,r)$, we have $\frac{1}{n}|f_B(y) - f_B(x)| < \frac{1}{n}|f_B(x)|$, hence $f_B(y) \neq 0$, hence $y \in B$, hence $y \in U$.

Finally, we show that $\mathscr{T}_d \subset T$. It suffices to show that, for any $x \in X$, $r > 0$, there exists $U \in \mathscr{T}$ such that $x \in U \subset B(x,r)$. Let $n_0 > 2/r$ be an integer. Since each $\mathscr{B}_n$ is locally finite, we find a neighborhood V of x which intersects only a finite number of Bs with $B \in \mathscr{B}_n$, $n \leq n_0$. Call these members $B_1, \ldots, B_k$. Choose $U \subset V$ such that

$$|f_{B_i}(y) - f_{B_i}(x)| < r \quad \forall\, y \in U,\ \forall\, i \in \{1,\ldots,k\}\,. \tag{6.2.1}$$

We claim that $U \subset B(x,r)$. That means that, for any $y \in U$, we have $\frac{1}{n}|f_B(y) - f_B(x)| < r$ for all $n \geq 1$ and $B \in \mathscr{B}_n$. If $n \geq n_0$ this is automatically satisfied since $|f_B| \leq 1$ and $2/n \leq 2/n_0 < r$. Assume now that $n \leq n_0$. If B is not one of the $B_1, \ldots, B_k$, then $U \cap B = \emptyset$ hence $f_B(y) = f_B(x) = 0$ and we are done. Finally, if $B = B_i$ for some i, then the desired inequality follows from (6.2.1).

6.3 Consequences: the compact case, the locally compact case, manifolds

Here are some consequences of the metrization theorems from the previous sections. First of all, since topological manifolds are paracompact (see e.g. 4.64), the Smirnov metrization theorem immediately implies

Theorem 6.3. *Any topological manifold is metrizable.*

This theorem follows also from the Urysohn metrization theorem (but note that the proof based on Smirnov's result is somehow more satisfactory: it uses paracompactness to pass from the local information to the global one; in particular, the Urysohn lemma is not used!). The Urysohn metrization theorem has however two more interesting consequences. First, for the compact case, we obtain:

Theorem 6.4. *If X is a compact Hausdorff space, then the following are equivalent*

1. *X is metrizable.*
2. *X is second countable.*

Using the one-point compactification, for locally compact spaces we will obtain the following (which provides another proof to Theorem 6.3).

Theorem 6.5. *Any locally compact Hausdorff and 2nd countable space is metrizable.*

In what follows, we will provide the missing proofs.

Proof. (of Theorem 6.4) The reverse implication follows from the Urysohn metrization theorem since compact spaces are normal (Corollary 4.29). We now prove $1 \Longrightarrow 2$. Let d be a metric inducing the topology of X. Since X is totally bounded (cf. Theorem 5.10), for each n we find a finite set F_n such that

$$X = \bigcup_{x \in F_n} B(x, \frac{1}{n}).$$

The set $A = \cup_n F_n$ is a countable union of finite sets, hence it is countable. We deduce that

$$\mathscr{B} = \left\{ B(a, \frac{1}{n}) : a \in A, n \geq 1 \text{ integer} \right\}$$

is a countable family of open sets of X. We claim it is a basis for the topology of X. Let U be an arbitrary open and $x \in U$. We have to prove that there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. Since $x \in U$, we find an integer n such that $B(x, \frac{1}{n}) \subset U$. Using the defining property for F_{2n}, we see that there exists $a \in A$ such that $x \in B(a, \frac{1}{2n})$. Using the triangle inequality, we deduce that for each $y \in B(a, \frac{1}{2n})$,

$$d(x,y) \leq d(x,a) + d(a,y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

hence $y \in B(x, \frac{1}{n}) \subset U$. In conclusion, $B = B(a, \frac{1}{2n}) \in \mathscr{B}$ satisfies $x \in B \subset U$.

Proof. (of Theorem 6.5) We apply the Theorem 6.4 to the one-point compactification (see Theorem 4.51) to deduce that X^+ is metrizable. Since X is a subspace of X^+, it is itself metrizable.

Chapter 7
The algebra of observables/spaces of functions

1. **The algebra $\mathscr{C}(X)$ of continuous functions**

2. **Approximations in $\mathscr{C}(X)$: the Stone-Weierstrass theorem**

3. **Recovering X from $\mathscr{C}(X)$: the Gelfand Naimark theorem**

4. **General function spaces $\mathscr{C}(X,Y)$**

    - Pointwise convergence, uniform convergence, compact convergence
    - Equicontinuity
    - Boundedness
    - The case when X is a compact metric spaces
    - The Arzela-Ascoli theorem
    - The compact-open topology

5. **More exercises**

7.1 The algebra $\mathscr{C}(X)$ of continuous functions

We start this chapter with a discussion of continuous functions from a Hausdorff compact space to the real or complex numbers. It makes no difference whether we work over $\mathbb{R}$ or $\mathbb{C}$, so let's just use the notation $\mathbb{K}$ for one of these base fields and we call it the field of scalars. For each $z \in \mathbb{K}$, we can talk about $|z| \in \mathbb{R}$- the absolute value of z in the real case, or the norm of the complex number z in the complex case.

For a compact Hausdorff space X, we consider the set of scalar-valued functions on X:

$$\mathscr{C}(X) := \{f : X \to \mathbb{K} : f \text{ is continuous}\}.$$

When we want to make a distinction between the real and complex case, we will use the more precise notations $\mathscr{C}(X,\mathbb{R})$ and $\mathscr{C}(X,\mathbb{C})$.

In this section we look closer at the "structure" that is present on $\mathscr{C}(X)$. First of all there are the algebraic operations which are inherited from the existing operations on $\mathbb{K}$: additions $f+g$ of functions and multiplication of functions f by scalars $\lambda \in \mathbb{K}$:

$$(f+g)(x) = f(x)+g(x), \quad (\lambda \cdot f)(x) = \lambda \cdot f(x).$$

In other words, $\mathscr{C}(X)$ has the structure of vector space. Next, one also has the multiplication $f \cdot g$ of functions:

$$(fg)(x) = f(x)g(x).$$

In this way $(C(X),+,\cdot)$ is also a ring. Rings which also happen to be vector spaces (in a compatible fashion) are called algebras:

Definition 7.1. A $\mathbb{K}$-algebra is a vector space A over $\mathbb{K}$ together with an operation

$$A \times A \to A, \quad (a,b) \mapsto a \cdot b$$

which is unital in the sense that there exists an element $1 \in A$ such that

$$1 \cdot a = a \cdot 1 = a \quad \forall\, a \in A,$$

and which is $\mathbb{K}$-bilinear and associative, i.e., for all $a,a',b,b',c \in A$, $\lambda \in \mathbb{K}$,

$$(a+a') \cdot b = a \cdot b + a' \cdot b,\ a \cdot (b+b') = a \cdot b + a \cdot b',$$

$$(\lambda a) \cdot b = \lambda (a \cdot b) = a \cdot (\lambda b),,$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c\ .$$

We say that A is commutative if $a \cdot b = b \cdot a$ for all $a,b \in \mathscr{A}$.

Example 7.2. The space of polynomials $\mathbb{K}[X_1,\ldots,X_n]$ with coefficients in $\mathbb{K}$ is a commutative algebra.

Corollary 7.3. *For any topological space X, $\mathscr{C}(X)$ with the operations discussed above is an algebra.*

Next, we move to the topological structures present on $\mathscr{C}(X)$. When X was an interval in $\mathbb{R}$, this was discussed in Section 3.9, Chapter 3 (with $n=1$ in the real case, $n=2$ in the complex one). As there, there is a metric

$$d_{\sup} : \mathscr{C}(X) \times \mathscr{C}(X) \longrightarrow \mathbb{R}, \quad d_{\sup}(f,g) := \sup\{|f(x)-g(x)| : x \in X\}.$$

Since f, g are continuous and X is compact, $d_{\sup}(f,g) < \infty$. As in Ch. 3, Sect. 3.9, $d_{\sup}$ is a metric and the induced topology is called **the uniform topology** on $\mathscr{C}(X)$. And, still as in the mentioned section:

Theorem 7.4. *For any compact Hausdorff space X, $(\mathscr{C}(X), d_{sup})$ is a complete metric space.*

Proof. Let (f_n) be a Cauchy sequence in $\mathscr{C}(X)$. The proof of Theorem 3.58 applies word by word to our X instead of the interval I, to obtain a function $f : X \to \mathbb{K}$ such that $d_{\sup}(f_n, f) \to 0$ when $n \to \infty$. Then similarly, the proof of Theorem 3.57 (namely that $\mathscr{C}(I, \mathbb{R}^n)$ is closed in $(\mathscr{F}(I, \mathbb{R}^n), \hat{d}_{\sup})$) applies word by word with I replaced by X to conclude that $f \in \mathscr{C}(X)$.

The metric on $\mathscr{C}(X)$ is of a special type: it comes from a norm. Namely, defining

$$||f||_{\sup} := \sup\{|f(x)| : x \in X\} \in [0, \infty) \tag{7.1.1}$$

for $f \in \mathscr{C}(X)$, we have

$$d_{\sup}(f,g) = ||f - g||_{\sup}.$$

We are now at the point at which the topological structure starts interacting with part of the algebraic one, namely the vector space structure. The discussion falls within the following general setting:

Definition 7.5. Let V be a vector space (over our $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A **norm** on V is a function

$$||\cdot|| : V \to [0, \infty), \ v \mapsto ||v||$$

such that

$$||v|| = 0 \iff v = 0$$

and is compatible with the vector space structure in the sense that:

$$||\lambda v|| = |\lambda| \cdot ||v|| \quad \forall\, \lambda \in \mathbb{K}, v \in V,$$

$$||v + w|| \leq ||v|| + ||w|| \quad \forall\, v, w \in V.$$

The metric associated to $||\cdot||$ is the metric $d_{||\cdot||}$ given by

$$d_{||\cdot||}(v,w) := ||v - w||.$$

A **Banach space** is a vector space V endowed with a norm $||\cdot||$ such that $d_{||\cdot||}$ is complete. When $\mathbb{K} = \mathbb{R}$ we talk about real Banach spaces, when $\mathbb{K} = \mathbb{C}$ about complex ones.

With these, we can now reformulate our discussion as follows:

Corollary 7.6. *For any compact Hausdorff space X, $(\mathscr{C}(X), ||\cdot||_{sup})$ is a Banach space.*

Next, we can also bring the full algebraic into discussion and discuss its compatibility with the topological structure. Again, the outcome fits into a more general setting:

Definition 7.7. A **Banach algebra** (over $\mathbb{K}$) is an algebra A equipped with a norm $||\cdot||$ which makes $(A, ||\cdot||)$ into a Banach space, such that the algebra structure and the norm are compatible, in the sense that,

$$||a \cdot b|| \leq ||a|| \cdot ||b|| \quad \forall\, a, b \in A.$$

Hence, with all these terminology, the full structure of $\mathscr{C}(X)$ is summarized in the following:

Corollary 7.8. *For any compact Hausdorff space, $\mathscr{C}(X)$ is a Banach algebra.*

There is a bit more one can say in the case when $\mathbb{K} = \mathbb{C}$: there is also the operation of conjugation, defined again pointwise:

$$\overline{f}(x) := \overline{f(x)}.$$

As before, this comes with an abstract definition:

Definition 7.9. A *-algebra is an algebra A over $\mathbb{C}$ together with an operation

$$(-)^* : A \to A,\ a \mapsto a^*,$$

which is an involution, i.e.

$$(a^*)^* = a \quad \forall\, a \in A,$$

and which satisfies the following compatibility relations with the rest of the structure:

$$(\lambda a)^* = \overline{\lambda} a^* \quad \forall\, a \in A, \lambda \in \mathbb{C},$$

$$1^* = 1, (a+b)^* = a^* + b^*,\ (ab)^* = b^* a^* \quad \forall\, a, b \in A.$$

Finally, a C^*-algebra is a Banach algebra $(A, ||\cdot||)$ endowed with a *-algebra structure, s.t.

$$||a^*|| = ||a||,\ \ ||a^* a|| = ||a||^2 \quad \forall\, a \in A.$$

Of course, $\mathbb{C}$ with its norm is the simplest example of C^*-algebra. To summarize our discussion in the complex case:

Corollary 7.10. *For any compact Hausdorff space, $\mathscr{C}(X, \mathbb{C})$ is a C^*-algebra.*

7.2 Approximations in $\mathscr{C}(X)$: the Stone-Weierstrass theorem

The Stone-Weierstrass theorem is concerned with density in the space $\mathscr{C}(X)$ (endowed with the uniform topology); the simplest example is Weierstrass's approximation theorem which says that, when X is a compact interval, the set of polynomial functions is dense in the space of all continuous functions. The general criterion makes use of the algebraic structure on $\mathscr{C}(X)$.

Definition 7.11. Given an algebra A (over the base field $\mathbb{R}$ or $\mathbb{C}$), a **subalgebra** is any vector subspace $B \subset A$, containing the unit 1 and such that

$$b \cdot b' \in B \quad \forall\, b, b' \in B.$$

When we want to be more specific about the base field, we talk about real or complex subalgebras.

Example 7.12. When $X = [0,1]$, the set of polynomial functions on $[0,1]$ is a unital subalgebra of $\mathscr{C}([0,1])$. Here the base field can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 7.13. Given a topological space X and a subset $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\mathscr{A}$ is **point-separating** if for any $x, y \in X$, $x \neq y$ there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$.

Example 7.14. When $X = [0,1]$, the subalgebra of polynomial functions is point-separating.

Here is the Stone-Weierstrass theorem in the real case ($\mathbb{K} = \mathbb{R}$).

Theorem 7.15. *(Stone-Weierstrass) Let X be a compact Hausdorff space. Then any point-separating real subalgebra $\mathscr{A} \subset \mathscr{C}(X, \mathbb{R})$ is dense in $(\mathscr{C}(X, \mathbb{R}), d_{sup})$.*

Proof. We first show that there exists a sequence $(p_n)_{n \geq 1}$ of real polynomials which, on the interval $[0,1]$, converges uniformly to the function $\sqrt{t}$. We construct p_n inductively by

$$p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n(t)^2), \quad p_1 = 0.$$

We first claim that $p_n(t) \leq \sqrt{t}$ for all $t \in [0,1]$. This follows by induction on n since

$$\sqrt{t} - p_{n+1}(t) = (\sqrt{t} - p_n(t))\left(1 - \frac{\sqrt{t} + p_n(t)}{2}\right).$$

and $p_n(t) \leq \sqrt{t}$ implies that $\sqrt{t} + p_n(t) \leq 2\sqrt{t} \leq 2$ (hence the right hand side is positive). Next, the recurrence relation implies that $p_{n+1}(t) \geq p_n(t)$ for all t. Then, for each $t \in [0,1]$, $(p_n(t))_{n \geq 1}$ is increasing and bounded above by 1, hence convergent; let $p(t)$ be its limit. By passing to the limit in the recurrence relation we find that $p(t) = \sqrt{t}$.

We still have to show that $p_n \rightrightarrows p$ on $[0,1]$. Let $\varepsilon > 0$ and we look for N such that $p(t) - p_n(t) < \varepsilon$ for all $n \geq N$ (note that $p - p_n$ is positive). Let $t \in [0,1]$. Since $p_n(t) \to p(t)$, we find $N(t)$ such that $p(t) - p_n(t) < \varepsilon/3$ for all $n \geq N(t)$. Since p and $p_{N(t)}$ are continuous, we find an open neighborhood $V(t)$ of t such that $|p(s) - p(t)| < \varepsilon/3$ and similarly for $p_{N(t)}$, for all $s \in V(t)$. Note that, for each $s \in V(t)$ we have the desired inequality:

$$p(s) - p_{N(t)}(s) = (p(s) - p(t)) + (p(t) - p_{N(t)}(t)) + (p_{N(t)}(t) - p_{N(t)}(s)) < 3\frac{\varepsilon}{3} = \varepsilon.$$

Varying t, $\{V(t) : t \in [0,1]\}$ will be an open cover of $[0,1]$ hence we can extract an open sub-cover $\{V(t_1),\dots,V(t_k)\}$. Then $N := \max\{N(t_1),\dots,N(t_k)\}$ does the job: for $n \geq N$ and $t \in [0,1]$, t belongs to some $V(t_i)$ and then

$$p(s) - p_n(s) \leq p(s) - p_{N(t_i)}(s) < \varepsilon.$$

We now return to the theorem and we denote by $\overline{\mathscr{A}}$ the closure of $\mathscr{A}$. We claim that:

$$f, g \in \overline{\mathscr{A}} \Longrightarrow \sup(f,g), \inf(f,g) \in \overline{\mathscr{A}},$$

where $\sup(f,g)$ is the function $x \mapsto \max\{f(x), g(x)\}$, and similarly $\inf(f,g)$. Since any $f \in \overline{\mathscr{A}}$ is the limit of a sequence in $\mathscr{A}$, we may assume that $f, g \in \mathscr{A}$. Since $\sup(f,g) = (f+g+|f-g|)/2$, $\inf(f,g) = (f+g-|f-g|)/2$ and $\mathscr{A}$ is a vector space, it suffices to show that, for any $f \in \mathscr{A}$, $|f| \in \overline{\mathscr{A}}$. Since any continuous f is bounded (X is compact!), by dividing by a constant, we may assume that $f \in \mathscr{A}$ takes values in $[-1,1]$. Using the polynomials p_n, since $\mathscr{A}$ is a subalgebra, $f_n := p_n(f^2) \in \mathscr{A}$, and it converges uniformly to $p(f^2) = |f|$. Hence $|f| \in \overline{\mathscr{A}}$.

We also claim that: for any $x, y \in X$ with $x \neq y$ and any $a, b \in \mathbb{R}$, there exists $f \in \mathscr{A}$ such that $f(x) = a$, $f(y) = b$. Indeed, by hypothesis, we find $g \in \mathscr{A}$ such that $g(x) \neq g(y)$; since $\mathscr{A}$ contains the unit (hence all the constants),

$$f := a + \frac{b-a}{g(y)-g(x)}(g - g(x))$$

will be in $\mathscr{A}$ and it clearly satisfies $f(x) = a$, $f(y) = b$.

Let now $h \in \mathscr{C}(X,\mathbb{R})$. Let $\varepsilon > 0$ and we look for $f \in \overline{\mathscr{A}}$ such that $d_{\sup}(h,f) \leq \varepsilon$. We first show that for any $x \in X$ there exists $f_x \in \overline{\mathscr{A}}$ such that $f_x(x) = h(x)$ and $f_x(y) < h(y) + \varepsilon$ for all $y \in X$. To that end, choose for each $y \in X$ function $f_{x,y} \in \mathscr{A}$ such that $f_{x,y}(x) = h(x)$ and $f_{x,y}(y) \leq h(y) + \varepsilon/2$ (possible due to the previous step). Using continuity, we find a neighborhood $V(y)$ of y such that $f_{x,y}(y') < h(y') + \varepsilon$ for all $y' \in V(y)$. From the cover $\{V(y) : y \in X\}$ we extract a finite subcover $\{V(y_1),\dots,V(y_k)\}$ and we set

$$f_x := \inf\left\{f_{x,y_1},\dots,f_{x,y_k}\right\}.$$

From the previous steps, $f_x \in \overline{\mathscr{A}}$; by construction, $f_x(y) < h(y) + \varepsilon$ for all $y \in X$ and $f_x(x) = h(x)$. Due to the last equality, we find an open neighborhood $W(x)$ of x such that $f_x(x') > h(x') - \varepsilon$ for all $x' \in W(x)$. We now let x vary and choose $x_1,\dots,x_l$ such that $\{W(x_1),\dots,W(x_l)\}$ cover X. Finally, we set $f := \sup\left\{f_{x_1},\dots,f_{x_l}\right\}$. By the discussion above, it belongs to $\overline{\mathscr{A}}$ while, by construction, $d_{\sup}(h,f) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary and $\overline{\mathscr{A}}$ is closed, $h \in \overline{\mathscr{A}}$. Thus $\overline{\mathscr{A}} = \mathscr{C}(X,\mathbb{R})$.

The previous theorem does not hold (word by word) over $\mathbb{C}$ instead of the reals. The appropriate complex-version of the Stone-Weierstrass theorem requires an extra-condition which refers precisely to the extra-structure present in the complex case: conjugation (hence the *-algebra structure on $\mathscr{C}(X,\mathbb{C})$).

Definition 7.16. Given a unital *-algebra A, a subalgebra $B \subset A$ is called a ***- subalgebra** if

$$b^* \in B \quad \forall\, b \in B.$$

With this, we have:

Corollary 7.17. *Let X be a compact Hausdorff space. Then any point-separating *-subalgebra $\mathscr{A} \subset \mathscr{C}(X,\mathbb{C})$ is dense in $(\mathscr{C}(X,\mathbb{C}), d_{sup})$.*

Proof. Let $\mathscr{A}_{\mathbb{R}} := \mathscr{A} \cap \mathscr{C}(X,\mathbb{R})$. Since for any $f \in \mathscr{A}$,

$$\mathrm{Re}(f) = \frac{f+\overline{f}}{2},\ \mathrm{Im}(f) = \frac{f-\overline{f}}{2i}$$

belong to $\mathscr{A}_{\mathbb{R}}$, it follows that $\mathscr{A}_{\mathbb{R}}$ separates points and is a unital subalgebra of $\mathscr{C}(X,\mathbb{R})$. From the previous theorem, $\mathscr{A}_{\mathbb{R}}$ is dense in $\mathscr{C}(X,\mathbb{R})$. Hence $\mathscr{A} = \mathscr{A}_{\mathbb{R}} + i\mathscr{A}_{\mathbb{R}}$ is dense in $\mathscr{C}(X,\mathbb{C})$.

7.3 Recovering X from $\mathscr{C}(X)$: the Gelfand Naimark theorem

The Gelfand-Naimark theorem says that a compact Hausdorff space can be recovered from its algebra $\mathscr{C}(X)$ of continuous functions (using only the algebra structure!!!). It makes no difference whether we work over $\mathbb{R}$ or $\mathbb{C}$; so let's say we work over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The key ingredient in recovering X from $\mathscr{C}(X)$ is the notion of maximal ideal.

Definition 7.18. Let A be an algebra. An **ideal of** A is any vector subspace $I \subset A$ satisfying

$$a \cdot x,\ x \cdot a \in I \quad \forall\, a \in A,\ x \in I.$$

The ideal I is called **maximal** if it is strictly contained in A and there is no other ideal J strictly containing I and different from A. We denote by M_A the set of all maximal ideals of A.

For instance, for $A = \mathscr{C}(X)$ (X a topological space), any subspace $A \subset X$ defines an ideal:

$$I_A := \{f \in \mathscr{C}(X) : f|_A = 0\}.$$

When $A = \{x\}$ is a point, we denote this ideal simply by I_x. Note that $I_A \subset I_x$ for all $x \in A$.

Proposition 7.19. *If X is a compact Hausdorff space, then I_x is a maximal ideal of $\mathscr{C}(X)$ for all $x \in X$, and any maximal ideal is of this type. In other words, one has a bijection*

$$\Psi : X \xrightarrow{\sim} M_{\mathscr{C}(X)},\ \ x \mapsto I_x.$$

Proof. Fix $x \in X$ and we show that I_x is maximal. Let I be another ideal strictly containing I_x; we prove $I = \mathscr{C}(X)$. Since $I \neq I_x$, we find $f \in I$ such that $f(x) \neq 0$. Since f is continuous, we find an open U such that $x \in U$, $f \neq 0$ on U. Now, $\{x\}$ and $X - U$ are two disjoint closed subsets of X hence, by Urysohn lemma, there exists $\eta \in \mathscr{C}(X)$ such that $\eta(x) = 0$, $\eta = 1$ outside U. Clearly, $\eta \in I_x \subset I$. Since I is an ideal containing f and η, $g := |f|^2 + \eta^2 \in I$. Note that $g > 0$: for $x \in U$, $f(x) \neq 0$, while for $x \in X - U$, $\eta(x) = 1$. But then any $h \in \mathscr{C}(X)$ is in I since it can be written as $g\frac{h}{g}$ with $g \in I$, $\frac{h}{g} \in \mathscr{C}(X)$. Hence $I = \mathscr{C}(X)$.

We still have to show that, if I is a maximal ideal, then $I = I_x$ for some x. It suffices to show that $I \subset I_x$ for some $x \in X$. Assume the contrary. Then, for any $x \in X$, we find $f_x \in I$ s.t. $f_x(x) \neq 0$. Since f_x is continuous, we find an open U_x s.t. $x \in U_x$, $f_x \neq 0$ on U_x. Now, $\{U_x : x \in X\}$ is an open cover of X. By compactness, we can select a finite subcover $\{U_{x_1}, \ldots, U_{x_k}\}$. But then

$$g := |f_{x_1}|^2 + \ldots + |f_{x_k}|^2 \in I$$

and $g > 0$ on X. By the same argument as above, we get $I = \mathscr{C}(X)$- contradiction!

The proposition shows how to recover X from $\mathscr{C}(X)$ as a set. To recover the topology, it is useful to slightly change the point of view and look at characters instead of maximal ideals.

Definition 7.20. Given an algebra A, a **character of** A is any $\mathbb{K}$-linear function $\chi : A \to \mathbb{K}$ which is not identically zero and satisfies

$$\chi(a \cdot b) = \chi(a)\chi(b) \quad \forall\ a, b \in A.$$

The set of characters of A is denoted by X_A and is called **the spectrum of** A. When we want to be more precise about $\mathbb{K}$, we talk about the real or the complex spectrum of A.

The previous proposition can be reformulated into:

Corollary 7.21. *If X is a compact Hausdorff space then, for any $x \in X$,*

$$\chi_x : \mathscr{C}(X) \to \mathbb{K},\ \chi_x(f) = f(x)$$

is a character of $\mathscr{C}(X)$, and any character is of this type. In other words, one has a bijection

$$\Phi : X \xrightarrow{\sim} X_{\mathscr{C}(X)}, \quad x \mapsto \chi_x. \tag{7.3.1}$$

Proof. It should be clear that any χ_x is a character. Let now χ be an arbitrary character. Let $I := \{f \in \mathscr{C}(X) : \chi(f) = 0\}$ (an ideal- check that!). We will make use of the remark that

$$f - \chi(f) \cdot 1 \in I \tag{7.3.2}$$

for all $f \in \mathscr{C}(X)$ (indeed, all these elements are killed by χ). We show that I is maximal. Let J be another ideal strictly containing I. Choosing $f \in J$ not belonging to I (i.e. $\chi(f) \neq 0$) and using (7.3.2) and $I \subset J$, we find that $1 \in J$ hence, as above, $J = \mathscr{C}(X)$. This proves that I is maximal. We deduce that it is of type I_x for some $x \in X$. But then, using (7.3.2) again we deduce that $(f - \chi(f) \cdot 1)(x) = 0$ for all f, i.e. $\chi = \chi_x$.

The advantage of characters and of X_A is that X_A carries a very natural topology.

Definition 7.22. Let A be an algebra and let X_A be its spectrum. For any $a \in A$, define

$$f_a : X_A \to \mathbb{K}, \quad f_a(\chi) := \chi(a).$$

We define $\mathscr{T}$ as the smallest topology on X_A with the property that all the functions $\{f_a : a \in A\}$ are continuous. The resulting topological space $(X_A, \mathscr{T})$ is called the **topological spectrum of** A.

Theorem 7.23. *Any compact Hausdorff space X is homeomorphic to the topological spectrum of $\mathscr{C}(X)$.*

Proof. Let $\mathscr{T}$ be the topology of X. We still have to show that the bijection Φ is a homeomorphism. We transport the topology of the spectrum to a topology $\mathscr{T}'$ on X using Φ; we have to show that $\mathscr{T}' = \mathscr{T}$. Notice that $\mathscr{T}'$ is the smallest topology with the property that all $f \in \mathscr{C}(X)$ are continuous as functions with respect to this new topology $\mathscr{T}'$. In particular, $\mathscr{T}' \subset \mathscr{T}$. For the reverse, we need the more explicit description of $\mathscr{T}'$: it is the topology generated by the subsets of X of type $f^{-1}(V)$ with $f \in \mathscr{C}(X)$ and $V \subset \mathbb{K}$ open. We fix $U \in \mathscr{T}$ and show that $U \in \mathscr{T}'$; it suffices to show that $\forall\, x \in U\ \exists\, f$ and V s.t. $x \in f^{-1}(V) \subset U$. But this follows from the Urysohn lemma: we find $f : X \to [0,1]$ continuous such that $f(x) = 0$ and $f = 1$ outside U. Taking $V = (-1,1)$ (or the unit disk if $\mathbb{K} = \mathbb{C}$),, we have the desired property.

Remark 7.24. (for the curious reader; $\mathbb{K} = \mathbb{C}$) The question that we did not answer is: which algebras A are of type $\mathscr{C}(X)$ for some compact Hausdorff X? What we did show is that X must be X_A. Furthermore, $a \mapsto f_a$ defines a map $\psi_A : A \to \mathscr{C}(X_A)$. Hence a possible answer is: algebras with the property that X_A is compact and Hausdorff, and ψ_A is an isomorphism (bijection). But this is clearly far from satisfactory. The complete answer is given by the full version of the Gelfand-Naimark theorem: it is the commutative C^*-algebras! This answer may seem a bit unfair since the notion of C^*-algebras seem to depend on data which is not algebraic (the norm!). However, a very special feature of C^*-algebras is that their norm can be recovered from the algebraic structure by the formula:

$$||a||^2 = \sup\{|\lambda| : \lambda \in \mathbb{C} \ \text{ such that } \lambda 1 - a^* a \text{ is not invertible}\}.$$

Sketch of proof that each commutative C^*-algebra is of type $\mathscr{C}(X)$: One first proves that X_A is compact and Hausdorff; then that $||\Psi_A(a)|| = ||a||\ \forall\, a \in A$; this implies that ψ_A is injective and its image is closed in $\mathscr{C}(X_A)$; finally, the Stone-Weierstrass implies that the image is also dense in $\mathscr{C}(X_A)$; hence ψ_A is bijective.

7.4 General function spaces $\mathscr{C}(X,Y)$

For any two topological spaces X and Y we denote by $\mathscr{C}(X,Y)$ the set of continuous functions from X to Y- a subset of the set $\mathscr{F}(X,Y)$ of all functions from X to Y. In general, there are several interesting topologies on $\mathscr{C}(X,Y)$. So far, in this chapter we were concerned with the uniform topology on $\mathscr{C}(X,Y)$ when X is compact and Hausdorff and $Y=\mathbb{R}$ or $\mathbb{C}$. In Section 3.9 , Chapter 3, in the case when $X\subset\mathbb{R}$ was an interval and $Y=\mathbb{R}^n$, we looked at the three topologies: of pointwise convergence, of uniform convergence, and of uniform convergence on compacts.

In this section we look at generalizations of these topologies to the case when X and Y are more general topological spaces. We assume throughout this entire section that

$$X-\text{is a locally compact topological space},\ Y-\text{is a metric space with a fixed metric } d. \tag{7.4.1}$$

These assumptions are not needed everywhere (e.g. for the pointwise topology on $\mathscr{C}(X,Y)$, the topology of X is completely irrelevant, etc etc). They are made in order to simplify the presentation.

Pointwise convergence, uniform convergence, compact convergence

Almost the entire Section 3.9, Chapter 3 goes through in this generality without any trouble ("word by word" most of the times). For instance, given a sequence $\{f_n\}_{n\geq 1}$ in $\mathscr{F}(X,Y)$, $f\in\mathscr{F}(X,Y)$, we will say that:

- f_n converges pointwise to f, and we write $f_n\xrightarrow{pt}f$, if $f_n(x)\to f(x)$ for all $x\in X$.
- f_n converges uniformly to f, and we write $f_n\rightrightarrows f$, if for any $\varepsilon>0$, there exists n_ε s.t.

$$d(f_n(x),f(x))<\varepsilon\quad\forall\, n\geq n_\varepsilon,\ \forall\, x\in X.$$

- f_n converges uniformly on compacts to f, and we write $f_n\xrightarrow{\text{cp}}f$ if, for any $K\subset X$ compact, $f_n|_K\rightrightarrows f|_K$.

And, as in *loc.cit* (with exactly the same proof), these convergences correspond to convergences with respect to the following topologies on $\mathscr{F}(X,Y)$:

- **the pointwise topology**, denoted $\mathscr{T}_{pt}$, is the topology generated by the family of subsets

$$S(x,U):=\{f\in\mathscr{F}(X,Y):f(x)\in U\}\subset\mathscr{F}(X,Y),$$

  with $x\in X$, $U\subset Y$ open.
- the uniform topology is induced by a sup-metric. For $f,g\in\mathscr{F}(X,Y)$, we define

$$d_{\sup}(f,g)=\sup\{d(f(x),g(x)):x\in X\}.$$

  Again, to overcome the problem that this supremum may be infinite (for some f and g) and to obtain a true metric, one considers

$$\hat{d}_{\sup}(f,g)=\min(d_{\sup}(f,g),1).$$

  The uniform topology is the topology associated to $\hat{d}_{\sup}$; it is denoted by $\mathscr{T}_{\text{unif}}$.
- **the topology of compact convergence**, denoted $\mathscr{T}_{cp}$, is the topology generated by the family of subsets

$$B_K(f,\varepsilon):=\left\{g\in\mathscr{F}(X,Y):\sup_{x\in K}d(f(x),g(x))<\varepsilon\right\},$$

  with $K\subset X$ compact, $\varepsilon>0$.

We will be mainly concerned with the restrictions of these topologies to the set $\mathscr{C}(X,Y)$ of continuous functions from X to Y. So, the part of Theorem 3.58 concerning continuous functions, with exactly the same proof, gives us the following:

Theorem 7.25. *If (Y,d) is complete, then $(\mathscr{C}(X,Y),\hat{d}_{sup})$ is complete.*

Equicontinuity

We continue in the setting (7.4.1). Recall that a function $f : X \to Y$ is continuous if and only if it is continuous at each point, i.e. if for each $x_0 \in X$ and any $\varepsilon > 0$ there exists a neighborhood U of x_0 such that

$$d(f(x), f(x_0)) < \varepsilon \quad \forall\, x \in U. \tag{7.4.2}$$

Definition 7.26. A subset $\mathscr{F} \subset \mathscr{C}(X,Y)$ is called **equicontinuous** if $\forall\, x_0 \in X$, $\forall\, \varepsilon > 0$, $\exists$ a neighborhood U of x_0 such that (7.4.2) holds $\forall\, f \in \mathscr{F}$.

When X is itself a metric space, then there is a "uniform" version of continuity and equicontinuity.

Definition 7.27. Assume that both (X,d) and (Y,d) are metric spaces. Then

1. A map $f : X \to Y$ is called **uniformly continuous** if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t.
$$d(f(x), f(y)) < \varepsilon \quad \forall\, x, y \in X \text{ with } d(x,y) < \delta. \tag{7.4.3}$$
2. A subset $\mathscr{F} \subset \mathscr{C}(X,Y)$ is called **uniformly equicontinuous** if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t. (7.4.3) holds $\forall$ $f \in \mathscr{F}$.

From the definitions we immediately see that, in general, the following implications hold:

$$\begin{array}{ccc} \mathscr{F} - \text{is uniformly equicontinuous} & \longrightarrow & \text{each } f \in \mathscr{F} \text{ is uniformly continuous} \\ \downarrow & & \downarrow \\ \mathscr{F} - \text{is equicontinuous} & \longrightarrow & \text{each } f \in \mathscr{F} \text{ is continuous} \end{array}$$

Proposition 7.28. *A sequence $\{f_n\}_{n\geq 1}$ is convergent in $(\mathscr{C}(X,Y), \mathscr{T}_{cp})$ if and only if $\{f_n\}_{n\geq 1}$ is convergent in $(\mathscr{C}(X,Y), \mathscr{T}_{pt})$ and it is equicontinuous as a subset of $\mathscr{C}(X,Y)$.*

Proof. For the direct implication we still have to show that, if $f_n \overset{\text{cp}}{\to} f$, then $\{f_n\}$ is equicontinuous. Let $x_0 \in X$. Since f is continuous, we find $V \in \mathscr{T}_X(x_0)$ s.t. $d(f(x), f(x_0)) < \varepsilon/3$ $\forall\, x \in V$. Since X is locally compact, we may assume that $\overline{V}$ is compact, hence $f_n|_{\overline{V}} \rightrightarrows f|_{\overline{V}}$. We then find n_ε such that $d(f_n(x), f(x)) < \varepsilon/3$ $\forall\, n \geq n_\varepsilon$, $x \in V$. Then

$$d(f_n(x), f_n(x_0)) \leq d(f_n(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), f_n(x_0)) < \varepsilon.$$

for all $x \in V$ and $n \geq n_\varepsilon$. By making V smaller if necessary, the previous inequality will also hold for all $n < n_\varepsilon$ (since there are a finite number of such n's, and each f_n is continuous).

We now prove the converse. Assume equicontinuity and assume that $f_n \to f$ pointwise. Let $K \subset X$ be compact; we prove that $f_n|_K \rightrightarrows f|_K$. Let $\varepsilon > 0$. For each $x \in K$, there is an open V_x containing x, such that

$$d(f_n(y), f_n(x)) < \varepsilon/3 \quad \forall\, y \in V_x,\ \forall\, n.$$

Since $\{V_x\}$ covers the compact K, we find a finite number of points $x_i \in K$ (with $1 \leq i \leq k$) such that the opens $V_i = V_{x_i}$ cover K. Since $f_n(x_i) \to f(x_i)$, we find n_ε such that $d(f_n(x_i), f(x_i)) < \varepsilon/3$ $\forall\, n \geq n_\varepsilon$, $\forall\, i \in \{1, \ldots, k\}$. In particular, $d(f(x_i), f(x)) = \lim_n d(f_n(x_i), f_n(x)) \leq \varepsilon/3$. Finally, for arbitrary $x \in K$, choosing i such that $x \in V_i$,

$$d(f_n(x), f(x)) \leq d(f_n(x), f_n(x_i)) + d(f_n(x_i), f(x_i)) + d(f(x_i), f(x)) < 3 \cdot \frac{\varepsilon}{3} = \varepsilon \quad \forall n \geq n_\varepsilon.$$

Boundedness

Let's also briefly discuss boundedness. For $\mathscr{F} \subset \mathscr{C}(X,Y)$ and $x \in X$ we use the notation

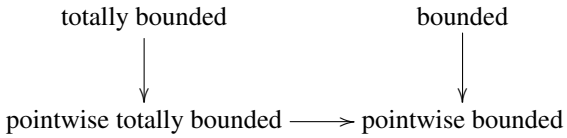
$$\mathscr{F}(x) := \{f(x) : f \in \mathscr{F}\}.$$

Boundedness conditions on $\mathscr{F}$ arise by using directly the metric d on Y, or by working in the metric space $\mathscr{C}(X,Y)$. One can also remember that that the notion of "bounded" in a metric space had an improved version of "totally bounded". Furthermore, on $\mathscr{C}(X,Y)$, while we had to adjust $d_{\sup}$ to make it into a metric, anything is bounded with respect to the resulting $\hat{d}_{\sup}$. All together, we end up with several interesting concepts of boundedness:

Definition 7.29. We say that $\mathscr{F} \subset \mathscr{C}(X,Y)$ is:

- **Bounded** if it is bounded with respect to $d_{\sup}$.
- **Totally bounded** if it is totally bounded in $(\mathscr{C}(X,Y), \hat{d}_{\sup})$.
- **Pointwise bounded** if $\mathscr{F}(x)$ is bounded in (Y,d) for all $x \in X$.
- **Pointwise totally bounded** if $\mathscr{F}(x)$ is totally bounded in (Y,d) for all $x \in X$.

From the definitions we immediately see that, in general, the following implications hold:



Total boundedness need not imply boundedness here: the finite family $\{0, \mathrm{id}_{\mathbb{R}}\} \subset \mathscr{C}(\mathbb{R},\mathbb{R})$ is totally bounded, but $d_{\sup}(0, \mathrm{id}_{\mathbb{R}}) = \infty$.

Example 7.30. For $Y = \mathbb{R}^n$ with the Euclidean metric d, since total boundedness and boundedness in $\mathbb{R}^n, d)$ are equivalent, we see that a subset $\mathscr{F} \subset \mathscr{C}(X,\mathbb{R}^n)$ is pointwise totally bounded if and only if it is pointwise bounded. However, it is not true that $\mathscr{F}$ is totally bounded if and only if it is bounded. What we have in general is:

Proposition 7.31. *If $\mathscr{F} \subset \mathscr{C}(X,Y)$ is totally bounded then it must be equicontinuous. Moreover, if each $f \in \mathscr{F}$ is uniformly continuous, then $\mathscr{F}$ is even uniformly equicontinuous.*

Proof. Fix $\varepsilon > 0$ small enough- e.g. $\varepsilon < 1$, and $x_0 \in X$. By assumption, we find $f_1, \ldots, f_k \in \mathscr{F}$ such that $\mathscr{F} \subset B(f_1, \varepsilon/3) \cup \ldots \cup B(f_k, \varepsilon/3)$, where the balls are the ones induced by $\hat{d}_{\sup}$. Since each f_i is continuous, we find a neighborhood U_i of x_0 such that

$$d(f_i(x), f_i(x_0)) < \varepsilon/3, \quad \forall\, x \in U_i.$$

Then $U = \cap_i U_i$ is a neighborhood of x_0. For $x \in U$, $f \in \mathscr{F}$, choosing i s.t. $f \in B(f_i, \varepsilon/3)$:

$$d(f(x), f(x_0)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0)),$$

which is $< \varepsilon$ (for all $x \in U$, $f \in \mathscr{F}$). This proves equicontinuity. For the second part the argument is completely similar (even simpler as all U_i will become X), where we use that each f_i is uniformly continuous.

Let us also recall that the notion of total boundedness was introduced in order to characterize compactness. Since $\hat{d}_{\sup}$ is complete whenever (Y,d) is (Theorem 7.25) we deduce:

Proposition 7.32. *When (Y,d) is complete, a subset $\mathscr{F} \subset \mathscr{C}(X,Y)$ is totally bounded if and only if it is relatively compact in $(\mathscr{C}(X,Y), \hat{d}_{sup})$.*

The case when X is a compact metric space

When X is a compact metric space the situation simplifies quite a bit. In some sense, the pointwise conditions imply the uniform ones (in the vertical implications from the previous two diagrams). To be more precise, let us combine the two diagrams and Proposition 7.31 together into a diagram of implications:

$$\begin{array}{ccccc} \text{totally bounded} & \longrightarrow & \text{bounded and uniformly equic.} & \longrightarrow & \text{uniformly equic.} \\ \downarrow & & \downarrow & & \downarrow \\ \text{pt. totally bounded and equic.} & \longrightarrow & \text{pt. bounded and equic.} & \longrightarrow & \text{equic.} \end{array}$$

where "pt" stands for pointwise and "equic" for equicontinuous (and we could have continued to the right with "each $f \in \mathscr{F}$ is (uniformly) continuous"). We now look for converses to the vertical implications.

Theorem 7.33. *If (X,d) is a compact metric space, $f : X \to Y$, $\mathscr{F} \subset \mathscr{C}(X,Y)$, then*

1. *(f is continuous) $\iff$ (f is uniformly continuous).*
2. *($\mathscr{F}$ is equicontinuous) $\iff$ ($\mathscr{F}$ is uniformly equicontinuous). In this case, moreover,*
   a. *($\mathscr{F}$ is pointwise bounded) $\iff$ ($\mathscr{F}$ is bounded).*
   b. *$\mathscr{T}_{unif}$ and $\mathscr{T}_{pt}$ induce the same topology on $\mathscr{F}$.*

In particular, if a sequence $(f_n)_{n\geq 1}$ is equicontinuous, then it is uniformly convergent (or bounded) iff it is pointwise convergent (or pointwise bounded, respectively).

Proof. For 1, 2 and (a) the nontrivial implications are the direct ones. For 1, assume that f is continuous. Let $\varepsilon > 0$. For each $x \in X$ choose V_x such that

$$d(f(y),f(x)) < \varepsilon/2 \quad \forall\, y \in V_x.$$

Apply now the Lebesgue lemma (Proposition 5.13) and let $\delta > 0$ be a resulting Lebesgue number. Then, for each $y,z \in X$ with $d(y,z) < \delta$ we find $x \in X$ such that $y,z \in V_x$, hence

$$d(f(y),f(z)) \leq d(f(y),f(x)) + d(f(z),f(x)) < \varepsilon.$$

This proves that f is uniformly continuous. Exactly the same proof applies to 2 (just add "for all $f \in \mathscr{F}$" everywhere). For (a), assume that $\mathscr{F}$ is equicontinuous and pointwise bounded. From the first condition we find an open cover $\{V_x\}_{x\in X}$ with $x \in V_x$ and $d(f(y),f(x)) < 1$ for all $y \in V_x$. Choose a finite subcover corresponding to $x_1,\ldots,x_k \in X$. Using that $\mathscr{F}(x_i)$ is bounded for each i, we find $M > 0$ such that

$$d(f(x_i),g(x_i)) < M \quad \forall\, f,g \in \mathscr{F},\ \ \forall\, 1 \leq i \leq k.$$

Then, for arbitrary $x \in X$, choosing i such that $x \in V_{x_i}$, we have

$$d(f(x),g(x)) \leq d(f(x),f(x_i)) + d(f(x_i),g(x_i)) + d(g(x_i),g(x)) < M+2,$$

for all $f,g \in \mathscr{F}$, showing that $\mathscr{F}$ is bounded. For (b), the non-obvious part is to show that $\mathscr{T}_{\text{unif}}|_{\mathscr{F}} \subset \mathscr{T}_{pt}|_{\mathscr{F}}$. Due to the definitions of these topologies, we start with $f \in \mathscr{F}$ and a ball

$$B_{\mathscr{F}}(f,\varepsilon) = \left\{g \in \mathscr{F} : d_{\sup}(g,f) < \varepsilon\right\},$$

and we are looking for $x_1,\ldots,x_k \in X$ and $\varepsilon_1,\ldots,\varepsilon_k > 0$ such that

$$\cap_i \left\{g \in \mathscr{F} : d(g(x_i),f(x_i)) < \varepsilon_i\right\} \subset B_{\mathscr{F}}(f,\varepsilon).$$

For that, choose as before a finite open cover $\{V_{x_i}\}$ of X such that $d(f(x),f(x_i)) < \varepsilon/6\ \forall\, f \in \mathscr{F},\ x \in V_{x_i}$. and, by the same inequalities as above, we find that the x_i and $\varepsilon_i = \varepsilon/3$ have the desired properties.

The Arzela-Ascoli theorem

The Arzela-Ascoli theorem has quite a few different looking versions. They all give compactness criteria for subspaces of $\mathscr{C}(X,Y)$ in terms of equicontinuity; sometimes the statement is a sequential one (giving criteria for an equicontinuous sequence to admit a convergent subsequence). The difference between the several versions comes either from the starting hypothesis on X and Y, or from the topologies one considers on $\mathscr{C}(X,Y)$. As in the last subsection, we restrict ourselves to the case that X and Y are metric and X is compact; the interesting topology on the space of functions will then be the uniform one.

Theorem 7.34. *(Arzela-Ascoli) Assume that (X,d) is a compact metric space, (Y,d) is complete. Then, for a subset $\mathscr{F} \subset \mathscr{C}(X,Y)$, the following are equivalent:*

1. *$\mathscr{F}$ is relatively compact in $(\mathscr{C}(X,Y),d_{sup})$.*
2. *$\mathscr{F}$ is equicontinuous and pointwise totally bounded.*

(note: when $Y=\mathbb{R}^n$ with the Euclidean metric, "pointwise totally bounded"="pointwise bounded").

Corollary 7.35. *Let X and Y be as above. Then any sequence $(f_n)_{n\geq 1}$ which is equicontinuous and pointwise totally bounded admits a subsequence which is uniformly convergent.*

Proof. The direct implication is clear now: if $\overline{\mathscr{F}}$ is compact, it must be totally bounded (cf. Theorem 5.10); this implies that $\overline{\mathscr{F}}$ (hence also $\mathscr{F}$) is equicontinuous and pointwise totally bounded. We now prove the converse. Let us assume for simplicity that $\mathscr{F}$ is also closed with respect to the uniform topology (otherwise replace it by its closure and, by the same arguments as before, show that equicontinuity and pointwise total boundedness hold for the closure as well). We show that $\mathscr{F}$ is compact. Using Theorem 5.10, it suffices to show that $\mathscr{F}$ is sequentially compact. So, let $(f_n)_{n\geq 1}$ be a sequence in $\mathscr{F}$ and we will show that it contains a convergent subsequence. Use Corollary 5.12 and consider

$$A=\{a_1,a_2,\ldots\}\subset X$$

which is dense in X. Since $(f_n(a_1))_{n\geq 1}$ is totally bounded, using Corollary 5.11, it follows that it has a convergent subsequence $(f_n(a_1))_{n\in I_1}$, where $I_1\subset\mathbb{Z}_+$. Let n_1 be the smallest element of I_1. Similarly, since $(f_n(a_2))_{n\in I_1}$ is totally bounded, we find a convergent subsequence $(f_n(a_2)_{n\in I_2}$ where $I_2\subset I_1$. Let n_2 be the smallest element of I_2 with $n_2>n_1$. Continue inductively to construct I_j and let n_j be its smallest element greater than n_{j-1}. Choosing $g_k=f_{n_k}$, this will be a subsequence of (f_n) which has the property that $(g_k(a_i))_{k\geq 1}$ is convergent for all i. We will show that (g_k) is Cauchy (hence convergent). Let $\varepsilon>0$. Since $\mathscr{F}$ is uniformly equicontinuous, we find δ such that

$$d(g_k(x),g_k(y))<\varepsilon/3 \quad \forall\, k \text{ and whenever } d(x,y)<\delta.$$

Since A is dense in X, the balls $B(a_i,\delta)$ cover X; since X is compact, we find some integer N such that X is covered by $B(a_i,\delta)$ with $1\leq i\leq N$. Since each of the sequences $(g_k(a_i))_{k\geq 1}$ is convergent for all $1\leq i\leq N$, we find n_ε such that

$$d(g_j(a_i),g_k(a_i))<\varepsilon/3 \quad \forall\, j,k\geq n_\varepsilon \ \ \forall\, 1\leq i\leq N.$$

Then, for all $x\in X$, $j,k\geq n_\varepsilon$, choosing $i\leq N$ such that $x\in B(a_i,\delta)$, we have

$$d(g_j(x),g_k(x))\leq d(g_j(x),g_j(a_i))+d(g_j(a_i),g_k(a_i))+d(g_k(a_i),g_k(x))<\varepsilon.$$

Finally, let us also mention the following more general version of the Arzela-Ascoli (see Munkres' book).

Theorem 7.36. *(Arzela-Ascoli) Assume that X is a locally compact Hausdorff space, (Y,d) is a complete metric space, $\mathscr{F}\subset\mathscr{C}(X,Y)$. Then $\mathscr{F}$ is relatively compact in $(\mathscr{C}(X,Y),\mathscr{T}_{cp})$ if and only if $\mathscr{F}$ is equicontinuous and pointwise totally bounded.*
Moreover, if X is also second countable, then any sequence $(f_n)_{n\geq 1}$ in $\mathscr{C}(X,\mathbb{R}^N)$ which is equicontinuous and pointwise totally bounded admits a subsequence which is uniformly convergent on compacts.

Chapter 8
Partitions of unity

1. **Some axioms for sets of functions**

2. **Finite partitions of unity**

3. **Arbitrary partitions of unity**

4. **The locally compact case**

5. **Urysohn's lemma**

6. **More exercises**

8.1 Some axioms for sets of functions

The theory of "partitions of unity" is the most important tool that allows one to pass "from local to global". As such, it is widely used in many fields of mathematics, most notably in many branches of Geometry and Analysis. The word "unity" stands for the constant function equal to 1, on some given space X. A "partition of unity" is a decomposition

$$\sum_i \eta_i = 1$$

of the constant function into a sum of continuous functions η_i. One is interested in such partitions of unity with the extra-requirement that each η_i is "concentrated in a given (usually very small) open U_i". The U_i's form a (given) open cover of X and one is interested in the existence of partitions of unity "subordinated" to the cover.

Let us also mention that, when it comes to applications to Geometry and Analysis, one deals with topological spaces that have extra-structure and the "partitions of unity" are required to be more than continuous (in most cases one can talk about differentiable functions, and the partitions are required to be so). Rather curiously, the existence of such "special" partitions of unity is actually easier to establish than the existence of the continuous partitions for general topological spaces. To include such applications, we will include in our discussion a given set $\mathscr{A}$ of continuous functions. To specify the axioms for $\mathscr{A}$, we consider the space of continuous functions on X:

$$\mathscr{C}(X) = \mathscr{C}(X,\mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}.$$

We have already discussed the algebraic structures present on $\mathscr{C}(X)$. Here we use the operation of taking sums of functions $f,g \in \mathscr{C}(X)$,

$$(f+g)(x) = f(x)+g(x)$$

as well as the fact that we can take quotients f/g, whenever g is nowhere vanishing:

$$\frac{f}{g}(x) := \frac{f(x)}{g(x)}.$$

Definition 8.1. Given a topological space X, we say that a subset $\mathscr{A} \subset \mathscr{C}(X)$:

- **is closed under finite sums** if $f+g \in \mathscr{A}$ whenever $f,g \in \mathscr{A}$.
- **is closed under quotients** if $f/g \in \mathscr{A}$ whenever $f,g \in \mathscr{A}$ and g is nowhere vanishing.

In many examples $\mathscr{A}$ is usually closed under all the algebraic operation present on $\mathscr{C}(X)$ i.e., $\mathscr{A}$ is a sub-algebra of $\mathscr{C}(X)$ (cf. Definition 7.11). However, the most important condition on $\mathscr{A}$ is the following topological condition, which should remind you of Definition 2.77.

Definition 8.2. Given a topological space X and $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\mathscr{A}$ is **normal** if for any two closed disjoint subsets $A,B \subset X$, there exists $f : X \to [0,1]$ which belongs to $\mathscr{A}$ and such that $f|_A = 0$, $f|_B = 1$.

Lemma 2.79 implies that the existence of a normal $\mathscr{A} \subset \mathscr{C}(X)$ forces X to be normal. Furthermore, Urysohn's lemma (Theorem 2.81) can be now reformulated as saying that $\mathscr{C}(X)$ is normal. All together, for Hausdorff spaces X one obtains that:

$$X \text{ is a normal space} \iff \mathscr{C}(X) \text{ is normal}$$

8.2 Finite partitions of unity

We start by giving a precise meaning to the statement that "a continuous function $\eta : X \to \mathbb{R}$ is concentrated in an open $U \subset X$". We will use the notation:

$$\{\eta \neq 0\} := \{x \in X : \eta(x) \neq 0\}.$$

Definition 8.3. Given a topological space X and $\eta : X \to \mathbb{R}$, define the **support of** η as

$$\operatorname{supp}(\eta) := \overline{\{\eta \neq 0\}} \subset X.$$

We say that η is supported in an open U if $\operatorname{supp}(\eta) \subset U$.

It is important that the support is defined as **the closure** of $\{\eta \neq 0\}$. This condition allows us to perform "globalization", as the following exercise indicates.

Exercise 8.4. Let $(X, \mathscr{T})$ be a topological space, $U \subset X$ open and $\eta \in \mathscr{C}(X)$ supported in U. Then, for any continuous map $g : U \to \mathbb{R}$,

$$(\eta \cdot g) : X \to \mathbb{R},\ (\eta \cdot g)(x) = \begin{cases} \eta(x)g(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}.$$

is continuous. Show that this statement fails if we only assume that $\{\eta \neq 0\} \subset U$.

Next we discuss finite partitions of unity.

Definition 8.5. Let X be a topological space, $\mathscr{U} = \{U_1, \ldots, U_n\}$ a finite open cover of X. A **partition of unity subordinated to** $\mathscr{U}$ is a family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\eta_1 + \ldots + \eta_n = 1, \ \operatorname{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an **$\mathscr{A}$-partition of unity** if $\eta_i \in \mathscr{A}$ for all i.

Exercise 8.6. Show that, given $\mathscr{A} \subset \mathscr{C}(X)$ with the property that $1 - f \in \mathscr{A}$ whenever $f \in \mathscr{A}$, the following are equivalent:

1. any 2-open cover $\mathscr{U} = \{U_1, U_2\}$ admits an $\mathscr{A}$-partition of unity subordinated to it.
2. $\mathscr{A}$ is normal (cf. Definition 8.2).

Theorem 8.7. *Assume that X is a topological space and $\mathscr{A} \subset \mathscr{C}(X)$ is normal and closed under finite sums and quotients. Then, for any finite open cover $\mathscr{U}$, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.*

Proof. The main topological ingredient in the proof is the following "shrinking lemma'.'

Lemma 8.8. *(the finite shrinking lemma) For any finite open covering $\mathscr{U} = \{U_i : 1 \leq i \leq n\}$ of a normal space X, there exists an open covering $\mathscr{V} = \{V_i : 1 \leq i \leq n\}$ such that*

$$\overline{V}_i \subset U_i, \ \ \forall\, i = 1, \ldots, n.$$

Proof. Let

$$A = X - (U_2 \cup \ldots \cup U_n), D = U_1.$$

Then A is closed, D is open, and $A \subset D$. By Lemma 2.82 from the end of Chapter 2, we find V_1 open such that

$$A \subset V_1 \subset \overline{V}_1 \subset D (= U_1).$$

This means that

$$\{V_1, U_2, \ldots, U_n\}$$

is a new open cover of X with $\overline{V}_1 \subset U_1$. In other words, we have managed to "refine U_1". Applying the same argument to this new cover (to refine U_2), we find a new open cover

$$\{V_1, V_2, U_3, \ldots, U_n\}$$

with $\overline{V}_1 \subset U_1$, $\overline{V}_2 \subset U_2$. Continuing this argument, we obtain the desired open cover $\mathscr{V}$.

We now prove the theorem. Let $\mathscr{U} = \{U_i\}$ be the given finite open cover. Apply the previous lemma twice and choose open covers $\mathscr{V} = \{V_i\}$, $\mathscr{W} = \{W_i\}$, with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. For each i, we use the separation property of $\mathscr{A}$ for the disjoint closed sets $(\overline{W}_i, X - V_i)$. We find $f_i : X \to [0,1]$ that belongs to $\mathscr{A}$, with $f_i = 1$ on $\overline{W}_i$ and $f_i = 0$ outside V_i. Note that

$$f := f_1 + \ldots + f_n$$

is nowhere zero. Indeed, if $f(x) = 0$, we must have $f_i(x) = 0$ for all i, hence, for all i, $x \notin W_i$. But this contradicts the fact that $\mathscr{W}$ is a cover of X. From the properties of $\mathscr{A}$, each

$$\eta_i := \frac{f_i}{f_1 + \ldots + f_n} : X \to [0,1]$$

belongs to $\mathscr{A}$. Clearly, their sum is 1. Finally, $\operatorname{supp}(\eta_i) \subset U_i$ because $\overline{V}_i \subset U_i$ and $\{x : \eta_i(x) \neq 0\} = \{x : f_i(x) \neq 0\} \subset V_i$.

8.3 Arbitrary partitions of unity

For arbitrary partitions of unity one has to deal with infinite sums $\sum_i f_i$ of continuous functions on X (indexed by some infinite set I). In such cases it is natural to require that, for each $x \in X$, the sum $\sum_i f_i(x)$ is finite (i.e. $f_i(x) = 0$ for all but a finite number of i's). Although the sum is then well defined as a function on X, to retain continuity, a slightly stronger notion is needed. All together, what we need here is the notion of "locally finite" introduced in Definition 4.60. We adapt it to families of functions as follows.

Definition 8.9. Given a topological space X, a family $\{\tilde{g}_i : i \in I\}$ of continuous functions $\tilde{g}_i : X \to \mathbb{R}$ is called a **locally finite family of continuous functions** if the family $\{\operatorname{supp}_X(\tilde{g}_i) : i \in I\}$ is locally finite in the sense of Definition 4.60.

We say that $\mathscr{A} \subset \mathscr{C}(X)$ is **closed under locally finite sums** if for any locally finite family $\{\tilde{g}_i : i \in I\}$ of functions from $\mathscr{A}$, the resulting sum $\sum \tilde{g}_i$ is in $\mathscr{A}$.

Exercise 8.10. Show that if $\{\tilde{g}_i : i \in I\}$ is a locally finite family of continuous functions, then

$$X \ni x \mapsto \sum_i \tilde{g}_i(x)$$

gives a well-defined continuous function $\sum_i \tilde{g}_i : X \to \mathbb{R}$.

Definition 8.11. Let X be a topological space, $\mathscr{U} = \{U_i : i \in I\}$ an open cover of X. A **partition of unity** subordinated to $\mathscr{U}$ is a locally finite family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\sum_i \eta_i = 1, \ \ \mathrm{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an **$\mathscr{A}$-partition of unity** if $\eta_i \in \mathscr{A}$ for all i.

Notice that if $\{\eta_i\}$ is a partition of unity subordinated to $\mathscr{U}$, then $\{\eta_i \neq 0\}$ is still an open covering of X, and it is a refinement of $\mathscr{U}$. This can be seen as a first indication of the relationship between partitions of unity and the paracompactness introduced in Definition 4.62. However, it would be even more fair to say that this remark may be seen as the original motivation for introducing the concept of paracompactness. However, the relationship is much stronger:

Theorem 8.12. *A Hausdorff space X is paracompact if and only if for any open cover $\mathscr{U}$ of X, there exists a partition of unity subordinated to $\mathscr{U}$.*

We are left with proving the existence of partitions of unity- that is, the direct implication. We will proceed in the slightly more general setting of $\mathscr{A}$-partitions of unity. As in the previous subsection, we will need a "shrinking lemma".

Lemma 8.13. *(the shrinking lemma) If X is a paracompact Hausdorff space then for any open cover $\mathscr{U} = \{U_i : i \in I\}$ there exists a locally finite open cover $\mathscr{V} = \{V_i : i \in I\}$ such that $\overline{V}_i \subset U_i$ for all $i \in I$.*

Proof. Consider $\mathscr{A} := \left\{V \subset X \text{ open} : \ \overline{V} \subset U_i \text{ for some } i \in I\right\}$. By Proposition 4.65, X is normal, hence we can apply Lemma 2.82. Remark now that that lemma implies that $\mathscr{A}$ is an open cover of X. Let $\mathscr{B} = \left\{B_j : j \in J\right\}$ be an open cover of X that is a locally finite refinement of $\mathscr{A}$. Then, for each $j \in J$, we find an element $f(j) \in I$ such that $\overline{B}_j \subset U_{f(j)}$ (and this defines a function $f : J \to I$). We define

$$V_i := \cup_{j \in f^{-1}(i)} B_j$$

(by convention, this is empty if $f^{-1}(i)$ is empty). Using Exercise 10.46, we have $\overline{V}_i \subset U_i$ for all i. Finally, remark that $\{V_i\}$ is locally finite: if a neighborhood of a point intersects V_i then it intersects B_j for some $j \in f^{-1}(i)$, hence if it intersects an infinite number of V_i's, then it would also intersect an infinite number of B_j's.

Theorem 8.14. *Let X be a paracompact Hausdorff space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal, closed under locally finite sums and closed under quotients.*
Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.

Proof. The proof is completely similar to the proof from the finite case. Apply the shrinking lemma twice to find coverings $\{V_i\}$ and $\{W_i\}$ with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. Then choose $\phi_i : X \to [0,1]$ such that $\phi_i = 1$ on $\overline{W}_i$ and 0 on $X - V_i$, with $\phi_i \in \mathscr{A}$. Finally, since our families are locally finite, $\eta_i = \phi_i / \sum_j \phi_j$ makes sense and is our desired partition of unity (fill in the details!).

8.4 The locally compact case

The locally compact Hausdorff case is nicer. First of all the condition on $\mathscr{A} \subset \mathscr{C}(X)$ to separate the closed subsets of X (which may be difficult to prove!) can be reduced to a local condition.

Theorem 8.15. *Let X be a Hausdorff paracompact space and $\mathscr{A} \subset \mathscr{C}(X)$ closed under locally finite sums and under quotients. If X is also locally compact, then the following are equivalent:*

1. *$\mathscr{A}$ is normal.*
2. *$\forall$ ($x \in U \subset X$ with U open), $\exists$ ($f \in \mathscr{A}$ nonegative, supported in U, with $f(x) > 0$).*

Proof. To prove that 1 implies 2, choose an open neighborhood V of x with $\overline{V} \subset U$, and apply normality of $\mathscr{A}$ to $\{x\}$ and $X - V$. This gives $f \in \mathscr{A}$ with values in $[0,1]$, equal to 1 at x and zero outside V, hence supported in U.

Assume 2. We claim that for any compact $C \subset X$ and any open U containing C, there exists a nonnegative $f \in \mathscr{A}$ supported in U such that $f|_C > 0$. Indeed, for each $c \in C$, condition 2 gives a nonnegative $f_c \in \mathscr{A}$ supported in U with $f_c(c) > 0$. The open sets $\{f_c > 0\}$ cover C, so finitely many, corresponding to $c_1, \ldots, c_k$, suffice. Then $f = f_{c_1} + \cdots + f_{c_k}$ has the desired properties: it belongs to $\mathscr{A}$, is positive on C, and its support is contained in the finite union of the supports of the f_{c_j}, hence in U.

To prove 1, let $A, B \subset X$ be two closed disjoint subsets. As terminology, $D \subset X$ is called relatively compact if $\overline{D}$ is compact. Since X is locally compact, any point has arbitrarily small relatively compact open neighborhoods (why?). For each $y \in X - A$, we choose such a neighborhood $D_y \subset X - A$. For each $a \in A$, since $a \in X - B$, by Lemma 8.13 and Lemma 2.82, we find an open D_a such that $a \in D_a \subset X - B$. Again, we may assume that $\overline{D}_a$ is relatively compact. Then $\{D_x : x \in X\}$ is an open cover of X; let $\mathscr{U} = \{U_i : i \in I\}$ be a locally finite refinement. We split the set of indices as $I = I_1 \cup I_2$, where I_1 contains those i for which $U_i \cap A \neq \emptyset$, while I_2 those for which $U_i \subset X - A$. Using Lemma 8.13 we also choose an open cover of X, $\mathscr{V} = \{V_i : i \in I\}$, with $\overline{V}_i \subset U_i$. Note that, by construction, each U_i (hence also each V_i) is relatively compact. Hence, by the claim above, we can find nonnegative $\eta_i \in \mathscr{A}$ such that

$$\eta_i|_{\overline{V}_i} > 0, \qquad \operatorname{supp}(\eta_i) \subset U_i.$$

Finally, we define

$$f(x) = \frac{\sum_{i \in I_1} \eta_i(x)}{\sum_{i \in I} \eta_i(x)}$$

Both sums are locally finite since $\operatorname{supp}(\eta_i) \subset U_i$. The denominator is everywhere positive because the V_i cover X and $\eta_i > 0$ on $\overline{V}_i$. Thus the closure properties of $\mathscr{A}$ give $f \in \mathscr{A}$, and nonnegativity of the η_i gives $0 \leq f \leq 1$. Also, $f|_A = 1$. Indeed, for $a \in A$, a cannot belong to the U_i's with $i \in I_2$ (i.e. those $\subset X - A$); hence $\eta_i(a) = 0$ for all $i \in I_2$, hence $f(a) = 1$. Finally, $f|_B = 0$. To see this, we show that $\eta_i(b) = 0$ for all $i \in I_1$, $b \in B$. Assume the contrary. We find $i \in I_1$ and $b \in B \cap U_i$. Now, from the construction of $\mathscr{U}$, $U_i \subset D_x$ for some $x \in X$. There are two cases. If $x = a \in A$, then the defining property for D_a, namely $D_a \cap B = \emptyset$, is in contradiction with our assumption ($b \in B \cap U_i$). If $x = y \in X - A$, then the defining property for D_y, i.e. $D_y \subset X - A$, is in contradiction with the fact that $i \in I_1$ (i.e. $U_i \cap A \neq \emptyset$).

Chapter 9
Embedding theorems

In this chapter we will describe a general method for attacking embedding problems. We will establish several results but, as the main final result, we state here the following:

Theorem 9.1. *Any compact n-dimensional topological manifold can be embedded in* $\mathbb{R}^{2n+1}$.

1. **Using function spaces**

2. **Using covers and partitions of unity**

3. **Dimension and open covers**

4. **More exercises**

9.1 Using function spaces

Throughout this section (X,d) is a metric space which is assumed to be compact and Hausdorff, and (Y,d) is a complete metric space (which, for the purpose of the chapter, you may assume to be $\mathbb{R}^n$ with the Euclidean metric). The associated embedding problem is: can X be embedded in $(Y,\mathscr{T}_d)$?. Since X is compact, this is equivalent to the existence of a continuous injective function $f : X \to Y$.

Definition 9.2. Given $f \in \mathscr{C}(X,Y)$, the injectivity defect of f is defined as

$$\delta(f) := sup\ \left\{d(x,x') : x,x' \in X \text{ such that } f(x) = f(x')\right\}.$$

For each $\varepsilon > 0$, we define the space of ε-approximate embeddings of X in Y as:

$$\mathrm{Emb}_\varepsilon(X,Y) := \{f \in \mathscr{C}(X,Y) : \delta(f) < \varepsilon\}$$

endowed with the topology of uniform convergence.

Proposition 9.3. *If $Emb_\varepsilon(X,Y)$ is dense in $\mathscr{C}(X,Y)$ with respect to the uniform topology, for all $\varepsilon > 0$, then there exists an embedding of X in Y.*

Proof. The space $\mathrm{Emb}(X,Y)$ of all embeddings of X in Y can be written as

$$\mathrm{Emb}(X,Y) = \cap_n \mathrm{Emb}_{1/n}(X,Y)$$

where the intersection is over all positive integers. Since (Y,d) is complete, Theorem 7.25 implies that $(\mathscr{C}(X,Y),d_{\sup})$ is complete. By Theorem 5.3, it has the Baire property. Hence, it suffices to show that the spaces $\mathrm{Emb}_\varepsilon(X,Y)$ are open in $\mathscr{C}(X,Y)$ (and then it follows not only that $\mathrm{Emb}(X,Y)$ is non-empty, but actually dense in $\mathscr{C}(X,Y)$).

So, let $\varepsilon > 0$ and we show that $\mathrm{Emb}_\varepsilon(X,Y)$ is open. Let $f \in \mathrm{Emb}_\varepsilon(X,Y)$ arbitrary; we are looking for δ s.t.

$$B_{d_{\sup}}(f,\delta) = \left\{g \in \mathscr{C}(X,Y) : d_{\sup}(g,f) < \delta\right\}$$

is inside $\mathrm{Emb}_\varepsilon(X,Y)$. We first claim that there exists δ such that

$$d(f(x),f(y)) < 2\delta \Longrightarrow d(x,y) < \varepsilon. \tag{9.1.1}$$

If no such δ exists, we would find sequences (x_n) and (y_n) in X with $d(f(x_n),f(y_n)) \to 0,\ \ d(x_n,y_n) \geq \varepsilon$. Hence (as we have already done several times by now), after eventually passing to convergent subsequences, we may assume that (x_n) and (y_n) are convergent, with limits denoted x and y. It follows that

$$d(f(x),f(y)) = 0,\ \ d(x,y) \geq \varepsilon,$$

which is in contradiction with $f \in \mathrm{Emb}_\varepsilon(X,Y)$. Hence we do find δ satisfying (9.1.1). We claim that δ has the desired property; hence let $g \in B_{d_{\sup}}(f,\delta)$ and we prove that $g \in \mathrm{Emb}_\varepsilon(X,Y)$. Note that

$$\delta(g) = \sup\left\{d(x,x') : (x,x') \in K(g)\right\},$$

where $K(g)$ consists of pairs (x,x') with $g(x) = g(x')$. Since g is continuous, $K(g)$ is closed in $X \times X$; since X is compact, $K(g)$ is compact. Hence the supremum is attained at some pair $(x,x') \in K(g)$. For this pair, the triangle inequality gives $d(f(x),f(x')) \leq d(f(x),g(x)) + d(g(x),g(x')) + d(g(x'),f(x')) < 2\delta$. By (9.1.1), $d(x,x') < \varepsilon$; hence $\delta(g) < \varepsilon$, as required.

9.2 Using covers and partitions of unity

In this section we assume that (X,d) is a compact metric space and $Y=\mathbb{R}^N$ is endowed with the Euclidean metric (where $N\geq 1$ is some integer). For the resulting embedding problem, we use the result of the previous section. We fix

$$f\in\mathscr{C}(X,\mathbb{R}^N),\ \varepsilon,\delta>0$$

and we search for $g\in\mathscr{C}(X,Y)$ with $\delta(g)<\varepsilon$, $d_{\sup}(f,g)<\delta$. The idea is to look for g of type

$$g(x)=\sum_{i=1}^{p}\eta_i(x)z_i, \tag{9.2.1}$$

where $\{\eta_i\}$ is a continuous partition of unity and $z_i\in\mathbb{R}^N$ some points. To control $\delta(g)$, the points z_i have to be chosen in "the most general" position.

Definition 9.4. We say that a set $\{z_1,\dots,z_p\}$ of points in $\mathbb{R}^N$ is in the general position if, for any $\lambda_1,\dots,\lambda_p\in\mathbb{R}$ from which at most $N+1$ are non-zero, one has:

$$\sum_{i=1}^{p}\lambda_i z_i=0,\ \sum_{i=1}^{p}\lambda_i=0\implies\lambda_i=0\ \ \forall\, i\in\{1,\dots,p\}.$$

We now return to our problem. Recall that, for a subset A of a metric space (X,d), $\operatorname{diam}(A)$ is $\sup\{d(a,b):a,b\in A\}$. For a family $\mathscr{A}=\{A_i:i\in I\}$, denote by $\operatorname{diam}(\mathscr{A})$ the supremum of $\{\operatorname{diam}(A_i):i\in I\}$. In the following, we control $\delta(g)$.

Lemma 9.5. *Let $\mathscr{U}=\{U_i\}$ be an open cover of X, $\{\eta_i\}$ a partition of unity subordinated to $\mathscr{U}$ and $\{z_i\}$ a set of points in $\mathbb{R}^N$ in general position, all indexed by $i\in\{1,\dots,p\}$. Assume that, for some integer m, each point in X lies in at most $m+1$ members of $\mathscr{U}$. If $N\geq 2m+1$ then the resulting function g given by (9.2.1) satisfies $\delta(g)\leq diam(\mathscr{U})$.*

Proof. Assume that $g(x)=g(y)$, i.e. $\sum_{i=1}^{p}(\eta_i(x)-\eta_i(y))z_i=0$. Now, x lies in at most $m+1$ members of $\mathscr{U}$, so at most $m+1$ numbers from $\{\eta_i(x):1\leq i\leq p\}$ are non-zero. Similarly for y. Hence at most $2(m+1)$ coefficients $\eta_i(x)-\eta_i(y)$ are non-zero. Note also that the sum of these coefficients is zero. Hence, since $\{z_1,\dots,z_p\}$ is in general position and $2(m+1)\leq N+1$, it follows that $\eta_i(x)=\eta_i(y)$ for all i. Choosing i such that $\eta_i(x)>0$, it follows that $x,y\in U_i$, hence $d(x,y)\leq\operatorname{diam}(U_i)$.

Next, we control $d_{\sup}(f,g)$. We use the notation $f(\mathscr{U})=\{f(U):U\in\mathscr{U}\}$.

Lemma 9.6. *Let $\mathscr{U}=\{U_i\}$ be an open cover of X, $\{\eta_i\}$ a partition of unity subordinated to $\mathscr{U}$ and $\{z_i\}$ a set of points in $\mathbb{R}^N$, all indexed by $i\in\{1,\dots,p\}$. Assume that, for some $r>0$,*

$$diam(f(\mathscr{U}))<r,\quad d(z_i,f(U_i))<r\ \ \forall\, i\in\{1,\dots,p\}.$$

Then the resulting function g given by (9.2.1) satisfies $d_{sup}(f,g)<2r$.

Proof. Since $d(z_i,f(U_i))<r$ we find $x_i\in U_i$ with $||z_i-f(x_i)||<r$. Writing

$$g(x)-f(x)=\sum_i\eta_i(x)(z_i-f(x_i))+\sum_i\eta_i(x)(f(x_i)-f(x)),$$

$$||g(x)-f(x)||\leq\sum_i\eta_i(x)||z_i-f(x_i)||+\sum_i\eta_i(x)||f(x_i)-f(x)||.$$

Here each $||z_i - f(x_i)|| < r$ by hypothesis, hence the first sum is $< r$. For the second sum note that, whenever $\eta_i(x) \neq 0$, we must have $x \in U_i$ hence, $||f(x_i) - f(x)|| < r$. Hence also the second sum is $< r$, proving that $||g(x) - f(x)|| < 2r$ for all $x \in X$. Since X is compact, we have $d_{\sup}(f,g) < 2r$.

Next, we show the existence of "small enough" covers of X and points in $\mathbb{R}^N$ in general position.

Proposition 9.7. *For ε, $\delta > 0$ there exists an open cover $\mathscr{U} = \{U_i : 1 \leq i \leq p\}$ of X with*

$$diam(\mathscr{U}) < \varepsilon, \;\; diam(f(\mathscr{U})) < \delta/2.$$

Moreover, for any such cover, there exist points $\{z_1, \ldots, z_p\}$ in $\mathbb{R}^N$ in general position such that

$$d(z_i, f(U_i)) < \delta/2 \quad \forall \;\; i \in \{1, \ldots, p\}.$$

In particular, g given by (9.2.1) satisfies $\delta(g) < \varepsilon$ and $d_{sup}(f,g) < \delta$, provided $\mathscr{U}$ has the property that each point in X lies in at most $m+1$ members of $\mathscr{U}$, where m satisfies $N \geq 2m+1$.

Proof. For the first part we use that f is uniformly continuous and choose $r < \varepsilon$ such that

$$d(x,y) < r \Longrightarrow d(f(x), f(y)) < \frac{\delta}{4}.$$

Consider then the open cover of X by balls of radius $r/3$ (or any other arbitrarily smaller radius) and choose a finite subcover $\mathscr{U}$. Each member has diameter at most $2r/3 < \varepsilon$, and its image under f has diameter at most $\delta/4 < \delta/2$, as desired.

For the second part, we choose $x_i \in U_i$ arbitrary and set $y_i = f(x_i) \in \mathbb{R}^N$. We prove that, in general, for any finite set $\{y_1, \ldots, y_p\}$ of points in $\mathbb{R}^N$ and any $r > 0$, there exists a set $\{z_1, \ldots, z_p\}$ of points in general position such that $d(z_i, y_i) < r$ for all i. We proceed by induction on p. Assume the statement holds up to p and we prove it for $p+1$. So, let $\{y_1, \ldots, y_{p+1}\}$ be points in $\mathbb{R}^N$. From the induction hypothesis, we may assume that $\{y_1, \ldots, y_p\}$ is already in general position. For each $I \subset \{1, \ldots, p\}$ of cardinality at most N we consider the "hyperplane"

$$\mathscr{H}_I := \left\{ \sum_{i \in I} \lambda_i y_i : \lambda_i \in \mathbb{R}, \sum_{i \in I} \lambda_i = 1 \right\}.$$

Since $|I| \leq N$, each such hyperplane has empty interior (why?), hence so does their union $\cup_I \mathscr{H}_I$ taken over all Is as above. Hence $B(y_{p+1}, r)$ will contain an element z_{p+1} which is not in this union. It is not difficult to check now that $\{y_1, \ldots, y_p, z_{p+1}\}$ is in general position.

9.3 Dimension and open covers

As in the previous section, we fix a compact metric space (X,d) and $Y=\mathbb{R}^N$ with the Euclidean metric. We assume that $N=2m+1$ for some integer m. Proposition 9.7 almost completes the proof of the existence of an embedding of X in $\mathbb{R}^{2m+1}$; what is missing is to make sure that the covers $\mathscr{U}$ from the proposition can be chosen so that each point in X lies in at most $m+1$ members of $\mathscr{U}$. Note however that this is an important demand. After all, all that we have discussed applies to any compact metric space X (e.g. S^5) and any $\mathbb{R}^N$ (e.g $\mathbb{R}$!); this extra-demand is the only one placing a condition on N in terms of the topology of X. Actually, this is about "the dimension" of X.

Definition 9.8. Let X be a topological space, $m\in\mathbb{Z}_{\geq 0}$. We say that X has **dimension** less or equal to m, and we write $dim(X)\leq m$, if any open cover $\mathscr{U}$ admits an open refinement $\mathscr{V}$ of multiplicity $\text{mult}(\mathscr{V})\leq m+1$, i.e. with the property that each $x\in X$ lies in at most $m+1$ members of $\mathscr{V}$.

The dimension of X is the smallest m with this property.

With this, Proposition 9.7 and Proposition 9.3 give us immediately:

Corollary 9.9. *Any compact metric space X with $dim(X)\leq m$ can be embedded in $\mathbb{R}^{2m+1}$.*

Of course, this nice looking corollary is rather cheap at this point: it looks like we just defined the dimension of a space, so that the corollary holds. However, the definition of dimension given above is not at all accidental. By the way, did you ever think how to define the (intuitively clear) notion of dimension by making use only of the topological information? What are the properties of the opens that make $\mathbb{R}$ one-dimensional and $\mathbb{R}^2$ two-dimensional? You may then discover yourself the previous definition. Of course, one should immediately prove that $\dim(\mathbb{R}^N)$ is indeed N or, more generally, that any m-dimensional topological manifold X has $\dim(X)=m$. These are all true, but they are not easy to prove right away. What we will show here is that:

Theorem 9.10. *Any compact m-dimensional manifold X satisfies $dim(X)\leq m$.*

This will be enough to apply the previous corollary and deduce Theorem 9.1 from the beginning of this chapter. The rest of this section is devoted to the proof of this theorem. First, we have the following metric characterization of dimension:

Lemma 9.11. *Let (X,d) be a compact metric space and m an integer. Then $dim(X)\leq m$ if and only if, for each $\delta>0$, there exists an open cover $\mathscr{V}$ with $diam(\mathscr{V})<\delta$ and $mult(\mathscr{V})\leq m+1$.*

Proof. For the direct implication, start with the cover by balls of radius $\delta/3$ and choose a refinement $\mathscr{V}$ as in Definition 9.8. Then $\text{diam}(\mathscr{V})\leq 2\delta/3<\delta$ and $\text{mult}(\mathscr{V})\leq m+1$. For the converse, let $\mathscr{U}$ be an arbitrary open cover and let δ be a Lebesgue number for $\mathscr{U}$ (see Proposition 5.13). An open cover $\mathscr{V}$ as in the statement then refines $\mathscr{U}$ and has multiplicity at most $m+1$.

Lemma 9.12. *Any compact subspace $K\subset\mathbb{R}^N$ has $dim(K)\leq N$.*

Proof. For simplicity in notations, we assume that $N=2$. We will use the previous lemma. First, we consider the following families of opens in the plane:

- $\mathscr{U}_0$ consisting of the open unit squares with vertices in the integral points (m,n) $(m,n\in\mathbb{Z})$.
- $\mathscr{U}_1$ consisting of the open balls of radius $\frac{1}{2}$ with centers in the integral points.
- $\mathscr{U}_2$ consisting of the open balls of radius $\frac{1}{4}$ with centers in the middles of the edges of the integral lattice.

Make a picture!Note that the members of each of the families $\mathscr{U}_i$ are disjoint. Hence

$$\mathscr{U}:=\mathscr{U}_0\cup\mathscr{U}_1\cup\mathscr{U}_2$$

is an open cover of $\mathbb{R}^2$ of multiplicity 3 with $\operatorname{diam}(\mathscr{U}) = \sqrt{2}$. To obtain similar covers of smaller diameter, we rescale. For each $\lambda > 0$, $\phi_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto \lambda v$ is a homeomorphism. The rescaling of $\mathscr{U}$ is

$$\mathscr{U}^\lambda = \{\phi_\lambda(U) : U \in \mathscr{U}\},$$

it has multiplicity 3 and diameter $\lambda\sqrt{2}$. Now, for $K \subset \mathbb{R}^2$ compact, we use the covers $\mathscr{U}^\lambda$ (and the compactness of K) to apply the previous lemma.

Lemma 9.13. *If X is a topological space and $X = \cup_{i=1}^p X_i$ where each X_i is closed in X with $\dim(X_i) \le m$, then $\dim(X) \le m$.*

Proof. Proceeding inductively, we may assume $p = 2$, i.e. $X = Y \cup Z$ with Y, Z-closed in X of dimension $\le m$. Let $\mathscr{U}$ be an arbitrary open cover of X; we prove that it has a refinement of multiplicity $\le m+1$. First we claim that $\mathscr{U}$ has a refinement $\mathscr{V}$ such that each $y \in Y$ lies in at most $m+1$ members of $\mathscr{V}$. To see this, note that $\{U \cap Y : U \in \mathscr{U}\}$ is an open cover of Y, hence it has a refinement (covering Y) $\{Y_a : a \in A\}$ (for some indexing set A). For each $a \in A$, write $Y_a = Y \cap V_a$ with $V_a \subset X$ open, and choose $U_a \in \mathscr{U}$ such that $Y_a \subset U_a$. Then

$$\mathscr{V} := \{V_a \cap U_a : a \in A\} \cup \{U - Y : U \in \mathscr{U}\}$$

is the desired refinement. Re-index it as $\mathscr{V} = \{V_i : i \in I\}$ (we assume that there are no repetitions, i.e. $V_i \neq V_{i'}$ whenever $i \neq i'$). Similarly, let $\mathscr{W} = \{W_j : j \in J\}$ be a refinement of $\mathscr{V}$ with the property that each $z \in Z$ belongs to at most $m+1$ members of $\mathscr{W}$. For each $j \in J$, choose $\alpha(j) \in I$ such that $W_j \subset V_{\alpha(j)}$. For each $i \in I$, define

$$D_i = \cup_{j \in \alpha^{-1}(i)} W_j.$$

Consider $\mathscr{D} = \{D_i : i \in I\}$. Since for each $j \in J$, $i \in I$

$$W_j \subset D_{\alpha(j)}, \;\; D_i \subset V_i,$$

$\mathscr{D}$ is an open cover of X, which refines $\mathscr{V}$ (hence also $\mathscr{U}$). It suffices to show that $\operatorname{mult}(\mathscr{D}) \le m+1$. Assume that there exist k distinct indices $i_1, \ldots, i_k$ with

$$x \in D_{i_1}, \ldots, D_{i_k}.$$

We have to show that $k \le m+1$. If $x \in Y$, since $D_i \subset V_i$ for all i, the defining property of $\mathscr{V}$ implies that $k \le m+1$. On the other hand, for each $a \in \{1, \ldots, k\}$, since $x \in D_{i_a}$, we find $j_a \in \alpha^{-1}(i_a)$ such that $x \in W_{j_a}$; hence, if $x \in Z$, then the defining property of $\mathscr{W}$ implies that $k \le m+1$.

Proof. (end of the proof of Theorem 9.10) Since X is a manifold, around each $x \in X$ we find a homeomorphism $\phi_x : U_x \to \mathbb{R}^n$ defined on an open neighborhood U_x of x. Let $V_x \subset U_x$ corresponding (by ϕ_x) to the open ball of radius 1. From the open cover $\{V_x : x \in X\}$, extract an open subcover, corresponding to $x_1, \ldots, x_k \in X$. Denoting

$$X_i = \overline{V_{x_i}} = \phi_{x_i}^{-1}(\overline{B_1}),$$

each X_i is a closed subset of X homeomorphic to a closed ball of radius 1 and $X = \cup_i X_i$, hence has $\dim(X_i) \le m$.

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Inleiding Topologie

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- Utrecht Lecture Notes 2026-2027 -

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Chapter 10
Exercises on the chapter "Topological spaces"

10.1 On topologies

Exercise 10.1. How many distinct topologies can there be defined on a set with two elements? But with three?

Exercise 10.2. Prove that, for any set X, $\mathscr{T}_{\text{cf}}$ and $\mathscr{T}_{\text{dis}}$ are indeed topologies.

Exercise 10.3. Given a topology $\mathscr{T}$ on $\mathbb{R}$, show that $\mathscr{T}$ is the discrete topology iff $\{r\} \in \mathscr{T}$ for all $r \in \mathbb{R}$.

Exercise 10.4. On $\mathbb{R}$ consider the family $\mathscr{T}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of type $(-\infty, r)$ with $r \in \mathbb{R}$. Show that $\mathscr{T}$ is a topology on $\mathbb{R}$ and compare it with the Euclidean topology.

Exercise 10.5. On $\mathbb{R}$ consider the family $\mathscr{B}$ consisting of $\emptyset$, $\mathbb{R}$ and all intervals of type $(-\infty, r]$ with $r \in \mathbb{R}$. Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$ and find the smallest topology containing $\mathscr{B}$. Is it larger or smaller than the topology from the previous exercise? But than the Euclidean topology?

Exercise 10.6. This is to recap the lower limit topology $\mathscr{T}_l$ on $\mathbb{R}$ from Example 2.9. Recall that it was defined as follows: a subset $D \subset \mathbb{R}$ belongs to $\mathscr{T}_l$ if and only if the following condition holds: for any $x \in D$ there exists an interval $[a, b)$ such that

$$x \in [a, b) \subset D.$$

Show that:

(a) $\mathscr{T}_l$ the smallest topology on $\mathbb{R}$ which contains all the intervals of type $[a, b)$ with $a, b \in \mathbb{R}$
(b) replacing the intervals of type $[a, b)$ by intervals of type $(a, b]$, one obtains a similar topology $\mathscr{T}_u$ (why u?).
(c) $\mathscr{T}_l$ and $\mathscr{T}_u$ are finer than $\mathscr{T}_{\text{eucl}}$, but $\mathscr{T}_l$ and $\mathscr{T}_u$ are not comparable.
(d) $\mathscr{T}_{\text{dis}}$ is the only topology on $\mathbb{R}$ which contains both $\mathscr{T}_l$ and $\mathscr{T}_u$.

Exercise 10.7. On $\mathbb{R}^2$ we define the topology $\mathscr{T}_l \times \mathscr{T}_l$ as the smallest topology which contains all subsets of type

$$[a, b) \times [c, d)$$

with $a, b, c, d \in \mathbb{R}$. Define similarly $\mathscr{T}_u \times \mathscr{T}_u$, $\mathscr{T}_l \times \mathscr{T}_u$ and $\mathscr{T}_u \times \mathscr{T}_l$. Show that any two of these are not comparable.

Exercise 10.8. Consider the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (the set of strictly positive integers to which we add the infinity) and, for each $n \in \mathbb{N}$ put

$$U_n := \left\{k \in \overline{\mathbb{N}} : k \geq n\right\} = \{n, n+1, \ldots\} \cup \{\infty\}.$$

Show that the following is a topology on $\overline{\mathbb{N}}$:

$$\mathscr{T}_{\text{seq}} := \{\emptyset, U_1, U_2, U_3, \ldots\}.$$

Exercise 10.9. Let (X,d) be a metric space, A, B subspaces of X such that

$$d(A,B) = 0.$$

(where $d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$). Is it true that A and B must have a common point (i.e. $A \cap B \neq \emptyset$)? What if we assume that both A and B are closed?

Exercise 10.10. Consider a set X, a set of indices I and, for each $i \in I$, a topology $\mathscr{T}_i$ on X. Show that $\mathscr{T} := \cap_i \mathscr{T}_i$ (i.e. the family consisting of subsets $U \subset X$ with the property that $U \in \mathscr{T}_i$ for all $i \in I$) is a topology on X.

Exercise 10.11. Given a set X and a family $\mathscr{S}$ of subsets of X, prove that there exists a topology $\mathscr{T}(\mathscr{S})$ on X which contains $\mathscr{S}$ and is the smallest with this property. (Hint: use the axioms to see what other subsets of X, besides the ones from $\mathscr{S}$, must $\mathscr{T}(\mathscr{S})$ contain.)

10.2 On induced topologies

Exercise 10.12. Consider the real line $\mathbb{R}$ as a subset of the plane $\mathbb{R}^2$. Show that the induced topology on $\mathbb{R}$ coincides with the Euclidean topology on the real line.

Exercise 10.13. To remove ambiguities regarding the Convention 2.13 show that, inside any topological space $(X, \mathscr{T})$, for all $A \subset Y \subset X$ one has

$$(\mathscr{T}|_Y)|_A = \mathscr{T}|_A.$$

To remove ambiguities regarding Convention 2.8 and 2.13 show that, for $A \subset \mathbb{R}^n$, the Euclidean topology on A coincides with the restriction to A of the Euclidean topology of $\mathbb{R}^n$.

Exercise 10.14. Let $(X, \mathscr{T})$ be a topological space and $B \subset A \subset X$.

(a) If A is open in X, show that B is open in X if and only if it is open in A.
(b) If A is closed in X, show that B is closed in X if and only if it is closed in A.

Exercise 10.15. Find an example of a topological space $(X, \mathscr{T})$ and $A \subset B \subset X$ such that A is closed in B, B is open in X, and A is neither open nor closed in X.

10.3 On continuity

Exercise 10.16. Consider

$$D := \{(x,y) : e^x > \sin(y)\cos(x)\},$$

$$A := \left\{(x,y) : x^7 - \sin(y^7) \geq \frac{1}{x^2+y^2+1}\right\}.$$

Show that D is open in $\mathbb{R}^2$, while A is closed in $\mathbb{R}^2$.

Exercise 10.17. Let $\mathbb{R}$ be endowed with the lower limit topology $\mathscr{T}_l$. Which one of the following functions $f:\mathbb{R}\to\mathbb{R}$ is continuous:

(a) $f(x)=x+1$
(b) $f(x)=-x$.
(c) $f(x)=x^2$.

Exercise 10.18. If (X,d) is a metric space and A is a subspace of X, then the function

$$d_A : X \to \mathbb{R},\ \ d_A(x) = d(x,A)$$

is continuous. Deduce that, for any closed subset A of a metric space X, there exists a continuous function $f:X\to[0,1]$ such that $A=f^{-1}(0)$.

Exercise 10.19. (from the 2021 retake exam) Let $(X,\mathscr{T}_X)$ and $(Y,\mathscr{T}_Y)$ be topological spaces and $A\subset Y$ be a subset equipped with the subspace topology; we denote by $i:A\to Y$ the inclusion. Show that a function $f:X\to A$ is continuous if and only if the composition $i\circ f:X\to Y$ is continuous.

Exercise 10.20. Given a topological space $(X,\mathscr{T})$ and $A\subset X$, show that the induced topology $\mathscr{T}|_A$ is the smallest topology on A with the property that the inclusion map $i:A\to X$ is continuous.

Exercise 10.21. (from the 2021 exam) Given a topological space $(X,\mathscr{T})$, and we use $[0,1]$ endowed with the Euclidean topology, show that if $\gamma_1,\gamma_2:[0,1]\to X$ are continuous and $\gamma_1(1)=\gamma_2(0)$, then

$$\gamma:[0,1]\to X,\ \ \gamma(t)=\begin{cases}\gamma_1(2t) & \text{if } t\in[0,\frac{1}{2}]\\ \gamma_2(2t-1) & \text{if } t\in(\frac{1}{2},1]\end{cases}$$

is continuous.

Exercise 10.22. (from the 2024 exam) Assume that $(X,\mathscr{T})$ is a topological space. We say that $X=A_1\cup A_2$ is a *decomposition* of $(X,\mathscr{T})$ if A_1,A_2 are subsets of X whose union is the entire X. For any such decomposition we endow A_1 and A_2 with the induced topology and we associate to it the following collection of subsets of X:

$$\mathscr{T}(A_1,A_2) := \{U\subset X : U\cap A_1 \text{ is open in } A_1 \text{ and } U\cap A_2 \text{ is open in } A_2\}.$$

(a) Show that $\mathscr{T}(A_1,A_2)$ is a topology on X and $\mathscr{T}\subset\mathscr{T}(A_1,A_2)$.

Next, a decomposition $X=A_1\cup A_2$ will be called *convenient* if $\mathscr{T}=\mathscr{T}(A_1,A_2)$. Please show that:

(b) If $X=A_1\cup A_2$ is a convenient decomposition of X then, for any function $f:X\to Y$ with values in another topological space Y, one has:

$$f \text{ is continuous} \iff f|_{A_1}:A_1\to Y \text{ and } f|_{A_2}:A_2\to Y \text{ are continuous.}$$

(c) If A_1 and A_2 are both open, or both closed, in X, then $X=A_1\cup A_2$ is a convenient decomposition.
(d) When $X=\mathbb{R}$ endowed with the Euclidean topology $\mathscr{T}_{\text{Eucl}}$, $\mathbb{R}=(-\infty,0)\cup[0,\infty)$ is not a convenient decomposition.

Similar exercises:

- *Exercise 16.26 from the 2021 exam*
- *Exercise 16.39 from the 2022 exam.*

Exercise 10.23. * Show that any continuous map

$$f : (\mathbb{R}, \mathscr{T}_{\text{Eucl}}) \to (\mathbb{R}, \mathscr{T}_l)$$

must be constant (recall that $\mathscr{T}_l$ is the lower limit topology- see Exercise 10.6).

10.4 On homeomorphisms and embeddings

Exercise 10.24. From the topologies that you found in Exercise 10.1, how many non homeomorphic ones are there?

Exercise 10.25. On $\mathbb{R}$ consider the topologies $\mathscr{T}_l$ and $\mathscr{T}_u$ (see Exercise 10.6). Show that $(\mathbb{R}, \mathscr{T}_l)$ and $(\mathbb{R}, \mathscr{T}_u)$ are homeomorphic.

Exercise 10.26. Show that $\mathbb{R}$ endowed with the Euclidean topology is not homeomorphic to $\mathbb{R}$ endowed with the topology from Exercise 10.4.

Exercise 10.27. Let $f\colon (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a continuous map between topological spaces. We say that f is

- *open* if for every open subset $U \subseteq X$, the subset $f(U) \subseteq Y$ is open.
- *closed* if for every closed subset $A \subseteq X$, the subset $f(A) \subseteq Y$ is closed.
- *local homeomorphism* if for each $x \in X$ there exists an open neighborhood U of x in X, and V of $f(x)$ in Y such that the restriction of f to U becomes a homeomorphism $f|_U : U \to V$.

Show that if $f\colon (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a continuous bijective map then

$$f \text{ is a homeomorphism} \iff f \text{ is an open map} \iff f \text{ is a closed map} \iff f \text{ is a local homeomorphism.}$$

Exercise 10.28. Show that, for any $n \geq 1$ integer,

$$\underbrace{S^1 \times \ldots \times S^1}_{n-\text{times}} \subset \underbrace{\mathbb{R}^2 \times \ldots \times \mathbb{R}^2}_{n-\text{times}} = \mathbb{R}^{2n}$$

can be embedded in $\mathbb{R}^{n+1}$. First prove it without using explicit formulas, then by explicit formulas.

Exercise 10.29. Exhibit an embedding $f : M \to M$ of the Moebius band into itself which is not surjective. What is the boundary of $f(M)$ in M?

Exercise 10.30. (from the 2016 exam) Consider

$$X = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, \text{ or } x = 0\right\},$$

$$Y = \left\{(x,y) \in \mathbb{R}^2 : xy = 0\right\},$$

both endowed with the Euclidean topology. Show that Y can be embedded in X, You do not have to provide explicit formulas for the embedding but please use pictures to describe the embedding and please provide explanations to support your claims.

Exercise 10.31. Show that for any $A \subset S^2$ nonempty, $S^2 \setminus A$ can be embedded in $\mathbb{R}^2$.

Exercise 10.32. Show that if n is an odd number, then $GL_n(\mathbb{R})$ is homeomorphic to $GL_1(\mathbb{R}) \times SL_n(\mathbb{R})$ (see Exercise 10.72).

10.5 The "removing a point trick"

The first part of the following exercise is extremely useful for some of the later exercises.

Exercise 10.33. If X and Y are homeomorphic, prove that for any $x \in X$ there exists $y \in Y$ such that $X - \{x\}$ is homeomorphic to $Y - \{y\}$.

Also explain that, "there exists $y \in Y$" cannot be replaced with "for any $y \in Y$".

10.6 On convergence

Exercise 10.34. Let $X = (0,1)$ endowed with the Euclidean topology. Is the sequence

$$x_n = \frac{1}{n+1}$$

convergent in the topological space X?

Exercise 10.35. Let $\mathbb{R}$ be endowed with the lower limit topology $\mathscr{T}_l$. Study the convergence of the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ where

$$x_n = \frac{1}{n},\ y_n = -\frac{1}{n}.$$

Exercise 10.36. On $X = \mathbb{R}$ consider the topology $\mathscr{T} = \{(-a,a) : 0 \leq a \leq \infty\}$. Is the sequence

$$x_n = (-1)^n + \frac{1}{n}$$

convergent in $(X, \mathscr{T})$? To what?

Exercise 10.37. What about convergence in $(X, \mathscr{T}_{\text{cf}})$. But about $(X, \mathscr{T}_{\text{cc}})$?

Exercise 10.38. We say that two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on X have the same convergence of sequences if, for a sequence $(x_n)_{n\geq 1}$ of elements of X, and $x \in X$, one has:

$$x_n \xrightarrow{\mathscr{T}_1} x \Longleftrightarrow x_n \xrightarrow{\mathscr{T}_2} x.$$

Show that

- If $\mathscr{T}_1$ and $\mathscr{T}_2$ satisfy the first countability axiom and have the same convergence of sequences, then $\mathscr{T}_1 = \mathscr{T}_2$.
- This is no longer true if one gives up the first countability axiom.

(Hint for the second part: This is difficult. Try $\mathscr{T}_{\text{discr}}$ and $\mathscr{T}_{\text{cf}}$. If it does not work, try to change one of them).

10.7 On closure, interior, etc

Exercise 10.39. Let $X = (-\infty,1)\cup(1,4)\cup[5,\infty)$. Find the closure, the interior and the boundary (in X) of $A = [0,1)\cup(1,2)\cup[3,4)\cup(5,6)$.

Exercise 10.40. Find the closure, the interior and the boundary of the following subsets of the plane:

(a) $\{(x,y) : x \geq 0, y \neq 0\}$.
(b) $\{(x,y) : x \in \mathbb{Q}, y > 0\}$.

Exercise 10.41. Show that the following three spaces are homeomorphic (giving the explicit homeomorphisms):

$$X = \left\{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 \leq 1\right\} \subset \mathbb{R}^2$$

$$Y = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2+y^2 \leq 1\right\} \subset \mathbb{R}^2$$

$$Z = S^1 \times (0,1] = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, 0 < z \leq 1\right\} \subset \mathbb{R}^3.$$

Then compute the interiors and the boundaries of X, Y (in $\mathbb{R}^2$) and Z (in $\mathbb{R}^3$). How comes that, although X, Y and Z are homeomorphic, their interiors and boundaries are quite different?

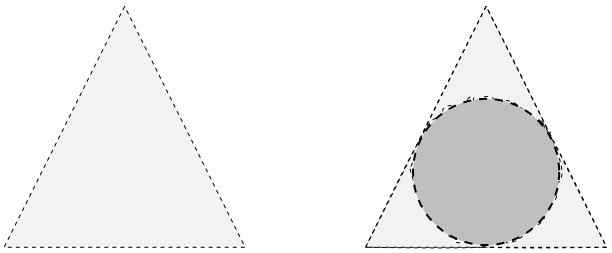
Exercise 10.42. Find the interior and the closure of $\mathbb{Q}$ in $\mathbb{R}$ in each of the cases: when $\mathbb{R}$ is endowed with

- the Euclidean topology.
- the discrete topology.
- the cofinite topology.
- the co-countable topology.

Exercise 10.43. Let $\mathscr{T}_l$ be the lower limit topology on $\mathbb{R}$. In the topological space $(\mathbb{R}, \mathscr{T}_l)$, find the closure, the interior and the boundary of each of the intervals

$$[0,1), (0,1], (0,1), [0,1], (0,1)\cup[2,3].$$

Exercise 10.44. (from a 2010 exam) Let X be the (interior of an) open triangle, as drawn in the picture (the edges are not part of X!), viewed as a topological space with the topology induced from $\mathbb{R}^2$. Let $A \subset X$ be the open disk drawn in the picture (tangent to the edges of the closed triangle). Compute the closure and the boundary of A in X.



X= an open triangle A= an open disk inside X

Fig. 10.1

Exercise 10.45. Compute the interior, the closure and the boundary of

$$A = (0,1] \times [0,1)$$

in the topological space $X = \mathbb{R}^2$ endowed with the topology $\mathscr{T}_l \times \mathscr{T}_l$ of Exercise 10.7.

Exercise 10.46. Let A, B and $\{A_i : i \in I\}$ denote subsets of a topological space X, where i runs in a set of indexes I. Prove that

(a) If $A \subset B$ then $\overline{A} \subset \overline{B}$.
(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) If I is finite then $\overline{\cup_i A_i} = \cup_i \overline{A}_i$.
(d) In general, when I is infinite, $\overline{\cup_i A_i} \supset \cup_i \overline{A}_i$, but the two may be different.
(e) We say that $\{A_i : i \in I\}$ is locally finite if any point $x \in X$ admits a neighborhood V which intersects only finitely many subsets A_is (i.e. such that $\{i \in I : A_i \cap V \neq \emptyset\}$ is finite). Under this assumption, show that $\overline{\cup_i A_i} = \cup_i \overline{A}_i$.

Exercise 10.47. Let A, B be two subsets of a topological space X. Recall that $\mathrm{Int}(A) = \overset{\circ}{A}$ denotes the interior of A. Prove that

(a) If $A \subset B$ then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.
(b) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.
(c) $\mathrm{Int}(A \cup B) \supset \mathrm{Int}(A) \cup \mathrm{Int}(B)$, but the equality may fail.

Then show that the closure and the interior are related as follows:

$$X \setminus \overline{A} = \mathrm{Int}(X \setminus A), \quad X \setminus \mathrm{Int}(A) = \overline{X \setminus A}$$

and, using this, relate the first items of this exercise to the ones from the previous exercise.

10.8 Density

Exercise 10.48. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 10.49. Show that $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$.

Exercise 10.50. Let T be the torus. Describe (on the picture) a continuous injection $f : \mathbb{R} \to T$ whose image is dense in T. Is f an embedding?

Exercise 10.51. Show that $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$ (see Exercise 10.72).

Exercise 10.52. Show that any continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, then f must be linear, i.e. there exists $a \in \mathbb{R}$ such that

$$f(x) = ax \quad \forall\, x \in \mathbb{R}.$$

(Hint: $a = f(1)$. For what x's can you prove that $f(x) = ax$? Then use continuity and the fact $\mathbb{Q}$ is dense in $\mathbb{R}$.)

10.9 On Hausdorffness, 2nd countability, separation

Exercise 10.53. How many from the topologies from Exercise 10.1 are Hausdorff?

Exercise 10.54. Is the topology from Exerc. 10.8 Hausdorff? But from 10.4? But from 10.5?

Exercise 10.55. Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Which of the following holds true:

(a) if $(X, \mathscr{T}_1)$ is Hausdorff then so is $(X, \mathscr{T}_2)$.
(b) if $(X, \mathscr{T}_2)$ is 2nd countable then so is $(X, \mathscr{T}_1)$.
(c) if $(X, \mathscr{T}_2)$ is 1st countable then so is $(X, \mathscr{T}_1)$.

Exercise 10.56. Show that, in any Hausdorff space X, all the subspaces with one element (i.e. of type $A = \{x\}$ with $x \in X$) are closed.

Example 10.57. Consider $\mathbb{R}$ endowed with the lower limit topology (see (2.1.2)). Show that

$$\mathscr{B}' := \{[p,q) : p,q \in \mathbb{Q}\}$$

is not a basis for $(\mathbb{R}, \mathscr{T}_l)$. Then try to prove that $(\mathbb{R}, \mathscr{T}_l)$ is not 2nd countable. Prove however that it is 1st countable.

Exercise 10.58. Consider $\mathbb{R}$ with the Euclidean topology. In each of the following cases, decide (and explain!) when A and B can be separated topologically or by continuous functions:

(a) $A = [0,1)$, $B = (1,2]$.
(b) $A = [0,1)$, $B = [1,2]$.
(c) $A = [0,1)$, $B = (2,3]$.

Exercise 10.59. (from the 2012 retake exam) Let X be a Hausdorff normal space. Show that, for $A, B \subset X$, A and B have the same closure in X if and only if, for any continuous function $f : X \to \mathbb{R}$, one has the equivalence

$$f|_A = 0 \iff f|_B = 0.$$

Exercise 10.60. Prove that the sphere S^n is an n-dimensional topological manifold. Show that it can be covered by two coordinate charts. Compute the change of coordinates.

Exercise 10.61. Let X be a 2nd countable space. Show that from any open cover $\mathscr{U}$ of X one can extract a countable subcover. In other words, for any collection $\mathscr{U} = \{U_i : i \in I\}$ consisting of opens in X such that $X = \cup_{i \in I} U_i$, one can find $i_1, i_2, \ldots \in I$ such that $X = \cup_k U_{i_k}$.

10.10 Mixed exercises

Exercise 10.62. Consider the following subsets of the plane:

(a) $A = \{(x,y) : x \geq 0\}$.

(b) $B = \{(x,y) : x = 0\}$.
(c) $C = \{(x,y) : x > 0, y < 5\}$.
(d) $D = \{(x,y) : xy < 1, x \geq 0\}$.
(e) $E = \{(x,y) : 0 \leq x < 5\}$.

Please answer the following:

(a) Which one of these is open in the plane?
(b) For each one of them, find the interior, the closure and the boundary in the plane.
(c) Notice that all of them are contained in A. Which ones are also open in A?
(d) For each one of them, find the interior, the closure and the boundary in A.

Exercise 10.63. Assume that $(X, \mathscr{T})$ is a topological space, $A \subset X$ and we endow A with the induced topology $\mathscr{T}|_A$. Show that:

1. if $(X, \mathscr{T})$ is Hausdorff, then so is $(A, \mathscr{T}|_A)$.
2. if $(X, \mathscr{T})$ is metrizable, then so is $(A, \mathscr{T}|_A)$.
3. if $(X, \mathscr{T})$ is second countable, then so is $(A, \mathscr{T}|_A)$.

Exercise 10.64. (from the 2010 exam) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathscr{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$.
- $\mathscr{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n, n+1, \ldots\}$ with $n \in \mathbb{N}$.

Questions:

1. Show that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on $\mathbb{N}$.
2. Show that the spaces $(\mathbb{N}, \mathscr{T}_1)$ and $(\mathbb{N}, \mathscr{T}_2)$ are not homeomorphic.

Exercise 10.65. (from the 2014 exam) For $\mathbb{R}$ we consider the family $\mathscr{S}$ of subsets consisting of all the intervals of type (m, M) with $m < M < 0$, all intervals of type (m, M) with $0 < m < M$ (m and M real numbers) and the interval $[-1, 1)$. Denote by $\mathscr{T}$ the smallest topology on $\mathbb{R}$ containing $\mathscr{S}$.

a. describe a basis $\mathscr{B}$ of $(\mathbb{R}, \mathscr{T})$ with $\mathscr{B} \neq \mathscr{T}$. Why taking $\mathscr{B} = \mathscr{S}$ does not work?
b. is $(\mathbb{R}, \mathscr{T})$ Hausdorff? Is it second countable?
c. find an interval of type $[a, b]$ whose closure inside $(\mathbb{R}, \mathscr{T})$ is not an interval.
d. find an interval of type (a, b) whose interior inside $(\mathbb{R}, \mathscr{T})$ is not an interval.
g. (+) consider

$$f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}),\ f(x) = \begin{cases} 0 \text{ if } x < -1 \\ 1 \text{ if } x \geq -1 \end{cases}$$

Is f continuous? Is f sequentially continuous?

Exercise 10.66. (from the 2024 exam) Let $\mathscr{T}_{\text{new}}$ be the topology on $\mathbb{R}$ consisting of those subsets $U \subset \mathbb{R}$ with the property that $U \cap (-\infty, 0)$ is open in $(-\infty, 0)$ and $U \cap [0, \infty)$ is open in $[0, \infty)$.

(a) Compute the closure of $A = (-1, 0)$ and the interior of $B = [0, 1]$ in $(\mathbb{R}, \mathscr{T}_{\text{new}})$.
(b) Is the sequence given by $x_n := \frac{(-1)^n}{n}$ convergent in the topological space $(\mathbb{R}, \mathscr{T}_{\text{new}})$?
(c) For what values of $\lambda \in \mathbb{R}$ is the following function

$$f_\lambda : (\mathbb{R}, \mathscr{T}_{\text{new}}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \quad f_\lambda(x) = \begin{cases} x & \text{if } x < 0 \\ \lambda & \text{if } x = 0 \\ x + \lambda^2 & \text{if } x > 0 \end{cases}$$

continuous?

(d) Does there exists λ such that f_λ is an embedding?

More exercises similar to the last two:

- *Exercise 16.13 from the 2024 exam, without items a and c.*
- *Exercise 16.43 from the 2016 exam, without items f, g and h.*
- *Exercise 11.41, without the first part of item a and without item c.*
- *Exercise 16.48 from the 2015 exam without item d.*
- *Exercise 16.83 from the 2012 exam*
- *Exercise 16.71 from the 2013 exam.*
- *Exercise 16.53 from the 2015 retake, where instead of items a and b you do: "Prove that for $k \geq 1$ the smallest topology on $\mathbb{R}$ containing $\mathscr{B}_k$ is the Euclidean topology" and $\mathscr{T}_0$ is the smallest topology containing $\mathscr{B}_0$. Also, ignore h.*
- *Exercise 16.100 from the 2011 exam A, all items except the first two (replace those by the question: prove that $\mathscr{T}$ is a topology) and except the question on connectedness.*

Exercise 10.67. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and let

$$f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$$

be a continuous map. Define $f^* \mathscr{T}_Y$ as the smallest topology on X with the property that

$$f : (X, f^* \mathscr{T}_Y) \to (Y, \mathscr{T}_Y)$$

is continuous. Show that

$$f^* \mathscr{T}_Y = \left\{ f^{-1}(V) : V \in \mathscr{T}_Y \right\}$$

and then, assuming that $(X, \mathscr{T}_X)$ is Hausdorff, show that

$$(f^* \mathscr{T}_Y = \mathscr{T}_X) \iff (f \text{ is an embedding}).$$

Similar exercise: Exercise 16.64 from the 2014 retake.

Exercise 10.68. Given a set X, show that any Hausdorff topology on X contains $\mathscr{T}_{\text{cf}}$. When is $\mathscr{T}_{\text{cf}}$ Hausdorff? When does it exist a smallest Hausdorff topology on X (i.e. a Hausdorff topology which is contained in all other Hausdorff topologies on X)?

Exercise 10.69. Show that if a space $(X, \mathscr{T})$ can be embedded in some Euclidean space $\mathbb{R}^n$, then $(X, \mathscr{T})$ is metrisable and must have the following three topological properties: it is Hausdorff, 2nd countable and normal.

Exercise 10.70. Let X be a Hausdorff normal space and let $A \subset X$ be a subspace with the property that any two continuous functions $f, g : X \to \mathbb{R}$ which coincide on A must coincide everywhere on X. Show that A is dense in X (i.e. the closure of A in X coincides with X).

Exercise 10.71. (from the 2022 exam) Give an example of a topological space $(X, \mathscr{T})$ such that for each $x \in X$ there exists a neighborhood U of x such that U is homeomorphic to $\mathbb{R}^2$, but X is not Hausdorff.

Exercise 10.72. The space $M_n(\mathbb{R})$, of $n \times n$ matrices with real coefficients can be identified with $\mathbb{R}^{n^2}$ and in this way has a natural topology (coming from the Euclidean metric). Prove that:

(a) the subspace $GL_n(\mathbb{R})$ of invertible matrices is open in $M_n(\mathbb{R})$.
(b) the subspace $SL_n(\mathbb{R})$ consisting of invertible matrices of determinant equal to 1 is closed in $GL_n(\mathbb{R})$.
(c) the subspace

$$O(n) := \{A \in GL_n(\mathbb{R}) : AA^* = \mathrm{Id}\}$$

is also closed (where A^* denotes the transpose of A and Id is the identity matrix).

More (similar) exercises can be found in previous exams as follows:

- On *topologies*: Exercise 16.37 from the 2022 exam.
- On *topologies, Hausdorffness*: Exercise 16.22 from the 2021 retake, item (a) and first half of item (b).
- On *topologies, homeomorphisms, interiors and closures*: Exercise 16.38 from the 2022 exam, except for the bulleted items.
- On *continuity, induced topologies*: Exercise 16.39 from the 2022 exam.

Chapter 11
Exercises on the chapter "Constructions of topological spaces"

11.1 General quotients/gluings

Exercise 11.1. Recall that a map $f\colon X \to Y$ is said to be open if for every open subset $U \subseteq X$, $f(U)$ is open in Y. Show that any surjective continuous open map is automatically a topological quotient map.

Exercise 11.2. Give an example of a topological quotient map which is not an open map.

Exercise 11.3. Start with the interval $[0,2]$ and glue the points 0, 1 and 2. Describe the equivalence relation R encoding this gluing and let $X = [0,2]/R$. Describe an embedding of X in $\mathbb{R}^2$.

Exercise 11.4. Let $X \subset \mathbb{R}^2$ like in Figure 11.1 (the shaded region). Describe an equivalence relation on X such that the result of the gluing is a sphere S^2. Then another one to get a torus. Then one that gives a disjoint union of a sphere and a torus.

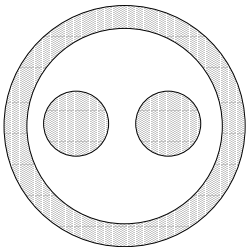


Fig. 11.1

Exercise 11.5. (part of an exercise from a 2010 exam) Let X be obtained by taking two disjoint copies of the interval $[0,2]$ (with the Euclidean topology) and gluing each t in the first copy with the corresponding t in the second copy, for all $t \in [0,2]$ different from the middle point. Explicitly, one may take the space

$$Y = [0,2] \times \{0\} \cup [0,2] \times \{1\} \subset \mathbb{R}^2$$

with the topology induced from the Euclidean topology, and X is the space obtained from Y by gluing $(t,0)$ to $(t,1)$ for all $t \in [0,2]$, $t \neq 1$. We endow X with the quotient topology.

(a) Is X Hausdorff?
(b) Show that X can also be obtained as a quotient of the circle S^1.

Exercise 11.6. (from the 2012 exam A) Let X be the space obtained from the sphere S^2 by gluing the north and the south pole (with the quotient topology). Show that X can be obtained from a square $[0,1]\times[0,1]$ by glueing some of the points on the *boundary* (note: you are not allowed to glue a point in the *interior* of the square to any other point). More precisely:

(a) Describe the equivalence relation R_0 on S^2 encoding the glueing that defines X.
(b) Make a picture of X in $\mathbb{R}^3$.
(c) Describe an equivalence relation R on $[0,1]\times[0,1]$ encoding a glueing with the required properties.
(d) Show that, indeed, X is homeomorphic to $[0,1]\times[0,1]/R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture).

Exercise 11.7. The aim of this exercise is to explain and show that $\mathbb{P}^2$ can be obtained from a Moebius band by gluing along its boundary circle a disk.

(a) Write the Moebius band as a union of two subspaces M and C where M is itself a Moebius band, C is a cylinder (i.e. homeomorphic to $S^1\times[0,1]$) and $M\cap C$ is a circle.
(b) Similarly, decompose $\mathbb{P}^2$ as the union of a Moebius band M and another subspace Q, such that Q is a quotient of the cylinder and $M\cap Q$ is a circle.
(c) Deduce that $\mathbb{P}^2$ can be obtained from a Moebius band and a disk D^2 by gluing them along their boundary circles.

Exercise 11.8. The aim of this exercise is to explain and show that the 3-sphere can be obtained by gluing two solid tori along their boundaries.

(a) First of all, by a solid torus we mean any space homeomorphic to $S^1\times D^2$, the product of the unit circle with the unit disk. Show that

$$f: S^1\times D^2\to\mathbb{R}^3,\ f((\alpha,\beta),(x,y))=((2-x)\alpha,(2-x)\beta,y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name solid torus).

Next, consider the 3-sphere S^3 viewed as a subspace $\mathbb{C}^2$,

$$S^3=\left\{(u,v): u,v\in\mathbb{C}, |u|^2+|v|^2=1\right\},$$

and, inside the sphere, we consider

$$X_1=\left\{(u,v)\in S^3: |u|<|v|\right\},\quad X_2=\left\{(u,v)\in S^3: |u|>|v|\right\},\quad A:=\left\{(u,v)\in S^3: |v|=\frac{\sqrt{2}}{2}\right\}.$$

(b) Show that $\overline{X}_i$ is a solid torus for $i\in\{1,2\}$.
(c) Show that $\overline{X}_1\cap\overline{X}_2=\partial(X_1)=\partial(X_2)=A$. (where the closures and boundaries are inside the space S^3).
(d) Show that the 3-sphere S^3 can be obtained from two disjoint copies of $S^1\times D^2$ (i.e. two solid tori) by gluing each point $(z_1,z_2)\in S^1\times S^1$ in the boundary of the first copy to the point (z_2,z_1) in the boundary of the second.

Exercise 11.9. Show that

(a) $\mathbb{P}^n$ is an n-dimensional topological manifold.
(b) The map $f:\mathbb{P}^n\to\mathbb{R}^{\frac{(n+1)(n+2)}{2}}$ which sends the line l_x through $x=(x_0,\dots,x_n)\in S^n$ to

$$f(l_x)=(x_0x_0,x_0x_1,\dots,x_0x_n,x_1x_1,x_1x_2,\dots,x_1x_n,\dots,x_nx_n)$$

is an embedding.

11.2 Some special quotients

Exercise 11.10. (from the 2021 exam) Consider the equivalence relation $\sim$ on $\mathbb{R}$, where $x \sim y$ iff x and y are both positive or both negative or both zero. Let $X = \mathbb{R}/\sim$ with the quotient topology. List (with proof) all closed subsets of X.

Exercise 11.11. Show that the space obtained from $\mathbb{R}$ by collapsing $[-1,1]$ to a point is homeomorphic to $\mathbb{R}$.

Exercise 11.12. Let X be the cone of the interval $(0,1)$. Construct an explicit continuous injection

$$f : X \to \mathbb{R}^2$$

and let C be its image. Is f a homeomorphism between X and C?

Exercise 11.13. Show that the space obtained from $\mathbb{R}$ by collapsing $(-1,1)$ to a point is not Hausdorff.

Exercise 11.14. Let $f : X \to Y$ be a surjective continuous map between two topological spaces, and consider the equivalence relation induced by f, $R_f = \{(x,x') \in X \times X : f(x) = f(x')\}$. By the general theory, we obtain a continuous map

$$\phi : X/R_f \to Y.$$

What are we using here, and what is the map explicitly? Then show that ϕ is a homeomorphism if and only if f is a topological quotient map.

11.3 Quotients modulo group actions

Exercise 11.15. (from the 2024 exam) Consider the action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \left\{\widehat{0},\ \widehat{1}\right\}$ (the groups of integers modulo 2) on the unit two-dimensional sphere S^2 given by

$$\widehat{k} \cdot (x,y,z) = \left(x,y,(-1)^k z\right) \qquad \left(\text{for } \widehat{k} \in \mathbb{Z}_2,\ (x,y,z) \in S^2\right).$$

Show that the resulting quotient space $S^2/\mathbb{Z}_2$ is homeomorphic to the closed unit disk D^2.

Similar exercise: Exercise 16.14 from the 2024 retake.

Exercise 11.16. (from the 2013 exam) Let $X = (-1,\infty)$.

(a) Find all the numbers $a,b \in \mathbb{R}$ with the property that

$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group $(\mathbb{Z},+)$ on X.

(b) For the a and b that you found, show that the resulting quotient space $X/\mathbb{Z}$ is homeomorphic to S^1.

Similar exercise: Exercise 16.61 from the 2014 exam.

Exercise 11.17. Let $X = (0,\infty)$. Show that

$$\phi_n(r) := 2^n r$$

defines an action of $\mathbb{Z}$ on X and $X/\mathbb{Z}$ is homeomorphic to S^1. (hint: see Example 3.34 and Exercise 1.3).

Exercise 11.18. Show that, for any action of a group Γ on a space X, the quotient map is automatically open. (Note: to appreciate this, recall that continuous surjective open maps are automatically topological quotient maps, but there exist topological quotient maps that are not open maps- see Exercise 11.1 and Exercise 11.2).

Exercise 11.19. Show that if Γ is a finite group acting on a Hausdorff space X, and if the action is free, then the quotient map $\pi : X \to X/\Gamma$ is a local homeomorphism.

Exercise 11.20. (from the 2011 exam) Consider the map

$$\pi\colon \mathbb{R}^2 \to S^1 \times \mathbb{R}, \qquad (x,y) \mapsto \Big(\big(\cos(x+y), \sin(x+y)\big), x-y \Big) \in S^1 \times \mathbb{R}.$$

- Describe an equivalence relation R on $\mathbb{R}^2$ such that $(S^1 \times \mathbb{R}, \pi)$ is a quotient of $\mathbb{R}^2$ modulo R.
- Find a group Γ and an action of Γ on $\mathbb{R}^2$ such that R is the equivalence relation induced by this action.
- Show that, indeed, $\mathbb{R}^2/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}$.

Exercise 11.21. Consider the unit circle in the complex plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Let n be a positive integer, consider the n-th root of unity

$$\xi = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \in \mathbb{C}$$

and let $\mathbb{Z}_n$ be the (additive) group of integers modulo n. Show that

$$\phi_{\hat{k}}(z) := \xi^k z$$

defines an action of $\mathbb{Z}_n$ on S^1, explain its geometric meaning, and show that $S^1/\mathbb{Z}_n$ is homeomorphic to S^1. What do you obtain when $n = 2$?

Exercise 11.22. Consider the 3-dimensional sphere interpreted as:

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}.$$

Consider the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ viewed as a group with respect to the multiplication of complex numbers. Consider the action of S^1 on S^3 given by $z \cdot (z_1, z_2) := (zz_1, \bar{z}z_2)$. We are interested in the resulting quotient S^3/S^1.

(a) Show that the maps $\tilde{f}, \tilde{g}, \tilde{h} : S^3 \to \mathbb{R}$ given by

$$\tilde{f}(z_1,z_2) = z_1z_2 + \bar{z}_1\bar{z}_2, \quad \tilde{g}(z_1,z_2) = i(z_1z_2 - \bar{z}_1\bar{z}_2), \quad \tilde{h}(z_1,z_2) = |z_1|^2 - |z_2|^2$$

induces continuous maps $f, g, h : S^3/S^1 \to \mathbb{R}$.

(b) Show that the maps $\tilde{f}, \tilde{g}, \tilde{h}$ put together, define a continuous surjective map

$$\text{Hopf} := (\tilde{f}, \tilde{g}, \tilde{h}) : S^3 \to S^2.$$

(c) Show that the Hopf is a topological quotient map.

(d) Deduce that we obtain a homeomorphism $(f,g,h) : S^3/S^1 \xrightarrow{\sim} S^2$.

Exercise 11.23. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}$ with the usual group operation

$$(m,n)+(m',n')=(m+m',n+n').$$

Show that

$$\phi_{m,n}(x,y) := (x+m, y+n)$$

defines an action of Γ on $\mathbb{R}^2$ and $\mathbb{R}^2/\Gamma$ is homeomorphic to the torus.

Exercise 11.24. Do the same for the same group but the new action:

$$\phi_{m,n}(x,y) := (x+my+n+\frac{m(m-1)}{2}, y+m).$$

Exercise 11.25. Consider the following groups:

(1) $\Gamma = \langle a,b; bab=a\rangle$ (the group in two generators a and b, subject to the relation $bab=a$).
(2) $\Gamma' = \mathbb{Z}\times\mathbb{Z}$ with the operation

$$(m,n)\circ(m',n') = (m+m', n+(-1)^m n').$$

(3) Γ'' is the subgroup of $(\mathrm{Homeo}(\mathbb{R}^2),\circ)$ (the group of homeomorphisms of the plane endowed with the composition of functions) generated by the transformations

$$\phi(x,y) = (x+1,-y), \psi(x,y) = (x,y+1).$$

Show that:

(a) these three groups are isomorphic.
(b) one obtains an action of these groups on $\mathbb{R}^2$; write it down explicitly.
(c) the resulting quotient is homeomorphic to the Klein bottle.

Exercise 11.26. Compose and solve a similar exercise in which the resulting quotient is homeomorphic to the Moebius band.

11.4 Product topology

Exercise 11.27. Show that

$$(\mathbb{R}^n, \mathscr{T}_{\mathrm{eucl}}) \times (\mathbb{R}^m, \mathscr{T}_{\mathrm{eucl}}) = (\mathbb{R}^{n+m}, \mathscr{T}_{\mathrm{eucl}}).$$

Exercise 11.28. Look at exercise 10.7 and show that there is no conflict in the notation; i.e. the topology $\mathscr{T}_l \times \mathscr{T}_l$ defined in that exercise does coincide with the product topology.

Exercise 11.29. Show that, if $(Z,\mathscr{T}_Z)$ is a third topological space, then a function

$$h = (f,g) : Z \to X\times Y,\ h(z) = (f(z),g(z))$$

is continuous if and only if its components $f: Z\to X$ and $g: Z\to Y$ are both continuous.

Exercise 11.30. Show that if X and Y are both Hausdorff, then so is $X\times Y$. Similarly for metrizability and first countability.

Exercise 11.31. (from the 2022 retake exam) Let $X = \{a,b\}$ be a set with two elements, endowed with the topology

$$\mathscr{T}_X = \{\emptyset, \{a\}, X\}.$$

Consider $X \times X$ endowed with the product topology.

1. Describe all the open subsets of $X \times X$.
2. Compute the interior and the closure of $A = \{(a,b)\} \subset X \times X$.

Exercise 11.32. Let X and Y be two topological spaces, $A \subset X$, $B \subset Y$. Then the following two topologies on $A \times B$ coincide:

(a) The product of the topology of A (induced from X) and that of B (induced from Y).
(b) The topology induced on $A \times B$ from the product topology on $X \times Y$.

Exercise 11.33. Show that a topological space X is Hausdorff if and only if

$$\Delta := \{(x,x) : x \in X\}$$

is closed in $X \times X$.

11.5 Topology bases

Exercise 11.34. Using Lemma 3.48, prove again that the Euclidean and the square metrics induce the same topology.

Exercise 11.35. Show that

$$\mathscr{B}_l := \{[a,b) : a,b \in \mathbb{R}\}$$

is a topology basis and $\mathscr{T}(\mathscr{B}_l)$ is the lower limit topology $\mathscr{T}_l$. Then use Lemma 3.48 to prove again that $\mathscr{T}_l$ is finer than $\mathscr{T}_{\text{Eucl}}$.

Exercise 11.36. Do again Exercise 10.7 using Lemma 3.48.

Exercise 11.37. Go back to exercise 10.5 and show that $\mathscr{B}$ there is a topology basis. Then using Lemma 3.48, show again that the resulting topology is larger than the topology from Exercise 10.4.

Exercise 11.38. Let $(X, \mathscr{T})$ be a topological space. For $f \in \mathscr{C}(X)$ we define

$$V(f) := \Big\{x \in X : f(x) \neq 0\Big\}.$$

Show that:

(a) The collection $\mathscr{B} := \{V(f) : f \in C(X)\}$ is a topology basis on X.
(b) In general, $\mathscr{T}(\mathscr{B}) \subset \mathscr{T}$.
(c) If X is Hausdorff and normal then $\mathscr{T}(\mathscr{B}) = \mathscr{T}$.

Exercise 11.39. Let k be a natural number ($k \in \{0,1,2,\ldots\}$). Let $\mathscr{B}_k$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and all the open intervals (a,b) which contain at most k integers (when $k = 0$, that means: intervals (a,b) that contain no integers). Show that:

1. each $\mathscr{B}_k$ is a topology basis on $\mathbb{R}$.

2. for $k \geq 1$, the topology induced by $\mathscr{B}_k$ on $\mathbb{R}$ is the Euclidean topology.
3. for $k = 0$, the topology induced by $\mathscr{B}_0$ is strictly smaller than the Euclidean topology.
4. Compute the interior and the closure of

$$D := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\} \subset \mathbb{R}^2$$

in $\mathbb{R}^2$ endowed with the product topology $\mathscr{T}_0 \times \mathscr{T}_{\text{Eucl}}$.

Exercise 11.40. (from the 2012 exam) Let $\mathscr{B}$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets

$$[n,a) := \{r \in \mathbb{R} : n \leq r < a\} \quad \text{with } n \in \mathbb{Z}, a \in \mathbb{R}.$$

1. Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $\mathscr{T}$ the associated topology.
2. Is $(\mathbb{R}, \mathscr{T})$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)?
3. compute the closure, the interior and the boundary of $A = [-\frac{1}{2}, \frac{1}{2}]$ in $(\mathbb{R}, \mathscr{T})$.

Similar exercises:

- *Exercise 16.43 from the 2016 exam, without items e, and g.*
- *Exercise 16.13 without item c.*
- *Exercise 16.100 from the 2011 exam A, without the question on connectedness.*
- *Exercise 16.48 from the 2015 exam, without item (d).*
- *Exercise 16.13 from the 2024 retake exam.*
- *Exercise 16.71 from the 2013 exam.*

Exercise 11.41. (from the 2014 retake exam) Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ with the property that

$$f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \quad f(x) = x^2$$

is continuous.

a. Describe a basis of $(\mathbb{R}, \mathscr{T})$ and show that any $U \in \mathscr{T}$ has the property that

$$-x \in U \quad \forall \ x \in U.$$

b. Find the closure and the interior of $(-1,2)$ in $(\mathbb{R}, \mathscr{T})$.
c. Is $(\mathbb{R}, \mathscr{T})$ Hausdorff? Can you find a sequence with two (distinct) limits?
g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R}, \mathscr{T})$?

Exercise 11.42. Consider the unit circle S^1 and the functions

$$\alpha, \beta : S^1 \to \mathbb{R}, \quad \alpha(x,y) = x, \beta(x,y) = y.$$

Show that:

(a) S^1, endowed with the smallest topology which makes α continuous, is not Hausdorff.
(b) the smallest topology on S^1 which makes both α and β continuous is the Euclidean one.

11.6 Spaces of functions

Exercise 11.43. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(nx)$. Show that (f_n) is uniformly convergent.

Exercise 11.44. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = x^n$. Is (f_n) pointwise convergent? But uniformly on compacts? But uniformly?

Exercise 11.45. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. Is (f_n) pointwise convergent? But uniformly on compacts? But uniformly?

Exercise 11.46. (from the 2011 exam) Consider the space $\mathscr{C}([0,1])$ of all continuous maps $f : [0,1] \to \mathbb{R}$, endowed with the sup-metric. Show that

$$A := \left\{ f \in \mathscr{C}([0,1]) : x^2 \leq e^{f(x)} + \sin(f(x)) \leq x \ \ \forall\, x \in [0,1] \right\}$$

is a closed subset of $\mathscr{C}([0,1])$, and that it is not bounded.

Exercise 11.47. (Dini's theorem) Let $f_n : I \to \mathbb{R}$ be an increasing sequence of continuous functions defined on a compact interval I, which converges pointwise to a continuous function f. Show that $f_n \rightrightarrows f$. Is the same true if we do not assume that f is continuous?

Exercise 11.48. Let $p_n \in \mathscr{C}([0,1],\mathbb{R})$ be the sequence of functions (even polynomials!) defined inductively by

$$p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n(t)^2), \ \ p_1 = 0.$$

Show that (p_n) converges uniformly to the function $f(t) = \sqrt{t}$.

(Hint: first show that $p_n(t) \leq \sqrt{t}$, then that $p_n(t)$ is increasing, then that it converges pointwise to $\sqrt{t}$, then look above).

Exercise 11.49. Now, did you know that there are continuous surjective functions $f : [0,1] \to [0,1] \times [0,1]$??? (yes, continuous curves in the plane which fill up an entire square!). Actually, you now have all the knowledge to show that (using Theorem 3.58); it is not obvious but, once you see the pictures, the proof is not too difficult; have a look at Munkres' book.

Chapter 12
Exercises on the chapter "Topological properties"

12.1 Connectedness

Exercise 12.1. Prove that connectedness is a topological property: if two spaces are homeomorphic, then one of them is connected if and only if the other one is connected.

Exercise 12.2. Let $\mathscr{T}_l$ be the lower limit topology on $\mathbb{R}$.

(a) Is $(\mathbb{R}, \mathscr{T}_l)$ connected? Deduce that $(\mathbb{R}, \mathscr{T}_l)$ and $(\mathbb{R}, \mathscr{T}_{\text{Eucl}})$ are not homeomorphic.
(b) But $[0,1]$ with the induced topology? Deduce that $[0,1]$ with the topology induced from $\mathscr{T}_l$ and $[0,1]$ with the topology induced from the Euclidean topology are not homeomorphic.
(c) But $(0,1]$? Can you deduce whether $(0,1]$ with the topology induced from $\mathscr{T}_l$ and $(0,1]$ with the topology induced from the Euclidean topology are not homeomorphic or not?

Exercise 12.3. Recall that by a circle we mean any topological space which is homeomorphic to S^1, and by a circle embedded in a topological space X we mean any subset $A \subset X$ which, when endowed with the induced topology, is a circle. Similarly, by a bouquet of two circles we mean any space which is homeomorphic to the space drawn in Figure 12.1, and we talk about embedded bouquets of circles. Let T be a torus.

1. Describe a circle C embedded in T such that its complement $T - C$ is connected.
2. Describe a bouquet of two circle B embedded in T such that $T - B$ is connected.

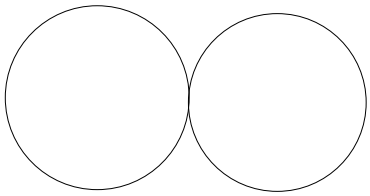


Fig. 12.1

Exercise 12.4. In Exercise 1.5 you showed that four of those spaces are homeomorphic to each other, but they did not seem to be homeomorphic to the fifth one. You now have the tool to prove the last assertion (so do it!).

Exercise 12.5. Prove that the following spaces are not homeomorphic:

1. S^1 and $[0,1)$.
2. $[0,1)$ and $\mathbb{R}$.
3. S^1 and S^2.
4. S^1 and a bouquet of two circles (the space from Figure 12.1).

Exercise 12.6. Show that the group $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices with real coefficients (see Exercise 10.72) is not connected.

Exercise 12.7. (from the 2022 retake exam) Consider the following three subsets of $\mathbb{R}^2$ equipped with the subspace topology:

- $A_1 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$
- $A_2 = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{4}x^2 + y^2 = 1\right\}$
- $A_3 = \left\{(x,y) \in \mathbb{R}^2 : y^2 = 1\right\}$

Give a sketch of all three subspaces. Then decide for each pair of subspaces whether they are homeomorphic or not: if they are, give a homeomorphism; if they are not, give an argument why they are not homeomorphic.

Exercise 12.8. (from the 2016 exam) Return to the spaces X and Y from Exercise 10.30. Show now that X cannot be embedded in Y.

Exercise 12.9. Show that any two of the three spaces drawn in Figure 12.2 are not homeomorphic.

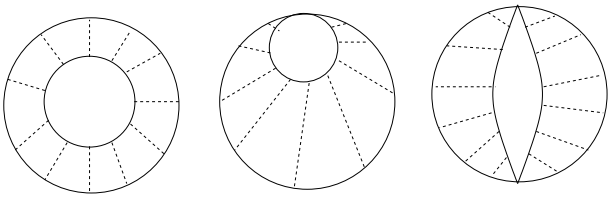


Fig. 12.2

Exercise 12.10. Show that there exists a real number $r \in (2,3)$ such that

$$r^7 - 2r^4 - r^2 - 2r = 2009.$$

Exercise 12.11. Show that any continuous function $f : [0,1] \to [0,1]$ admits at least one fixed point (i.e. there exists $t \in [0,1]$ such that $f(t) = t$). (Hint: $g(t) = f(t) - t$ positive or negative?)

Exercise 12.12. Assuming that the temperature on the surface of the earth is a continuous function, prove that, at any moment in time, on any great circle of the earth, there are two antipodal points with the same temperature.

Exercise 12.13. For a topological space $(X, \mathscr{T})$, show that the following are equivalent:

(1) X is connected.
(2) $\emptyset$ and X are the only subsets of X which are both open and closed.
(3) the only continuous functions $f : X \to \{0,1\}$ are the constant ones (where $\{0,1\}$ is endowed with the discrete topology).

Exercise 12.14. Prove that if $X \subset \mathbb{R}^n$ is connected then so is its closure, but its interior may fail to be connected.

Exercise 12.15. Show that the cone of any space is connected.

Exercise 12.16. (from the 2015 exam) Let X be a topological space.

a. If $\gamma_1, \gamma_2 : [0,1] \to X$ are continuous and $\gamma_1(1) = \gamma_2(0)$, show that

$$\gamma : [0,1] \to X, \quad \gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is continuous.

b. If there exist $A, B \subset X$ path connected such that $X = A \cup B$ and $A \cap B \neq \emptyset$, then X is connected.

Exercise 12.17. Prove that if X and Y are homeomorphic, then they have the same number of connected components.

Exercise 12.18. (from the 2021 exam) Let $X = \mathscr{C}([0,1], \mathbb{R})$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. We consider on it the topology induced by the metric $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$. Show that X is path-connected.

12.2 Compactness

Exercise 12.19. Prove that compactness is a topological property: if two spaces are homeomorphic, then one of them is compact if and only if the other one is compact.

Exercise 12.20. Let $\mathscr{T}_l$ be the lower limit topology on $\mathbb{R}$.

(a) Is $[0,1]$, with the topology induced from $\mathscr{T}_l$, compact?
(b) Answer again the question from Exercise 12.2, (b), on homeomorphisms, with a different proof now.
(c) Is $(0,1]$ with the topology induced from $\mathscr{T}_l$ compact? Try now again the very last question from Exercise 12.2.

Exercise 12.21. (from the 2022 retake exam) Consider

$$X = \left\{(x,y) \in \mathbb{R}^2 : xy = 1\right\}, \quad Y = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1.\right\}$$

endowed with the induced by the Euclidean topology on $\mathbb{R}^2$.

1. Draw X and Y.
2. Show that X and Y are not homeomorphic. Provide two proofs, each one using another topological property.

Exercise 12.22. Show that the boundary of any compact subspace $X \subset \mathbb{R}^n$ is compact.

Exercise 12.23. Find a topological space $(X, \mathscr{T})$ which is compact but is not Hausdorff. In the example that you found, exhibit a compact set $A \subset X$ such that A is not closed.

Exercise 12.24. Find an example which shows that the Tube Lemma (Lemma 4.31) fails if Y is not compact.

Exercise 12.25. Let X be a topological space and $A, B \subset X$. If A and B are compact, show that $A \cup B$ is compact.

Exercise 12.26. Let $(X, \mathscr{T})$ be a Hausdorff space, $A, B \subset X$. If A and B are compact, show that $A \cap B$ is compact. Is the Hausdorffness assumption on X essential?

Exercise 12.27. (from the 2014 retake exam) Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Show that:

(a) if $(X, \mathscr{T}_2)$ is connected then so is $(X, \mathscr{T}_1)$;
(b) if $(X, \mathscr{T}_2)$ is compact then so is $(X, \mathscr{T}_1)$;
(c) if $(X, \mathscr{T}_2)$ is sequentially compact then so is $(X, \mathscr{T}_1)$;

Exercise 12.28. For the notion of closed map we refer to Exercise 10.27. Show that, if X is a compact space and Y is a Hausdorff space, then any continuous map $f : X \to Y$ is automatically a closed map. What do you obtain if you combine this with item Exercise 10.27?

Exercise 12.29. Assume that X is a topological space and $(x_n)_{n\geq 1}$ is a sequence in X, convergent to $x \in X$. Show that

$$A = \{x, x_1, x_2, x_3, \ldots\}$$

is compact.

Exercise 12.30. Given a set X, when is $(X, \mathscr{T}_{\text{cf}})$ compact?

Exercise 12.31. Recall that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$Gr(f) = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

If f is bounded, show that f is continuous if and only if $Gr(f)$ is closed in $\mathbb{R}^2$. What if f is not bounded. (hint: for the first part, use sequences; for the last part: $1/x$).

Exercise 12.32. Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Show that if $(X, \mathscr{T}_2)$ is compact and $(X, \mathscr{T}_1)$ is Hausdorff then $\mathscr{T}_1$ and $\mathscr{T}_2$ must coincide.

Exercise 12.33. Let $(X, \mathscr{T})$ be a second countable topological space. Show that X is compact if and only if for any decreasing sequence of nonempty closed subsets

$$\ldots \subset F_3 \subset F_2 \subset F_1 \subset X,$$

one has $\cap_{n=1}^{\infty} F_n \neq \emptyset$. (Hint: restate the property in terms of opens; then try to use Exercise 10.61).

Exercise 12.34. If X is a Hausdorff space, and $A \subset X$ is compact, show that the quotient X/A (obtained from X by collapsing A to a point) is Hausdorff.

Exercise 12.35. Show that the cone and the suspension of any compact topological space are compact.

Exercise 12.36. Go back to exercise 11.8 and provide all the arguments (making use of compactness) and solve part (a) as well.

Exercise 12.37. Show that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$ (see Exercise 1.27 in the first chapter). Then do again Exercise 3.39 from Chapter 3.

Exercise 12.38. (from the 2015 exam) Consider the group of integers modulo 2, $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$. Define the following action of $\mathbb{Z}_2$ on the closed unit disk $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$:

$$\hat{0} \cdot z = z, \ \hat{1} \cdot z = -z \quad (\text{for all } z \in D^2).$$

Prove that the resulting quotient $D^2/\mathbb{Z}_2$ is homeomorphic to D^2.

Exercise 12.39. If X is connected and compact and $f : X \to \mathbb{R}$ is continuous, show that $f(X) = [m, M]$ for some $m, M \in \mathbb{R}$.

Exercise 12.40.

(a) Is it true that any continuous surjective map $f : S^1 \to S^1$ is a homeomorphism?
(b) Show that S^1 cannot be embedded in $\mathbb{R}$.
(c) Show that any continuous injective map $f : S^1 \to S^1$ is a homeomorphism.

Exercise 12.41. (from the 2011 exam) Prove that there is no continuous injective map $f : S^1 \vee S^1 \to S^1$, where $S^1 \vee S^1$ is a bouquet of two circles (two copies of S^1, tangent to each other).

Exercise 12.42. (from the 2012 exam) Prove that there is no continuous map $g : \mathbb{C} \to \mathbb{C}$ with the property that $g(z)^2 = z$ for all $z \in \mathbb{C}$.

Exercise 12.43. (from the 2013 exam) Prove that there is no continuous injective map $f : S^2 \to S^1$. *(1p)*
(warning: there are injective maps from S^2 to S^1!).

Exercise 12.44. Show that the sequence $x_n = 2011^{\sin(5n)}$ (in $\mathbb{R}$ with the Euclidean topology) has a convergent subsequence.

Exercise 12.45. Show that for any sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$, there exists a subsequence $(x_{n_k})_{k\geq 1}$ and a sequence of integers $(a_k)_{k\geq 1}$ such that $(x_{n_k} + a_k)_{k\geq 1}$ is convergent.

Exercise 12.46. (from the 2012 exam) Let X be a topological space and let $\gamma : [0,1] \to X$ be a continuous function. Assume that γ is locally injective, i.e. that, for any $t \in [0,1]$, there exists a neighborhood V of t in $[0,1]$ such that

$$\gamma|_V : V \to X$$

is injective. Show that, for any $x \in X$, the set

$$\gamma^{-1}(x) := \{t \in [0,1] : \gamma(t) = x\}$$

is finite.

Exercise 12.47. (from the 2011 exam) Let X be the space of continuous maps $f : [0,1] \to [0,1]$ with the property that $f(0) = f(1)$. We endow it with the sup-metric $d_{\sup}$ and the induced topology (recall that $d_{\sup}(f,g) = sup\{|f(t) - g(t)| : t \in [0,1]\}$). Prove that:

(a) X is bounded and complete.
(b) X is not compact.

12.3 Connectedness and compactness for some exotic topologies

Many of the concepts that we introduced were tested on $\mathbb{R}$ endowed with various exotics topologies, starting with the lower limit topology $\mathscr{T}_l$. For connectedness and compactness on $(\mathbb{R}, \mathscr{T}_l)$ we already had Exercise 12.2 and Exercise 12.20. It is now time to test them on a couple of other exotic (i.e., non Euclidean) topologies that we have seen.

Exercise 12.48. Go back to Exercise 10.65 and please continue with the following:

e. find an interval $[a,b]$ with the property that, together with the topology induced from $(\mathbb{R}, \mathscr{T})$, is not compact.
f. find an interval (a,b) with the property that, together with the topology induced from $(\mathbb{R}, \mathscr{T})$, is not connected.

Exercise 12.49. Go back to Exercise 11.41 and answer also:

d. Is $[-1,1]$ (with the topology induced from $\mathscr{T}$) compact? But connected?
e. Is $[-1,1)$ (with the topology induced from $\mathscr{T}$) compact?
f. Is $[-3,1)\cup(1,3]$ (with the topology induced from $\mathscr{T}$) connected?
g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R},\mathscr{T})$?

Exercise 12.50. Go back to Exercise 10.66 and answer also:

d. Is $C=[-1,1]$, with the induced topology $\mathscr{T}_{\text{new}}|_C$, compact? But connected?

Here are a couple more similar exercises:

- in Exercise 11.39 add the item: "Is $(\mathbb{R},\mathscr{T}_0)$ compact?"
- in Exercise 10.64: for each of the spaces $(\mathbb{N},\mathscr{T}_1)$ and $(\mathbb{N},\mathscr{T}_2)$, decide whether the space is Hausdorff, connected or compact.

Similar items appear also in some of the exam exercises that did not include in the main text, such as:

- in Exercise 16.48 from the 2015 exam: look at part of item (d).
- in Exercise 16.43 from the 2016 retake: look at items (f), (g) and (h). Then try to use those to do (i).
- in Exercise 16.13 from the 2024 retake: look at item (c)
- in Exercise 16.22 from the 2021 retake: look at item (b).
- in Exercise 16.100 from the 2011 exam A: look at item 4.

12.4 Local compactness and the one-point compactification

Exercise 12.51. What is the one-point compactification of $X=(0,1)\cup(2,3)$?

Exercise 12.52. What is the one-point compactification of $\mathbb{R}^2$?

Exercise 12.53. Show that the following subspace of $\mathbb{R}^2$ is locally compact and find its one-point compactification:

$$X=S^1\cup((0,2)\times\{0\})\subset\mathbb{R}^2.$$

Exercise 12.54. (from the 2010 exam) Let X, Y and Z be the spaces drawn in Figure 12.3.

(a) Show that any two of them are not homeomorphic.
(b) Compute their one-point compactifications X^+, Y^+ and Z^+.
(c) Which two of the spaces X^+, Y^+ and Z^+ are homeomorphic and which are not?

Exercise 12.55. (from the 2024 exam) Which of the following spaces are homeomorphic to each other (and why):

(a) $C_0=(S^1\times[0,1))^+$, the one-point compactification of $S^1\times[0,1)$.
(b) $C_1=\text{Cone}(S^1)$, the abstract cone associated to the space S^1.
(c) the concrete cone $C_2=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=(1-z)^2,\ 0\le z\le 1\}$.

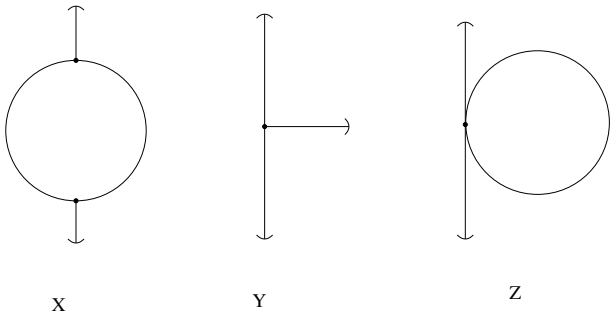


Fig. 12.3

Exercise 12.56. Consider

$$X = [0,1] \times [0,1)$$

with the topology induced from $\mathbb{R}^2$. Prove that X is a locally compact Hausdorff space and describe its one-point compactification (use a picture). What happens if we replace X by $Y = X \cup \{(1,1)\}$?

Exercise 12.57. (from the 2016 exam) Return to the spaces X and Y from Exercise 10.30:

$$X = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, \text{ or } x = 0\right\},$$

$$Y = \left\{(x,y) \in \mathbb{R}^2 : xy = 0\right\}.$$

In exercises 10.30 and 12.8 we have seen that Y can be embedded in X but X cannot be embedded in Y. Show that:

c) the one-point compactifications of X and Y are homeomorphic.

Again, please use pictures but back it up by explanations (please explain why the picture you draw is indeed the one-point compactification).

Exercise 12.58. (from the 2024 retake) Given a topological space X, by *a two-point compactification* of X we mean a pair (Y,i) consisting of a compact Hausdorff space together with a map $i : X \to Y$ which is an embedding and such that Y contains exactly two elements that not belong to the image of i. For $X = (0,1) \cup (2,3)$ endowed with the Euclidean topology:

(a) Give an example of such a two-point compactification, call it Y_1 (I am going to ignore the map i_1 from the notation, although you should indicate it on the picture).
(b) Find now another two-point compactification Y_2 of X such that Y_1 and Y_2 are not homeomorphic.
(c) Find two more, Y_3 and Y_4, not homeomorphic to Y_1 or Y_2 and not homeomorphic to each other.

Clarification: you can describe the spaces Y_1, Y_2, etc and the maps i_1, i_2, etc using pictures. But please provide the arguments/propositions that justify/support the claims that you make- e.g: "Y_1 is compact because ...", or "to see that Y_1 and Y_2 are not homeomorphic we use the X-method as follows ...".

Exercise 12.59. (from the 2011 exam) Consider

$$X_1 = \left\{(x,y,z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = y = 0, z \geq 0)\right\},$$

$$X_2 = \left\{(x,y,z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = 0, y^2 + z^2 = 1, z \geq 0)\right\}.$$

$$X_3 = \left\{(x,y,z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 = 1) \text{ or } (y = 0, z = 0, \frac{1}{2} < |x| < 1)\right\},$$

(a) Show that X_1, X_2, X_3 are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures).
(b) Show that the one-point compactifications of X_1, X_2 and X_3 are homeomorphic to each other.

Exercise 12.60. (from the 2013 exam) Consider

$$X = \left\{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 = 1\right\} \setminus \{(0,0,1),(0,0,-1)\},$$

$$Y = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 2\sqrt{x^2+y^2}\right\} \setminus \{(0,0,0)\}.$$

$$\pi : X \to \mathbb{R}^3,\ \pi(u,v,w) = (2u\sqrt{u^2+v^2}, 2v\sqrt{u^2+v^2}, 2w\sqrt{u^2+v^2}).$$

(a) Compute the closure $\overline{Y}$ of Y in $\mathbb{R}^3$ and prove that it is compact.
(b) Prove that the one-point compactification of X is homeomorphic to $\overline{Y}$.
(c) Draw a picture of Y and explain the map π on the picture.

Exercise 12.61. (from the 2012 exam) Let X be a bouquet of two circles:

$$X = \left\{(x,y) \in \mathbb{R}^2 : ((x-1)^2+y^2-1)((x+1)^2+y^2-1) = 0\right\}.$$

We say that a space Y is an *exam space* if there exist three distinct points $p,q,r \in X$ such that Y is homeomorphic to the one-point compactification of $X - \{p,q,r\}$.

Find the largest number l with the property that there exist exam spaces $Y_1,\ldots,Y_l$ with the property that any two of them are not homeomorphic (prove all the statements that you make!).

Exercise 12.62. Describe an embedding of the cylinder $S^1 \times [0,1]$ into the space X, show that its complement is locally compact, and find the one-point compactification of the complement, in each of the cases:

(a) X is a torus.
(b) X is the plane $\mathbb{R}^2$.

Exercise 12.63. (from the 2021 exam) Let $p,q \in S^2$ be the points $(1,0,0)$ and $(-1,0,0)$. Let $X = S^2/\{p,q\}$ with the quotient topology.

(a) Show that X is not a 2-dimensional manifold.
(b) Show that $S^2 \setminus \{p,q\}$ is not homeomorphic to $\mathbb{R}^2$.

Exercise 12.64. (from the 2014 retake exam) Assume that X is a sphere S^2 minus n points and Y is a sphere minus m points, where $m,n \geq 0$ are integers. Show that if X is homeomorphic to Y, then $n = m$.

Exercise 12.65. For each natural number n we consider a space X_n that is obtained by removing n distinct points from $\mathbb{R}^2$. We consider the 1-point compactification X_n^+ and we denote by $\infty_n \in X_n^+$ the point at infinity (so that $X_n^+ = X_n \cup \{\infty_n\}$). Show that

a. X_n^+ can be embedded in $\mathbb{R}^3$ (here you **do not** have to write down explicit formulas for the embedding, but please explain your reasoning using pictures and mention what result(s) you use in order to reach the final conclusion). *(0.5 points)*
b* If X_n and X_m are homeomorphic, then $n = m$. *(1 point)*

Exercise 12.66. Show that, for any circle C embedded in the Klein bottle K, $K-C$ is locally compact. Then describe such a circle C such that $(K-C)^+$ is homeomorphic to $\mathbb{P}^2$.

Exercise 12.67. Find two locally compact Hausdorff spaces X and Y which are not homeomorphic but which have homeomorphic one-point compactifications.

Exercise 12.68. For any continuous map $f : S^1 \to T^2$ we define

$$X_f := T^2 - f(S^1).$$

(a) Is it true that, for any continuous function f, X_f is compact? But locally compact? But metrizable? But connected?
(b) Describe two embeddings $f_1, f_2 : S^1 \to T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic.
(c) Describe the one-point compactifications $X_{f_1}^+$ and $X_{f_2}^+$.
(d) Describe $f : S^1 \to T^2$ continuous such that X_f^+ is homeomorphic to S^2.

Exercise 12.69. (from the 2010 exam) Let M be the Moebius band. For any continuous function $f : S^1 \to M$ we denote by M_f the complement of its image:

$$M_f := M - f(S^1)$$

and we denote by M_f^+ the one-point compactification of M_f.

(a) Show that for any f, M_f is open in M, it is locally compact but, if nonempty, it is not compact.
(b) Find an example of f such that M_f^+ is homeomorphic to D^2. Then one for which it is homeomorphic to S^2. And then one for $\mathbb{P}^2$.

Exercise 12.70. Show that there exists a Hausdorff space X and an embedded circle C in X, such that the one-point compactification of $X-C$ is homeomorphic to X.

Exercise 12.71. (from the 2022 retake exam) Let $N \in \mathbb{Z}_{>0}$ and let $(X, \mathscr{T})$ be a compact topological space together with an embedding $\Phi : X \to \mathbb{R}^N$, where $\mathbb{R}^N$ is endowed with the Euclidean topology. Prove that $\Phi(X) \subset \mathbb{R}^N$ cannot be open if X is non-empty.

Exercise 12.72. Let X be a connected, locally compact, Hausdorff space, $X \neq \emptyset$. Show that X is compact if and only if X^+ is not connected.

Exercise 12.73. (from the 2014 retake exam) Assume that $(X, \mathscr{T}_X)$ is a compact Hausdorff space and A is nonempty a closed subset of X. We consider the complement of A in X,

$$Y := X - A = \{x \in X : x \notin A\},$$

we denote by X/A the space obtained from X by collapsing A to a point and we consider the canonical projection:

$$\pi : X \to X/A$$

(recall that X/A is endowed with the topology $\pi_* \mathscr{T}_X$). Show that:

a. Y is locally compact and Hausdorff.

b. For any open U in X with the property that $U \cap A = \emptyset$ or $A \subset U$, one has that $\pi(U)$ is open in X/A.
c. X/A is a compact Hausdorff space.
d. The one-point compactification of Y is homeomorphic to X/A.

Exercise 12.74. (from the 2014 exam) Note that this exercise shows, in particular, that the open Moebius band and the open cylinder are not homeomorphic.

Let X be the open cylinder $(-1,1) \times S^1$ and let Y be the open Moebius band (i.e. the Moebius band discussed in the lectures, from which the boundary circle was removed).

a. Describe the 1-point compactification X^+ as a subspace of $\mathbb{R}^3$.
b. (+) Describe $X^+ \subset \mathbb{R}^3$ by explicit formulas and write down an explicit embedding

$$f : X \to \mathbb{R}^3$$

so that X^+ is the image of f together with the extra-point $(0,0,0)$.
c. Show that the 1-point compactification of Y is homeomorphic to the projective plane $\mathbb{P}^2$.
d. Show that X and Y are not homeomorphic.

Chapter 13
Exercises on the chapter "Topological properties versus metric ones"

Exercise 13.1. Let (X,d) be a metric space. Show that any sequence $(x_n)_{n\geq 1}$ in X with the property that

$$d(x_{n+1},x_n) \leq \frac{d(x_n,x_{n-1})}{2}$$

for all n, is Cauchy.

Exercise 13.2. Let (X,d) be a complete metric space and let $f : X \to X$ be a map with the property that there exists $\lambda \in (0,1)$ such that

$$d(f(x),f(y)) \leq \lambda d(x,y)$$

for all $x,y \in X$. Show that f has a unique fixed point (i.e. $a \in X$ with $f(a) = a$).
(Hint: the difficult part is the existence. Start with any x_0 and consider $x_{n+1} = f(x_n)$).

Exercise 13.3. We say that a topological space X is separable if there exists $A \subset X$ countable and dense in X.

1. Show that if X is 2nd countable, then it is separable.
2. Show that a metric space is 2nd countable if and only if it is separable.
3. Deduce that $(\mathbb{R}, \mathscr{T}_l)$ is not metrizable.

Chapter 14
Exercises on the chapter "The algebra of observables/spaces of functions"

14.1 On Stone-Weierstrass

Exercise 14.1. Show that $\mathscr{C}^\infty([0,1])$ (the space of real-valued functions on $[0,1]$ which are infinitely many times differentiable) is dense in $\mathscr{C}([0,1])$.

Exercise 14.2. On the sphere S^2 we consider the real-valued functions

$$f(x,y,z) = x+y+z, \quad g(x,y,z) = xy+yz+zx.$$

Does $\{f,g\}$ separate points? What if we add the function $h(x,y,z) = xyz$?

Exercise 14.3. Let X be a compact topological space. Show that if a finite set of continuous functions

$$\{f_1,\dots,f_n\} \subset \mathscr{C}(X)$$

separates points, then X can be embedded in $\mathbb{R}^n$.

A nice instance of this situation is the one from Exercise 11.22: then $X = S^3/S^1$, $n = 3$, with the set of functions $\{f,g,h\}$ from that exercise. You should check that, indeed, this set does separate points. The resulting embedding (f,g,h) allows us to identify S^3/S^1 with the 2-sphere $S^2 \subset \mathbb{R}^3$. Another instance is described in the next exercise.

Exercise 14.4. Consider the projective plane $\mathbb{P}^2$ interpreted as $S^2/\mathbb{Z}_2$ and we use the notation $[x,y,z] \in S^2/\mathbb{Z}_2$ for the orbit through (=equivalence class of) points $(x,y,z) \in S^2$ (recall that each such class encodes the line in $\mathbb{R}^3$ through the origin and (x,y,z)). Consider the maps

$$g,h,k : \mathbb{P}^2 \to \mathbb{R}, \quad g([x,y,z]) = xy, \quad h([x,y,z]) = xz, \quad k([x,y,z]) = yz.$$

Does $\{g,h,k,\}$ separate points? What if we add $f([x,y,z]) = x^2 - y^2$? Combine now with Exercise 14.3

Exercise 14.5. (from the 2015 retake exam) Let X be a compact Hausdorff space. Let $f_1,\dots,f_n : X \to \mathbb{R}$ be continuous functions, and consider "the sub-algebra of $\mathscr{C}(X)$ generated by $f_1,\dots,f_n$":

$$\mathscr{A}(f_1,\dots,f_n) := \Big\{ f \in \mathscr{C}(X) : f = P(f_1,\dots,f_n) \ \text{ for some polynomial } P \in \mathbb{R}[X_1,\dots,X_n] \Big\}. \tag{14.1.1}$$

(we use quotes because calling it a (sub)-algebra doesn't make it into an actual (sub)-algebra before you actually prove it is!). Show that $\mathscr{A}(f_1,\dots,f_n)$ is dense in $\mathscr{C}(X)$ if and only if

$$(f_1,\dots,f_n) : X \to \mathbb{R}^n$$

is an embedding.

Exercise 14.6. Combine the previous exercises 14.3 and 14.5 into a single one and aim at formulating an equivalence of four items, one related to embeddings, one related to densities, and two related to point separation. Try to first guess such a complete statement, then find the optimal way to prove it.

Exercise 14.7. If K is a compact subspace of $\mathbb{R}^n$, show that the subspace $\mathrm{Pol}(K) \subset \mathscr{C}(K)$ of polynomial functions on K is dense in $\mathscr{C}(K)$ in the uniform topology. (Here, a function $f : K \to \mathbb{R}$ is called polynomial if there exists a polynomial $P \in \mathbb{R}[X_1,\dots,X_n]$ such that $f(x) = P(x)$ for all $x \in K$).

Exercise 14.8. Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic of period 2π, then f can be realized as the limit of a sequence of functions of type

$$T(x) = a_0 + \sum_{k=1}^{N} (a_k \cos(kx) + b_k \sin(kx)).$$

Exercise 14.9. (from the 2015 exam) Let X be compact Hausdorff space and let $C(X)$ be the space of real-valued continuous functions on X, endowed with the topology induced by the *sup*-metric.

Prove that if $\mathscr{A} \subset C(X)$ is a dense subset of $C(X)$, then $\mathscr{A}$ must be point-separating.

Exercise 14.10. Show that, for any compact smooth manifold X, $\mathscr{C}^\infty(X)$ is dense in $\mathscr{C}(X)$.

Exercise 14.11. This is about adapting Stone-Weierstrass to the non-compact spaces. More precisely, we assume that X is a locally compact Hausdorff space. The first adaptation is to replace $\mathscr{C}(X)$ by $\mathscr{C}_0(X)$, which is the collection of continuous functions $f : X \to \mathbb{R}$ which vanish at ∞ in the sense that for each $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that

$$|f(x)| < \varepsilon, \quad \forall\ x \in X - K_\varepsilon.$$

This condition ensures that the sup-norm of f is finite. The second adaptation is to allow for non-unital algebras and non-unital sub-algebras, which means that we are giving up on any requirement regarding units. Show that:

(a) $C_0(X)$ is a non-unital algebra.
(b) The sup-norm (7.1.1) is a well-defined norm on $C_0(X)$ and makes $C_0(X)$ into a non-unital Banach algebra.
(c) If $\mathscr{A}$ is a non-unital subalgebra of $C_0(X)$ which is point separating and satisfies the condition that for any $x \in X$ there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$, then $\mathscr{A}$ is dense in $C_0(X)$.

(Note: if only interested in (c), item (b) can be skipped.)

14.2 On Gelfand-Naimark-more theoretical

Exercise 14.12. (from the 2015 retake exam) Let A be an algebra over $\mathbb{R}$ with unit denoted $\mathbf{1}_A$, assume that $\chi : A \to \mathbb{R}$ is a character and let $I := \mathrm{Ker}(\chi) = \{a \in A : \chi(a) = 0\}$. Show that:

(a) I is an ideal of A.
(b) $a - \chi(a) \cdot \mathbf{1}_A \in I$ for all $a \in A$.
(c) I is a maximal ideal.

Exercise 14.13. This exercise is about "geometric ideals" in the algebra $\mathscr{C}(X)$ of a space X, i.e., those of type

$$I_Y := \{f \in \mathscr{C}(X) : f|_Y = 0\} \subset \mathscr{C}(X),$$

associated to subspaces $Y \subset X$. Assume now that X is a compact Hausdorff space and show that:

(a) For any $Y \subset X$, I_Y is indeed an ideal in $\mathscr{C}(X)$.
(b) Any ideal of this type corresponds to a unique closed subset of X. More precisely:

a. for any $Y \subset X$, closed or not, one has $I_Y = I_{\overline{Y}}$, where $\overline{Y}$ is the closure in X.
b. If $Y_1, Y_2 \subset X$ are closed and $I_{Y_1} = I_{Y_2}$, then $Y_1 = Y_2$.

(c*) An ideal $I \subset \mathscr{C}(X)$ is of this type if and only if it is closed in $\mathscr{C}(X)$.

Exercise 14.14. Show that the spectrum X_A of any commutative algebra A over $\mathbb{R}$ is Hausdorff.
(Note: all you need to know about the topology on X_A is: for any $a \in A$, $f_a : X_A \to \mathbb{R}$, $f_a(\chi) = \chi(a)$, is continuous!)

Exercise 14.15. Let A be a commutative algebra over $\mathbb{R}$. Recall that, for any $a \in A$ we consider that function

$$f_a : X_A \to \mathbb{R},\ f_a(\chi) = \chi(a),$$

and the topology on X_A was defined as the smallest topology on X_A with the property that all the functions f_a, with $a \in A$, are continuous. Show that, for any other topological space X and function $f : X \to X_A$, one has:

$$f : X \to X_A \text{ is continuous} \quad \Longleftrightarrow \quad f_a \circ f : X \to \mathbb{R} \text{ is continuous } \forall\, a \in A$$

Exercise 14.16. Let X be a compact, Hausdorff topological space, and consider a sub-algebra $\mathscr{A} \subset C(X)$.

(a) Show that for any $x \in X$, $\chi_x|_{\mathscr{A}} : \mathscr{A} \to \mathbb{R}$, $a \mapsto a(x)$, is a character on $\mathscr{A}$. Consider now the resulting map

$$F : X \to X_{\mathscr{A}}, \quad F(x) = \chi_x|_{\mathscr{A}}$$

(b) Show that F injective if and only if $\mathscr{A}$ is point separating.
(c) Find example of X and $\mathscr{A}$ for which F is not surjective.
(d) Show that F is always continuous.
(e*) Show that if $\mathscr{A}$ is separating and satisfies: $\sqrt{f} \in \mathscr{A}$ for all $f \in \mathscr{A}$ with $f > 0$, then F is a homeomorphism.

(Note: If you do not manage to do item (d), have a look at Exercise 14.15. For item (e) see Exercise 14.30)

Exercise 14.17. Let X and Y be two compact Hausdorff spaces and assume that

$$\Phi : \mathscr{C}(Y) \to \mathscr{C}(X)$$

is an algebra homomorphism in the sense that it is an $\mathbb{R}$-linear map, sends the constant function 1 to the constant function 1, and $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in \mathscr{C}(Y)$.

(a) For $x \in X$ show that

$$\widehat{\chi}_x : \mathscr{C}(Y) \to \mathbb{R}, \quad \widehat{\chi}_x(g) = \Phi(g)(x) \quad \forall\, g \in \mathscr{C}(Y),$$

or, equivalently, the composition $\mathscr{C}(Y) \xrightarrow{\Phi} \mathscr{C}(X) \xrightarrow{\chi_x} \mathbb{R}$, is a character on $\mathscr{C}(Y)$.
(b) Using the Gelfand-Naimark bijection (7.3.1), deduce that for any $x \in X$ there exists $y \in Y$ such that

$$\Phi(g)(x) = g(y) \quad \forall\, g \in \mathscr{C}(Y).$$

c) deduce that there exists a map $\varphi : X \to Y$ such that

$$\Phi(f) = f \circ \varphi \quad \forall\, f \in \mathscr{C}(Y).$$

(d) Prove that ϕ it is continuous.

(Hint: one way to do the last item is to try to use Exercise 14.15 ... for $A = \mathscr{C}(Y)$)

14.3 On Gelfand-Naimark- "computing" spectra

Exercise 14.18. Consider the algebra $A = \mathbb{R}[t]$ of polynomials in one variable t. Show that X_A is homeomorphic to $\mathbb{R}$. What if you take more variables?

Exercise 14.19. Consider the algebra $A = \mathbb{R}[t]/(t^3)$ (remainders modulo t^3). Compute X_A.

Exercise 14.20. Consider the quotient of the ring of polynomials in two variables, modulo the ideal generated by $f = X^2 + Y^2 - 1$

$$A = \mathbb{R}[X,Y]/(X^2+Y^2-1).$$

Show that A an algebra over $\mathbb{R}$ and show that X_A is homeomorphic to S^1. (Hint: Let $\alpha, \beta \in A$ corresponding to X and Y. Then a character χ is determined by $u = \chi(\alpha)$ and $v = \chi(\beta)$. What do they must satisfy? Also, have a look back at Exercise 11.42).

Note: If you are not comfortable yet with such quotients, take as definition A :=the algebra of polynomial functions on $S^1 \subset \mathbb{R}^2$ and then α and β in the hint above are the first and second projection.

Exercise 14.21. (from the 2011exams) Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathscr{R}_p$ the set of remainders modulo p. In other words,

$$\mathscr{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) :\ \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

For $q \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denoted by $[q] \in \mathscr{R}_p$ the induced equivalence class. Show that:

(a) The following three operations are well defined and make $\mathscr{R}_p$ into an algebra.

$$[q_1] + [q_2] := [q_1 + q_2], \quad [q_1] \cdot [q_2] := [q_1 \cdot q_2], \quad \lambda [q] := [\lambda q]$$

(b) For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathscr{R}_p$ has only one point.
(c) For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathscr{R}_p$ is homeomorphic to S^n.
(d) What is the spectrum when $p = x_0 x_1 \ldots x_n$?

Exercise 14.22. (from the 2024 exam) On S^2 we consider the functions $f, g, h : S^2 \to \mathbb{R}$ given by:

$$f(x,y,z) = x, \quad g(x,y,z) = y, \quad h(x,y,z) = z.$$

We also consider the collection of all polynomial expressions generated on f, g and h:

$$\mathscr{A}(f,g,h) := \Big\{P(f,g,h) :\ P \in \mathbb{R}[X,Y,Z]\Big\} = \left\{ \sum_{0 \le i,j,k \le d} c_{i,j,k} f^i g^j h^k :\ d \in \mathbb{N},\ c_{i,j,k} \in \mathbb{R} \right\} \subset C(S^2).$$

(a) Show that $\mathscr{A}(f,g,h)$ is dense in $C(S^2)$.
(b) Show that for any character $\chi : \mathscr{A}(f,g,h) \to \mathbb{R}$, the numbers $u := \chi(f), v := \chi(g), w := \chi(h)$ must satisfy

$$u^2 + v^2 + w^2 = 1.$$

(c) Show that, conversely, for any $u,v,w \in \mathbb{R}$ satisfying this equation, there exists a character $\chi : \mathscr{A}(f,g,h) \to \mathbb{R}$ with $\chi(f) = u$, $\chi(g) = v$, $\chi(h) = w$, and this character is unique.
(d) The previous two items tell us that we have a bijection

$$\Psi : X_{\mathscr{A}(f,g,h)} \to S^2, \quad \chi \mapsto (\chi(f), \chi(g), \chi(h)).$$

Show that this is actually a homeomorphism.

Exercise 14.23. (from the 2012 retake) Let A be a commutative algebra over $\mathbb{R}$ and assume that $a_1, \ldots, a_n \in A$ generate A, i.e. that any $a \in A$ can be written as $a = P(a_1, \ldots, a_n)$ for some $P \in \mathbb{R}[X_1, \ldots, X_n]$. Let X_A be the topological spectrum of A. The first aim is to show that

$$f : X_A \to \mathbb{R}^n, \;\; f(\chi) = (\chi(a_1), \ldots, \chi(a_n))$$

is an embedding. To that end we denote the components of f by $f_i : X_A \to \mathbb{R}$, and we ask you to show that:

(a) f is continuous.
(b) $\chi(P(a_1, \ldots, a_n)) = P(\chi(a_1), \ldots, \chi(a_n))$ for any character $\chi \in X_A$ and any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$.
(c) f is injective.
(d) the topology of X_A is the smallest topology on X_A with the property that all the functions f_i are continuous.
(e) f is an embedding.

We now take for A the algebra $A_K = \mathrm{Pol}(K)$ of real-valued polynomial functions on a subset $K \subset \mathbb{R}^n$ and for a_i the polynomial functions

$$a_i : K \to \mathbb{R}, \;\; a_i(x_1, \ldots, x_n) = x_i, \quad (1 \leq i \leq n),$$

and we consider the resulting function $f : X_{\mathrm{Pol}(K)} \to \mathbb{R}^n$. Please do the following:

(f) Show that $\mathrm{Pol}(K)$ is finitely generated with generators $a_1, \ldots, a_n$.
(g) Show that the image of f contains K.
(h) For the sphere $S^n \subset \mathbb{R}^{n+1}$, deduce that the spectrum of the algebra $\mathrm{Pol}(S^n)$ is homeomorphic to S^n.
(i) For which subspaces $K \subset \mathbb{R}^n$ can one use a similar argument to deduce that $X_{\mathrm{Pol}(K)}$ is homeomorphic to K?

Exercise 14.24. If A and B are two commutative algebras over $\mathbb{R}$, a map

$$\Phi : A \to B$$

is called algebra homomorphism if it is $\mathbb{R}$-linear, $\Phi(a_1 a_2) = \Phi(a_1)\Phi(a_2)$ for all $a_1, a_2 \in A$, and $\Phi(1_A) = 1_B$, where $1_A \in A$ and $1_B \in B$ are the units. It is called an isomorphism of algebras if it is also bijective.

(a) Show that for any algebra homomorphism $\Phi : A \to B$ between two algebras,

$$\Phi^* : X_B \to X_A, \;\; \Phi^*(\chi) = \chi \circ \Phi$$

is a continuous map,
(b) If Φ is an isomorphism of algebras then Φ^* is a homeomorphism.
(c) Compute the spectrum of

$$A := \left\{ f : \mathbb{R}^2 \to \mathbb{R} \text{ continuous} : f(x+1,y) = f(x,y+1) = f(x,y), \; \forall\, x,y \in \mathbb{R} \right\}.$$

Exercise 14.25. (from the 2024 retake) Consider

$$\mathscr{A} = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous such that } f(x) = f(x+1) \text{ for all } x \in \mathbb{R}\}.$$

(a) Show that $\mathscr{A}$ is a sub-algebra of $C(\mathbb{R})$.
(b) Is $\mathscr{A}$ dense in $C(\mathbb{R})$?
(c) Show that the spectrum of $\mathscr{A}$ is homeomorphic to S^1.

Exercise 14.26. (This is from the retake exam of 2014) Let $\mathscr{C}(D^2)$ be the algebra of real-valued continuous functions on the unit disk D^2 and let $\mathscr{A}$ be the subset consisting of those $f \in \mathscr{C}(D^2)$ with the property that they are constant on the boundary circle S^1.

a. is $\mathscr{A}$ a sub-algebra of $\mathscr{C}(D^2)$?
b. is $\mathscr{A}$ dense in $\mathscr{C}(D^2)$?
c. show that the spectrum of $\mathscr{A}$ is homeomorphic to S^2.

Exercise 14.27. Consider the algebra

$$A = \{f \in \mathscr{C}(S^n) : f(z) = f(-z) \ \ \forall\, z \in S^n\}.$$

Show that X_A is homeomorphic to $\mathbb{P}^n$.

14.4 On Gelfand-Naimark- a few more exercises

Exercise 14.28. While our discussion of Gelfand-Naimark focused on relating topological spaces with algebras, the discussion can be improved by showing that it interacts nicely with the concept of "morphism" on each side: continuous maps on the side of topological spaces, and algebra homomorphism on the side of algebras. In some sense, this is what Exercises 14.17 and 14.24 were about. Let us put everything together into one single story (and one single, hopefully simpler, exercise): show that:

(a) for any continuous map $\varphi : X \to Y$ between two topological spaces,

$$\varphi^* : \mathscr{C}(Y) \to \mathscr{C}(X), \quad \varphi^*(f) := f \circ \varphi$$

is an algebra homomorphism. Moreover, φ^* is an isomorphism if φ is a homeomorphism.
(b) for any algebra homomorphism $F : A \to B$ between two algebras,

$$F^* : X_B \to X_A,\ F^*(\chi) = \chi \circ F$$

is a continuous map. Moreover, F^* is a homeomorphism if F is an isomorphism.
(c) If X and Y are two compact Hausdorff spaces, than any algebra homomorphism

$$F : \mathscr{C}(Y) \to \mathscr{C}(X)$$

must be of type $F = \varphi^*$, for some continuous map $\varphi : X \to Y$, and φ is unique.
(d) Two compact Hausdorff spaces X and Y are homeomorphic iff the algebras $\mathscr{C}(X)$ and $\mathscr{C}(Y)$ are isomorphic.

Exercise 14.29. This is a reverse of Exercise 14.13, starting on the algebra side. Show that:

(a) For any algebra A and any ideal $I \subset A$, the following is a closed subspace of X_A:

$$X_{A,I} := \{\chi \in X_A : \chi(a) = 0 \ \forall\, a \in I\}$$

(b) When $A = \mathscr{C}(X)$ with X compact and Hausdorff, and $I = I_Y$ corresponding to some closed subset $Y \subset X$, then the (Gelfand-Naimark) homeomorphism between X_A and X, restricts to one between $X_{A,I}$ and Y.
(c) On the other hand, in (a), notice that $X_{A,I}$ is just a description of the spectrum of the quotient A/I without any reference to the quotient itself. Make this precise and explicit! After that, re-think item (b).

Exercise 14.30. In this exercise we provide a step-by-step way to tackle item (e) from Exercise 14.16. With those notations, and under those assumptions, do the following:

(e1) Show that if $f, g \in \mathscr{A}$ and $f \geq g$, then $\chi(f) \geq \chi(g)$.
(e2) Show that for all $f \in \mathscr{A}$ one has $|\chi(f)| \leq ||f||_{\sup}$.
(e3) Deduce that for any sequence $(f_n)_{n\geq 1}$ of elements in $\mathscr{A}$, convergent to some $f \in C(X)$, the sequence $(\chi(f_n))_{n\geq 1}$ is convergent.
(e4) Deduce that there exists an extension of $\chi : \mathscr{A} \to \mathbb{R}$ to a continuous map

$$\tilde{\chi} : C(X) \to \mathbb{R}.$$

(e5) Then show that $\tilde{\chi}$ is a character.
(e6) Conclude that F is a homeomorphism between X and X_A.

Exercise 14.31. (from the 2014 exam) In this exercise we work over $\mathbb{R}$ (hence we consider real-valued functions and real algebras). Let X and Y be compact Hausdorff spaces. For $u \in \mathscr{C}(X)$, $v \in \mathscr{C}(Y)$, define $u \otimes v \in \mathscr{C}(X \times Y)$ given by

$$u \otimes v : X \times Y \to \mathbb{R}, \ \ (u \otimes v)(x,y) := u(x)v(y),$$

and we denote by $\mathscr{A} \subset \mathscr{C}(X \times Y)$ the set of functions of type

$$\sum_{i=1}^{k} u_i \otimes v_i \quad \text{with } k \in \mathbb{N}, u_1, \ldots, u_k \in \mathscr{C}(X), v_1, \ldots, v_k \in \mathscr{C}(Y)$$

Show that:

a. (+) $\mathscr{A}$ is a dense subalgebra of $\mathscr{C}(X \times Y)$.
b. For any $\chi \in X_{\mathscr{A}}$, $\chi_1 : \mathscr{C}(X) \to \mathbb{R}$ and $\chi_2 : \mathscr{C}(Y) \to \mathbb{R}$ given by

$$\chi_1(u) := \chi(u \otimes 1), \ \ \chi_2(v) := \chi(1 \otimes v)$$

are characters.
c. (+) $X_{\mathscr{A}}$ is homeomorphic to $X \times Y$.

Exercise 14.32. (from the 2016 exam) Here we work over the field $\mathbb{R}$ of real numbers. Let A be an arbitrary algebra, assume that A is commutative (i.e. $a_0 a_1 = a_1 a_0$ for any $a_0, a_1 \in A$) and denote by $\mathbf{0_A}$ and $\mathbf{1_A}$ the zero and the unit element of A, respectively. Fix an element $\alpha \in A$. Consider

$$B = A \times A = \{(a_0, a_1) : a_o, a_1 \in A\}$$

with the operations

$$(a_0, a_1) + (a_0', a_1') = (a_0 + a_0', a_1 + a_1'),$$
$$\lambda \cdot (a_0, a_1) = (\lambda a_0, \lambda a_1),$$
$$(a_0, a_1) \bullet (a_0', a_1') = (a_0 a_0' + \alpha a_1 a_1', a_0 a_1' + a_0' a_1).$$

a) Show that B, together with the previous operations (written in red), is an algebra (don't forget the unit!).

b) Assume that $\chi : B \to \mathbb{R}$ is a character and consider

$$\overline{\chi} : A \to \mathbb{R}, \quad \overline{\chi}(a) = \chi(a, \mathbf{0_A}),$$

$$r := \chi(\mathbf{0_A}, \mathbf{1_A}) \in \mathbb{R}.$$

Show that:

b1) $\overline{\chi}$ is a character on A.
b2) $\chi(a_0, a_1) = \overline{\chi}(a_0) + r\,\overline{\chi}(a_1)$ for any $(a_0, a_1) \in B$.
b3) $\overline{\chi}(\alpha) = r^2$.

c) Conversely, for any character $\overline{\chi}$ on A and any $r \in \mathbb{R}$ satisfying the equation from b3), show that χ defined by the formula from b2) is a character on B.

d) When $A := \mathbb{R}[X_0, X_1]$ is the algebra of polynomials in two variables and $\alpha := 1 - (X_0)^2 - (X_1)^2$, show that the topological spectrum of B is *homeomorphic* to S^2.

Chapter 15
Exercises on the chapter "Partitions of unity"

15.1 More exercises

Exercise 15.1. Let $\mathscr{A}$ be the following collection of subsets of $\mathbb{R}$:

$$\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}.$$

Which of the following collections refines $\mathscr{A}$?

$$\mathscr{B} = \{(x, x+1) : x \in \mathbb{R}\},$$

$$\mathscr{C} = \left\{(n, n+\frac{3}{2}) : n \in \mathbb{Z}\right\},$$

$$\mathscr{D} = \left\{(x, x+\frac{3}{2}) : x \in \mathbb{R}\right\}.$$

Exercise 15.2. Which of the collections from the previous exercise is locally finite?

Exercise 15.3. (from the 2012 exam) Consider the following open cover of $\mathbb{R}$:

$$\mathscr{U} := \left\{(r,s) : r,s \in \mathbb{R}, |r-s| < \frac{1}{3}\right\}.$$

Describe a locally finite subcover of $\mathscr{U}$.

Exercise 15.4. (from the 2011 exam) Show that the family of open intervals

$$\mathscr{U} := \{(q, q+1) : q \in \mathbb{R}\}$$

forms an open cover of $\mathbb{R}$ which admits no finite sub-cover, but which admits a locally finite sub-cover.

Exercise 15.5. (from the 2013 exam) Let $X = (0, \infty)$ and consider the open cover of X

$$\mathscr{U} = \{U_1, U_2, U_3, \ldots\} \quad \text{with } U_n = (0, n).$$

(a) Show that $\mathscr{U}$ does not admit a subcover which is locally finite.
(b) Describe a locally finite refinement of $\mathscr{U}$.

Exercise 15.6. Show that if a family $\{p_i : i \in I\}$ of non-zero polynomial functions $p_i : \mathbb{R} \to \mathbb{R}$ is locally finite in (cf. Definition 8.9), then it must be finite.

Exercise 15.7. Let $\mathscr{P} \subset \mathscr{C}(\mathbb{R})$ be the space of all polynomial functions on $\mathbb{R}$. Is $\mathscr{P}$ normal?

Exercise 15.8. Show that the space $\mathscr{C}^1(\mathbb{R}) \subset \mathscr{C}(\mathbb{R})$ of functions of class C^1 is normal. What do you conclude from this?

Exercise 15.9. Do the same for the space $\mathscr{C}^\infty(\mathbb{R})$ of smooth (i.e. infinitely differentiable) functions on $\mathbb{R}$.

Exercise 15.10. Now do the same for $\mathscr{C}^\infty(\mathbb{R}^n)$.

It is very tempting now to talk about smooth manifolds . These are manifolds on which we can talk about smoothness. More precisely, a smooth manifold is a topological manifold X together with a specified family of coordinate charts $\{\chi_i : U_i \to \mathbb{R}^n\}$, such that $\{U_i\}$ is an open cover of X, $c_{i,j} := \chi_i \circ \chi_j^{-1}$ is a smooth function. Here, $c_{i,j}$ plays the role of the "change of coordinates" since

$$\chi_i(x) = c_{i,j}(\chi_j(x)).$$

Also, $c_{i,j}$ is a function defined on an open in $\mathbb{R}^n$ (namely $\chi_j(U_i \cap U_j)$) with values in $\mathbb{R}^n$; hence it makes sense to talk about its smoothness. Given such a smooth manifold, a function $f : X \to \mathbb{R}$ is called smooth if its representation in each chart, i.e. each $f \circ \chi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth. Denote by $\mathscr{C}^\infty(X)$ the space of smooth functions on X; of course, $\mathscr{C}^\infty(X) \subset \mathscr{C}(X)$. Once you get used to all these definitions, the following should not be too difficult now:

Exercise 15.11. Show that, for any smooth manifold X, $\mathscr{C}^\infty(X)$ is normal. Deduce that any open cover admits a smooth partition of unity subordinated to it.

In this context, a map $f : X \to \mathbb{R}^N$ is called smooth if all its components are smooth. Adapting the proof of Theorem 4.41 and using Exercise 15.10 above, one can now try a more difficult exercise:

Exercise 15.12. Show that, for any smooth compact manifold X, there exists a smooth embedding $f : X \to \mathbb{R}^N$, for N large enough.

Exercise 15.13. Let X and Y be two topological spaces and assume that Y is Hausdorff and normal. Show that a function

$$f : X \to Y$$

is continuous if and only if, for each continuous function $\phi : Y \to \mathbb{R}$, the function $\phi \circ f : X \to \mathbb{R}$ is continuous.

Exercise 15.14. (from the 2021 exam)

(a) Consider the open cover $\mathscr{U} = \{(-2,1),(-1,2)\}$ of the interval $(-2,2)$ with the Euclidean topology. Give an example of a partition of unity subordinate to $\mathscr{U}$.
(b) Let $X = \{x,y\}$ be a set with two elements. Give an example of a topology on X such that every open cover[1] has a subordinate partition of unity, but the topology is not Hausdorff.

Exercise 15.15. Let X be a Hausdorff paracompact space and let $f : X \to [0,\infty)$ be a function which is assumed to be locally bounded i.e., which satisfies:

$$\forall\ a \in X,\ \ \exists \text{ neighborhood } V \text{ of } a, \exists\, M > 0 \text{ s.t. } \ f|_V \leq M.$$

Show that there exists $g : X \to (0,\infty)$ continuous such that $f \leq g$ on X. What can you add if X is compact? (Hint: partition of unity).

Exercise 15.16. Let X be a Hausdorff, paracompact space. Show that for any open U in $X \times \mathbb{R}$ containing $X \times \{0\}$,

$$X \times \{0\} \subset U \subset X \times \mathbb{R},$$

there exists a continuous function $r : X \to (0, \infty)$ such that U contains the r-tube, i.e.,

$$\Big\{(x,t) \in X \times \mathbb{R} : t \in (-r(x), r(x))\Big\} \subset U. \tag{15.1.1}$$

When X is compact deduce the tube lemma (which says that r can be chosen to be constant!).
(Hint start with small enough opens V_i and numbers r_i such that $V_i \times (-r_i, r_i) \subset U$. And a partition of unity.)

Exercise 15.17. Let X be a Hausdorff paracompact space. We denote by $\mathscr{C}_c(X)$ the subset of $\mathscr{C}(X)$ consisting of continuous functions $f : X \to \mathbb{R}$ with compact support. A **positive integral** on X is any linear map

$$I : \mathscr{C}_c(X) \to \mathbb{R}$$

with the property that, for any $f \in \mathscr{C}_c(X)$, one has $f \geq 0 \Longrightarrow I(f) \geq 0$. If, furthermore, the right hand side is zero only when $f = 0$, we say that I is strictly positive. The set of positive integrals on X and the set of strictly positive ones are denoted, respectively,

$$\mathscr{I}(X),\ \mathscr{I}^*(X).$$

Show that if X admits an open cover $\{U_i : i \in I\}$ such that $\mathscr{I}^*(U_i) \neq \emptyset$ for all $i \in I$, then $\mathscr{I}^*(X) \neq \emptyset$.

(Hint: notice first that $I_\eta(f) := I(\eta \cdot f)$ defines an element $I_\eta \in \mathscr{I}(X)$ whenever $I \in \mathscr{I}(U)$ and $\eta \in \mathscr{C}(X)$ positive and supported in U. And use a partition of unity.)

Exercise 15.18. As in the previous exercise, we assume that X is a Hausdorff paracompact space and we denote by $\mathscr{I}(X)$ and $\mathscr{I}^*(X)$ the set of positive integrals on X, respectively the set of strictly positive ones. Show that:

(a) For $U \subset X$ open and $f \in \mathscr{C}_c(U)$, after extending f by zero outside U one obtains a continuous function $\tilde{f} \in \mathscr{C}_c(X)$.
(b) For $U \subset X$ open and $I \in \mathscr{I}(X)$ one obtains a positive integral on U, denoted $I|_U \in \mathscr{I}(U)$ and defined as follows:

$$I|_U : \mathscr{C}_c(U) \to \mathbb{R}, \quad I|_U(f) := I(\tilde{f}).$$

(c) If $I \in \mathscr{I}(X)$ and there exists an open cover $\{U_i : i \in I\}$ of X such that $I|_{U_i} = 0$ for all i, then $I = 0$.
(d) Given an open cover $\{U_i : i \in I\}$ of X together with positive integrals $I_i \in \mathscr{I}(U_i)$, the following are equivalent:

   a. there exists $I \in \mathscr{I}(X)$ such that $I|_{U_i} = I_i$ for all $i \in I$.
   b. the I_is coincide on the overlaps, i.e., $I_i|_{U_i \cap U_j} = I_j|_{U_i \cap U_j}$ for all $i, j \in I$.

Chapter 16
Previous exams

16.1 The 2025 retake exam (April 17, 2025)

Notes: Please write down your name and the student no. Your solutions can be in Dutch or English. PLEASE MOTIVATE ALL YOUR ANSWERS!!!! EXPLAIN WHAT YOU DO, WHAT YOU USE, WHY YOUR CLAIMS ARE TRUE. There are 14 collectable points, the final mark is at most 10.

Exercise 16.1 (10x0.5= 5pts). On $X = \mathbb{R}$ we consider the family of subsets consisting of all intervals of type $(-\infty, -r]$ with $r \geq 0$, all intervals of type $[r, 2r]$ with $r \geq 0$ and the set $\{0\}$:

$$\mathscr{S} := \Big\{(-\infty, -r] : r \geq 0\Big\} \cup \Big\{[r, 2r] : r \geq 0\Big\} \cup \Big\{\{0\}\Big\}.$$

We denote by $\mathscr{T}$ the smallest topology on X containing $\mathscr{S}$.

(a) Show that $\mathscr{S}$ is not a topology basis and describe a basis for $\mathscr{T}$.

(b) Show that $\mathscr{T}$ is neither larger nor smaller than the Euclidean topology.

(c) Is $(X, \mathscr{T})$ connected?

(d) Is $(X, \mathscr{T})$ compact?

(e) Is $(X, \mathscr{T})$ Hausdorff?

(f) Is $(X, \mathscr{T})$ second countable?

(g) Compute the interior and the closure of $(-1, 2)$ in $(X, \mathscr{T})$.

(h) Is the sequence $(x_n)_{n\geq 1}$ given by $x_n = -\frac{1}{n}$ convergent in $(X, \mathscr{T})$?

(i) What about $y_n = 1 + \frac{1}{n}$?

(j) What about $z_n = -1 - \frac{1}{n}$?

Exercise 16.2 (5x0.5= 2.5pts). In this exercise X, Y and Z are topological spaces, and the set $\{0, 1\}$ will be endowed with the discrete topology, i.e. consisting of $\emptyset$, $\{0\}$, $\{1\}$ and $\{0, 1\}$.

(a) Show that if $f : X \to Y$ and $g : Y \to Z$ are continuous maps then so is $g \circ f$.

(b) Show that if $f : X \times Y \to Z$ is continuous then for any $y_0 \in Y$ the function

$$f^{y_0} : X \to Z, \quad f^{y_0}(x) := f(x, y_0)$$

is continuous.

(c) Show that if X is connected then any continuous function $f : X \to \{0, 1\}$ must be constant.

(d) Formulate and prove the converse of the previous item.

(e) Using the previous items prove that a product of two connected spaces is again connected.

Exercise 16.3 (1+1.5= 2.5pts). Consider $X := \mathbb{R} \times [-1, 1]$ endowed with the Euclidean topology.

(a) Show that the following defines an action of $(\mathbb{Z},+)$ on X:

$$n\cdot(x,t) := (x+2n\pi,(-1)^n t) \qquad (n\in\mathbb{Z}).$$

(b) Show that $X/\mathbb{Z}$ can be embedded in $\mathbb{R}^3$.

Exercise 16.4 (5x0.5= 2.5pts). Given a compact Hausdorff space X and A a closed nonempty subset,

(a) Show that $X\setminus A$ is locally compact;
(here $X\setminus A$ is the complement of A in X)

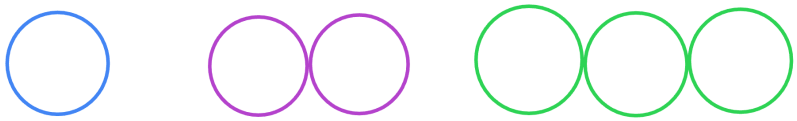
(b) Show that the one-point compactification of $X\setminus A$ is homeomorphic to X/A;
(here X/A is the quotient of X obtained by collapsing A to a point)

(c) Show that $\mathscr{A} := \{f\in C(X) : f|_A \text{ is constant}\}$ is a sub-algebra of $C(X)$;
(here $C(X)$ denotes the algebra of real-valued continuous functions on X)

(d) Show that the spectrum of $\mathscr{A}$ is homeomorphic to X/A;

(e) Show that if A has at least two points, then $\mathscr{A}$ is not dense in $C(X)$.

Exercise 16.5 (1.5pts). Which of the three spaces in the figure is/are homeomorphic to the one-point compactification of a subset of $\mathbb{R}$ (with the Euclidean topology)?



NOTES:

- Your solutions can be in Dutch or English (or a mixture of those).
- Please write down your name and the student no, and make sure your name is on each sheet.
- In total you can collect up to 14 points. The final mark is $\min\{10, \text{number of collected points}\}$.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! Please do not just write down the final result (that will not count!) but explain it/provide the details. In particular:
  - in Exercise 1, questions (c), (d), (e), (f), (h), (i), (j) please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)).
  - the same in Exercise 5: if the answer is yes describe the subspace of $\mathbb{R}$ and prove that its one-point compactification is what you claimed; if the answer is no, prove it.
  - if you do understand some parts intuitively but cannot write it down properly, try to use pictures (that is not enough for a proof, but it may be worth points, especially if the pictures are backed up by some theoretical explanations).
- To motivate your answers you can use any of the theorems, propositions, etc that we discussed in the class or that are in the dictaat.

16.2 The 2025 exam (January 30, 2025)

Notes: In total you can collect up to 13 points. The final mark is $\min\{10, \text{number of collected points}\}$. Your solutions can be in Dutch or English. PLEASE MOTIVATE ALL YOUR ANSWERS!!!! More notes on the other side.

Exercise 16.6 (7x0.5= 3.5pts). Let $\mathscr{B}$ be the collection of subsets of S^1 of type

$$S^1_{[a,b)} := \left\{(\cos 2\pi t, \sin 2\pi t) \in S^1 : t \in [a,b)\right\},$$

one for each $a, b \in \mathbb{R}$. Let $\mathscr{T} = \mathscr{T}(\mathscr{B})$ be the induced topology.

(a) Show that $\mathscr{B}$ is not a topology on S^1, but it is a topology basis.

(b) Show that $\mathscr{T}$ is strictly larger than the Euclidean topology on S^1.

(c) Is $(S^1, \mathscr{T})$ Hausdorff?

(d) Is $(S^1, \mathscr{T})$ connected?

(e) Is $(S^1, \mathscr{T})$ compact?

(f) Is the sequence $x_n = \left(\cos \frac{(n-1)\pi}{n}, \sin \frac{(n-1)\pi}{n}\right)$ convergent in $(S^1, \mathscr{T})$?

(g) Compute the interior and the closure, in the topological space $(S^1, \mathscr{T})$, of

$$A := \left\{(x,y) \in S^1 : y > 0\right\}.$$

Exercise 16.7 (1pt). Prove that the space $(S^1, \mathscr{T})$ from the previous exercise is homeomorphic to a quotient of $(\mathbb{R}, \mathscr{T}_l)$ modulo the action of a group. [Recall: $\mathscr{T}_l$ stands for the lower limit topology on $\mathbb{R}$.]

Exercise 16.8 (2pts). Show that any open continuous map $f : X \to Y$ from a compact space $X \neq \emptyset$ to a Hausdorff connected space Y is surjective. [Recall: f open means: $\forall\, U \subset X$ open, $f(U)$ is open in Y.]

Exercise 16.9 (1pt). Prove that if X and Y are path-connected spaces then $X \times Y$ is connected.

Exercise 16.10 (1+ 1+ 1+ 0.5= 3.5 pts). We consider the closed unit disk

$$D^2 := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$$

and, inside it, the subset:

$$A := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/2\right\}.$$

We endow these spaces with the Euclidean topology.

(a) Prove D^2/A, i.e. the space obtained from D^2 by collapsing A to a point, is homeomorphic to D^2. Provide explicit formulas.

(b) Prove that the 1-point compactification of the complement $D^2 - A$ of A in D^2 is homeomorphic to D^2. Provide explicit formulas.

(c) Show that $\mathscr{A} := \left\{f \in C(D^2) : f|_A \text{ is constant}\right\}$ is a sub-algebra of $C(D^2)$ and the spectrum of $\mathscr{A}$ is homeomorphic to D^2.

(d) Show that $\mathscr{A}$ is not dense in $C(D^2)$.

Exercise 16.11 (2pt). Prove that the product $S^1 \times D^2$ of the unit circle with the unit disk (with their Euclidean topologies) can be embedded in $\mathbb{R}^3$. [Hint: $S^1 \times D^2$ is also called "a full torus".]

NOTES:

- Your solutions can be in Dutch or English (or a mixture of those).
- Please write down your name and the student no, and make sure your name is on each sheet.
- In total you can collect up to 13 points. The final mark is $\min\{10, \text{number of collected points}\}$.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! Please do not just write down the final result (that will not count!) but explain it/provide the details. In particular:
  - in Exercise 1, questions (c), (d), (e), (f), please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)).
  - in Exercise 2, please specify the group, the action, and don't forget to prove your claims.
  - etc
  - in Exercise 6, please prove that the map you describe (explicitly) is actually an embedding.

16.3 The 2024 retake exam (April 15, 2024)

Exercise 16.12 (2.5=0.5+ 1+ 1 pts). Assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a set X such that

$$\mathscr{T}_1 \subset \mathscr{T}_2.$$

Show that:

(a) if $(X,\mathscr{T}_1)$ is Hausdorff then so is $(X,\mathscr{T}_2)$;
(b) if $(X,\mathscr{T}_2)$ is compact then so is $(X,\mathscr{T}_1)$;
(c) if $(X,\mathscr{T}_2)$ is compact and $(X,\mathscr{T}_1)$ is Hausdorff then $\mathscr{T}_1$ and $\mathscr{T}_2$ must coincide.

Exercise 16.13 (4.5= 1+ 1+ 0.5+ 1+ 1 pts). On $X=\mathbb{R}$ we consider the family of subsets consisting of all intervals of type $[r,\infty)$ with $r\geq 0$, all intervals of type $(-\infty,-r]$ with $r\geq 0$ and the set $\{0\}$:

$$\mathscr{B} := \{(-\infty,-r] : r\geq 0\}\cup\{[r,\infty) : r\geq 0\}\cup\{\{0\}\}.$$

We denote by $\mathscr{T}$ the smallest topology on X containing $\mathscr{B}$.

(a) Show that $\mathscr{B}$ is a topology basis but not a topology.
(b) Compute the interior and the closure of $(-1,2)$ in $(X,\mathscr{T})$.
(c) Is $(X,\mathscr{T})$ connected?
(d) Is the sequence $(x_n)_{n\geq 1}$ given by $x_n = 1+\frac{(-1)^n}{n}$ convergent in $(X,\mathscr{T})$? To what?
(e) Describe a non-constant continuous function from $(X,\mathscr{T})$ to $\mathbb{R}$ endowed with the Euclidean topology. Then try to describe all of them (don't forget to prove all your claims).

Exercise 16.14 (2= 0.5+ 1.5 pts). Let $X=[-1,1]\times[-1,1]$ endowed with the Euclidean topology. Consider the group Γ that is the product of the group of integers modulo two, $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}=\left\{\widehat{0},\ \widehat{1}\right\}$, with itself:

$$\Gamma := \mathbb{Z}_2\times\mathbb{Z}_2 = \left\{(\widehat{0},\widehat{0}),\ (\widehat{0},\widehat{1}),(\widehat{1},\widehat{0}),(\widehat{1},\widehat{1})\right\}.$$

(a) show that the following defines an action of Γ on X:

$$(\widehat{k},\widehat{l})\cdot(x,y) := \left((-1)^k x,(-1)^l y\right)$$

(b) show that the resulting quotient space X/Γ is homeomorphic to X.

Exercise 16.15 (2.5= 0.5+1+1). Given a topological space X, by *a two-point compactification* of X we mean a pair (Y,i) consisting of a compact Hausdorff space together with a map $i:X\to Y$ which is an embedding and such that Y contains exactly two elements that not belong to the image of i. For $X=(0,1)\cup(2,3)$ endowed with the Euclidean topology:

(a) Give an example of such a two-point compactification, call it Y_1 (I am going to ignore the map i_1 from the notation, although you should indicate it on the picture).
(b) Find now another two-point compactification Y_2 of X such that Y_1 and Y_2 are not homeomorphic.
(c) Find two more, Y_3 and Y_4, not homeomorphic to Y_1 or Y_2 and not homeomorphic to each other.

Clarification: you can describe the spaces Y_1, Y_2, etc and the maps i_1, i_2, etc using pictures. But please provide the arguments/propositions that justify/support the claims that you make- e.g: "Y_1 is compact because ...", or "to see that Y_1 and Y_2 are not homeomorphic we use the X-method as follows ...".

Exercise 16.16 (2= 0.5+ 0.5+ 1 pts). Consider

$$\mathscr{A} = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous such that } f(x) = f(x+1) \text{ for all } x \in \mathbb{R}\}.$$

(a) Show that $\mathscr{A}$ is a sub-algebra of $C(\mathbb{R})$.
(b) Is $\mathscr{A}$ dense in $C(\mathbb{R})$?
(c) Show that the spectrum of $\mathscr{A}$ is homeomorphic to S^1.

NOTES:

- Please do not panic because of the length of the exam: in total you can collect up to 13.5 points, but you only need 10 to obtain the maximum mark.
- However, if you do feel overwhelmed by the length, please do not jump too much from one exercise to another trying to answer all of them, but rather focus on a couple that you choose carefully.
- I am kindly asking you to pay some attention to the handwriting. E.g., please make sure you use a proper pen/pencil/etc, and please try to avoid writing in very very small letters.
- Please write down your name and the student no, and make sure your name is on each sheet.
- your solutions can be in Dutch or English (or a mixture of those).
- Reminder: you are allowed to use four A3 pages (e.g. two sheets front and back) that you prepared in advance (handwritten or typed). No other material is allowed.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please do not just write down the final result (that will not count!) but explain it/provide the details, and please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well).

16.4 The 2024 exam (February 2nd, 2024), English

NOTES:

- Reminder: you are allowed to use four A3 pages (e.g. two sheets front and back) that you prepared in advance (handwritten or typed). No other material is allowed.
- your solutions can be in Dutch or English (or a mixture of those).
- Please write down your name and the student no, and make sure your name is on each sheet.
- In total you can collect up to 12 points. The final mark is $\min\{10, \text{number of collected points}\}$.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please do not just write down the final result (that will not count!) but explain it/provide the details, and please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)).

Exercise 16.17 (*This exercise is worth 6 points: 1pt for each of the items (1), (5) and (6), the others 0.5pts*). Assume that $(X,\mathscr{T})$ is a topological space. We say that $X = A_1 \cup A_2$ is a *decomposition* of $(X,\mathscr{T})$ if A_1, A_2 are subsets of X whose union is the entire X. For any such decomposition we endow A_1 and A_2 with the induced topology and we associate to it the following collection of subsets of X:

$$\mathscr{T}(A_1,A_2) := \{U \subset X : U \cap A_1 \text{ is open in } A_1 \text{ and } U \cap A_2 \text{ is open in } A_2\}.$$

(1) Show that $\mathscr{T}(A_1,A_2)$ is a topology on X and $\mathscr{T} \subset \mathscr{T}(A_1,A_2)$.

Next, a decomposition $X = A_1 \cup A_2$ will be called *convenient* if $\mathscr{T} = \mathscr{T}(A_1,A_2)$. Please show that:

(2) If $X = A_1 \cup A_2$ is a convenient decomposition of X then, for any function $f : X \to Y$ with values in another topological space Y, one has:

$$f \text{ is continuous} \iff f|_{A_1} : A_1 \to Y \text{ and } f|_{A_2} : A_2 \to Y \text{ are continuous.}$$

(3) If A_1 and A_2 are both open, or both closed, in X, then $X = A_1 \cup A_2$ is a convenient decomposition.

(4) When $X = \mathbb{R}$ endowed with the Euclidean topology $\mathscr{T}_{\text{Eucl}}$,

$$\mathbb{R} = (-\infty, 0) \cup [0, \infty)$$

is not a convenient decomposition.

Next, we denote by $\mathscr{T}_{\text{new}}$ the topology on $\mathbb{R}$ associated to the decomposition from item (4); hence

$$\mathscr{T}_{\text{new}} := \{U \subset \mathbb{R} : U \cap (-\infty,0) \text{ is open in } (-\infty,0) \text{ and } U \cap [0,\infty) \text{ is open in } [0,\infty)\},$$

where we emphasise that "open" refers to the Euclidean topology on $(-\infty,0)$ and $[0,\infty)$.

(5) Compute the closure of $A = (-1,0)$ and the interior of $B = [0,1]$ in $(\mathbb{R},\mathscr{T}_{\text{new}})$.

(6) Is $C = [-1,1]$, with the induced topology $\mathscr{T}_{\text{new}}|_C$, compact? But connected?

(7) Is the sequence given by $x_n := \frac{(-1)^n}{n}$ convergent in the topological space $(\mathbb{R},\mathscr{T}_{\text{new}})$?

(8) For what values of $\lambda \in \mathbb{R}$ is the following function

$$f_\lambda : (\mathbb{R},\mathscr{T}_{\text{new}}) \to (\mathbb{R},\mathscr{T}_{\text{Eucl}}), \quad f_\lambda(x) = \begin{cases} x & \text{if } x < 0 \\ \lambda & \text{if } x = 0 \\ x + \lambda^2 & \text{if } x > 0 \end{cases}$$

continuous?

(9) Does there exists λ such that f_λ is an embedding?

Exercise 16.18. *(2 pts)* Consider the action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \left\{\widehat{0},\ \widehat{1}\right\}$ (the groups of integers modulo 2) on the unit two-dimensional sphere S^2 given by

$$\widehat{k}\cdot(x,y,z) = \left(x,y,(-1)^k z\right) \qquad \left(\text{for } \widehat{k}\in\mathbb{Z}_2,\ (x,y,z)\in S^2\right).$$

Show that the resulting quotient space $S^2/\mathbb{Z}_2$ is homeomorphic to the closed unit disk D^2.

Exercise 16.19. *(2 pts)* Which of the following spaces are homeomorphic to each other (and why):

(a) $C_0 = (S^1\times[0,1))^+$, the one-point compactification of $S^1\times[0,1)$.
(b) $C_1 = \mathrm{Cone}(S^1)$, the abstract cone associated to the space S^1.
(c) the concrete cone $C_2 = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 = (1-z)^2,\ 0\le z\le 1\right\}$.

Exercise 16.20. *(2 pts)* On S^2 we consider the functions $f,g,h : S^2\to\mathbb{R}$ given by:

$$f(x,y,z) = x, \quad g(x,y,z) = y, \quad h(x,y,z) = z.$$

We also consider the collection of all polynomial expressions generated on f, g and h:

$$\mathscr{A} := \left\{ \sum_{0\le i,j,k\le d} c_{i,j,k} f^i g^j h^k : d\in\mathbb{N}, c_{i,j,k}\in\mathbb{R} \right\} \subset C(S^2).$$

1. Show that $\mathscr{A}$ is dense in $C(S^2)$.
2. Show that for any character $\chi : \mathscr{A}\to\mathbb{R}$, the numbers
$$u := \chi(f), \quad v := \chi(g), \quad w := \chi(h)$$
must satisfy $u^2+v^2+w^2 = 1$.
3. Show that for any $u,v,w\in\mathbb{R}$ satisfying $u^2+v^2+w^2 = 1$, there exists a character $\chi : \mathscr{A}\to\mathbb{R}$ with $\chi(f) = u$, $\chi(g) = v$, $\chi(h) = w$, and this character is unique.
4. The previous two items tell us that we have a bijection
$$\Psi : X_{\mathscr{A}} \to S^2, \quad \chi\mapsto(\chi(f),\chi(g),\chi(h)).$$
Show that this is actually a homeomorphism.
(Recall here that $X_{\mathscr{A}}$ (the topological spectrum of the algebra $\mathscr{A}$) is the set of all characters on $\mathscr{A}$ and its topology is the smallest one with the property that, for each $a\in\mathscr{A}$, the map $F_a : X_{\mathscr{A}}\to\mathbb{R}$, $F_a(\chi) = \chi(a)$ is continuous.)

AGAIN: PLEASE EXPLAIN /GIVE AS MANY DETAILS AS POSSIBLE ...

... REMEMBER THAT SOMEBODY ELSE (WHO CANNOT READ YOUR MIND) WILL EVALUATE YOUR SOLUTIONS

... and: GOOD LUCK!

16.5 The 2023 retake exam (March 16 , 2023)

Retake Exam Inleiding Topologie, March 16, 2023

- Every claim you make must be justified, either by argument or by citing a result from class. For example, if you are asked whether a certain space is a manifold, you have to prove that it is a manifold or prove that it is not a manifold – unless the example was explicitly shown in class to be or not to be a manifold, in which case you can cite it. Likewise: if you are asked for an example with a certain property, you have to prove that your example has this property, etc.
- Even if you cannot solve (a part of) a problem, you are allowed to use it in a later (part of a) problem.
- You can answer in Dutch or English. In any case, write readable!

Problem 1 (3+4 points)**.** *Consider the set $\mathbb{Z}$ of integers. Define $\mathcal{T}$ as the collection of all subsets of $\mathbb{Z}$ containing 0.*

(i) Show that $\mathcal{T}$ is not a topology, but one can make it into a topology by adding a single subset of $\mathbb{Z}$.

(ii) Decide whether the resulting topological space is Hausdorff and whether it is second-countable. (If it is second-countable, you should give explictly a countable basis of topology; if it is not, you should show that this is impossible.)

Problem 2 (3+3 points)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Recall that $X \sqcup Y$ denotes their disjoint union. This is a set such that X and Y are subsets of $X \sqcup Y$, and we have $X \cup Y = X \sqcup Y$ and $X \cap Y = \varnothing$. Recall further that we equip $X \sqcup Y$ with the disjoint union topology $\mathcal{T}_{X \sqcup Y}$, which is defined as $\{U \subseteq X \sqcup Y : U \cap X \in \mathcal{T}_X$ and $U \cap Y \in \mathcal{T}_Y\}$.*

(i) Let $(Z, \mathcal{T}_Z)$ be a further topological space and $f \colon X \sqcup Y \to Z$ a function. Show that f is continuous if and only if $f|_X \colon X \to Z$ and $f|_Y \colon Y \to Z$ are continuous. (Use nothing but the definition of continuity and the axioms of a topological space.)

(ii) Show that if X and Y are compact, so is $X \sqcup Y$.

2

Problem 3 (1+4+12)**.** *Consider the subspaces*

- $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,
- $A_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$,
- $A_3 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$,
- $A_4 = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1 \text{ or } (x+1)^2 + y^2 \leq 1\}$.

of $\mathbb{R}^2$ with the Euclidean topology.

(i) Sketch $A_1, \ldots, A_4$.

(ii) Describe the interiors of $A_1, \ldots, A_4$. Which of these are connected?

(iii) For every $1 \leq i < j \leq 4$ decide whether A_i and A_j are homeomorphic.

Problem 4 (6 points)**.** *Give examples of two manifolds X and Y such that the one-compactification X^+ of X is a manifold, while the one-point compactification Y^+ of Y is not a manifold.*

Problem 5 (2+6)**.** *Let M be the Möbius strip, i.e. the quotient of $[0,1]^2$ by the smallest equivalence relation $\sim$ on $[0,1]^2$ such that $(0,t) \sim (1,1-t)$.*

(i) Define an embedding $f\colon S^1 \to M$ such that the image $f(S^1)$ equals the set of equivalence classes of the points (x,y) with $x = 0$ or 1, and show that it is an embedding.

(ii) Consider $X = (S^1 \times [0,1]) \sqcup M$ with the disjoint union topology and let $\simeq$ be the smallest equivalence relation on X such that $(x,0) \simeq f(x)$ for $(x,0) \in S^1 \times [0,1]$. Denote by Y the quotient $X/\simeq$ with the quotient topology. (Informally, this is glueing a cylinder and a Möbius strip along a boundary curve.) Show that Y is homeomorphic to M.

Problem 6 (6 points)**.** *Let (X,d) be a compact metric space. Let $U_1, \ldots, U_n \subseteq X$ be an open cover of X. Show that there exists a $\delta > 0$ such that for every subset $A \subseteq X$ with $d(a,b) < \delta$ for all $a,b \in A$, there exists an index i with $A \subseteq U_i$.*

16.6 The 2023 exam (February 2 , 2023)

Exam Inleiding Topologie, February 2, 2023

- Every claim you make must be justified, either by argument or by citing a result from class. For example, if you are asked whether a certain space is a manifold, you have to prove that it is a manifold or prove that it is not a manifold – unless the example was explicitly shown in class to be or not to be a manifold, in which case you can cite it. Likewise: if you are asked for an example with a certain property, you have to prove that your example has this property, etc.
- Even if you cannot solve part (i) of a problem, you are allowed to use it in part (ii).
- You can answer in Dutch or English. In any case, write readable!

Problem 1 (1+5 points)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.*

(i) For $A \subseteq X$, state the definition of the subspace topology on A.

(ii) Let $A, B \subseteq X$ be open subsets with $A \cup B = X$. Let $f\colon X \to Y$ be a map such that $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous, where we equip A and B with the subspace topology. Prove that f is continuous.

Problem 2 (3+4 points)**.** *Consider the set $\mathbb{N}$ of positive integers. Define $\mathcal{T}$ as the collection of all subsets of $\mathbb{N}$ of the form $U_n = \{k \in \mathbb{N} : k \geq n\}$ for $n \in \mathbb{N}$.*

(i) Show that $\mathcal{T}$ is not a topology, but one can make it into a topology by adding a single subset of $\mathbb{N}$.

(ii) Decide whether the resulting topological space is Hausdorff and whether it is second-countable (i.e. has a countable basis of topology).

2

Problem 3 (1+9+5 points). *Consider the subspaces*

- $A_1 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$,
- $A_2 = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$,
- $A_3 = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$,
- $A_4 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

of $\mathbb{R}^2$ with the Euclidean topology.

(i) Sketch $A_1, \dots, A_4$.

(ii) Which A_i are manifolds?

(iii) For every $1 \leq i < j \leq 4$ decide whether A_i and A_j are homeomorphic.

Problem 4 (6 points). *Equip the set $\mathcal{C}([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the topology of uniform convergence, induced by the metric*

$$d(f,g) = \max_{x\in[0,1]} (|f(x) - g(x)|).$$

Show that there is no compact subset $K \subseteq \mathcal{C}([0,1],\mathbb{R})$ such that the functions

$$f_n \colon [0,1] \to \mathbb{R}, \qquad x \mapsto nx$$

are contained in K for all $n \in \mathbb{N}$.

Problem 5 (4+4 points). *Let $\mathbb{P}^2 = D^2/\sim$ be the projective plane, with $\sim$ being the smallest equivalence relation such that $x \sim (-x)$ for all $x \in S^1 = \partial D^2$.*

(i) Give an example of an embedding $f \colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus f(S^1)$ is path-connected.

(ii) Give examples of embeddings $f, g \colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus f(S^1)$ is not homeomorphic to $\mathbb{P}^2 \setminus g(S^1)$.

Problem 6 (8 points). *Let M be the Möbius strip, i.e. the quotient of $[0,1]^2$ by the smallest equivalence relation $\sim$ on $[0,1]^2$ such that $(0,t) \sim (1,1-t)$. Define an action of the group $(\mathbb{Z},+)$ on $\mathbb{R} \times [0,1]$ such that the quotient of $\mathbb{R} \times [0,1]$ by this group action is homeorphic to M. (Here, we equip $\mathbb{R} \times [0,1] \subset \mathbb{R}^2$ with the subspace topology of the Euclidean topology.)*

16.7 The 2022 retake exam (April 2022)

Below you find first an English, then a Dutch version of the exam. You can write all solutions in Dutch or English.
Notes:

- Write your solutions by hand. Write legible. Don't write close to the margins.
- After the exam you have 40 minutes to scan and upload your solutions **as a pdf in one file**. Check their readability and that you have scanned all pages.
- All claims have to be proven (if they are not known from the lecture or exercises). In particular if you are asked for an example you have to **prove** that the example satisfies the requirements.
- Read through all problems before you start to work.
- If you are unable to solve the first part of a question, you can still use the result in the second part.

Exercise 16.21. Consider the following three subsets of $\mathbb{R}^2$ equipped with the subspace topology:

- $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- $A_2 = \{(x,y) \in \mathbb{R}^2 : \frac{1}{4}x^2 + y^2 = 1\}$
- $A_3 = \{(x,y) \in \mathbb{R}^2 : y^2 = 1\}$

Give a sketch of all three subspaces. Then decide for each pair of subspaces whether they are homeomorphic or not: if they are, give a homeomorphism; if they are not, give an argument why they are not homeomorphic.

Exercise 16.22. Let $\mathbb{N} = \{0,1,2,\dots\}$ be the natural numbers and $X = \{p \in \mathbb{N} : p = 0 \text{ or } p \text{ is prime}\}$. For every $k \in \mathbb{N}$ with $k > 0$ define $D_k = \{p \in X : p \text{ does not divide } k\}$, where we use the convention that 0 only divides itself. Let $\mathscr{T}$ be the smallest topology on X such that all D_k are in $\mathscr{T}$.

(a) Show a subspace $A \subset X$ is closed if and only if $A = X$ or A is finite and does not contain 0.
(b) Decide (with argument) whether $\mathscr{T}$ is Hausdorff and whether $\mathscr{T}$ is compact.

Exercise 16.23. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $A \subset Y$ be a subset equipped with the subspace topology; we denote by $i\colon A \to Y$ the inclusion. Show that a function $f\colon X \to A$ is continuous if and only if the composition $i \circ f\colon X \to Y$ is continuous.

Exercise 16.24. [6 points] Give examples $i, j\colon S^1 \to M$ of embeddings of the circle into the Möbius strip such that $M \setminus i(S^1)$ is not homeomorphic to $M \setminus j(S^1)$.

The rest of the exercises are missing.

16.8 The 2022 exam (January 2022)

Below you find first an English, then a Dutch version of the exam. You can write all solutions in Dutch or English. Notes:

- Write your solutions by hand. Write legible. Don't write close to the margins.
- After the exam you have 40 minutes to scan and upload your solutions **as a pdf in one file**. Check their readability and that you have scanned all pages.
- All claims have to be proven (if they are not known from the lecture or exercises). In particular if you are asked for an example you have to **prove** that the example satisfies the requirements.
- Read through all problems before you start to work.
- If you are unable to solve the first part of a question, you can still use the result in the second part.

Exercise 16.25. [5 points] Consider the equivalence relation $\sim$ on $\mathbb{R}$, where $x \sim y$ iff x and y are both positive or both negative or both zero. Let $X = \mathbb{R}/\sim$ with the quotient topology. List (with proof) all closed subsets of X.

Exercise 16.26. [10 points] Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f\colon X \to Y$ a function. Let further $(C_i)_{i\in I}$ be a cover of X by closed subsets $C_i \subseteq X$, i.e. $\bigcup_{i\in I} C_i = X$. Assume that $f|_{C_i}\colon C_i \to Y$ is a continuous function for each $i \in I$, where we equip C_i with the subspace topology.

(a) Give an example of X, Y, f and $(C_i)_{i\in I}$ as above, where f is *not* continuous.
(b) Show that f is necessarily continuous if the indexing set I is finite.

Exercise 16.27. [6 points] Let $X = \mathscr{C}([0,1], \mathbb{R})$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. We consider on it the topology induced by the metric $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Show that X is path-connected.

Exercise 16.28. [6 points] Give examples $i, j\colon S^1 \to T$ of embeddings of the circle into the torus such that $T \setminus i(S^1)$ is not homeomorphic to $T \setminus j(S^1)$.

Exercise 16.29. [10 points]

(a) Consider the open cover $\mathscr{U} = \{(-2,1), (-1,2)\}$ of the interval $(-2,2)$ with the Euclidean topology. Give an example of a partition of unity subordinate to $\mathscr{U}$.
(b) Let $X = \{x, y\}$ be a set with two elements. Give an example of a topology on X such that every open cover[1] has a subordinate partition of unity, but the topology is not Hausdorff.

Exercise 16.30. [13 points] Let $p, q \in S^2$ be the points $(1,0,0)$ and $(-1,0,0)$. Let $X = S^2/\{p,q\}$ with the quotient topology.

(a) Show that X is not a 2-dimensional manifold.
(b) Show that $S^2 \setminus \{p,q\}$ is not homeomorphic to $\mathbb{R}^2$. (Hint: Use one-point compactifications.)

16.9 The 2021 retake exam (April 2021)

Exercise 16.31. [6 punten] Bekijk de verzamelingen

$$X = \left\{(x,y) \in \mathbb{R}^2 : xy = 1\right\}$$

en

$$Y = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1.\right\}$$

met de deelruimtetopologie van de Euclidische topologie op $\mathbb{R}^2$.

1. Schets X en Y. (1 pt)
2. Zijn X en Y homeomorf? Geef een bewijs. (5 pt)

Exercise 16.32. [7 punten] Zij $X = \{c,o\}$ een verzameling met twee elementen. Bekijk de topologie $\mathscr{T}_X = \{\varnothing, \{o\}, X\}$ op X. Bekijk $X \times X$ met de producttopologie.

1. Noem alle open deelverzamelingen van $X \times X$. (3pt)
2. Zij $A = \{(c,o)\} \subset X \times X$. Bepaal het inwendige en de afsluiting van A. (4pt)

Exercise 16.33. [12 punten] Geef een voorbeeld van een topologische ruimte die

1. tweedeaftelbaar, maar niet Hausdorff is (4pt);
2. Hausdorff, maar niet tweedeaftelbaar is (4pt);
3. compact, maar niet samenhangend is (4pt).

Geef in ieder voorbeeld een bewijs voor de eigenschappen (tenzij al bekend uit het hoorcollege).

Exercise 16.34. [7 punten] Zij $\mathbb{R}^+ = \mathbb{R} \cup \{\infty\}$ de één-punt-compactificatie van $\mathbb{R}$ met de topologie

$$\mathscr{T}^+ = \mathscr{T}_{\text{eucl}} \cup \left\{\mathbb{R}^+ \setminus K : K \subset \mathbb{R} \text{ compact}\right\},$$

waar $\mathscr{T}_{\text{eucl}}$ de Euclidische topologie op $\mathbb{R}$ is.
Bewijs dat er geen continue functie $f\colon \mathbb{R}^+ \to \mathbb{R}$ is met $f(x) = x$ voor alle $x \in \mathbb{R}$.

Exercise 16.35. [9 punten]

1. Geef een voorbeeld van een inbedding $i\colon S^1 \to T$ van een cirkel naar een torus zodat $T \setminus i(S^1)$ niet samenhangend is. (4pt)
2. Geef voorbeelden van inbeddingen $i, j\colon S^1 \to T$ zodat $T \setminus i(S^1)$ en $T \setminus j(S^1)$ niet homeomorf met elkaar zijn. (5pt)

Geef voor ieder voorbeeld een bewijs.

Exercise 16.36. [9 punten] Zij $N \in \mathbb{Z}_{>0}$ een positief getal en $(X, \mathscr{T})$ een compacte topologische ruimte met een inbedding $\Phi\colon X \to \mathbb{R}^N$, waar $\mathbb{R}^N$ de euclidische topologie heeft. Bewijs dat $\Phi(X) \subset \mathbb{R}^N$ niet open kan zijn als X niet-leeg is.

16.10 The 2021 exam (January 2021)

Schrijf leesbaar. Als je een deel van een opgave niet kunt oplossen, kun je het resultaat niettemin bij de volgende delen gebruiken. Veel succes!

Exercise 16.37. [4 punten] Zij $\mathscr{T}$ de verzameling

$$\{[a,b] : a < b \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$$

van deelverzamelingen van $\mathbb{R}$. Is $\mathscr{T}$ een topologie? Geef een bewijs.

Exercise 16.38. [11 punten] Bekijk de verzamelingen

$$X = (-1,0) \cup (0,1) \subset \mathbb{R}$$

en

$$Y = \left\{(x,y) \in \mathbb{R}^2 : d((0,0),(x,y)) \leq 1 \text{ of } d((2,0),(x,y)) \leq 1.\right\}$$

1. Schets X en Y. (1 pt)
2. Zijn X en Y homeomorf? Geef een bewijs. (4 pt)
3. Bereken voor X en Y het inwendige en de afsluiting. Geef voor al deze vier ruimtes aan of ze

   - samenhangend en/of
   - compact en/of
   - eerstaftelbaar

   zijn. Je hoeft bij deze deelopgave geen bewijzen te geven. (6 pt)

Exercise 16.39. [6 punten] Laat $(X, \mathscr{T}_X)$ en $(Y, \mathscr{T}_Y)$ topologische ruimtes zijn en $\mathscr{U} = \{U_i\}_{i \in I}$ een open dekking van X. Zij $f\colon X \to Y$ een afbeelding zodat de beperking $f|_{U_i}\colon U_i \to Y$ continu is voor iedere $i \in I$ (waar we U_i met de deelruimtetopologie bekijken). Bewijs dat f continu is.

Exercise 16.40. [7 punten] Geef een voorbeeld van een topologische ruimte $(X, \mathscr{T})$ zodat er voor iedere $x \in X$ een omgeving U bestaat zodat U en $\mathbb{R}^2$ homeomorf zijn, maar X niet Hausdorff is. Geef een bewijs. (Hint: Neem een quotient van twee kopieën van $\mathbb{R}^2$.)

Exercise 16.41. [10 punten] Zij K de kleinfles. Beantwoord iedere vraag met ja of nee en geef een bewijs.

1. Bestaat er een continue inbedding $g\colon S^1 \to K$ zodat $K \setminus g(S^1)$ samenhangend is? (5 pt)
2. Bestaat er een surjectieve continue afbeelding $f\colon [0,1] \to K$? (5 pt)

Exercise 16.42. [12 punten]

1. Bewijs dat er geen inbedding $S^1 \to \mathbb{R}$ bestaat. (6 pt)
2. Bewijs dat iedere injectieve continue afbeelding $S^1 \to S^1$ een homeomorfisme is. (6 pt)

16.11 The 2020 retake exam

Hertentamen Inleiding Topologie, WISB243 2020-04-14, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise! Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 40. The exam grade E will be obtained from your total score T by rounding off $\min(T/4, 10)$ to one decimal accuracy.
- You are free to write the solutions either in English, or in Dutch.

Succes !

10 pt total **Exercise 1.** Let $\mathscr{B}$ be the family of subsets of the form

$$\mathscr{B} = \{(a,b) : \ -\infty < a < b < \infty\} \bigcup \{\mathbb{Q} \cap (a,b) : \ -\infty < a < b < \infty\}.$$

2 pt (a) Show that $\mathscr{B}$ is a topology basis on $\mathbb{R}$.

2 pt (b) Let $\mathscr{T}$ be the topology generated by $\mathscr{B}$. Show that $(\mathbb{R}, \mathscr{T})$ is a Hausdorff space.

2 pt (c) Show that $\mathbb{R} \setminus \mathbb{Q}$ is closed in $(\mathbb{R}, \mathscr{T})$.

2 pt (d) Show that the topological space $(\mathbb{R}, \mathscr{T})$ is not normal.

2 pt (e) Let $F : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{eucl})$ be a continuous map with $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ and where $\mathscr{T}_{eucl}$ is the euclidean topology on $\mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

9 pt total **Exercise 2.** Let $X = \{(x,y) : \ y \geq 0, \ (x,y) \neq (0,0)\}$. We understand X as a topological space with the induced subspace topology from $X \subset \mathbb{R}^2$.

3 pt (a) Show that the subsets $E = \{(x,0) : \ x < 0\}$ and $F = \{(x,0) : \ x > 0\}$ are closed subsets in X.

3 pt (b) Construct explicitly a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ if $x \in E$ and $f(x) = 0$ if $x \in F$.

3 pt (c) Is X a normal topological space? (give a proof for your answer).

9 pt total **Exercise 3.** Let (X,d_1) and (X,d_2) be two metric spaces and assume that (X,d_1) is a compact topological space (with the topology indued by the metric). Let $f : X \to Y$ be a continuous map. Show that f is uniformly continuous, i.e. that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_1(x,y) < \delta \Rightarrow d_2(f(x),f(y)) < \varepsilon.$$

12 pt total **Exercise 4.** Let X,Y be topological spaces. We call a function $f : X \to Y$ proper if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$.

3 pt (a) Let X,Y be topological spaces and $f : X \to Y$ a continuous map. Show that $f(C) \subset Y$ is compact for every $C \subset X$ compact

3 pt (b) Let X,Y be locally compact Hausdorff spaces with one-point compactifications X_∞ and Y_∞. For a map $f : X \to Y$ we define $\hat{f} : X_\infty \to Y_\infty$ by setting $\hat{f}(x) = f(x)$ for $x \in X$ and $\hat{f}(\infty) = \infty$. Show that $\hat{f} : X_\infty \to Y_\infty$ is continuous if and only if the map $f : X \to Y$ is continuous and proper.

3 pt (c) Let X be a locally compact Hausdorff space and $C \subset X$ a closed subset. Show that $C \cup \{\infty\} \subset X_\infty$ is closed. Moreover show that a subset $C' \subset X_\infty$ is closed if and only if C' is compact.

3 pt (d) Let X,Y be locally compact Hausdorff spaces and $f : X \to Y$ a continuous and proper map. Use the previous parts of this exercise to show that if $C \subset X$ is closed, then $f(C) \subset Y$ is closed.

16.12 The 2020 exam (January 28, 2020)

Tentamen Inleiding Topologie, WISB243 2020-01-28, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise! Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 48.
- You are free to write the solutions either in English, or in Dutch.

Succes !

10 pt total **Exercise 1.** Let $\mathbb{N} = \{1,2,3,\dots,\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathscr{B}$ be the family of subsets of $\mathbb{N}$ of the form $U_{a,b} = \{a+nb : n \in \mathbb{N}_0\}$ where $a,b \in \mathbb{N}$ are coprime, i.e. $\gcd(a,b) = 1$.

5 pt (a) Show that $\mathscr{B}$ is a topology basis on $\mathbb{N}$.

5 pt (b) Let $\mathscr{T}$ be the topology on $\mathbb{N}$ generated by the topology basis $\mathscr{B}$. Show that the topological space $(\mathbb{N}, \mathscr{T})$ is Hausdorff.

9 pt total **Exercise 2.** Let $(X, \mathscr{T})$ be a locally compact Hausdorff space.

3 pt (a) Let $U \subset X$ be an open subset. Show that U is locally compact (with the induced subspace topology).

3 pt (b) Let $C \subset X$ be closed. Show that C is locally compact (with the induced subspace topology).

3 pt (c) Let $U \subset X$ open and $C \subset X$ closed. Show that $U \cap C$ is locally compact.

11 pt total **Exercise 3.** Let $J = [-1,1]$, let $S := \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $X := S \times J$ be equipped with the product topology. Let $\Gamma = \{1,-1\}$ be the multiplicative group of two elements. We consider the action φ of Γ by homeomorphisms on the space X given by

$$\varphi : \Gamma \times X \to X, \quad (\gamma, (z_1,z_2,t)) \mapsto (\gamma z_1, \gamma z_2, \gamma t).$$

2 pt (a) Show that every orbit for this action consists of precisely two points.

We equip the quotient X/Γ with the quotient topology, and denote the canonical projection by $p : X \to X/\Gamma$. We consider the map $\sigma : [0,1]^2 = [0,1] \times [0,1] \to X/\Gamma$ given by

$$\sigma(s,t) = p(\cos \pi s, \sin \pi s, 2t-1), \quad (0 \leq s,t \leq 1).$$

3 pt (b) Show that σ is surjective.

We define the equivalence relation $\sim$ on $[0,1]^2$ by $(s,t) \sim (s',t') \iff \sigma(s,t) = \sigma(s',t')$. The quotient space $[0,1]^2/\sim$ is equipped with the quotient topology. We denote the associated natural projection by $q : [0,1]^2 \to [0,1]^2/\sim$.

5 pt (c) Prove that there exists a homeomorphism

$$\bar{\sigma} : [0,1]^2/\sim \ \longrightarrow \ X/\Gamma$$

such that $\bar{\sigma} \circ q = \sigma$.

1 pt (d) Which of the following assertions is correct: (1) X/Γ is homeomorphic to the cylinder; (2) X/Γ is homeomorphic to the Möbius band. (Here you are not required to give any justification.)

10 pt total **Exercise 4.** We call a topological space extremally disconnected space if for every two disjoint open subsets U and V one has $\overline{U} \cap \overline{V} = \emptyset$. Here $\overline{U}$ and $\overline{V}$ are the closures of the sets U and V.

4 pt (a) Prove the following statement: A topological space is extremally disconnected if and only if the closure of every open subset U is open.

3 pt (b) Show that an extremally disconnected Hausdorff space with at least two elements is not connected.

3 pt (c) Give an example of a topological space X that has the following properties

- X has at least two points
- X is connected
- X is extremally disconnected
- for every pair of distinct points, each has a neighborhood not containing the other point.

8 pt total **Exercise 5.** Let X be a compact Hausdorff space and $x_0 \in X$.

3 pt (a) Show that $Y = X \setminus \{x_0\}$ is a locally compact Hausdorff space.

5 pt (b) Let Y_∞ be the one-point compactification of Y. Show that Y is homeomorphic to X.

16.13 The 2019 retake exam (April 15, 2019)

Herkansing Inleiding Topologie, WISB243 2019-04-15, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Aldo and Francesco, 2: Maarten) and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise!
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The grade will be obtained from your total score T by rounding off $\min(T/4, 10)$ to a half integer above 6 or an integer below 6.5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** For $\mathbb{R}$ we consider the collection $\mathscr{B}$ of all subsets of the form

$$[p,q) := \{x \in \mathbb{R} \mid p \leq x < q\}, \qquad \text{with } p,q \in \mathbb{Q}, p < q.$$

3 pt (a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. Denote by $\mathscr{T}$ the topology generated by $\mathscr{B}$.

2 pt (b) Show that $\mathscr{T}$ contains the Euclidean topology.

2 pt (c) Is $(\mathbb{R}, \mathscr{T})$ first countable, second countable? Is it Hausdorff?

3 pt (d) Show that $[0,1)$ is open and closed in $(\mathbb{R}, \mathscr{T})$. Is $(\mathbb{R}, \mathscr{T})$ connected?

3 pt (e) Show that $[0,1)$ and $[0,1]$ are not compact in $(\mathbb{R}, \mathscr{T})$.

9 pt total **Exercise 2.** Let $X := [0,1] \times [-2,2]$ and the subset $B := [0,1] \times [-1,1] \subset X$ both be equipped with the topology induced by the Euclidean topology on $\mathbb{R}^2$.

We equip X with the equivalence relation R whose equivalence classes are given by $\{(s,t)\}$ for $0 < s < 1$ and $-2 < t < 2$, $\{(s,-2),(s,2)\}$ for $0 < s < 1$; $\{(0,t),(1,-t)\}$ for $-2 < t < 2$ and, finally, $\{(0,\pm 2),(1,\pm 2)\}$. Accordingly, X/R equipped with the quotient topology, is the Klein bottle (which is known to be a Hausdorff space). The associated quotient map is denoted by $\pi : X \to X/R$.

3 pt (a) The restriction of R to B is the relation R_B defined by $b_1 R_B b_2 \iff \pi(b_1) = \pi(b_2)$, for $b_1, b_2 \in B$. Show that R_B is an equivalence relation on B and explicitly determine the associated equivalence classes in B.

1 pt (b) The quotient B/R_B is equipped with the quotient topology. To which well known space is this quotient homeomorphic? (You need not justify your answer.)

Let $\pi_B : B \to B/R_B$ be the associated quotient map.

2 pt (c) Show that there exists a unique map $f : B/R_B \to X/R$ such that for all $b \in B$ we have $f(\pi_B(b)) = \pi(b)$.

3 pt (d) Prove that the map f is an embedding.

11 pt total **Exercise 3.** Let M be a topological space, and assume that $\gamma \mapsto \varphi_\gamma$ is an action of the group $\mathbb{Z}_2 = \{-1, 1\}$ on M by homeomorphisms. Let $M/\mathbb{Z}_2$ be the associated quotient (equipped with the quotient topology), and $\pi : M \to M/\mathbb{Z}_2$ the quotient map.

2 pt (a) Given a subset $V \subset M$, show that $\pi^{-1}(\pi(V)) = V \cup \varphi_{-1}(V)$.

2 pt (b) Show that for $V \subset M$ open, the set $\pi(V)$ is open in $M/\mathbb{Z}_2$.

3 pt (c) Let $\{U_i\}_{i \in I}$ be an open cover of M. For every $m \in M$ let $i_m, j_m \in I$ be indices such that $m \in U_{i_m}$ and $\varphi(m) \in U_{j_m}$. Show that there exists an open neighborhood V_m of m such that $V_m \subset U_{i_m}$ and $\varphi(V_m) \subset U_{j_m}$.

4 pt (d) Show that M is compact if and only if $\pi(M)$ is compact. Hint: for one of the implications consider the collection $\{\pi(V_m)\}$ resulting from (c).

11 pt total **Exercise 4.** For M a locally compact Hausdorff space we denote by $C_c(M)$ the real linear space of continuous functions $M \to \mathbb{R}$ with compact support. If U is an open subset of M, we put $C_c(U) := \{f \in C_c(M) \mid \mathrm{supp} f \subset U\}$.

2 pt (a) If U is open in M, $f \in C_c(M)$, $\psi \in C(M)$ and $\mathrm{supp}\,\psi \subset U$, show that $\psi f \in C_c(U)$.

By a positive integral on an open subset U of M we mean a linear map $I : C_c(U) \to \mathbb{R}$ such that for all $f \in C_c(U)$ we have:

$$f \geq 0 \;\Rightarrow\; I(f) \geq 0.$$

A positive integral I on U is said to be strictly positive if for all $f \in C_c(U)$ we have

$$f \geq 0,\, I(f) = 0 \;\Rightarrow\; f = 0.$$

1 pt (b) Prove the following result. If I is a positive integral on an open subset U of M and $\psi : M \to \mathbb{R}$ is continuous function with $\psi \geq 0$ and $\mathrm{supp}\psi \subset U$ then $I_\psi : f \mapsto I(\psi f)$ is a positive integral on M.

5 pt (c) Assume that M is second countable, and that for every point $m \in M$ there exists an open neighborhood $U_m \ni m$ and a strictly positive integral $I_m : C_c(U_m) \to \mathbb{R}$ on U_m. Show that there exists a strictly positive integral on M.

In the next item you may use without proof that the map $J : C_c((0,1)^n) \to \mathbb{R}$ defined by the n-fold repeated Riemann integral $J(f) = \int_0^1 \cdots \int_0^1 f(x)\, dx_1 \cdots dx_n$, is a strictly positive integral on the open subset $(0,1)^n$ of $\mathbb{R}^n$.

3 pt (d) If M is a topological manifold of dimension $n \geq 1$, show that there exists a strictly positive integral on M.

16.14 The 2019 exam (January 29, 2019)

Tentamen Inleiding Topologie, WISB243 2019-01-29, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Aldo and Francesco, 2: Maarten) and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise!
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The exam grade E will be obtained from your total score T by rounding off $\min(T/4, 10)$ to one decimal accuracy.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** For $\mathbb{R}$ we consider the collection $\mathscr{B}$ of all subsets of the form

$$(a,n] := \{x \in \mathbb{R} \mid a < x \leq n\},$$

with $n \in \mathbb{Z}$ and $a \in \mathbb{R}$ such that $a < n$.

2 pt (a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. Denote by $\mathscr{T}$ the topology generated by $\mathscr{B}$.

4 pt (b) Is $(\mathbb{R}, \mathscr{T})$ second countable? Is it Hausdorff? Is it metrizable?

Consider the set $A := [\frac{1}{2}, 1]$.

3 pt (c) Determine the closure of A.

4 pt (d) Show that A is compact, and that $[0,1]$ is not compact.

10 pt total **Exercise 2.**

1 pt (a) Show that $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected for the topology induced by the Euclidean topology.

4 pt (b) Show that there exists no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

5 pt (c) If M is a two-dimensional topological manifold, show that there exists no continuous injective map $g : M \to S^1$. Hint: use (b).

11 pt total **Exercise 3.** Let X be topological space, and $A \subset X$ a subset. The space X/A obtained by collapsing A to a point is equipped with the quotient topology. The quotient map is denoted by $\pi : X \to X/A$.

1 pt (a) Show that $\pi^{-1}(\pi(A)) = A$.

2 pt (b) If $S \subset X - A$ show that $\pi^{-1}(\pi(S)) = S$.

2 pt (c) If $T \subset X$ satisfies $T \supset A$, show that $\pi^{-1}(\pi(T)) = T$.

From now on we assume that A is connected for the induced topology.

4 pt (d) If X is the union of two disjoint non-empty open subsets U and V, show that $\pi(U)$ and $\pi(V)$ are disjoint open subsets of X/A.

2 pt (e) Show that X is connected if and only if X/A is connected.

10 pt total **Exercise 4.** Let X be a paracompact topological space and A a closed subset of X. The goal of this exercise is to show that A is paracompact for the induced topology. For this, we assume that $\mathscr{V} = \{V_i\}_{i \in I}$ is an open covering of A.

1 pt (a) Show that there exists a family $\mathscr{U} = \{U_i\}_{i \in I}$ of open subsets of X such that $V_i = U_i \cap A$ for all $i \in I$.

2 pt (b) Show that $\mathscr{U}^* := \mathscr{U} \cup \{X - A\}$ is an open covering of X.

1 pt (c) Show that there exists a locally finite refinement $\mathscr{W} = \{W_j \mid j \in J\}$ of $\mathscr{U}^*$.

3 pt (d) Let $J_0 = \{j \in J \mid W_j \subset X - A\}$ and put $J_1 = J - J_0$. Show that $\mathscr{W}' = \{W_j \cap A\}_{j \in J_1}$ is an open covering of A which is subordinate to $\mathscr{V}$.

3 pt (e) Show that A is paracompact.

16.15 The 2018 retake exam (April 16, 2018)

Retake Inleiding Topologie, WISB243 2018-04-16, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Lauran, 2: Luca) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score T by rounding off $\min(T/4, 10)$ in the usual way.
- You are free to write the solutions either in English, or in Dutch.

Succes !

11 pt total **Exercise 1.** Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ such that

(1) $[0,1) \in \mathscr{T}$;

(2) the map $V : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$ is continuous with respect to $\mathscr{T}$;

(3) for each $c \in \mathbb{R}$ the map $T_c : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + c$ is continuous with respect to $\mathscr{T}$.

2 pt (a) Show that for every $U \in \mathscr{T}$ and $a \in \mathbb{R}$ the set $U + a = \{x + a \mid x \in U\}$ belongs to $\mathscr{T}$.

3 pt (b) Show that the collection $\mathscr{B} := \{[a, a + 2^{-n}) \mid a \in \mathbb{R},\ n \in \mathbb{N}\}$ is a subset of $\mathscr{T}$.

3 pt (c) Show that $\mathscr{B}$ is a topology basis.

3 pt (d) Show that the topology generated by $\mathscr{B}$ equals $\mathscr{T}$.

11 pt total **Exercise 2.** Let $J = [-1,1]$, let $S := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let $X := S \times J$ be equipped with the product topology. Let $\Gamma = \{1, -1\}$ be the multiplicative group of two elements. We consider the action φ of Γ by homeomorphisms on the space X given by

$$\varphi : \Gamma \times X \to X, \quad (\gamma, (z_1, z_2, t)) \mapsto (\gamma z_1, \gamma z_2, \gamma t).$$

2 pt (a) Show that every orbit for this action consists of precisely two points.

We equip the quotient X/Γ with the quotient topology, and denote the canonical projection by $p : X \to X/\Gamma$. We consider the map $\sigma : [0,1]^2 = [0,1] \times [0,1] \to X/\Gamma$ given by

$$\sigma(s,t) = p(\cos \pi s, \sin \pi s, 2t-1), \quad (0 \leq s,t \leq 1).$$

3 pt (b) Show that σ is surjective.

We define the equivalence relation $\sim$ on $[0,1]^2$ by $(s,t) \sim (s',t') \iff \sigma(s,t) = \sigma(s',t')$. The quotient space $[0,1]^2/\sim$ is equipped with the quotient topology. We denote the associated natural projection by $q : [0,1]^2 \to [0,1]^2/\sim$.

5 pt (c) Prove that there exists a homeomorphism

$$\bar{\sigma} : [0,1]^2/\sim \ \longrightarrow \ X/\Gamma$$

such that $\bar{\sigma} \circ q = \sigma$.

1 pt (d) Which of the following assertions is correct: (1) X/Γ is homeomorphic to the cilinder; (2) X/Γ is homeomorphic to the Möbius band. (Here you are not required to give any motivation.)

11 pt total **Exercise 3.** We consider a (non-empty) topological space Y, a set Z and a map $f : Y \to Z$. The map f is said to be locally constant if for every $y \in Y$ there exists an open neighborhood U of y in Y such that the restriction $f|_U$ is constant. Equivalently, this means that $f(U)$ consists of a single element.

3 pt (a) Show that f is locally constant if and only if every fiber $f^{-1}(\{z\})$, for $z \in Z$, is open in Y.

3 pt (b) Show that $f : Y \to Z$ is locally constant if and only if f is continuous for the discrete topology on Z.

3 pt (c) If Y is connected and $f : Y \to Z$ locally constant, show that f is constant.

2 pt (d) If Y is not connected, show that there exists a set Z and a locally constant function $f : Y \to Z$ which is not constant.

11 pt total **Exercise 4.** Let X be a Hausdorff topological space and $f : X \to (0,\infty)$ a function such that that for every $a \in X$ there exists an open neighborhood U of a and a constant $m > 0$ such that $f(x) \geq m$ for all $x \in U$.

3 pt (a) If X is compact, show that there exists a constant $m_X > 0$ such that $f(x) \geq m_X$ for all $x \in X$.

We now assume that X is paracompact.

3 pt (b) Show that there exists a locally finite covering $\{V_i \mid i \in I\}$ of X such that for every $i \in I$ there exists a constant $m_i > 0$ such that $f(x) \geq m_i$ for all $x \in V_i$.

5 pt (c) Show that there exists a continuous function $\mu : X \to (0,\infty)$ such that $f(x) \geq \mu(x)$ for all $x \in X$.

16.16 The 2018 exam (January 29, 2018)

Exam Inleiding Topologie, WISB243 2018-01-29, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Lauran, 2: Luca) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 55. The final grade will be obtained from your total score through division by 5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

15 pt total **Exercise 1.** On $\mathbb{R}$ we consider the collection $\mathscr{B} \subset \mathscr{P}(\mathbb{R})$ consisting of $\mathbb{R}$ and all intervals of the form $[m,a)$, with $m \in \mathbb{Z}$ and $a \in \mathbb{R}$ (for convenience we agree that $[m,a) = \emptyset$ for $m \geq a$).

3 pt (a) Show that $\mathscr{B}$ is a topology basis, but not a topology.

Let $\mathscr{T}$ be the topology on $\mathbb{R}$ generated by $\mathscr{B}$.

2 pt (b) Is $\mathscr{T}$ Hausdorff?

2 pt (c) Is $\mathscr{T}$ second countable?

5 pt (d) Determine the interior and the closure of the set $A := [-\frac{1}{2}, \frac{1}{2}]$ with respect to $\mathscr{T}$.

3 pt (e) Determine for which $r > 0$ the subset $[0,r]$ is connected for the subspace topology (induced by $\mathscr{T}$). Prove the validity of your answer.

10 pt total **Exercise 2.** Let X and Y be non-empty topological spaces. We equip $X \times Y$ with the product topology.

5 pt (a) Show: if X and Y are Hausdorff spaces, then $X \times Y$ is a Hausdorff space.

5 pt (b) Show that the converse is also true: if $X \times Y$ is Hausdorff, then both X and Y are Hausdorff.

11 pt total **Exercise 3.** We consider a second countable locally compact Hausdorff space X. A function $\varphi : X \to [0,\infty)$ is said to be locally bounded if and only if for every $a \in X$ there exists a neighborhood V of a and a constant $M > 0$ such that $\varphi \leq M$ on V.

Let $f : X \to [0,\infty)$ be a function. Prove that the following assertions are equivalent.

(1) The function f is locally bounded.

(2) There exists a continuous function $g : X \to [0,\infty)$ such that $f \leq g$ on X.

4 pt for the proof that '(2) $\Rightarrow$ (1)'.

7 pt for the proof that '(1) $\Rightarrow$ (2)'.

19 pt total **Exercise 4.** Let S^1 be the unit circle in $\mathbb{R}^2$. Let Γ be the group $\{1, g\}$ of two elements, with group law given by $11 = g^2 = 1$ and $1g = g1 = g$.

We consider two actions α, β of Γ by homeomorpisms on S^1. These actions are given by $\alpha_1 = \beta_1 = \mathrm{id}_{S^1}$ and

$$\alpha_g(x) = (-x_1, -x_2), \quad \beta_g(x) = (x_1, -x_2), \qquad (x = (x_1, x_2) \in S^1).$$

In the following you may use without proof that α and β are indeed actions by homeomorphisms. For $\gamma \in \Gamma$ we define $\rho_\gamma : S^1 \times S^1 \to S^1 \times S^1$ by

$$\rho_\gamma((x,y)) = (\alpha_\gamma(x), \beta_\gamma(y)), \quad ((x,y) \in S^1 \times S^1).$$

2 pt (a) Show that ρ is an action of Γ on $S^1 \times S^1$ by homeomorphisms.

1 pt (b) For each point $p = (x,y)$ of the torus $S^1 \times S^1$, determine the orbit Γp.

In the following, q denotes the natural quotient map $S^1 \times S^1 \to S^1 \times S^1/\Gamma$ associated with the action ρ. We define the map $f : [0,1] \times S^1 \to S^1 \times S^1$ by $f(s,y) = (\cos \pi s, \sin \pi s, y)$.

3 pt (c) Show that f is a topological embedding.

3 pt (d) Show that $F := q \circ f$ is surjective from $[0,1] \times S^1$ onto the quotient $S^1 \times S^1/\Gamma$.

Let $\sim$ be the equivalence relation on $[0,1] \times S^1$ determined by $z \sim z' \iff F(z) = F(z')$.

4 pt (e) Prove that $S^1 \times S^1/\Gamma$ is homeomorphic to $[0,1] \times S^1/\sim$.

3 pt (f) Show that F is bijective from $[0,1) \times S^1$ onto $S^1 \times S^1/\Gamma$.

3 pt (g) Calculate the equivalence classes of $\sim$ in $[0,1] \times S^1$. Use a picture to indicate why $[0,1] \times S^1/\sim$ is homeomorphic to the Klein bottle.

16.17 The 2017 retake exam (April 18, 2017)

Retake Inleiding Topologie, WISB243 2017-04-18, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 55. The final grade will be obtained from your total score through division by 5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

14 pt total **Exercise 1.** Let $\mathscr{T}$ be the collection of sets $U \subset \mathbb{R}$ such that $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite.

2 pt (a) Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.

2 pt (b) Is $(\mathbb{R}, \mathscr{T})$ Hausdorff?

1 pt (c) Determine the closure of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathscr{T}$.

2 pt (d) Determine the interior of $[0,1]$ with respect to $\mathscr{T}$.

3 pt (e) Show that every subset S of $\mathbb{R}$ is compact for the topology induced by $\mathscr{T}$.

4 pt (f) Let $A \subset \mathbb{R}$ be a subset of at least two elements. Show that A is not connected with respect to the topology induced by $\mathscr{T}$ if and only if A is finite.

9 pt total **Exercise 2.** Let X and Y be topological spaces, and $f : X \to Y$ a continuous map. We assume that X is compact and that for every $x \in X$ there exists an open neighborhood U of x such that the restriction $f|_U : U \to Y$ is injective.

3 pt (a) Show that there exists an open covering $\{U_i \mid i \in I\}$ of X such that for every $y \in Y$ and $i \in I$ the intersection $U_i \cap f^{-1}(\{y\})$ consists of at most one point.

6 pt (b) Show that there exists an $N > 0$ such that for every $y \in Y$ the fiber $f^{-1}(\{y\})$ consists of at most N elements.

11 pt total **Exercise 3.** Let X be a topological space, and let $\{U_i \mid i \in I\}$ be an open cover of X such for each $i \in I$ the set U_i is connected for the induced topology. Let $A \subset X$ be open and closed.

2 pt (a) Show that for each $i \in I$ we have $A \cap U_i = \emptyset$ or $A \cap U_i = U_i$.

2 pt (b) Let $i, j \in I$ be such that $U_i \cap U_j \neq \emptyset$. Show that either A contains both U_i and U_j or is disjoint from both.

If $i, j \in I$, then we will write $i \sim j$ to indicate that there exists a sequence $i_0, \dots i_n$ in I such that $i_0 = i$ and $i_n = j$ and $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ for all $k \in \{1, \dots, n\}$. It is readily seen that $\sim$ defines an equivalence relation on I. You may use this without proof.

2 pt (c) If $i, j \in I$ are $\sim$-equivalent, show that either A contains both U_i and U_j or is disjoint from both.

2 pt (d) Show that if all elements of I are $\sim$-equivalent then X is connected.

3 pt (e) Conversely, if X is connected, show that all elements of I are $\sim$-equivalent.

11 pt total **Exercise 4.** We consider the closed unit disk in $\mathbb{C}$ given by

$$\bar{D} = \{Z \in \mathbb{C} \mid |z| \leq 1\}.$$

We consider the map $\varphi : \bar{D} \to \mathbb{C}^2$ given by

$$\varphi(z) = ((1 - \|z\|)z, z^2).$$

We define the equivalence relation R on $\bar{D}$ by $zRw :\iff \varphi(z) = \varphi(w)$.

4 pt (a) Show that $\bar{D}/R$ is homeomorphic to $\mathbb{P}^2(\mathbb{R})$.

4 pt (b) Show that there exists a topological embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

3 pt (c) We consider the algebra A of continuous functions $f : \bar{D} \to \mathbb{C}$ such that $f(z) = f(-z)$ for all $z \in \partial D$. Show that the topological spectrum $\mathbf{X}_A$ of A is homeomorphic to $\mathbb{P}^2(\mathbb{R})$.

10 pt total **Exercise 5.** Let X and Y be locally compact Hausdorff spaces. Their one-point compactifications are denoted by $\widehat{X} = X \cup \{\infty_X\}$ and $\widehat{Y} = Y \cup \{\infty_Y\}$. Let $f : X \to Y$ be a continuous map. We define $\widehat{f} : \widehat{X} \to \widehat{Y}$ by $\widehat{f} = f$ on X and $\widehat{f}(\infty_X) = \infty_Y$.

4 pt (a) Show: if $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous then for every compact set $K \subset Y$ the set $f^{-1}(K)$ is compact in X.

6 pt (b) Show that the converse implication is also valid: if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is compact in X, then $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous.

16.18 The 2017 exam (January 30, 2017)

Exam Inleiding Topologie, WISB243 2017-01-30, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Francesco, 2: Panagiotis) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score through division by 4.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** We consider the Euclidean topology $\mathscr{T}_{\text{eucl}}$ on $\mathbb{R}$ and also the topology $\mathscr{T}_l$ generated by the intervals $[a,b)$, for $a,b \in \mathbb{R},\ a < b$.

1 pt (a) Show that the intervals $[a,b)$ form a basis for the topology $\mathscr{T}_l$.

2 pt (b) Let $p,q \in \mathbb{R}\ p > 0$. Show that the map $T : \mathbb{R} \to \mathbb{R},\ x \mapsto px+q$ is a homeomorphism of the topological space $(\mathbb{R}, \mathscr{T}_l)$ to itself.

2 pt (c) Show that $\mathscr{T}_{\text{eucl}} \subset \mathscr{T}_l$.

1 pt (d) Show that $(\mathbb{R}, \mathscr{T}_l)$ is Hausdorff.

1 pt (e) Suppose $S \subset \mathbb{R}$ is compact for the topology induced by $\mathscr{T}_l$. Show that S is also compact for the topology induced by $\mathscr{T}_{eucl}$.

2 pt (f) Suppose $S \subset \mathbb{R}$ is compact for (the topology induced by) $\mathscr{T}_l$. Show that for every $a \in \mathbb{R}$ the set $(-\infty, a) \cap S$ is also compact in $(\mathbb{R}, \mathscr{T}_l)$.

2 pt (g) Show that $[0,1]$ is not compact in $(\mathbb{R}, \mathscr{T}_l)$.

2 pt (h) Show that $(\mathbb{R}, \mathscr{T}_l)$ is not locally compact.

9 pt total **Exercise 2.** Let Γ be the group of strictly positive real numbers, with the ordinary multiplication. We equip $\mathbb{R}$ with the Euclidean topology and consider that action of Γ on $\mathbb{R}$ given by $(\gamma, x) \mapsto \gamma x$. The quotient space $Y := \mathbb{R}/\Gamma$ is equipped with the quotient topology. Let $\pi : \mathbb{R} \to Y$ be the natural map.

2 pt (a) Show that Y consists of the three points $a := \pi(-1)$, $b := \pi(0)$ and $c := \pi(1)$.

4 pt (b) Determine the quotient topology $\mathscr{T}_Y$ of $Y = \{a,b,c\}$ explicitly, i.e., give a list of all open subsets of Y.

3 pt (c) Is Y Hausdorff, connected or compact? Prove the validity of each answer.

12 pt total **Exercise 3.** Let X be a locally compact Hausdorff topological space which is the disjoint union of two non-empty open subsets $X_1, X_2 \subset X$. Let $X^+ = X \cup \{\infty\}$ be the one point compactification of X.

3 pt (a) Show that the implication '$(1) \Rightarrow (2)$' holds for the following statements:

(1) At least one of the sets X_1 and X_2 is compact.

(2) X^+ is not connected.

In the rest of the exercise, assume that both X_1 and X_2 are connected.

3 pt (b) If U is non-empty, open and closed in X show that U contains one of the sets X_1 and X_2.

4 pt (c) Show that the implication '$(2) \Rightarrow (1)$' holds as well.

2 pt (d) Give an example of a non-connected locally compact Hausdorff space X such that X^+ is connected.

10 pt total **Exercise 4.** Let D be the closed unit disk in $\mathbb{R}^2$ and let $X := D \times \{-1, 1\} \subset \mathbb{R}^3$ be equipped with the restriction of the Euclidean topology on $\mathbb{R}^3$.

1 pt (a) Show that X is compact Hausdorff.

Let $\varphi : X \to \mathbb{R}^3$ be the map given by $\varphi(x, \pm 1) = (x, \pm\sqrt{1 - \|x\|^2})$, where $\|\cdot\|$ is the Euclidean norm. It is clear that φ is continuous.

2 pt (b) Show that the image of φ is the unit sphere S^2 in $\mathbb{R}^3$.

Let A be the set of continuous functions $f : X \to \mathbb{R}$ such that $f(x, -1) = f(x, +1)$ for all $x \in \partial D$. Here ∂D denotes the boundary of D in $\mathbb{R}^2$.

1 pt (c) Show that A is a subalgebra of $C(X)$.

6 pt (d) Show that the topological spectrum $\mathbf{X}_A$ of A is homeomorphic to S^2.

16.19 The 2016 exam (January 24, 2016)

Note: All the questions are worth 0.5 points (adding up to a total of 11 points). Please motivate all your answers. For instance, in Exercise 1, point c) do not just answer with "yes" or "no", but provide a proof. Also, in d) of the same exercise, please do not just write down the final result, but also explain how you found it. The most difficult questions are 4-d) and especially 3-c). You may write the answers in English or Dutch.

Exercise 16.43. For $\mathbb{R}$ we consider the family $\mathscr{S}$ of subsets consisting of all the intervals of type $(a,b]$ with $a<b\leq 0$ and all intervals of type $[c,d)$ with $0\leq c<d$. Let also $\mathscr{B}$ be the family which is $\mathscr{S}$ together with the subset $\{0\}$ of $\mathbb{R}$. We endow $\mathbb{R}$ with the smallest topology containing $\mathscr{S}$- denote it by $\mathscr{T}$.

a) Show that $\mathscr{S}$ is not a topology basis and that $\mathscr{B}$ is the smallest topology basis containing $\mathscr{S}$.
b) Show that $(\mathbb{R},\mathscr{T})$ is Hausdorff.
c) Is the sequence $x_n=\frac{1}{n}$ convergent in $(\mathbb{R},\mathscr{T})$?
d) Compute the closure and the interior of

$$A=(-3,-2]\cup(-1,0)\cup(0,1)\cup(2,3)$$

in the topological space $(\mathbb{R},\mathscr{T})$
e) Show that the identity map $\mathrm{Id}:(\mathbb{R},\mathscr{T})\to(\mathbb{R},\mathscr{T}_{\mathrm{Eucl}})$ is continuous.
f) Show that $[0,1]$, with the topology induced from $(\mathbb{R},\mathscr{T})$, is not compact.
g) Show that $(\mathbb{R},\mathscr{T})$ is not connected.
h) Describe all the connected subsets of $(\mathbb{R},\mathscr{T})$.
i) Show that any continuous function $f:(\mathbb{R},\mathscr{T}_{\mathrm{Eucl}})\to(\mathbb{R},\mathscr{T})$ is constant.

Exercise 16.44. Consider

$$X=\left\{(x,y)\in\mathbb{R}^2:x^2+y^2=1,\ \text{or}\ x=0\right\},$$
$$Y=\left\{(x,y)\in\mathbb{R}^2:xy=0\right\},$$

both endowed with the Euclidean topology. Show that:

a) Y can be embedded in X,
b) X cannot be embedded in Y,
c) that the one-point compactifications of X and Y are homeomorphic.

In this exercise you do not have to provide explicit formulas for the embedding and the homeomorphism from a) and c), respectively. But please use pictures to describe those functions and please provide all the arguments to support your claims (e.g., if you draw a picture for the one-point compactification of X, please explain why the picture you draw is indeed the one-point compactification).

Exercise 16.45. Show that:

a) If X is a compact Hausdorff space and A is nonempty closed in X, then the space X/A obtained from X by collapsing A to a point is compact and Hausdorff.
b) If $U\subset\mathbb{R}^n$ is a bounded nonempty open subset of $\mathbb{R}^n$ (with the Euclidean topology), considering $X=\overline{U}$ (the closure of U in $\mathbb{R}^n$) and $A=\partial(U)$ (the boundary of U in $\mathbb{R}^n$) show:

b1) X is compact and A is closed in X.
b2) the one-point compactification of U is homeomorphic to X/A.

c) If $U\subset\mathbb{R}^n$ is a bounded open subset of $\mathbb{R}^n$ then its one-point compactification can be embedded in $\mathbb{R}^{n+1}$. (a *complete* solution here is worth 1 point).

Exercise 16.46. Here we work over the field $\mathbb{R}$ of real numbers. Let A be an arbitrary algebra, assume that A is commutative (i.e. $a_0a_1 = a_1a_0$ for any $a_0, a_1 \in A$) and denote by $\mathbf{0_A}$ and $\mathbf{1_A}$ the zero and the unit element of A, respectively. Fix an element $\alpha \in A$. Consider

$$B = A \times A = \{(a_0, a_1) : a_0, a_1 \in A\}$$

with the operations

$$(a_0, a_1) + (a_0', a_1') = (a_0 + a_0', a_1 + a_1'),$$
$$\lambda \cdot (a_0, a_1) = (\lambda a_0, \lambda a_1),$$
$$(a_0, a_1) \bullet (a_0', a_1') = (a_0 a_0' + \alpha a_1 a_1', a_0 a_1' + a_0' a_1).$$

a) Show that B, together with the previous operations (written in red), is an algebra (don't forget the unit element of B!).
b) Assume that $\chi : B \to \mathbb{R}$ is a character and consider

$$\overline{\chi} : A \to \mathbb{R}, \quad \overline{\chi}(a) = \chi(a, \mathbf{0_A}),$$
$$r := \chi(\mathbf{0_A}, \mathbf{1_A}) \in \mathbb{R}.$$

Show that:

b1) $\overline{\chi}$ is a character on A.
b2) $\chi(a_0, a_1) = \overline{\chi}(a_0) + r\overline{\chi}(a_1)$ for any $(a_0, a_1) \in B$.
b3) $\overline{\chi}(\alpha) = r^2$.

c) Conversely, for any character $\overline{\chi}$ on A and any $r \in \mathbb{R}$ satisfying the equation from b3), show that χ defined by the formula from b2) is a character on B.
d) When $A := \mathbb{R}[X_0, X_1]$ is the algebra of polynomials in two variables and $\alpha := 1 - (X_0)^2 - (X_1)^2$, show that the topological spectrum of B is *homeomorphic* to S^2.

16.20 The 2015 exam (January 28, 2015)

Note: In all the questions below, **please explain/prove your answers** (e.g., in Exercise 1, part a., b., etc, please **do not just say "yes" or "no"**. Also, in part e. or g. of the same exercise, **do not just write down the final result**, but also explain how you found it- e.g. by explaining your reasoning using pictures). There are three questions marked with a "*"; they are probably more difficult than the rest. The final mark for the exam is the number of points that you collect, except for the case in which you collect more than 10 points, when the final mark will be 10 (**in total, the exercises below are worth 11.5 points**).

Exercise 16.47. Prove that, for any Hausdorff space $(X,\mathscr{T})$, any finite subset $F\subset X$ is closed in X.
(1 **point)**

Exercise 16.48. Consider the family $\mathscr{B}$ of subsets of $\mathbb{R}^2$ consisting of all the subsets of type $(a,b)\times(a,b)$ with $a<b$ real numbers:

$$\mathscr{B}=\{(a,b)\times(a,b):a,b\in\mathbb{R},a<b\}.$$

Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}^2$ containing $\mathscr{B}$. We also consider

$$A=[0,1]\times[0,2]\subset\mathbb{R}^2.$$

a. Is $(\mathbb{R}^2,\mathscr{T})$ second countable? *(0.5 points)*
b. Is $(\mathbb{R}^2,\mathscr{T})$ Hausdorff? *(0.5 points)*
c. Is the identity map $\mathrm{Id}:\mathbb{R}^2\to\mathbb{R}^2$ continuous as a map from $(\mathbb{R}^2,\mathscr{T})$ to $(\mathbb{R}^2,\mathscr{T}_{\mathrm{Eucl}})$? But as a map from $(\mathbb{R}^2,\mathscr{T}_{\mathrm{Eucl}})$ to $(\mathbb{R}^2,\mathscr{T})$? *(0.5 points)*
d. Is A, with the topology induced from $\mathscr{T}$, connected? Is it compact? *(0.5 points)*
e. Let $x=(0,2)\in\mathbb{R}^2$ (the point of coordinates 0 and 2). Compute the closure of $\{x\}$ in $(\mathbb{R}^2,\mathscr{T})$. *(0.5 points)*
f. Show that the sequence $(x_n)_{n\geq1}$ given by

$$x_n=(\sin^2(n),\cos^4(n+2015))\in\mathbb{R}^2$$

is convergent in $(\mathbb{R}^2,\mathscr{T})$ and has more than one limit. *(0.5 points)*
g. Compute the interior and the closure of A in $(\mathbb{R}^2,\mathscr{T})$. *(1 point)*
h* Show that any continuous map $f:(\mathbb{R}^2,\mathscr{T})\to\mathbb{R}$ must be constant. *(1 point)*

(total: $6\times0.5+2\times1=$ **5 points**)

Exercise 16.49. Consider the group of integers modulo 2, $\mathbb{Z}_2=\{\hat{0},\hat{1}\}$. Define the following action of $\mathbb{Z}_2$ on the closed unit disk $D^2=\{z\in\mathbb{C}:|z|\leq1\}$:

$$\hat{0}\cdot z=z,\ \hat{1}\cdot z=-z\quad(\text{for all } z\in D^2).$$

Prove that the resulting quotient $D^2/\mathbb{Z}_2$ is homeomorphic to D^2 (provide a complete argument; a "proof" based only on pictures is not enough to get the entire 1 point for this exercise).
(**1 point**)

Exercise 16.50. Let X be compact Hausdorff space and let $C(X)$ be the space of real-valued continuous functions on X, endowed with the topology induced by the *sup*-metric.

Prove that if $\mathscr{A} \subset C(X)$ is a dense subset of $C(X)$, then $\mathscr{A}$ must be point-separating.
(1 **point)**

Exercise 16.51. For any algebra A over $\mathbb{R}$ and any ideal I of A, we define

$$X(A,I) := \{\chi \in X_A : \chi(x) = 0 \ \ \forall\, x \in I\} \subset X_A$$

and we endow it with the topology induced from the topology on the spectrum X_A of A. Show that:

a. For any algebra A and any ideal I of A, $X(A,I)$ is closed in X_A. *(0.5 points)*
b. Applied to $A = \mathbb{R}[x,y]$ (polynomials in two variables) and I the ideal consisting of polynomials that are divisible by x^2+y^2-1, $X(A,I)$ is homeomorphic to S^1. *(0.5 points)*
c* Assume now that X is a compact Hausdorff space, $Y \subset X$ and set

$$A = C(X), \ \ I := \{f \in C(X) : f(y) = 0 \ \ \forall\, y \in Y\}.$$

Show that $X(A,I)$ is homeomorphic to the closure $\overline{Y}$ of Y in X (where the last space is endowed with the topology induced from X). *(1 point)*

(total: $2 \times 0.5 + 1 =$ **2 points**)

Exercise 16.52. For each natural number n we consider a space X_n that is obtained by removing n distinct points from $\mathbb{R}^2$. We consider the 1-point compactification X_n^+ and we denote by $\infty_n \in X_n^+$ the point at infinity (so that $X_n^+ = X_n \cup \{\infty_n\}$). Show that

a. X_n^+ can be embedded in $\mathbb{R}^3$ (here you **do not** have to write down explicit formulas for the embedding, but please explain your reasoning using pictures and mention what result(s) you use in order to reach the final conclusion). *(0.5 points)*
b* If X_n and X_m are homeomorphic, then $n = m$. *(1 point)*

(total: $0.5 + 1 =$ **1.5 points**)

16.21 The 2015 retake exam (March 11, 2015)

Note: Please explain/prove your answers. E.g., in Exercise 1, parts c., d., f. please **do not just say "yes" or "no"** but also prove your claims. Also, in part e. or h., **do not just write down the final result**, but also explain how you found it. Also, in Exercise 2, do not forget to prove that the examples you give are not homeomorphic; etc. etc.

Exercise 16.53. Let k be a natural number ($k \in \{0,1,2,\ldots\}$). Let $\mathscr{B}_k$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and all the open intervals (a,b) which contain at most k integers (when $k=0$, that means: intervals (a,b) that contain no integers). Show that:

a. $\mathscr{B}_k$ is a topology basis on $\mathbb{R}$. *(0.5 points)*
b. for $k \geq 1$, the topology induced by $\mathscr{B}_k$ on $\mathbb{R}$ is the Euclidean topology. *(0.5 points)*

Next, we assume that $k=0$ and we look at the topology $\mathscr{T}_0 := \mathscr{T}(\mathscr{B}_0)$ induced by $\mathscr{B}_0$.

c. Is $(\mathbb{R}, \mathscr{T}_0)$ Hausdorff? *(0.5 points)*
d. Is $(\mathbb{R}, \mathscr{T}_0)$ compact? *(0.5 points)*
e. Compute the interior and the closure of $A = [0,2)$ in $(\mathbb{R}, \mathscr{T}_0)$. *(0.5 points)*
f. Is the sequence $(x_n)_{n\geq 1}$ given by

$$x_n = \frac{n^{2015} + \sin(n)}{n^2}$$

convergent in $(\mathbb{R}, \mathscr{T}_0)$? *(0.5 points)*
g. Show that any continuous function $f : (\mathbb{R}, \mathscr{T}_0) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}})$ is constant. *(0.5 points)*
h. Compute the interior and the closure of

$$D := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\} \subset \mathbb{R}^2$$

in $\mathbb{R}^2$ endowed with the product topology $\mathscr{T}_0 \times \mathscr{T}_{\text{Eucl}}$. *(1 point)*

Exercise 16.54. Find two locally compact Hausdorff spaces X and Y which are not homeomorphic but which have homeomorphic one-point compactifications. *(0.75 pts)* Can you find such X and Y such that all the spaces X, Y, X^+ and Y^+ are connected? *(0.75 pts)*

Exercise 16.55. Let $\Gamma = \mathbb{R}_{>0}$ be the group of strictly positive real numbers, endowed with the usual multiplication. Let $X = \mathbb{R}^n \setminus \{0\}$. Show that

$$\Gamma \times X \to X, \;\; (r,x) \mapsto rx$$

(the usual multiplication of vectors x by scalars r) defines an action of Γ on X and **prove** that X/Γ is homeomorphic to S^{n-1} (the $n-1$-dimensional sphere).

Exercise 16.56. Let X be a topological space.

a. If $\gamma_1, \gamma_2 : [0,1] \to X$ are continuous and $\gamma_1(1) = \gamma_2(0)$, show that

$$\gamma : [0,1] \to X, \;\; \gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is continuous. *(0.5 points)*
b. If there exist $A, B \subset X$ path connected such that $X = A \cup B$ and $A \cap B \neq \emptyset$, then X is connected. *(0.5 points)*

Exercise 16.57. Let A be an algebra over $\mathbb{R}$, assume that $\chi : A \to \mathbb{R}$ is a character and let

$$I := \mathrm{Ker}(\chi) \ \ (= \{a \in A : \chi(a) = 0\}).$$

Show that:

a. I is an ideal of A. *(0.5 points)*
b. $a - \chi(a) \cdot \mathbf{1}_A \in I$ for all $a \in A$ (where $\mathbf{1}_A$ is the unit of A). *(0.5 points)*
c. I is a maximal ideal. *(0.5 points)*

Exercise 16.58. Let X be a compact Hausdorff space,

$$F = (F_1, \ldots, F_n) : X \to \mathbb{R}^n$$

a continuous function and consider

$$\mathscr{A}_F = \{f \in C(X) : f = P \circ F \ \text{ for some polynomial function } P : \mathbb{R}^n \to \mathbb{R}\}.$$

Show that F is an embedding if and only if $\mathscr{A}_F$ is dense in $C(X)$. *(1.5 points)*
(Recall: $C(X)$ denotes the space of all continuous functions from X to $\mathbb{R}$, and it is a topological space with the topology induced by the sup-norm).

16.22 The 2014 exam (January 29, 2014)

Note: The questions marked with "+" are worth 1 point, the rest 0.5 points. As agreed, you are allowed to use during the exam the three sheets of A4 paper (= six pages) containing definitions, theorems, etc from the course- that you prepared at home. **For some explanations/hints, please see the end of the exam!!!!!!!!!!!!**.

Exercise 16.59. For $\mathbb{R}$ we consider the family $\mathscr{S}$ of subsets consisting of all the intervals of type (m,M) with $m < M < 0$, all intervals of type (m,M) with $0 < m < M$ (m and M real numbers) and the interval $[-1,1)$. Denote by $\mathscr{T}$ the smallest topology on $\mathbb{R}$ containing $\mathscr{S}$.

a. show that $\mathscr{S}$ is not a topology basis and describe a basis of $(\mathbb{R},\mathscr{T})$.
b. is $(\mathbb{R},\mathscr{T})$ Hausdorff? Is it second countable?
c. find an interval of type $[a,b]$ whose closure inside $(\mathbb{R},\mathscr{T})$ is not an interval.
d. find an interval of type (a,b) whose interior inside $(\mathbb{R},\mathscr{T})$ is not an interval.
e. find an interval of type $[a,b]$ with the property that, together with the topology induced from $(\mathbb{R},\mathscr{T})$, is not compact.
f. find an interval of type (a,b) with the property that, together with the topology induced from $(\mathbb{R},\mathscr{T})$, is not connected.
g. (+) consider

$$f:(\mathbb{R},\mathscr{T})\to(\mathbb{R},\mathscr{T}_{\text{Eucl}}),\ f(x)=\begin{cases}0 \text{ if } x<-1\\ 1 \text{ if } x\geq -1\end{cases}$$

Is f continuous? Is f sequentially continuous?

Exercise 16.60. Let X be the open cylinder $(-1,1)\times S^1$ and let Y be the open Moebius band (i.e. the Moebius band discussed in the lectures, from which the boundary circle was removed).

a. Describe the 1-point compactification X^+ as a subspace of $\mathbb{R}^3$.
b. (+) Describe $X^+\subset\mathbb{R}^3$ by explicit formulas and write down an explicit embedding

$$f:X\to\mathbb{R}^3$$

so that X^+ is the image of f together with the extra-point $(0,0,0)$.
c. Show that the 1-point compactification of Y is homeomorphic to the projective plane $\mathbb{P}^2$.
d. Show that X and Y are not homeomorphic.

Exercise 16.61. Let $X=[-1,1]\times\mathbb{R}$.

a. (+) Find all the numbers $\lambda,a,b\in\mathbb{R}$ with the property that

$$n\cdot(x,y):=(\lambda^n x, a+by+\lambda n)$$

defines an action of the additive group $(\mathbb{Z},+)$ on X.
b. Show that, in all cases, $X/\mathbb{Z}$ is compact.
c. For the values of λ,a,b that you found compute the resulting quotients $X/\Gamma/$

Exercise 16.62. In this exercise we work over $\mathbb{R}$ (hence we consider real-valued functions and real algebras). Let X and Y be compact Hausdorff spaces. For $u \in \mathscr{C}(X)$, $v \in \mathscr{C}(Y)$, define $u \otimes v \in \mathscr{C}(X \times Y)$ given by

$$u \otimes v : X \times Y \to \mathbb{R}, \ \ (u \otimes v)(x,y) := u(x)v(y),$$

and we denote by $\mathscr{A} \subset \mathscr{C}(X \times Y)$ the set of functions of type

$$\sum_{i=1}^{k} u_i \otimes v_i \quad \text{with } k \in \mathbb{N}, u_1, \ldots, u_k \in \mathscr{C}(X), v_1, \ldots, v_k \in \mathscr{C}(Y)$$

Show that:

a. (+) $\mathscr{A}$ is a dense subalgebra of $\mathscr{C}(X \times Y)$.
b. For any $\chi \in X_{\mathscr{A}}$, $\chi_1 : \mathscr{C}(X) \to \mathbb{R}$ and $\chi_2 : \mathscr{C}(Y) \to \mathbb{R}$ given by

$$\chi_1(u) := \chi(u \otimes 1), \ \ \chi_2(v) := \chi(1 \otimes v)$$

are characters.
c. (+) $X_{\mathscr{A}}$ is homeomorphic to $X \times Y$.

Notes/hints:

1. Please motivate all your answers. For instance, in Exercise 1, for b. do not just give an yes/no answer, for, c.-f. prove why the intervals that you found do satisfy the required conditions, at point g. explain/prove why f is, or isn't, continuous or sequentially continuous, and similarly for the other exercises.
2. For items a. and c. of Exercise 2, and item c. of Exercise 3, you do not have to give explicit formulas; pictures are enough, provided they are properly explained.
3. Exercise 2: you may want to remember the models $T_{R,r}$ of the torus:

$$\begin{aligned} T_{R,r} &= \left\{ (x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2 \right\} = \\ &= \left\{ \big((R + r\cos(a))\cos(b), (R + r\cos(a))\sin(b), r\sin(a)\big) : a, b \in [-\pi, \pi] \right\} \end{aligned}$$

(where, to obtain a torus, one has to assume $R > r > 0$). For d.: you may want to remember that $\mathbb{P}^2$ is a 2-dimensional topological manifold (in particular, each point has a neighborhood homeomorphic to $\mathbb{R}^2$).
4. In exercise 3, if you encounter 0^0, take it to be 1 (say by convention).

16.23 The 2014 retake exam (March 12, 2014)

Exercise 16.63. Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ with the property that

$$f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{\text{Eucl}}), \quad f(x) = x^2$$

is continuous.

a. Describe a basis of $(\mathbb{R}, \mathscr{T})$ and show that any $U \in \mathscr{T}$ has the property that

$$-x \in U \quad \forall \ x \in U.$$

b. Find the closure and the interior of $(-1,2)$ in $(\mathbb{R}, \mathscr{T})$.
c. Is $(\mathbb{R}, \mathscr{T})$ Hausdorff? Can you find a sequence with two (distinct) limits?
d. Is $[-1,1]$ (with the topology induced from $\mathscr{T}$) compact? But connected?
e. Is $[-1,1)$ (with the topology induced from $\mathscr{T}$) compact?
f. Is $[-3,1) \cup (1,3]$ (with the topology induced from $\mathscr{T}$) connected?
g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R}, \mathscr{T})$?

Exercise 16.64. Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a continuous function between two *Hausdorff* topological spaces. Define $f^* \mathscr{T}_Y$ as the smallest topology on X with the property that

$$f : (X, f^* \mathscr{T}_Y) \to (Y, \mathscr{T}_Y)$$

is continuous, and define $f_* \mathscr{T}_X$ as the largest topology on Y with the property that

$$f : (X, \mathscr{T}_X) \to (Y, f_* \mathscr{T}_X)$$

is continuous. Show that

a. $f^* \mathscr{T}_Y = \left\{ f^{-1}(V) : V \in \mathscr{T}_Y \right\}$ and $f_* \mathscr{T}_X = \left\{ V \subset Y : f^{-1}(V) \in \mathscr{T}_X \right\}$.
b. If f is a homeomorphism then $f_* \mathscr{T}_X = \mathscr{T}_Y$ and $f^* \mathscr{T}_Y = \mathscr{T}_X$.
c. If $f_* \mathscr{T}_X = \mathscr{T}_Y$ and $(Y, \mathscr{T}_Y)$ is connected, then f is surjective.
d. If $f^* \mathscr{T}_Y = \mathscr{T}_X$ then f is injective.
e. $f^* \mathscr{T}_Y = \mathscr{T}_X$ holds if and only if f is an embedding.

Exercise 16.65. Assume that X is a sphere S^2 minus n points and Y is a sphere minus m points, where $m, n \geq 0$ are integers. Show that if X is homeomorphic to Y, then $n = m$.

Exercise 16.66. Assume that $(X, \mathscr{T}_X)$ is a compact Hausdorff space and A is a nonempty closed subset of X. We consider the complement of A in X,

$$Y := X - A = \{x \in X : x \notin A\},$$

we denote by X/A the space obtained from X by collapsing A to a point and we consider the canonical projection:

$$\pi : X \to X/A$$

(recall that X/A is endowed with the topology $\pi_* \mathscr{T}_X$). Show that:

a. Y is locally compact and Hausdorff.

b. For any open U in X with the property that $U \cap A = \emptyset$ or $A \subset U$, one has that $\pi(U)$ is open in X/A.
c. X/A is a compact Hausdorff space.
d. The one-point compactification of Y is homeomorphic to X/A.

Exercise 16.67. Let $\mathscr{C}(D^2)$ be the algebra of real-valued continuous functions on the unit disk D^2 and let $\mathscr{A}$ be the subset consisting of those $f \in \mathscr{C}(D^2)$ with the property that they are constant on the boundary circle S^1.

a. is $\mathscr{A}$ a sub-algebra of $\mathscr{C}(D^2)$?
b. is $\mathscr{A}$ dense in $\mathscr{C}(D^2)$?
c. show that the spectrum of $\mathscr{A}$ is homeomorphic to S^2.

Notes:

1. You may give your answers in Dutch or English.
2. All the questions a., b. etc are worth 0.5 points. Exercise 3 is worth 1.5 points (note also that the sum of all the points is 11 ...).
3. As before, you are allowed to use during the exam the three sheets of A4 paper (= six pages) containing definitions, theorems, etc from the course- that you prepared at home.
4. PLEASE MOTIVATE ALL YOUR ANSWERS!!!!!!!!!! (give details, explain your reasoning, use pictures whenever appropriate, etc etc).

16.24 The 2013 exam A (April 17, 2013)

Exercise 16.68. (*1 pt*) Let X and Y be two topological spaces. For $A \subset X$, $B \subset Y$, we consider $A \times B$ as a subset of $X \times Y$. Show that:

$$\mathrm{Int}(A \times B) = \mathrm{Int}(A) \times \mathrm{Int}(B)$$

(the interior of $A \times B$ inside $X \times Y$ (with respect to the product topology)= the product of the interior of A in X with the interior of B in Y).

Exercise 16.69. (*1 pt*) Give an example of a connected, bounded, open subset of $\mathbb{R}^2$ which cannot be written as a finite union of balls (here we use the Euclidean metric and topology on $\mathbb{R}^2$).

Exercise 16.70. Let $X = (-1, \infty)$.

(a) (*1 pt*) Find all the numbers $a, b \in \mathbb{R}$ with the property that

$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group $(\mathbb{Z}, +)$ on X.

(b) (*1 pt*) For the a and b that you found, show that the resulting quotient space $X/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 16.71. On $X = \mathbb{Z}$ we consider the family $\mathscr{B}$ of subsets of X consisting of the empty set and the subsets of type

$$N_{a,b} := a + b\mathbb{Z} = \{a + bn : n \in \mathbb{Z}\} \subset \mathbb{Z},$$

with $a, b \in \mathbb{Z}$, $b > 0$.

(a) (*0.25 pts*) Show that, for $a, a', b, b_1, b_2 \in \mathbb{Z}$ with $b, b_1, b_2 > 0$:

$$N_{a,1} = N_{a,-1} = \mathbb{Z},$$

$$N_{a,b_1 b_2} \subset N_{a,b_1} \cap N_{a,b_2}$$

and one has the following equivalences:

$$a' \in N_{a,b} \iff a \in N_{a',b} \iff N_{a,b} = N_{a',b}.$$

(b) (*0.5 pts*) Show that $\mathscr{B}$ is not a topology on X.

(c) (*0.5 pts*) Show that $\mathscr{B}$ is a topology basis on X. Let $\mathscr{T}$ be the induced topology.

(d) (*1 pt*) Compute the interior and the closure of $A := \{-1, 1\}$ in $(X, \mathscr{T})$.

(e) (*0.5 pts*) Show that $(X, \mathscr{T})$ is Hausdorff.

(f) (*0.25 pts*) Show that, for any $b \in \mathbb{Z}$, $b > 0$, $\mathbb{Z}$ can be written as a union of b nonempty subsets that belong to $\mathscr{B}$, each two of them being disjoint.

(g) (*0.5 pts*) Show that any subset of type $N_{a,b}$ is both open and closed in $(X, \mathscr{T})$.

(h) (*1 pt*) Show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p-\text{prime number}} N_{0,p}$$

and then, using (g) and (e), deduce that the set of prime numbers is infinite.

Exercise 16.72. Consider the 3-sphere S^3 viewed as a subspace of $\mathbb{C}^2$:

$$S^3 = \left\{(u,v) : u,v \in \mathbb{C}, |u|^2 + |v|^2 = 1\right\}.$$

Inside the sphere we consider

$$A := \left\{(u,v) \in S^3 : |v| = \frac{\sqrt{2}}{2}\right\}.$$

(a) (*1 pt*) Show that $S^3 \setminus A$ has two connected components.
(b) (*0.5 pts*) Show that the two connected components, denoted X_1 and X_2, satisfy:

$$\overline{X}_1 \cap \overline{X}_2 = \partial(X_1) = \partial(X_2) = A.$$

(where the closures and boundaries are inside the space S^3).
(c) (*1 pt*) Consider the unit circle and the closed unit disk

$$S^1 = \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\right\}, \quad D^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

By a solid torus we mean any space homeomorphic to $S^1 \times D^2$. Show that

$$f : S^1 \times D^2 \to \mathbb{R}^3, \; f((\alpha,\beta),(x,y)) = ((2-x)\alpha, (2-x)\beta, y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name "solid torus").
(d) (*1 pt*) Show that $\overline{X}_i$ is a solid torus for $i \in \{1,2\}$.
(e) (*1 pt*) Deduce that the 3-sphere can be obtained from two disjoint copies of $S^1 \times D^2$ (i.e. two solid tori) by gluing any point $(z_1,z_2) \in S^1 \times S^1$ in the boundary of the first copy with the point (z_2,z_1) in the boundary of the second.

Note 1: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 13 points in the game!).

Note 2: please MOTIVATE ALL YOUR ANSWERS (e.g., in Exercise 2, do not just give the example, but also explain/prove why it has the required properties).

16.25 The 2013 exam B (June 26, 2013)

Exercise 16.73. Prove that there is no continuous injective map $f : S^2 \to S^1$. (*1p*)
(warning: there are injective maps from S^2 to S^1!).

Exercise 16.74. Let $X = (0,\infty)$ and consider the open cover of X

$$\mathscr{U} = \{U_1, U_2, U_3, \ldots\} \quad \text{with } U_n = (0,n).$$

(a) Show that $\mathscr{U}$ does not admit a subcover which is locally finite. (*0.5p*)
(b) Describe a locally finite refinement of $\mathscr{U}$. (*0.5p*)

Exercise 16.75. Consider

$$X = \left\{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 = 1\right\} \setminus \{(0,0,1),(0,0,-1)\},$$

$$Y = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 2\sqrt{x^2+y^2}\right\} \setminus \{(0,0,0)\}.$$

$$\pi : X \to \mathbb{R}^3,\ \pi(u,v,w) = (2u\sqrt{u^2+v^2}, 2v\sqrt{u^2+v^2}, 2w\sqrt{u^2+v^2}).$$

(a) Compute the closure $\overline{Y}$ of Y in $\mathbb{R}^3$ and prove that it is compact. (*1p*)
(b) Prove that the one-point compactification of X is homeomorphic to $\overline{Y}$. (*1p*)
(c) Draw a picture of Y and explain the map π on the picture. (*1p*)

Exercise 16.76. Let X be a Hausdorff, locally compact, 2nd countable topological space and assume that U is an open in $X \times \mathbb{R}$ containing $X \times \{0\}$:

$$X \times \{0\} \subset U \subset X \times \mathbb{R}.$$

The aim of this exercise is to prove that there exists a continuous function $f : X \to (0,\infty)$ such that U contains

$$U_f := \{(x,t) \in X \times \mathbb{R} : |t| < f(x)\}.$$

Consider

$$r : X \to \mathbb{R},\ \ r(x) = \sup\{r \in (0,1] : \{x\} \times (-r,r) \subset U\}.$$

(a) Show that one can find an open cover $\{V_i : i \in I\}$ of X and a family $\{r_i : i \in I\}$ of strictly positive real numbers (for some indexing set I) such that

$$r(y) > r_i \quad \forall\, y \in V_i,\ \forall\, i \in I. \quad (0.5p)$$

(note: depending on the argument that you find, I that you construct may be countable, but it may also be "very large"- e.g. "as large as X").
(b) For $\{V_i : i \in I\}$, $\{r_i : i \in I\}$ as above, use a partition of unity argument to build a continuous function $f : X \to (0,\infty)$ such that $U_f \subset U$. (*1.5p*)
(c) Deduce that if X is actually compact, then f may be chosen to be constant. (*0.5p*)

Exercise 16.77. In this exercise we work over $\mathbb{R}$. Let X be a compact, Hausdorff topological space, $C(X)$ the space of real-valued continuous functions on X and let

$$\mathscr{A} \subset C(X).$$

be a point-separating subalgebra. The aim of this exercise is to understand whether the spectrum $X_{\mathscr{A}}$ is homeomorphic to X. The homeomorphism will be, in principle, provided by the map:

$$F : X \to X_{\mathscr{A}},\ F(x) = \chi_x|_{\mathscr{A}}$$

(the restriction of $\chi_x : C(X) \to \mathbb{R}$ to $\mathscr{A}$, where we recall that χ_x sends f to $f(x)$).

(a) Show that F is continuous. (*0.5p*)
(b) Show that F is injective. (*0.5p*)
(c) Give an example of X and A for which F is not surjective. (*0.5p*)

The next five steps are to prove that the image of F consists of those characters $\chi \in X_{\mathscr{A}}$ with the property that χ is continuous, where $\mathscr{A}$ is endowed with the topology induced by the sup-metric. Let χ be such a character.

(d) Show that, if $f, g \in \mathscr{A}$ and $f \geq g$, then $\chi(f) \geq \chi(g)$. (*0.5p*)
(Hint: recall that, in the proof of the Stone-Weierstrass theorem we showed that, for $f \in \mathscr{A}$ with $f \geq 0$, one has $\sqrt{f} \in \overline{\mathscr{A}}$).
(e) Show that for all $f \in \mathscr{A}$ one has $|\chi(f)| \leq ||f||_{\sup}$. (*0.5p*)
(f) Deduce that for any sequence $(f_n)_{n\geq 1}$ of elements in $\mathscr{A}$, convergent to some $f \in C(X)$, the sequence $(\chi(f_n))_{n\geq 1}$ is convergent. (*0.5p*)
(g) Deduce that there exists an extension of $\chi : \mathscr{A} \to \mathbb{R}$ to a continuous map

$$\tilde{\chi} : C(X) \to \mathbb{R}. \quad (0.5p)$$

(h) And then show that $\tilde{\chi}$ is a character. (*0.5p*)

Finally:

(i) Conclude that F is a homeomorphism between X and the space of continuous characters, endowed with the topology induced from X_A. (*0.5p*)

Notes:

- English or Dutch (or both)- doesn't matter, but please write clearly (thanks!).
- Please motivate your answers. For instance, in exercise 4, part (b), do not forget, after you write the function f, to check that f is well defined, continuous and that it has the desired property.
- In this exam, the sub-points of any exercise do not fully depend on each other. For instance, in the last exercise, you may do (e) without doing (d) (... but using it).

16.26 The 2013 retake exam (August, 2013)

Exercise 16.78. On $X=[0,\infty)$ we consider the family $\mathscr{B}$ of subsets consisting of the empty set and all intervals of type $[r,\infty)$ with $r\in\mathbb{R}_+$ and we denote by $\mathscr{T}$ the smallest topology on X containing $\mathscr{B}$.

(a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. (*0.5p*)
(b) Is $(X,\mathscr{T})$ Hausdorff? Is it compact? Is it connected? (*0.75p*)
(c) Is the sequence $(x_n)_{n\geq 1}$ given by

$$x_n=1+\frac{(-1)^n}{n}$$

convergent in $(X,\mathscr{T})$? To what? (*0.5p*)
(d) Compute the interior and the closure of $[1,2)\times(1,2]$ in $X\times X$ endowed with the product topology. (*0.5p*).

Exercise 16.79. Decide (and explain) which of the following statements hold true:

(a) $S^1\times\mathbb{R}$ can be embedded in $\mathbb{R}^2$. (*0.5p*)
(b) $S^1\times S^1\times S^1$ can be embedded in $\mathbb{R}^4$. (*0.5p*)
(c) S^2 can be embedded in S^1. (*0.5p*)
(d) the Moebius band can be embedded into the projective space P^2. (*0.5p*)
(e) the projective space P^3 can be embedded in $\mathbb{R}^6$. (*0.5p*)

Exercise 16.80. By a solid torus we mean any space homeomorphic to $S^1\times D^2$.

(a) Show that

$$f:S^1\times D^2\to\mathbb{R}^3,\ f((\alpha,\beta),(x,y))=((2-x)\alpha,(2-x)\beta,y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name solid torus). (*1p*)

Next, consider the 3-sphere S^3 viewed as a subspace $\mathbb{C}^2$:

$$S^3=\left\{(u,v):u,v\in\mathbb{C},|u|^2+|v|^2=1\right\}.$$

Inside the sphere we consider

$$A:=\left\{(u,v)\in S^3:|v|=\frac{\sqrt{2}}{2}\right\}.$$

(b) Show that $S^3\setminus A$ has two connected components; denote them by X_1 and X_2. (*0.5p*)
(c) Show that $\overline{X}_i$ is a solid torus for $i\in\{1,2\}$. (*0.5p*)
(d) Show that the 3-sphere S^3 can be obtained from two disjoint copies of $S^1\times D^2$ (i.e. two solid tori) by gluing any point $(z_1,z_2)\in S^1\times S^1$ in the boundary of the first copy with the point (z_2,z_1) in the boundary of the second.(*1p*)

Exercise 16.81. Give an example of two connected, Hausdorff, locally compact spaces X and Y which are not homeomorphic but have the property that their one-point compactifications are homeomorphic. (*1p*)

Exercise 16.82. If A and B are two commutative algebras over $\mathbb{R}$, a map

$$F:A\to B$$

is called an algebra homomorphism if it is $\mathbb{R}$-linear and satisfies

$$F(a_1a_2) = F(a_1)F(a_2)$$

for all $a_1, a_2 \in A$, and $F(1_A) = 1_B$, where $1_A \in A$ and $1_B \in B$ are the units. It is called an isomorphism of algebras if it is also bijective.

1. Show that for any algebra homomorphism $F : A \to B$, one has an induced continuous map

$$F^* : X_B \to X_A,\ F^*(\chi) = \chi \circ F.$$

   Moreover, if F is an isomorphism, show that F^* is a homeomorphism. (*0.5p*)
2. If X and Y are two compact Hausdorff spaces, show that for any algebra homomorphism

$$F : C(Y) \to C(X)$$

   one finds a continuous function $f : X \to Y$ such that

$$F(\phi) = \phi \circ f \quad \forall\, \phi \in C(Y). \quad (1p)$$

3. Compute the spectrum of

$$A := \left\{ f : \mathbb{R}^2 \to \mathbb{R} \text{ continuous} : f(x+1,y) = f(x,y+1) = f(x,y),\ \forall\, x,y \in \mathbb{R} \right\}. \quad (0.5p)$$

Note: As usual, please motivate/explain/prove all your statements!!!

16.27 The 2012 exam A (April 18, 2012)

Exercise 16.83. Let $\mathscr{B}$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets

$$[n,a) := \{r \in \mathbb{R} : n \leq r < a\} \quad \text{with } n \in \mathbb{Z}, a \in \mathbb{R}.$$

1. Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $\mathscr{T}$ the associated topology. (*1p*)
2. Is $(\mathbb{R}, \mathscr{T})$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)? (*1p*)
3. compute the closure, the interior and the boundary of $A = [-\frac{1}{2}, \frac{1}{2}]$ in $(\mathbb{R}, \mathscr{T})$. (*1.5p*)

Exercise 16.84. Prove directly that the abstract torus T_{abs} is homeomorphic to $S^1 \times S^1$. More precisely, define an explicit map

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^4$$

whose image is

$$S^1 \times S^1 = \left\{(x,y,z,t) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + t^2 = 1\right\}$$

and which induces a homeomorphism $f : T_{\text{abs}} \to \mathbb{S}^1 \times S^1$. Provide all the arguments. (*1.5p*).

Exercise 16.85. Let X be the space obtained from the sphere S^2 by gluing the north and the south pole (with the quotient topology). Show that X can be obtained from a square $[0,1] \times [0,1]$ by glueing some of the points on the *boundary* (note: you are not allowed to glue a point in the *interior* of the square to any other point). More precisely:

1. Describe the equivalence relation R_0 on S^2 encoding the glueing that defines X. (*0.25p*)
2. Make a picture of X in $\mathbb{R}^3$. (*0.25p*)
3. Describe an equivalence relation R on $[0,1] \times [0,1]$ encoding a glueing with the required properties. (*1p*)
4. Show that, indeed, X is homeomorphic to $[0,1] \times [0,1]/R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture). (*0.5p*)

Exercise 16.86. Show that:

1. There exist continuous surjective maps $f : S^1 \to S^1$ which are not injective. (*0.5p*)
2. Any continuous injective map $f : S^1 \to S^1$ is surjective. (*1p*)

Exercise 16.87. Show that any continuous map

$$f : (\mathbb{R}, \mathscr{T}_{\text{Eucl}}) \to (\mathbb{R}, \mathscr{T}_l)$$

must be constant (recall that $\mathscr{T}_l$ is the lower limit topology- i.e. the one generated by intervals of type $[a,b)$). (*1.5p*)

Note 1: Motivate all your answers. Whenever you use a Theorem or Proposition, please make that clear (e.g. by stating it). Please write clearly (English or Dutch).

Note 2: The final mark is $\min\{10, 1+p\}$ where p is the number of points you collect from the exercises.

16.28 The 2012 exam B (June 27, 2012)

Exercise 16.88. (*1p*) Show that

$$K := \left\{ (x,y) \in \mathbb{R}^2 : x^{2012} + y^{2012} \leq 10 \sin(e^x + e^y + 1000) + e^{\cos(x^2+y^2)} \right\}.$$

is compact.

Exercise 16.89. (*1.5p*) Let X be a bouquet of two circles:

$$X = \left\{ (x,y) \in \mathbb{R}^2 : ((x-1)^2 + y^2 - 1)((x+1)^2 + y^2 - 1) = 0 \right\}.$$

We say that a space Y is an *exam space* if there exist three distinct points $p,q,r \in X$ such that Y is homeomorphic to the one-point compactification of $X - \{p,q,r\}$.

Find the largest number l with the property that there exist exam spaces $Y_1, \ldots, Y_l$ with the property that any two of them are not homeomorphic (prove all the statements that you make!).

Exercise 16.90. (*1p*) Let X be a topological space and let $\gamma : [0,1] \to X$ be a continuous function. Assume that γ is locally injective, i.e. that, for any $t \in [0,1]$, there exists a neighborhood V of t in $[0,1]$ such that

$$\gamma|_V : V \to X$$

is injective. Show that, for any $x \in X$, the set

$$\gamma^{-1}(x) := \{ t \in [0,1] : \gamma(t) = x \}$$

is finite.

Exercise 16.91. (*1p*) Let X be a Hausdorff normal space and let $A \subset X$ be a subspace with the property that any two continuous functions $f, g : X \to \mathbb{R}$ which coincide on A must coincide everywhere on X. Show that A is dense in X (i.e. the closure of A in X coincides with X).

Exercise 16.92. (*1p*) Consider the following open cover of $\mathbb{R}$:

$$\mathscr{U} := \left\{ (r,s) : r,s \in \mathbb{R}, |r-s| < \frac{1}{3} \right\}.$$

Describe a locally finite subcover of $\mathscr{U}$.

Exercise 16.93. (each of the sub-questions is worth *0.5 p*) Let A be a commutative algebra over $\mathbb{R}$. Assume that it is finitely generated, i.e. there exist $a_1, \ldots, a_n \in A$ (called generators) such that any $a \in A$ can be written as

$$a = P(a_1, \ldots, a_n),$$

for some polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$. Recall that X_A denotes the topological spectrum of A; consider the functions

$$f_i : X_A \to \mathbb{R}, \;\; f_i(\chi) = \chi(a_i) \quad 1 \leq i \leq n,$$
$$f = (f_1, \ldots, f_n) : X_A \to \mathbb{R}^n.$$

Show that

(a) f is continuous.
(b) For any character $\chi \in X_A$ and any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$,

$$\chi(P(a_1, \ldots, a_n)) = P(\chi(a_1), \ldots, \chi(a_n)).$$

(c) f is injective.
(d) the topology of X_A is the smallest topology on X_A with the property that all the functions f_i are continuous.
(e) f is an embedding.

Next, for a subspace $K \subset \mathbb{R}^n$, we denote by $\mathrm{Pol}(K)$ the algebra of real-valued polynomial functions on K and let $a_1, \ldots, a_n \in \mathrm{Pol}(K)$ be given by

$$a_i : K \to \mathbb{R}, \;\; a_i(x_1, \ldots, x_n) = x_i.$$

Show that

(f) $\mathrm{Pol}(K)$ is finitely generated with generators $a_1, \ldots, a_n$.
(g) Show that the image of f (from the previous part) contains K.

Finally:

(h) For the $(n-1)$ sphere $K = S^{n-1} \subset \mathbb{R}^n$, deduce that f induces a homeomorphism between the spectrum of the algebra $\mathrm{Pol}(K)$ and K.
(i) For which subspaces $K \subset \mathbb{R}^n$ can one use a similar argument to deduce that the spectrum of $\mathrm{Pol}(K)$ is homeomorphic to K?

Note: Motivate all your answers; give all details; please write clearly (English or Dutch). The mark is given by the formula:

$$\min\{10, 1 + p\},$$

where p is the number of points you collect from the exercises.

16.29 The 2012 retake exam (August 22, 2012)

Exercise 16.94. On $X = \mathbb{R}$ consider the topology:

$$\mathscr{T} = \{(-a,a) : 0 \leq a \leq \infty\}.$$

(a) Is $(X,\mathscr{T})$ metrizable? (*0.5 p*)
(b) Is $(X,\mathscr{T})$ 1st countable? (*0.5 p*)
(c) Is $(X,\mathscr{T})$ connected? (*0.5 p*)
(d) Is the sequence $x_n = (-1)^n + \frac{1}{n}$ convergent in $(X,\mathscr{T})$? To what? (*0.5 p*)
(e) Find the interior and the closure of $A = (-1,2)$ in $(X,\mathscr{T})$. (*0.5 p*)
(f) Show that any continuous function $f : X \to \mathbb{R}$ is constant. (*0.5 p*)

Exercise 16.95. Let M be the Moebius band. For a continuous map $f : S^1 \to M$ we denote

$$X_f := M - f(S^1).$$

(a) Show that for any f, X_f is open in M, it is locally compact but, if nonempty, it is not compact. (*0.5 p*)
(b) Is there a function f such that X_f^+ is homeomorphic to the projective space $\mathbb{P}^2$? (*0.5 p*)

Exercise 16.96.

(a) Let A be a commutative algebra over $\mathbb{R}$ and assume that $a_0, a_1, \ldots, a_n \in A$ generate A, i.e. that any $a \in A$ can be written as

$$a = P(a_0, \ldots, a_n),$$

for some $P \in \mathbb{R}[X_0, \ldots, X_n]$. Let X_A be the topological spectrum of A. Show that

$$f : X_A \to \mathbb{R}^{n+1}, \;\; f(\chi) = (\chi(a_0), \ldots, \chi(a_n))$$

is an embedding. (*1.5 p*)
(b) If $A = \mathrm{Pol}(K)$ is the algebra of real-valued polynomial functions on a subset $K \subset \mathbb{R}^{n+1}$ and a_i are the polynomial functions

$$a_i : K \to \mathbb{R}, \;\; a_i(x_0, \ldots, x_n) = x_i, \quad (0 \leq i \leq n),$$

show that the image of the resulting f contains K. (*0.5 p*)
(c) For the sphere $S^n \subset \mathbb{R}^{n+1}$, deduce that the spectrum of the algebra $\mathrm{Pol}(S^n)$ is homeomorphic to S^n. (*1 p*)

Exercise 16.97. (*1 p*) Prove that there is no continuous map $g : \mathbb{C} \to \mathbb{C}$ satisfying $g(z)^2 = z$ for all $z \in \mathbb{C}$.

Exercise 16.98. (*1 p*) On $(0,\infty)$ we define the action of the group $(\mathbb{Z},+)$ by:

$$\mathbb{Z} \times (0,\infty) \to (0,\infty), \;\; (n,r) \mapsto 2^n r.$$

Show that the quotient $(0,\infty)/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 16.99. (*1 p*) Let X be a Hausdorff normal space. Show that, for $A, B \subset X$, A and B have the same closure in X if and only if, for any continuous function $f : X \to \mathbb{R}$, one has the equivalence

$$f|_A = 0 \Longleftrightarrow f|_B = 0.$$

16.30 The 2011 exam A (April 20, 2011)

Notes:

- justify all your answers!!!
- the marking starts from 1 point. By solving the exercises, you can earn 9.5 more points. Your mark for this exam will be the minimum between your total number of points and 10.

Exercise 16.100. On $X = \mathbb{R}$ consider the family of subsets:

$$\mathscr{B} := \{(-p,p) : p \in \mathbb{Q}, p > 0\}, \ \mathscr{T} = \{(-a,a) : 0 \leq a \leq \infty\}.$$

- Show that $\mathscr{B}$ is a topology basis (*0.5 pt*).
- Show that $\mathscr{T}$ is the topology associated to $\mathscr{B}$. (*0.5 pt*).
- Is the sequence $x_n = (-1)^n + \frac{1}{n}$ convergent in $(X, \mathscr{T})$? To what? (*0.5 pt*).
- Find the interior and the closure of $A = (-1,2)$ in $(X, \mathscr{T})$ (*0.5 pt*).
- Show that any continuous function $f : X \to \mathbb{R}$ is constant (*0.5 pt*).
- For the topological space $(X, \mathscr{T})$, decide whether it is:

  1. Hausdorff (*0.5 pt*).
  2. 1st countable (*0.5 pt*).
  3. Metrizable (*0.5 pt*).
  4. Connected (*0.5 pt*).

Exercise 16.101. Which of the following spaces are homeomorphic and which are not:

- $(1,\infty)$ and $(0,\infty)$ (*0.5 pt*).
- $\mathbb{R}^2 - D^2$ and $\mathbb{R}^2 - \{0\}$ (*0.5 pt*).
- $(0,1)$ and $[0,1)$ (*0.5 pt*).
- $S^1 \times (\mathbb{R}^2 - \{0\})$ and $T^2 \times \mathbb{R}^*$ (*0.5 pt*).
- $S^1 \times (\mathbb{R}^2 - \{0\})$ and $T^2 \times \mathbb{R}^*_+$ (*0.5 pt*).

(here D^2 denotes the closed unit disk, T^2 denotes the torus, $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^*_+ = (0,\infty)$).

Exercise 16.102. Consider the map

$$\pi \colon \mathbb{R}^2 \to S^1 \times \mathbb{R}, \qquad (x,y) \mapsto \Big(\big(\cos(x+y), \sin(x+y) \big), x-y \Big) \in S^1 \times \mathbb{R}.$$

- Describe an equivalence relation R on $\mathbb{R}^2$ such that $(S^1 \times \mathbb{R}, \pi)$ is a quotient of $\mathbb{R}^2$ modulo R (*0.5 pt*).
- Find a group Γ and an action of Γ on $\mathbb{R}^2$ such that R is the equivalence relation induced by this action (*0.5 pt*).
- Show that, indeed, $\mathbb{R}^2/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}$ (*0.5 pt*).

Exercise 16.103. Show that, if a topological space X is Hausdorff, then the cone $\mathrm{Cone}(X)$ of X is Hausdorff (*0.5 pt*).

Exercise 16.104. Show that any continuous function $f : [0,1] \to [0,1]$ admits a fixed-point, i.e. there exists $t_0 \in [0,1]$ such that $f(t_0) = t_0$. (*0.5 pt*).

16.31 The 2011 exam B (June 29, 2011)

Exercise 16.105. Consider

$$X_1 = \left\{(x,y,z) \in \mathbb{R}^3 : (z=0) \text{ or } (x=y=0, z \geq 0)\right\},$$

$$X_2 = \left\{(x,y,z) \in \mathbb{R}^3 : (z=0) \text{ or } (x=0, y^2+z^2=1, z \geq 0)\right\}.$$

$$X_3 = \left\{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2+z^2=1) \text{ or } (y=0, z=0, \frac{1}{2} < |x| < 1)\right\},$$

(a) Show that X_1, X_2, X_3 are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures). (*0.5 p*)
(b) Show that the one-point compactifications of X_1, X_2 and X_3 are homeomorphic to each other. (*1 p*)

Exercise 16.106. Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathscr{R}_p$ the set of remainders modulo p. In other words,

$$\mathscr{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) : \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

We also denote by $\pi_p : \mathbb{R}[X_0, X_1, \ldots, X_n] \to \mathscr{R}_p$ the resulting quotient map. Show that:

(a) There is a unique algebra structure on $\mathscr{R}_p$ (i.e. unique operations $+$, $\cdot$ and multiplications by scalars, defined on $\mathscr{R}_p$) with the property that π_p is a morphism of algebras, i.e.

$$\pi_p(q_1+q_2) = \pi_p(q_1) + \pi_p(q_2),\ \pi_p(q_1 \cdot q_2) = \pi_p(q_1) \cdot \pi_p(q_2),\ \lambda \pi_p(q) = \pi_p(\lambda q)$$

for all $q_1, q_2 \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, $\lambda \in \mathbb{R}$. (*0.5 p*)
(b) For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathscr{R}_p$ has only one point. (*1 p*)
(c) For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathscr{R}_p$ is homeomorphic to S^n (*1 p*) .
(d) What is the spectrum for $p = x_0 x_1 \ldots x_n$? (*0.5 p*)

Exercise 16.107. Let X be the space of continuous maps $f : [0,1] \to [0,1]$ with the property that $f(0) = f(1)$. We endow it with the sup-metric $d_{\sup}$ and the induced topology (recall that $d_{\sup}(f,g) = sup\{|f(t) - g(t)| : t \in [0,1]\}$). Prove that:

(a) X is bounded and complete. (*1 p*)
(b) X is not compact. (*0.5 p*)

Exercise 16.108. Show that:

(a) The product of two sequentially compact spaces is sequentially compact. (*1 p*)
(b) Deduce that the product of two compact metric spaces is a compact space. (*1 p*)

Exercise 16.109. Show that the family of open intervals

$$\mathscr{U} := \{(q, q+1) : q \in \mathbb{R}\}$$

forms an open cover of $\mathbb{R}$ which admits no finite sub-cover, but which admits a locally finite sub-cover. (*1.5 p*)

Exercise 16.110. Prove that there is no continuous injective map $f : S^1 \vee S^1 \to S^1$, where $S^1 \vee S^1$ is a bouquet of two circles (two copies of S^1, tangent to each other). (*1.5 p*)

Note: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 11 points in the game!).

16.32 The 2011 retake exam (August 24, 2011)

Exercise 16.111. Show that the equation

$$x^5 + 7x^2 - 30x + 1 = 0$$

has at least two solutions $x_0, x_1 \in (0,2)$. (*1 p*)

Exercise 16.112. Consider the space $\mathscr{C}([0,1])$ of all continuous maps $f : [0,1] \to \mathbb{R}$, endowed with the sup-metric. Show that

$$A := \left\{ f \in \mathscr{C}([0,1]) : x^2 \leq e^{f(x)} + \sin(f(x)) \leq x \ \ \forall\, x \in [0,1] \right\}$$

is a closed subset of $\mathscr{C}([0,1])$, and that it is not bounded. (*1 p*)

Exercise 16.113. Describe a subspace $X \subset \mathbb{R}^2$ which is connected, whose closure (in $\mathbb{R}^2$) is compact, but with the property that X is not locally compact. (*1 p*)

Exercise 16.114. Let $G = (0,\infty)$ be the group of strictly positive reals, endowed with the usual product. Find an action of G on $\mathbb{R}^4 \setminus \{0\}$ with the property that $(\mathbb{R}^4 \setminus \{0\})/G$ is homeomorphic to S^3. (*1 p*)

Exercise 16.115. Let $X = \mathbb{R}^2$ endowed with the product topology $\mathscr{T}_l \times \mathscr{T}_l$, where $\mathscr{T}_l$ is the lower limit topology on $\mathbb{R}$.

a. Describe a topology basis for the topological space X. (*0.5 p*)
b. Find a sequence $(x_n)_{n\geq 1}$ of points in $\mathbb{R}^2$ which converges to $(0,0)$ with respect to the Euclidean topology, but which has no convergent subsequence in the topological space X. (*0.5 p*)
c. Compute the interior, the closure and the boundary (in X) of

$$A = [0,1) \times (0,1]. \quad (1p)$$

(please use pictures!).

Exercise 16.116. Decide (and explain) which of the following statements hold true:

a. $S^1 \times S^1 \times S^1$ can be embedded in $\mathbb{R}^4$. (*0.5 p*)
b. S^1 can be embedded in $(0,\infty)$. (*0.5 p*)
c. the cylinder $S^1 \times [0,1]$ can be embedded in the Klein bottle. (*0.5 p*)
d. The Moebius band can be embedded into the projective space $\mathbb{P}^2$. (*0.5 p*)
e. the projective space $\mathbb{P}^3$ can be embedded in $\mathbb{R}^6$. (*0.5 p*)

Exercise 16.117. Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathscr{R}_p$ the set of remainders modulo p. In other words,

$$\mathscr{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) : \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

For $q \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denoted by $[q] \in \mathscr{R}_p$ the induced equivalence class. Show that:

a. The operations (on $\mathscr{R}_p$) $+$, $\cdot$ and multiplications by scalars given by

$$[q_1] + [q_2] := [q_1 + q_2],\ [q_1] \cdot [q_2] := [q_1 \cdot q_2],\ \lambda [q] := [\lambda q]$$

are well-defined and make $\mathscr{R}_p$ into an algebra. (*0.5 p*)

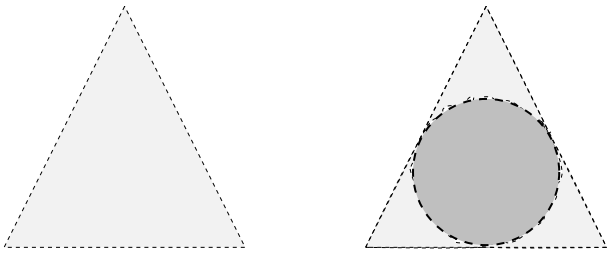
b. For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathscr{R}_p$ has only one point. (*0.5 p*)

c. For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathscr{R}_p$ is homeomorphic to S^n (*1 p*) .

16.33 The 2010 exam (April 21, 2010)

Important: motivate/prove your answers to the questions from the exercises. When making pictures, try to make them as clear as possible. When using a result from the lecture notes, please give a clear reference.

Exercise 16.118. Let X be the (interior of an) open triangle, as drawn in the picture (the edges are not part of X!), viewed as a topological space with the topology induced from $\mathbb{R}^2$. Let $A \subset X$ be the open disk drawn in the picture (tangent to the edges of the closed triangle). Compute the closure and the boundary of A in X. (*1p*)



X= an open triangle A= an open disk inside X

Fig. 16.1

Exercise 16.119. Let X be obtained by taking two disjoint copies of the interval $[0,2]$ (with the Euclidean topology) and gluing each t in the first copy with the corresponding t in the second copy, for all $t \in [0,2]$ different from the middle point. Explicitly, one may take the space

$$Y = [0,2] \times \{0\} \cup [0,2] \times \{1\} \subset \mathbb{R}^2$$

with the topology induced from the Euclidean topology, and X is the space obtained from Y by gluing $(t,0)$ to $(t,1)$ for all $t \in [0,2]$, $t \neq 1$. We endow X with the quotient topology.

(a) Is X Hausdorff? But connected? But compact? (*1.5p*)
(b) Can you find $A, B \subset X$ which, with the topology induced from X, are compact, but such that $A \cap B$ is not compact? (*1p*)
(c) Show that X can also be obtained as a quotient of the circle S^1. (*0.5p*)

Exercise 16.120. Let X, Y and Z be the spaces drawn in Figure 16.2.

(a) Show that any two of them are not homeomorphic. (*1.5p*)
(b) Compute their one-point compactifications X^+, Y^+ and Z^+. (*1p*)
(c) Which two of the spaces X^+, Y^+ and Z^+ are homeomorphic and which are not? (*1p*)

Exercise 16.121. Let M be the Moebius band. For any continuous function $f : S^1 \to M$ we denote by M_f the complement of its image:

$$M_f := M - f(S^1)$$

and we denote by M_f^+ the one-point compactification of M_f.

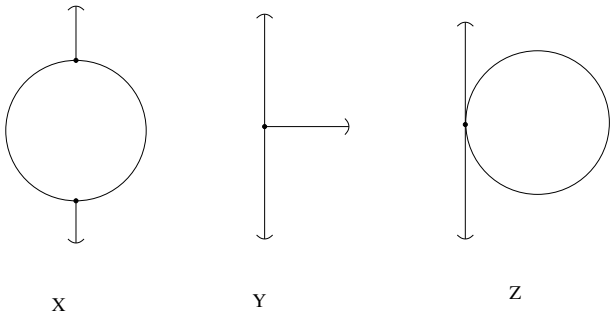


Fig. 16.2

(a) Show that for any f, M_f is open in M, it is locally compact but, if nonempty, it is not compact. (*1p*)
(b) Find an example of f such that M_f^+ is homeomorphic to D^2. Then one for which it is homeomorphic to S^2. And then one for $\mathbb{P}^2$. (*1.5p*)

16.34 The 2010 retake exam (August 27, 2010)

Exercise 16.122. Let $\mathbb{N} = \{0,1,2,\ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathscr{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0,1,\ldots,n\}$ with $n \in \mathbb{N}$.
- $\mathscr{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n,n+1,\ldots\}$ with $n \in \mathbb{N}$.

Questions:

1. Show that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on $\mathbb{N}$. (*1p*)
2. Show that the spaces $(\mathbb{N},\mathscr{T}_1)$ and $(\mathbb{N},\mathscr{T}_2)$ are not homeomorphic. (*0.5p*)
3. for each of the spaces $(\mathbb{N},\mathscr{T}_1)$ and $(\mathbb{N},\mathscr{T}_2)$ decide whether the space is Hausdorff, connected or compact. (*1.5p*)

Exercise 16.123. Let $\mathscr{T}_r$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathscr{B}_r := \{(a,b] : a,b \in \mathbb{R}, a < b\}.$$

Compute the interior, the closure and the boundary of

$$A := \left(\left(0,\frac{1}{3}\right) \cup \left[\frac{1}{2},1\right]\right) \times [0,1)$$

inside the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathscr{T}_r \times \mathscr{T}_r$. (*2p*).

Exercise 16.124. Let X be the connected sum of a Moebius band and a torus. Show how one can obtain X from a disk by gluing some of the points on the boundary of the disk (*2p*). Then describe on the picture a continuous map $f : S^1 \to X$ such that the one-point compactification of $X \setminus f(S^1)$ is homeomorphic to a sphere. (*0.5p*)

Exercise 16.125. Let X be the one-point compactification of the space obtained by removing two points from the torus. Show that:

1. X can be embedded in $\mathbb{R}^3$.(*1p*)
2. X is not homeomorphic to S^2. (*1p*)

16.35 The 2009 exam part A (April, 2009)

Exercise 16.126. (*3.25p*) Let $\mathscr{T}_l$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathscr{B}_l := \{[a,b) : a,b \in \mathbb{R}, a < b\}.$$

(a) With the topology induced from $(\mathbb{R}, \mathscr{T}_l)$, is $[0,1)$ connected? But compact? (*1p*)
(b) The same questions for $(0,1]$. (*1p*)
(c) Are $[0,1)$ and $(0,1]$ (with the induced topologies) homeomorphic? (*0.25p*)
(d) Compute the interior, the closure and the boundary of

$$A = (0,1] \times [0,1)$$

in the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathscr{T}_l \times \mathscr{T}_l$. (*1p*)

Exercise 16.127. (*2.5p*) Describe an embedding of the cylinder $S^1 \times [0,1]$ into the space X, show that its complement is locally compact, and find the one-point compactification of the complement, in each of the cases:

(a) X is a torus. (*1.5p*)
(b) X is the plane $\mathbb{R}^2$. (*1p*)

Exercise 16.128. (*2.5p*)

(a) Write the Moebius band as a union of two subspaces M and C where M is itself a Moebius band, C is a cylinder (i.e. homeomorphic to $S^1 \times [0,1]$) and $M \cap C$ is a circle. (*1p*)
(b) Similarly, decompose $\mathbb{P}^2$ as the union of a Moebius band M and another subspace Q, such that Q is a quotient of the cylinder and $M \cap Q$ is a circle. (*1p*)
(c) Deduce that $\mathbb{P}^2$ can be obtained from a Moebius band and a disk D^2 by gluing them along their boundary circles. (*0.5p*)

Exercise 16.129. (*1.75p*) Let X be a topological space and let $\mathrm{Cone}(X)$ be its cone. Show that:

(a) If X is compact then so is $\mathrm{Cone}(X)$. (*1p*)
(b) If X is path connected then so is $\mathrm{Cone}(X)$. (*0.5p*)
(c) If X is connected then so is $\mathrm{Cone}(X)$. (*0.25p*)

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

Overall hint: At the end of each question, you can see its "point value" (10 points in total). The more difficult questions of each exercise are the ones for which you get less points! So, try not to spend too much time with the more difficult questions before you answered the others. Finally: use pictures!

16.36 The 2009 retake exam (August, 2009)

Exercise 16.130. (*2.5p*) Let X be the (interior of an) open triangle, as drawn in the picture (the edges are not part of X!), viewed as a topological space with the topology induced from $\mathbb{R}^2$. Let $A \subset X$ be the open disk drawn in the picture (tangent to the edges of the closed triangle).

(a) Compute the closure and the boundary of A in X. (*1p*)
(b) Show that both of them are locally compact but not compact. (*0.5p*)
(c) For each one of them, compute their one-point compactification. (*1p*)

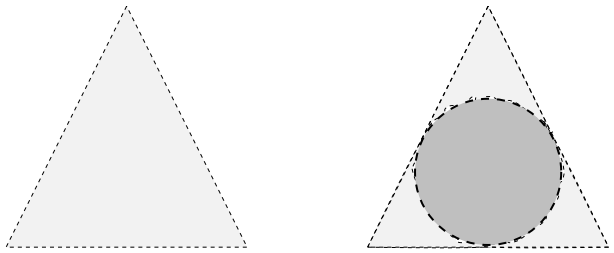


Fig. 16.3

Exercise 16.131. (*3.5p*) In each of the following examples, decide (and prove) whether X can be embedded in Y:

(a) $X = [0,1]$, $Y = [0,1)$. (*0.5p*)
(b) $X = \mathbb{R}$, $Y = S^1$. (*0.5p*)
(c) $X = S^1 \times \mathbb{R}$, $Y = \mathbb{R}^2$. (*0.5p*)
(d) $X = S^1 \times S^1$, $Y = \mathbb{R}^3$. (*0.5p*)
(e) $X =$ Moebius band, $Y =$ Klein bottle. (*0.5p*)
(f) $X =$ a bouquet of two circles, $Y = S^1$. (*1p*)

Exercise 16.132. (*4p*) Given two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on a set X, show that:

(a) $\mathscr{T}_1 \cap \mathscr{T}_2 := \{U \subset X : U \in \mathscr{T}_1, U \in \mathscr{T}_2\}$ is a topology on X. (*0.5p*)
(b) If $(X, \mathscr{T}_1)$ or $(X, \mathscr{T}_2)$ is connected, then so is $(X, \mathscr{T}_1 \cap \mathscr{T}_2)$. (*0.5p*)
(c) If $(X, \mathscr{T}_1)$ or $(X, \mathscr{T}_2)$ is compact, then so is $(X, \mathscr{T}_1 \cap \mathscr{T}_2)$. (*0.5p*)

Next, let $X = \mathbb{R}$, let $\mathscr{T}_l$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathscr{B}_l := \{[a,b) : a,b \in \mathbb{R}, a < b\}$$

and let $\mathscr{T}_u$ be defined similarly, using intervals of type $(a,b]$.

(d) What is $\mathscr{T}_l \cap \mathscr{T}_u$? (*1p*)
(e) Show that the discrete topology is the only topology on $\mathbb{R}$ containing $\mathscr{T}_l \cup \mathscr{T}_u$. (*0.5p*)
(f) Finally, show that the converses of (b) and (c) above do not hold in general. (*1p*)

16.37 The 2008 exam A (April 15, 2008)

Exercise 16.133. We consider the collection of subsets of $\mathbb{R}$:

$$\mathscr{B} := \{[a,b) : a,b \in \mathbb{R}, a \leq b\} \cup \{\mathbb{R}\}.$$

(a) Show that $\mathscr{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. (*0.5p*)
(b) Let $\mathscr{T}$ be the smallest topology on $\mathbb{R}$ containing $\mathscr{B}$. Show that $\mathscr{T}$ is larger than the Euclidean topology $\mathscr{T}_{\text{eucl}}$. (*1p*)
(c) In the topological space $(\mathbb{R}, \mathscr{T})$, find the closure, the interior and the boundary of

$$A = (0,1) \cup [2,3]. \qquad (1.5p)$$

(d) Show that $(\mathbb{R}, \mathscr{T})$ and $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$ are not homeomorphic.(*0.5p*)

Exercise 16.134. (*1.5p*) Show that the torus T contains a subspace C homeomorphic to a bouquet of two circles such that $T - C$ is homeomorphic to the open 2-disk. Similarly for the double torus and a bouquet of four circles.

Exercise 16.135. (*1.5p*) Consider the group $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$ of remainders modulo n (with the addition modulo n), and the action of Z_n on the circle S^1 given by

$$k \bullet (\cos(t), \sin(t)) = \left(\cos\left(t + \frac{2k\pi}{n}\right), \sin\left(t + \frac{2k\pi}{n}\right)\right)$$

(for $k \in \mathbb{Z}_n$, $(\cos(t), \sin(t)) \in S^1$). Show that the resulting quotient $S^1/\mathbb{Z}_n$ is homeomorphic to S^1. What can you say when $n = 2$?

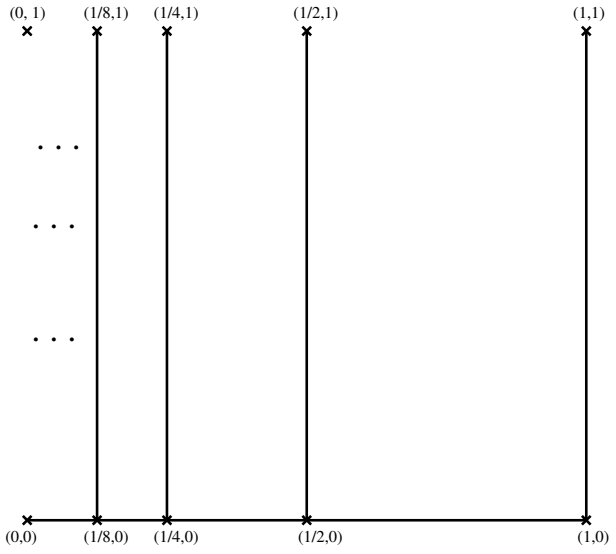


Fig. 16.4

Exercise 16.136. Consider the following subset of $\mathbb{R}^2$:

$$X = \bigcup_{n\geq 1 \text{ integer}} \left\{\frac{1}{2^n}\right\} \times [0,1] \quad \cup \quad [0,1]\times\{0\} \quad \cup \quad \{(0,1)\}$$

(see the picture), with the induced topology. Explain which of the following properties are true for X.

(a) it is Hausdorff. (*0.5p*)
(b) it is compact. (*1p*)
(c) it is locally compact. (*0.5p*)
(d) it is connected. (*0.5p*)
(e) it is path connected. (*0.5p*)

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

16.38 The 2007 exam A (April 16, 2007)

Exercise 16.137. Show that, in any topological space X, for any subspace $A \subset X$, one has

$$\partial(A) = \partial(X - A).$$

(1.5p)

Exercise 16.138.

(a) Is it true that any continuous surjective map $f : S^1 \to S^1$ is a homeomorphism? (0.75p)
(b) Show that S^1 cannot be embedded in $\mathbb{R}$. (1p)
(c) Show that any continuous injective map $f : S^1 \to S^1$ is a homeomorphism. (0.75p)

Exercise 16.139. Let (X,d) be a metric space and A,B closed subsets of X such that

$$d(A,B) = 0.$$

(Recall that $d(A,B) =: \inf\{d(a,b) : a \in A, b \in B\}$).

(a) Is it true that A and B must have a common point (i.e. $A \cap B \neq \emptyset$)? (1p)
(b) What if we assume that both A and B are compact? (0.75p)
(c) What if we assume that either A or B is compact? (0.75p)

Exercise 16.140. For any continuous map $f : S^1 \to T^2$ we define

$$X_f := T^2 - f(S^1).$$

(a) Is it true that, for any continuous function f, X_f is compact? But locally compact? But metrizable? But normal? But connected? (0.5p)
(b) Describe two embeddings $f_1, f_2 : S^1 \to T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic. (.75p)
(c) Describe the one-point compactifications $X_{f_1}^+$ and $X_{f_2}^+$. (0.75p)
(d) Describe $f : S^1 \to T^2$ continuous such that X_f^+ is homeomorphic to S^2. (0.5p)

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

16.39 The 2006 exam A (April 16, 2006)

Exercise 16.141. Let $(X, \mathscr{T})$ be a topological space and assume that $A, B \subset X$ are compact subsets. Prove that $A \cup B$ is compact. Prove that the same holds for $A \cap B$ if X is Hausdorff.

Exercise 16.142. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathscr{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$.
- $\mathscr{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n, n+1, \ldots\}$ with $n \in \mathbb{N}$.

Questions:

1. Show that $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on $\mathbb{N}$.
2. Show that the spaces $(\mathbb{N}, \mathscr{T}_1)$ and $(\mathbb{N}, \mathscr{T}_2)$ are not homeomorphic.
3. for each of the spaces $(\mathbb{N}, \mathscr{T}_1)$ and $(\mathbb{N}, \mathscr{T}_2)$ decide whether the space is Hausdorff, connected or compact.

Exercise 16.143. Questions:

1. Is $[0, 1]$ homeomorphic to $[0, 1)$?
2. Does there exist a continuous surjective map $f : [0, 1] \to [0, 1)$? But a continuous injective one?
3. Does there exist a continuous surjective map $f : S^1 \to [0, 1)$? But an injective one?

Exercise 16.144. Consider

$$X = [0, 1] \times [0, 1)$$

with the topology induced from $\mathbb{R}^2$. Prove that X is a locally compact Hausdorff space and describe the one-point compactification (use a picture). What happens if we replace X by $Y = X \cup \{(1, 1)\}$?

There was also an "exam B", with exercises that are no longer covered by these lecture notes.

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