Homework 1, November 19, 2025

We consider the following collections of subsets of $\mathbb{R}$:

- $\mathcal{T}_1$: consisting of $\emptyset$, $\mathbb{R}$ and all intervals of the form (r, ∞) with $r \in \mathbb{R}$.
- $\mathcal{T}_2$: defined similarly but using $(-\infty, r)$ instead of (r, ∞).
- $\mathcal{B}_1$: consisting of $\emptyset$, $\mathbb{R}$ and all intervals of the form $[r, \infty)$ with $r \in \mathbb{R}$.
- $\mathcal{B}_2$: defined similarly but using $(-\infty, r]$ instead of $[r, \infty)$.

Please prove that:

(1) $\mathcal{T}_1$ is a topology on $\mathbb{R}$.

(2) $\mathcal{B}_1$ is not a topology on $\mathbb{R}$.

(3) any topology on $\mathbb{R}$ that contains both $\mathcal{T}_1$ and $\mathcal{T}_2$ must contain $\mathcal{T}_{\text{Eucl}}^{\mathbb{R}}$.

(4) $\mathcal{T}_{\text{Eucl}}^{\mathbb{R}}$ is the smallest topology on $\mathbb{R}$ that contains both $\mathcal{T}_1$ and $\mathcal{T}_2$.

(5) $\mathcal{T}_1$ is smaller than any topology that contains $\mathcal{B}_1$.

(6) there is only one topology on $\mathbb{R}$ that contains both $\mathcal{B}_1$ and $\mathcal{B}_2$.