Def: X = set (1)

- a topology on X : a collection $\mathcal{T}$ of subsets of X s.t.:

(T1) $\emptyset, X \in \mathcal{T}$

(T2) intersection of two members of $\mathcal{T}$ is in $\mathcal{T}$:

$U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}$

(T3) union of any collection from $\mathcal{T}$ is in $\mathcal{T}$:

$U_i \in \mathcal{T}$ $\forall i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$.

(I: any indexing set)

- a topological space : a set X & a topology $\mathcal{T}$ on X $\rightarrow (X, \mathcal{T})$

Given $(X, \mathcal{T})$:

→ Call the $U \in \mathcal{T}$ opens of the space $(X, \mathcal{T})$

$\mathcal{T}$ is providing the set X with "a collection of opens"

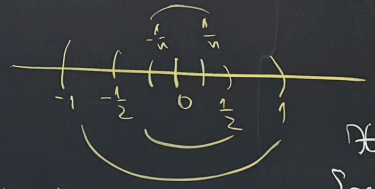
(T1) $\emptyset, X$ are open in $(X, \mathcal{T})$

(T2) U_1, U_2 - open $\Rightarrow U_1 \cap U_2$ - open

(T3) arbitrary union of opens is open (in the space $(X, \mathcal{T})$)

$\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

(T2 with infinite $\cap$ is not a good idea.



$x \in U$

$\{x\} \subseteq U$

→ a subset $A \subseteq X$ called : closed in $(X, \mathcal{T})$ if $X \setminus A$ = open (2)

(1)

Def: X = set

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$\mathcal{T}$ is

wit

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(T

$\bigcap_{n \in \mathbb{N}} (-\frac{1}{n}$

(T21

(2)

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→ for $x \in X$: notation $\mathcal{T}(x) = \{ U \in \mathcal{T} \mid x \in U \} \ni U$

open neighborhood of x

→ Given a sequence $(x_n)_{n \geq 1}$ in X

say that it is **convergent to x in $(X, \mathcal{T})$**

if $x \in X$ and : $(\forall) U \in \mathcal{T}(x)$

$(\exists) n_U \in \mathbb{N}$ s.t. $x_n \in U \ (\forall) n \geq n_U$

→ Given top. spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ a continuous map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is any function $f : X \to Y$ satisfying:

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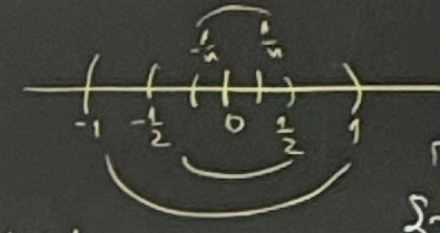

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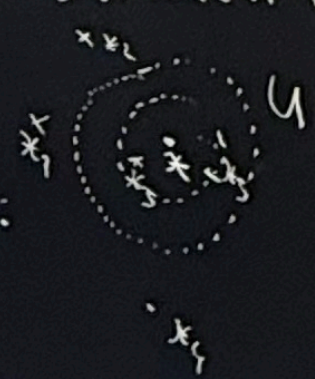

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X s.t.:

$\mathcal{T}$ on X

Examples: X = arbitrary set ③

1. $\mathcal{T}_{triv}$ consists of $\emptyset, X$ only.
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Example 5: metric topologies: Start with ④ a metric space (X, d).

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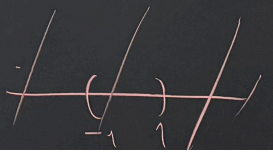
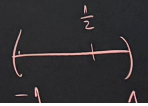
$B_d(x, r)$ r

$\mathbb{R}$ $X = U$



$X = (-1, 1)$

$d(x, y) = |x - y|$



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$B_d(x, r)$ r

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→ Given a sequence $(x_n)_{n \geq 1}$ in X say that it is convergent to x in $(X, \mathcal{T})$ if $x \in X$ and: $(\forall) U \in \mathcal{T}(x)$ $(\exists) n_U \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq n_U$

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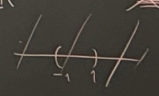
$\mathbb{R}$ $\supseteq U$



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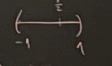


$d(x, y) = |x - y|$

$B_d(0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2})$

$B_d(\frac{1}{2}, \frac{1}{3}) = (\frac{1}{6}, \frac{5}{6})$

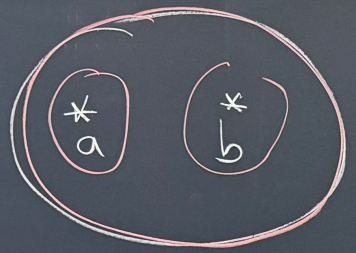
$B_d(0, 2) = \{y \in (-1, 1) : d(y, 0) < 2\} = (-1, 1)$

$B_d(\frac{2}{3}, \frac{4}{3}) = (0, 1)$

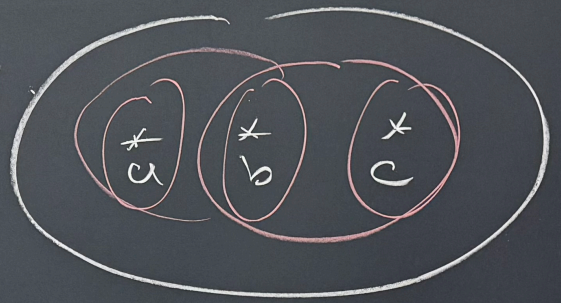
$(0, \frac{4}{3})$

$F \subseteq X$ at most countable ← the cocountable topology

$\mathcal{T}_d = \{ U \subseteq X \mid U \text{ is open w.r.t. } d \}$



$X = \{a, b\}$



many possibilities

$\mathcal{T}_1 = \{\emptyset, X\}$

$\mathcal{T}_2 = \{\emptyset, X, \{a\}\}$

$\mathcal{T}_3 = \{\emptyset, X, \{b\}\}$

$\mathcal{T}_4 = \{\emptyset, X, \{a\}, \{b\}\}$

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For $x \in X$, $\varepsilon > 0$: $B_d(x, \varepsilon) := \{y \in X : d(y,x) < \varepsilon\} \subseteq X$

A set $U \subseteq X$ is called open w.r.t. d if:

$(\forall)\, x \in U \quad \exists\, \varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subseteq U$ (*)

The collection

$$\mathcal{T}_d = \{U \subseteq X \mid U \text{ is open w.r.t. } d\}$$

called: the topology induced by d.

- the cofinite topology on X: $F \subseteq X$ finite
- the cocountable topology: $F \subseteq X$ at most countable

Prop: Given (X, d): (a) $\mathcal{T}_d$ is a topology on X

(b) $\mathcal{T}_d$ is the smallest topology on X that contains all $B_d(x, \varepsilon)$ $(\forall) x \in X, (\forall) \varepsilon > 0$.

than $\mathcal{T}_2$ / ...or than $\mathcal{T}_1$

Rk: for $U \subseteq X$: $U \in \mathcal{T}_d \iff U$ is a union of balls

---

Pf of Prop: (a). Check (T_2) for $\mathcal{T}_d$: Assume $U_1, U_2 \in \mathcal{T}_d$. To prove: (5) $U_1 \cap U_2 \in \mathcal{T}_d$

i.e.; by (*):

$(\forall) x_1 \in U_1, \exists\, \varepsilon_1 > 0$ s.t. $B(x_1, \varepsilon_1) \subseteq U_1$

$(\forall) x_2 \in U_2, \exists\, \varepsilon_2 > 0$ s.t. $B(x_2, \varepsilon_2) \subseteq U_2$

I know

i.e.: to prove

$(\forall) x \in U_1 \cap U_2, \exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U_1 \cap U_2$

Prove what we have to prove. Let $x \in U_1 \cap U_2$, i.e. $x \in U_1$ & $x \in U_2$

Use what we know for $x_1 = x$, $x_2 = x$:

$\Rightarrow \exists\, \varepsilon_1$ s.t. $B(x, \varepsilon_1) \subseteq U_1$

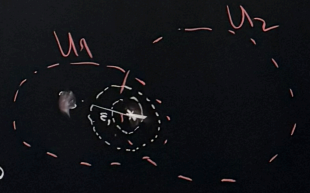
$\exists\, \varepsilon_2$ s.t. $B(x, \varepsilon_2) \subseteq U_2$

Take now $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ Then

$B(x, \varepsilon) \subseteq B(x, \varepsilon_1) \subseteq U_1$

$\subseteq B(x, \varepsilon_2) \subseteq U_2$

$\Rightarrow$ we found $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U_1 \cap U_2$.



Rest of axioms: similar.

(x, ε) $(\forall) x, \varepsilon$ $\exists$ $U \in \mathcal{T}$.

& $\emptyset$ itself. at most countable

topology

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Rk: for $U \subseteq X$: $U \in \mathcal{T}_d \iff U$ is a union of balls

---

proof of remark: "$\Rightarrow$" (6) $U \in \mathcal{T}_d$.

$\Rightarrow (\forall) x \in U \ \exists \varepsilon_x > 0$ s.t. $B(x,\varepsilon_x) \subseteq U$

Now: $(\forall) x \in U$:

$$\{x\} \subseteq B(x,\varepsilon_x) \subseteq U.$$

take now $\bigcup_{x\in U}$:

$$U = \bigcup_{x\in U} \{x\} \subseteq \bigcup_{x\in U} B(x,\varepsilon_x) \subseteq U.$$

$\Rightarrow U = \bigcup_{x\in U} B(x,\varepsilon_x)$ ... an union of balls.

for (b): If $\mathcal{T}$ is another top containing all $B(x,\varepsilon)$

To prove: $\mathcal{T}_d \subseteq \mathcal{T}$ $\Rightarrow U = U$ open balls $\Rightarrow U \in \mathcal{T}$ ((3) $(\forall) x, \varepsilon$, $B \in \mathcal{T}$)

i.e.: $?\ U \in \mathcal{T}_d \Rightarrow U \in \mathcal{T}$?

~~2d of Prop~~: (a). Check (T_2) for $\mathcal{T}_d$: Assume $U_1, U_2 \in \mathcal{T}_d$. To prove: (5) $U_1 \cap U_2 \in \mathcal{T}_d$

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Prove what we have to prove. Let $x \in U_1 \cap U_2$, i.e. $\begin{cases} x \in U_1 \\ \text{i.e.} \\ x \in U_2 \end{cases}$

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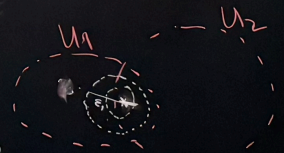

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$\mathcal{T}_1$ is smaller than $\mathcal{T}_2$

Example 6-7-8: topologies on $\mathbb{R}$: lots of topologies (7): $\mathcal{T}_{triv}$, $\mathcal{T}_{disc}$, $\mathcal{T}_{cf}$, $\mathcal{T}_{cc}$

More natural one, call it $\mathcal{T}_{Eucl} := $ the topology $\mathcal{T}_{d_{Eucl}}$ induced by the Euclidean distance $d_{Eucl}(x,y) = |x-y|$

Therefore, for $U \subseteq \mathbb{R}$:

$U \in \mathcal{T}_{Eucl} \overset{RN}{\Longleftrightarrow}$ U is an union of intervals of type $(a,b) \subseteq \mathbb{R}$.

$\overset{(*)}{\Longleftrightarrow}$ $(\forall)\, x \in U$, $\exists$ open interval (a,b) s.t. $x \in (a,b) \subseteq U$

Variation: instead of (a,b) use intervals of type $[a,b)$

& define $\mathcal{T}_\ell$ (lower limit topology) by

$U_i \in \mathcal{T}$ ($\forall$) $i \in I$ $\Rightarrow$ $\bigcup_{i \in I} U_i \in \mathcal{T}$.

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$\forall p$ $\|\cdot\|_p$ $d(x, y) = \|x - y\|$ $\mathcal{T}_d$

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$\overset{(*)}{\Leftarrow}$ ($\forall$) $x \in U$, $\exists$ open interval $[a, b)$ s.t. $x \in (a, b) \subseteq U$ (**)

Variation: instead of (a, b) use intervals of type $[a, b)$ & define $\mathcal{T}_\ell$ by (**). Check/prove: $\mathcal{T}_\ell$ is a topology.

$\mathcal{T}_\ell$: lower limit topology

Now: $\mathcal{T}_{Eucl} \subseteq \mathcal{T}_\ell$

$(0, 1) = \bigcup \underbrace{\left[\frac{1}{n}, 1\right)}_{\in \mathcal{T}_\ell} \in \mathcal{T}_\ell$

$(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b) \in \mathcal{T}_\ell$


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1. $\mathcal{T}_{triv}$ consists of $\emptyset, X$ only
2. $\mathcal{T}_{disc}$ —"— all subsets of X
3. $\mathcal{T}_{cf}$ —"— all complements $X \setminus F$ with $F \subseteq X$ finite & $\emptyset$ itself.
4. $\mathcal{T}_{cc}$ —"— all complements $X \setminus F$, $F \subseteq X$ at most countable & $\emptyset$ itself.
Given $\mathcal{T}_1, \mathcal{T}_2$ topologies on X, say: $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$ or $\mathcal{T}_2$ is larger than $\mathcal{T}_1$
if $\mathcal{T}_1 \subseteq \mathcal{T}_2$
Example 5: metric topologies: Start with a metric space (X, d)
For $x \in X$, $\varepsilon > 0$: $B_\varepsilon(x) = \{y \in X : d(y,x) < \varepsilon\}$
A set $U \subseteq X$ is called open wrt d if
$\forall x \in U$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$ (!)
The collection
$\mathcal{T}_d = \{U \subseteq X \mid U \text{ is open wrt } d\}$
called the topology induced by d
the discrete topology on X
the cocountable topology
Exp: Given (X, d): a) $\mathcal{T}_d$ is a topology on X
b) $\mathcal{T}_d$ is the smallest topology on X that contains all $B_\varepsilon(x)$ for each $x \in X$, $\varepsilon > 0$
Rk: for $U \subseteq X$: $U \in \mathcal{T}_d \Leftrightarrow U$ is a union of balls
$\|\cdot\|_p$ $d(x,y) = \|x-y\|_p$ $\mathcal{T}_d$
$B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x-y\|_p < \varepsilon\}$
Example 6-7-8: topologies on $\mathbb{R}$: lots of topologies: $\mathcal{T}_{triv}$, $\mathcal{T}_{disc}$, $\mathcal{T}_{cf}$, $\mathcal{T}_{cc}$
More natural one, call it $\mathcal{T}_{Eucl}$:= the topology $\mathcal{T}_{d_{Eucl}}$ induced by the Euclidean distance $d_{Eucl}(x,y) = |x-y|$
Therefore, for $U \subseteq \mathbb{R}$:
$U \in \mathcal{T}_{Eucl} \overset{RN}{\Leftrightarrow}$ U is an union of intervals of type $[a,b) \subseteq \mathbb{R}$
$\overset{(*)}{\Leftrightarrow}$ $(\forall) x \in U, \exists$ open interval $[a,b)$ s.t. $x \in (a,b) \subseteq U$ (**)
Variation. instead of (a,b) use intervals of type [a,b)
& define $\mathcal{T}_\ell$ by (**). Check/prove: $\mathcal{T}_\ell$ is a topology.
lower limit topology
Now: $\mathcal{T}_{Eucl} \subseteq \mathcal{T}_\ell$
$(0,1) = \bigcup \overbrace{\left[\frac{1}{n}, 1\right)}^{\in \mathcal{T}_\ell} \in \mathcal{T}_\ell$
$[a,b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b) \in \mathcal{T}_\ell$