COURSE	main OBJECTS	MAPS ①	ISOMORPHISMS	Key Words
Linear algebra	vector spaces (or just $\mathbb{R}^n$)	linear maps f (or matrices)	linear maps f bijective	dimension, basis, linear independence, subspace etc etc
Group Theory	Groups $(G,\cdot)$	Group homomorphism $f: G_1 \to G_2$	Bijective homomorphisms	cardinality, order of elts quotients, abelianity, etc
Ring Theory	Rings $(R,+,\cdot)$	Ring homomorphism $f: R_1 \to R_2$	Bijective homomorphism	Ideals, etc
Analysis	$\mathbb{R}^n$ opens $U \subseteq \mathbb{R}^n$	smooth maps $f: U \to V$	diffeomorphism	.....
TOPOLOGY	TOPOLOGICAL SPACES $(X,\mathcal{T})$	Continuous maps $f:(X_1,\mathcal{T}_1)\to(X_2,\mathcal{T}_2)$	homeomorphisms (i.e. bijective f & f, f^{-1} are both continuous)	

When are two o

First atten

Hence $d:$ X

c

"Space"?

define:

Exercise

Temporary definition:
- "space" X: a set + extra-structure that allows us to talk about convergen
- a map $f: X \to Y$ between "spaces" is "continuous" if: $\left(\text{if } x_k \to x \text{ in } X\right) \to \left(f\right.$
- a homeomorphism between "spaces" is a bijective f s.t. f, f^{-1} are continuo

$\mathbb{Z}_2 \not\simeq \mathbb{Z}_3$ (cardin)

$\mathbb{Z}_2 \times \mathbb{Z}_2 \not\simeq \mathbb{Z}_4$ (order)

$\mathbb{Z}_6 \simeq \mathfrak{S}_3$ (commutativ

When are two objects isomorphic? (2)

First attempt to make precise sense of "spaces": metric spaces (X, d)

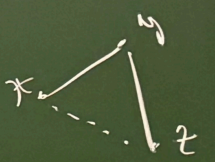
Hence $d : X \times X \longrightarrow \mathbb{R}$, $(x,y) \longmapsto d(x,y)$ s.t.

(M1) $d(x,y) \geq 0 \quad (\forall)\ x, y \in X$

(M2) $d(x,y) = 0 \iff x = y$

(M3) $d(x,y) = d(y,x)$

(M4) $d(x,y) + d(y,z) \geq d(x,z)$



"Space"? Yes:

define: $x_k \to x$ in (X,d) if $\lim_{k \to \infty} d(x_k, x) = 0$ (limit in $\mathbb{R}$!)

Exercise: the resulting notion of continuity is equivalent to the one you saw before.

about convergence of sequences $(x_n)_{n \geq 1}$, with $x_n \in X$.

x in $X) \longrightarrow (f(x_k) \to f(x)$ in $Y)$

are continuous

Ex 1: $X = \mathbb{R}^n$ or subset of $\mathbb{R}^n$, the Euclidean metric

$$d_{Eucl}(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$x = (x_1, \dots, x_n)$
$y = (y_1, \dots, y_n) \in \mathbb{R}^n$

Ex 2: $X = \mathbb{R}^n$ (or subset of it), the sup-max metric:

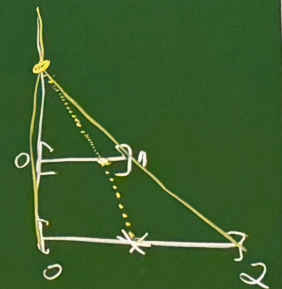
$$\rho(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

Rk/exercise: For a sequence $(x^k)_{k \geq 1}$ in $\mathbb{R}^n$, one has:

$$x^k \to x \text{ in } (X, d_{Eucl}) \iff x^k \to x \text{ in } (\mathbb{R}^n, \rho)$$

Ex 3: On any set X we have the discrete metric:

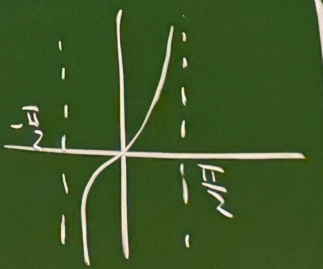
$$d_{discr}(x,y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \neq y \end{cases}$$



$f(x) = x$

Ex 4:
- $[a,b]$, $a, b \in \mathbb{R}$, $a < b$ are all "spaces"... all homeomorphic (e.g. $[0,1] \simeq [0,2]$)
- (a,b) with $a < b$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$ ... all homeomorphic (e.g. $(-\frac{\pi}{2}, \frac{\pi}{2}) \simeq \mathbb{R}$)

tg & arctg.



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TOPOLOGY	TOPOLOGICAL SPACES $(X,\mathcal{T})$	Continuous maps $f:(X_1,\mathcal{T}_1)\to(X_2,\mathcal{T}_2)$	homeomorphisms (i.e bijective s.t f, f^{-1} are both cont.ous)	

Temporary definition:
- "space" X: is a set + extra structure that allows us to talke about convergence of sequences $(x_n)_{n\geq 1}$, with $x \in X$
- a map $f: X \to Y$ between "spaces" is "continuous" if: $\left(\text{if } x_n \to x \text{ in } X\right) \Rightarrow \left(f(x_n) \to f(x) \text{ in } Y\right)$
- a homeomorphism between "spaces" is a bijective f s.t. f, f^{-1} are continuous

② When are two objects isomorphic?

First attempt to make precise sense of "spaces": metric spaces (X,d)

Hence $d: X \times X \to \mathbb{R}$, $(x,y) \longmapsto d(x,y)$ s.t.

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(M3) $d(x,y)=d(y,x)$

(M4) $d(x,y)+d(y,z) \geq d(x,z)$

"Space"? Yes.

define: $x_n \to x$ in (X,d) if $\left(\lim_{n\to\infty} d(x_n,x)=0 \text{ (limit in } \mathbb{R}!)\right)$

Exercise: the resulting notion of continuity is equivalent to the one you saw before

③

Ex1: $X=\mathbb{R}^n$ or subset of $\mathbb{R}^n$, the Euclidean metric

$$d_{Eucl}(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \qquad x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n) \in \mathbb{R}^n$$

Ex2: $X=\mathbb{R}^n$ (or subset of it), the "sup metric":

$$S(x,y)=\max\{|x_1-y_1|,\dots,|x_n-y_n|\}$$

Rk/exercise: For a sequence $(x^k)_{k\geq 1}$ in $\mathbb{R}^n$, one has

$x^k \to x$ in $(\mathbb{R}^n, d_{Eucl}) \iff x^k \to x$ in $(\mathbb{R}^n, S)$

Ex3: On any set X we have the discrete metric

$$d_{discr}(x,y)=\begin{cases}0 & \text{when } x=y\\ 1 & \text{when } x\neq y\end{cases}$$

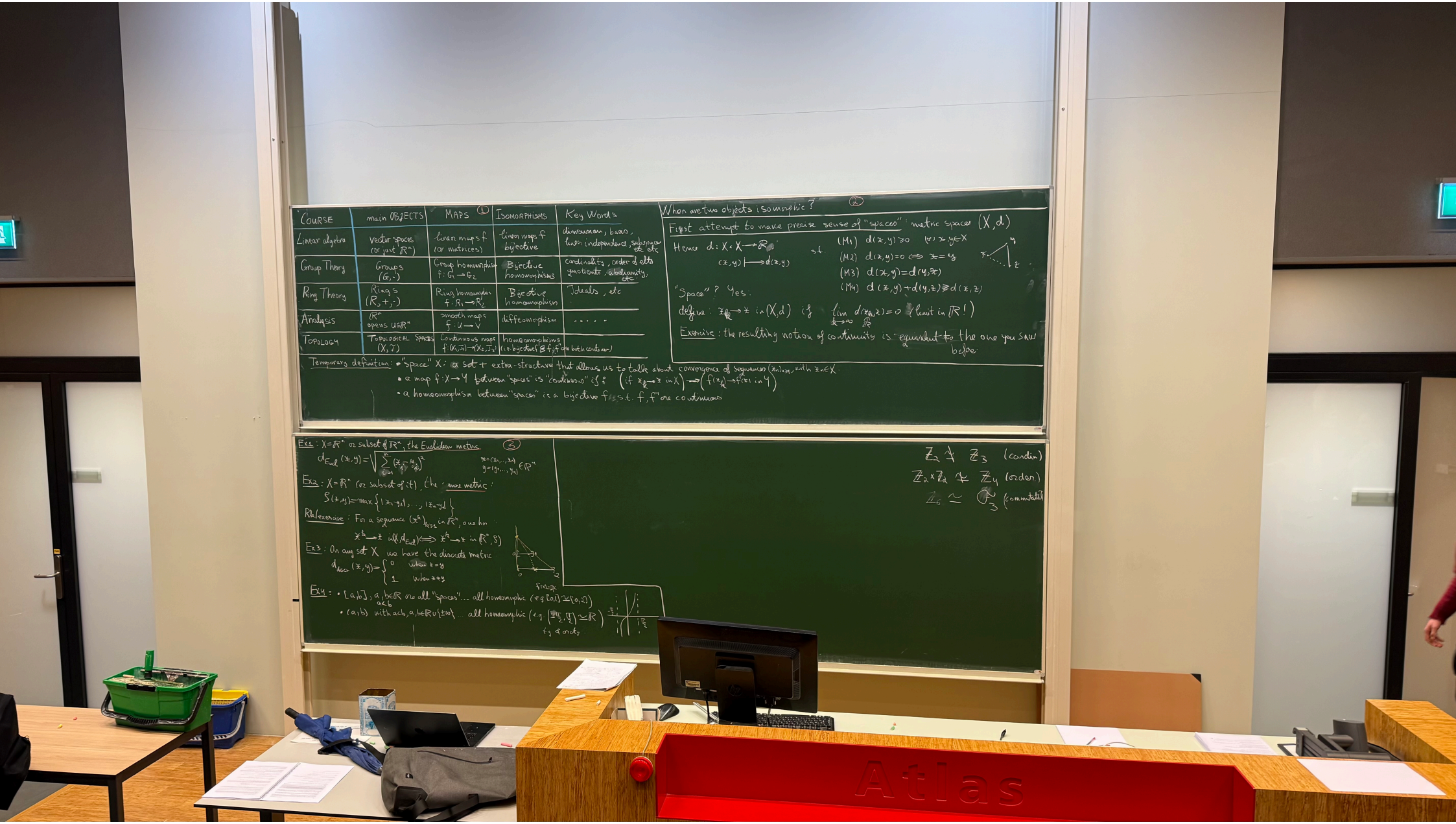
Fix x

Ex4:
- $[a,b]$, $a,b \in \mathbb{R}$ are all "spaces"... all homeomorphic (e.g $[0,1] \cong [a,b]$ via $t \mapsto a+t(b-a)$)
- (a,b) with $a<b$, $a,b \in \mathbb{R} \cup \{\pm\infty\}$ ... all homeomorphic (e.g. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \simeq \mathbb{R}$ via $\tan$)

$\mathbb{Z}_2 \not\cong \mathbb{Z}_3$ (cardin)

$\mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$ (order)

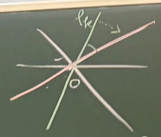
$\mathbb{Z}_6 \simeq$ (commutativity)



(3)

Ex 5 (of "space"): the projective space $\mathbb{P}^n := \{ \ell \subseteq \mathbb{R}^{n+1} \mid \ell \text{ is a line through the origin} \}$
($\ell \subseteq \mathbb{R}^{n+1}$ a 1-dim. vector subspace)

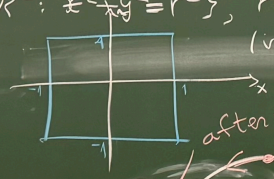
(4)



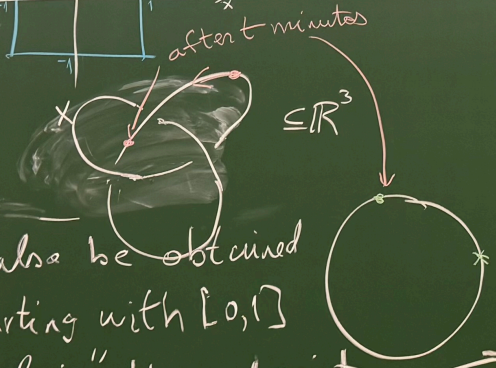
Ex 6: Circles: any "space" which is homeomorphic to $S^1 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} = \{(\cos t, \sin t) : t \in \mathbb{R}\}$ ($\subseteq \mathbb{R}^2$) (the unit circle)

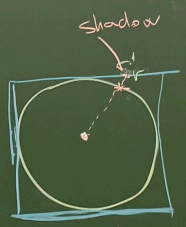
E.g.: • $S^1_r = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=r^2\}$, $r>0$

• the unit square (no inside!!)

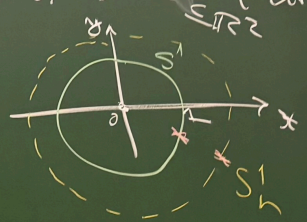


• knotted circles





$f : S^1_r \to [-1,1] \times [-1,1]$



$f(x) = rx$

Ex: $[0,1)$ ~~homeom~~ to S^1

$f : [0,1) \to S^1$



$f(t) = (\cos 2\pi t, \sin 2\pi t)$

Continuous, bijective but f^{-1} not continuous.

• can also be obtained by starting with $[0,1]$ and "gluing" the end-points

$\Rightarrow$



$\Rightarrow$



(get it from $[0,1)$ but not homeomorphic to it!!!)

method to produce new "spaces"

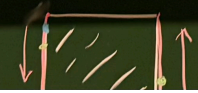
SMS | Key Words

dimension, basis,

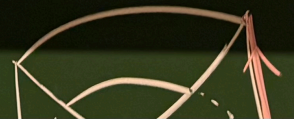
(5)

Ex: Moebius band(s)

① By gluing



$\Rightarrow$



Moebius band

f" not continuous.

"method to produce new spaces"

Words

…nsion, bases,
…independence, subspace etc etc
…nality, order of elts
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…
…continuous)
…lows us to talk about
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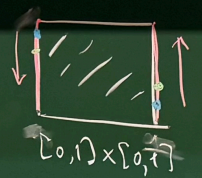
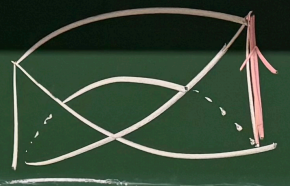
Ex: Moebius band(s) ⑤

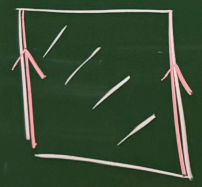
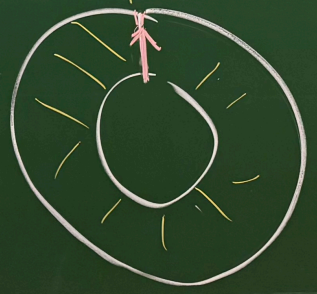
① By gluing: 

$\Rightarrow$ 
Moebius band
(or twist 3 times)

② geometrically: $R > r > 0$

$$M_{R,r} = \left\{ \left(\left(R + t\sin\frac{a}{2}\right)\cos a,\ \left(R + t\sin\frac{a}{2}\right)\sin a,\ t\cos\frac{a}{2} \right) \in \mathbb{R}^3 : a \in [0, 2\pi],\ t \in [-r, r] \right\}$$

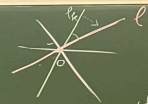
using e.g. d_{Eucl} $\Rightarrow$ get a space.

still in the plane!

Ex 5 (of "space"): the projective space $\mathbb{P}^n := \{ \ell \subseteq \mathbb{R}^{n+1} \mid \ell \text{ is a line through the origin} \}$ (9)

($\ell \subseteq \mathbb{R}^{n+1}$ a 1-dim. vector subspace)



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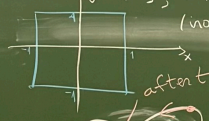

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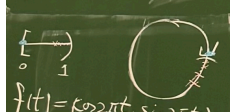

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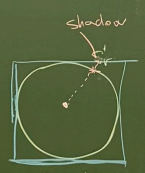

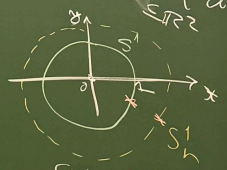

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…um)

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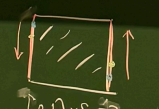
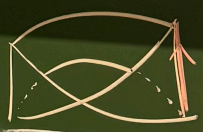
(if $x_n \to x$ in …

Ex: Moebius band(s) (5)

① By gluing



$\Longrightarrow$



Moebius band (or twist 3 times)

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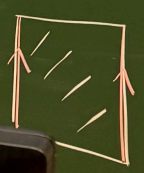

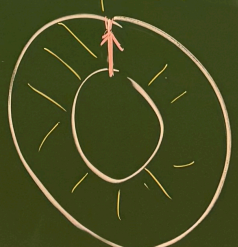

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