Lago di Bolsena (Italy): view of growing Cumulus clouds over the mountains to the west of the lake on 25 July 2010, 17:23 LT.

Lago di Bolsena (Italy): view of a thunderstorm over the mountains to the west of the lake on 25 July 2010, 18:24 LT. The rain shaft is visible below the darkest clouds and a shield of high Cirrus clouds (the anvil of the thunderstorm) is visible aloft.
Lecture notes for the course on *Dynamical Meteorology* (NS-MO402M), *Climate Dynamics* (NS-363B) and *Boundary Layers, Transport and Mixing* (NS-MO412M) at Utrecht University in the academic year 2014-2015
By Aarnout van Delden
Office: Room 615, Buys Ballot Gebouw (BBG), Princetonplein 5, 3584CC Utrecht, Netherlands
E-mail: a.j.vandelden@uu.nl
Website: [http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm](http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm)
© This update: February 2015

![Diagram of typical flow pattern](image)

Typical flow pattern between two isobaric surfaces (1000 hPa is near the Earth’s surface; 500 hPa is at about 5 km above sea level) in a mid-latitude baroclinically unstable disturbance in the northern hemisphere. The warm air rises along very slanted trajectories, giving rise to layered clouds, as shown in the figure below. See **Figure 4 (Box 1.7)**.

![Layered clouds background](image)

Layered clouds as background of “The Scream” by Edvard Munch.
# Chapter titles of these lecture notes

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>0) Preface</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>1)</td>
<td>Introduction to the Atmosphere</td>
<td>14</td>
</tr>
<tr>
<td>2)</td>
<td>Energy Balance</td>
<td>205</td>
</tr>
<tr>
<td>3)</td>
<td>Hydrostatic Balance</td>
<td>341</td>
</tr>
<tr>
<td>4)</td>
<td>Convection</td>
<td>377</td>
</tr>
<tr>
<td>5)</td>
<td>Geostrophic Balance</td>
<td>405</td>
</tr>
<tr>
<td>6)</td>
<td>Orographic Effects</td>
<td>426</td>
</tr>
<tr>
<td>7)</td>
<td>Zonal Mean State of the Atmosphere</td>
<td>454</td>
</tr>
<tr>
<td>8)</td>
<td>Thermal wind Balance and Cross-Frontal Circulations</td>
<td>507</td>
</tr>
<tr>
<td>9)</td>
<td>Baroclinic Waves, Cyclogenesis and Frontogenesis</td>
<td>527</td>
</tr>
<tr>
<td>10)</td>
<td>Numerical Simulation of the Life-Cycle of Unstable Baroclinic Waves</td>
<td>582</td>
</tr>
<tr>
<td>11)</td>
<td>Planetary Waves, Wave Drag and Meridional Transport</td>
<td>616</td>
</tr>
<tr>
<td>12)</td>
<td>Diabatic-Dynamical Interaction in the General Circulation</td>
<td>654</td>
</tr>
<tr>
<td>13)</td>
<td>Zonal asymmetries in the General Circulation</td>
<td>718</td>
</tr>
</tbody>
</table>

The pdf’s of the chapters and of the full document can be downloaded from [http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm](http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm)
Preface

On understanding a complicated problem

Anyone who wants to analyze the properties of matter in a real problem might want to start by writing down the fundamental equations and then try to solve them mathematically. Although there are people who try to use such an approach, these people are the failures in this field; the real successes come to those who start from a physical point of view, people who have a rough idea where they are going and then begin by making the right kind of approximations, knowing what is big and what is small in a given complicated situation. These problems are so complicated that even an elementary understanding, although inaccurate and incomplete, is worth while having, and so the subject will be one that we shall go over again and again, each time with more accuracy...


i Introduction to atmospheric dynamics 4
ii List of books 7
iii Weather maps, observational data and reanalysis data 10
iv Acknowledgements 12

i Introduction to atmospheric dynamics

These lecture notes have grown out of many years of giving the one-semester courses on *Geophysical Fluid Dynamics, Dynamical Meteorology* and *Climate Dynamics* at Utrecht University. I have now given the lecture notes the more natural title *Atmospheric Dynamics*, which covers two subjects that are traditionally called *Dynamical Meteorology* and *Synoptic Meteorology*, as well as part of the subject of *Climate Dynamics*.

**Climate Dynamics** deals with the theory of the annual average or seasonal average and/or zonal average\(^1\) state of the atmosphere, and usually also involves the ocean, the cryosphere (the glaciers and ice caps) and the biosphere.

**Dynamical Meteorology** deals with the theory of circulations associated with atmospheric weather phenomena such as cyclones, anticyclones, fronts, sea breezes, tornadoes, thunderstorms, lee-waves and severe downslope windstorms.

**Synoptic Meteorology** deals with the analysis of observations of these weather phenomena. This analysis is practically impossible without "conceptual models" of weather phenomena, or "synoptic models" as they are called on page 231 of Godske et al. (1957)\(^2\). The two subjects (dynamical meteorology and synoptic meteorology) are of course intimately related and here we will not make any further attempt to separate them.

More than 2000 years ago meteorology was an integral part of physics. However, only little more than half a century ago the position of meteorology in relation to the mother

---

\(^1\) Zonal average is the average with respect longitude.

discipline (physics) was less privileged. This is illustrated very well by the following words from the *Feynman Lectures on Physics* (1963).³

"We turn now to earth sciences, or geology. First, meteorology and the weather. Of course the intruments of meteorology are physical instruments, and the development of experimental physics made these instruments possible, as was explained before. However, the theory of meteorology has never been satisfactorily worked out by the physicist. "Well," you say, "there is nothing but air, and we know the equations of motions of air". Yes we do. "So, if we know the condition of air today, why can't we figure out the condition of the air tomorrow?". First, we do not really know what the condition is today, because the air is swirling and twisting everywhere. It turns out to be very sensitive, and even unstable. If you have ever seen water run smoothly over a dam, and turn into a large number of blobs and drops as it falls, you will understand what I mean by unstable. You know the condition of the water before it goes over the spillway; it is perfectly smooth; but the moment it begins to fall, where do the drops begin? What determines how big the lumps are going to be? That is not known, because the water is unstable. Even a smooth moving mass of air, in going over a mountain turns into complex whirlpools and eddies. In many fields we find this situation of turbulent flow that we cannot analyze today. Quickly we leave the subject of weather..."⁴

Nowadays things have changed again, despite the fact that turbulent flow is still a seemingly intractable problem. Precisely during the period when Feynman was giving his lectures, research in the fields of dynamical meteorology and climate dynamics was going through phase of explosive development. We would now not consider meteorology to be a part of geology, but principally as an application of physics and, to a lesser extent, chemistry. In fact, W.J. Humphreys of the U.S. Weather Bureau writes in the issue of 30 August 1935 of *Science* (p. 197) that meteorology awoke to alertness with a “galvanic shock” by the invention of the telegraph in the nineteenth century. Humphreys states that,

“Some twenty-two centuries ago meteorology already was accounted an important science; so important indeed that the discriminating Aristotle wrote a sizeable book on it. Then for two millennia the rains came and the winds blew with never an explanation of how or why - curiosity was inhibited by faith and inquiry estopped by authority. Slowly came a drowsy awakening…
Not long ago meteorology was a descriptive subject that required no preparation to study and but a few weeks’ time to master, whereas today it ranges nearly the whole field of classical physics with all the mathematics that implies.”

Humphreys did not know that meteorology would receive another enormous boost within 15 years by the invention of the computer in the 1940’s. Results of experiments with the first numerical weather prediction model that could keep up with the actual weather were published in 1950⁵. Computers made possible a thorough exploration of the solutions of the very complex coupled nonlinear differential equations that govern the thermodynamical behaviour of a fluid such as the atmosphere. These equations were set forth, beginning more than 2000 years ago, by the Greeks philosophers (conservation of mass, based on the

⁴ These words of Feynman make clear that the subject of Atmospheric Dynamics is difficult. The laws on which the subject is based have been known since the nineteenth century. However, these laws have provided solutions to atmospheric problems only very slowly, precisely because of this property of “flow-instability”.
philosophy that "nothing comes from nothing"), by Isaac Newton in 1687 (definition of, and relation between, force and the product of mass and velocity, called momentum, embodied in Newton's second law), by Émile Clapeyron in 1834 (the equation of state/ideal gas law) and by James Prescott Joule, Rudolf Clausius and Lord Kelvin (conservation of energy, now referred to as the first law of thermodynamics) in 1850. The expression of these laws in the form of mathematical (differential) equations was complete at some point in the second half of the nineteenth century due to the work of, notably, Pierre-Simon Laplace, William Ferrel, Horace Lamb and Wilhelm von Bezold. Cleveland Abbe in 1901 and Vilhelm Bjerknes in 1904 summarized these equations, showing that they form a closed set that can in principle be solved if the state of the atmosphere is known at a certain point in time. Abbe and Bjerknes were too optimistic about the feasibility of this project, considering the fact that the theory of radiation (emission, absorption and transfer) was still being developed at the time of their writing. It was Lewis Fry Richardson who showed in a remarkable book, entitled “Weather prediction by numerical process”, published in 1922, how this project could be executed numerically. Richardson’s ideas, specifically on radiative transfer of energy and turbulent transfer of momentum, energy and humidity, proved very fruitful when they were put to test after 1950. Over the past 60 years the set of general laws of fluid motion, thermodynamics and radiation, which were only a skeleton in the days of Cleveland Abbe and Vilhelm Bjerknes, came alive. The solutions of these equations are sometimes puzzling and complex, while the myriad phenomena that they describe are so rich that one can devote many lifetimes to their investigation. Maybe this is what frightened Richard Feynman and, unfortunately for meteorology, made him turn away to focus on other problems in physics.

Following John Dutton, the basic question of these lecture notes is phrased as follows.

The basic problem of atmospheric dynamics revolves around the question of why the observed responses are those that are “chosen”.

In order to start looking for an answer to this question, we have to define our point of departure. We have to define the prerequisites, i.e. the required knowledge to start the study of the dynamics of the atmosphere. You (the reader or student) should be familiar with the basic concepts (momentum, mass energy) and equations that describe the movement as well as the thermodynamic state (temperature and pressure) of an air parcel in the atmosphere of a rotating planet, i.e. the concepts and equations that were summarized in 1904 by Bjerknes. You should also be familiar with the basic laws of radiation due to Lambert, Planck, Stefan, Boltzmann and Kirchhoff. This basic knowledge is usually attained in introductory courses in mechanics (equation of motion, Coriolis effect), hydrodynamics

Bjerknes, V., 1904a: Das Problem der Wettervorhersage, betrachtet vom Standpunkte der Mechanik und der Physik. Met. Zeit., 21, 1-7. The translation of this paper into English by E. Volken and S. Brönniman, with an interpretation by G. Gramelsberger, was published in Met Zelt (NF),18, 663-673.
7 In fact, Bjerknes’s dream was “broken” to a certain degree in 1963 when Edward Lorenz (see problem 0.2) questioned its feasibility in an article entitled, “Deterministic nonperiodic flow”, published in Journal of the Atmospheric Sciences, vol. 20, 130-141, which showed that the solutions of nonlinear systems with forcing and dissipation can be “chaotic”. This means that the solutions depend sensitively on initial conditions.
(equation of motion, continuity equation), geophysical fluid dynamics (equation of motion, Coriolis effect), classical thermodynamics (equation of state and energy conservation) and radiative transfer. Any person with this basic knowledge of physics should be able to read and understand these notes. Nevertheless, I think some familiarity with meteorology (e.g. from the book by McIlveen (2010) or the book by Wallace and Hobbs (2006), and the first two chapters of Holton (2004)) will be extremely helpful if not essential as a basis to understand the subject matter in these lecture notes (see the list of books in the next section).

ii List of books

The fundamentals of fluid dynamics are documented very well in the book by Batchelor (1970) (see the list below). Textbooks that treat the basic principals of atmospheric dynamics are those by Holton (1972-2012), Gill (1982), Dutton (1986), James (1994), Salby (2012), Vallis (2006), Martin (2006) and Lackmann (2011). The four editions of the book by Holton are probably the most suitable. The book by Martin (2006) is very suitable for learning the basic principles, although this book is much more limited in scope compared to Holton’s book. However, a substantial part of the theory treated in these notes cannot be found in any of the above-mentioned books. Wherever this is true, the references to the pertinent journal articles are given. The relatively recent book by Lackmann (2011) contains many examples of weather systems in the United States. These notes are more biased to presenting European examples. The book by Markowski and Richardson (2010) gives the best overview of the dynamics of mesoscale weather systems, while the books by Dutton (1986), James (1994), Holton (2004) and Vallis (2006) are the most theoretical. Andrews et al. (1987) (also very theoretical) is focussed on the middle atmosphere (the atmosphere above 10 km above sea-level). Peixoto and Oort (1992) is a “classic” on the physics of the present climate of Earth’s atmosphere. The books by Andrews (2010) and Salby (2012) are broadest in scope, treating also the radiation balance of the atmosphere, a subject that is usually considered to be part of a separate course on Radiation or on Climate Dynamics (chapter 2 of these lecture notes).

So, we have the following list of recommended textbooks on Atmospheric Dynamics (the books listed in red and underlined are especially recommended):


In view of this list, one may ask: what justifies the existence and use of these lecture notes when there are so many textbooks available? Although this may sound a bit arrogant, the fact is that all the books listed above are outdated in one way or another. For instance, there is at this moment only one textbook (Lackmann, 2011) that contains a relatively thorough discussion of the theory behind the very important concept of potential vorticity and its use in describing the structure and dynamics of the atmosphere. Adjustment to hydrostatic balance and the interaction between radiation and dynamics are not considered worthy of much discussion in any textbook on Atmospheric Dynamics either, even though a discussion of the latter topic is required in order to explain the characteristic thermal layering of the atmosphere, i.e. the division of the atmosphere into a troposphere and a stratosphere. In these lecture notes, these topics occupy a prominent position.

At the start of the one-semester masters course on Dynamical Meteorology at Utrecht University it is assumed that the student is more or less familiar with the basics of geophysical fluid dynamics. Students, whose knowledge is deficient in this respect, are strongly advised to do the parallel third year bachelors course on Geophysical Fluid Dynamics, which is based on the book by Cushman-Roisin and Beckers (2011).
The Masters course on **Dynamical Meteorology** at Utrecht University consists of 16 lectures and practical sessions, starting in September. The first 10 lectures cover chapter 1 of these notes, which is by far the longest chapter. In fact, chapter 1 treats the full subject of dynamical meteorology at an introductory level, while all subsequent chapters are focused on a more in depth treatment of a specific important topic that is introduced in chapter 1. After 10 weeks all students should be familiar with the application of the principals of geophysical fluid dynamics to the atmosphere and with the most important conceptual models of weather phenomena, and, thus, will be able to perform a case study, using observational data or “reanalysis data”, which should be presented in oral and written form in January. The latter part of the course (from the middle of November to end of January) is devoted to a more detailed treatment of a limited choice of the following topics: high frequency waves and hydrostatic adjustment (chapter 3), convection and thunderstorms (chapter 4), adjustment to geostrophic balance (chapter 5), the influence of orography (chapter 6), the zonal mean (average around latitude circles) state of the atmosphere, including the jets (chapter 7), fronts, frontogenesis and adjustment to thermal wind balance (chapter 8), instability of low frequency waves and cyclogenesis (chapter 9) and numerical simulation of the life cycle of mid-latitude cyclones (chapter 10). The material in chapter 2 (Energy balance) is part of a third year one-semester Physical Science Bachelors course that comes under the title **Climate Dynamics**. Chapter 11 is concerned with the equator to pole transport of thermal energy (sometimes called “sensible heat”) and momentum by planetary waves and large scale vortices (“eddies”) and the influence of this transport on the zonal mean state. Chapter 12 is concerned with the interaction between diabatic processes (e.g. radiation) and adiabatic processes (e.g. transport of heat and momentum associated with dynamics) in shaping the zonal mean state of the atmosphere. The material in chapters 10, 11 and 12 is part of a one-semester graduate course on the transport and mixing in the atmosphere under the title **Boundary Layers, Transport and Mixing**. Finally, chapter 13 is
concerned with zonal asymmetries (waves and circulations) in the tropics and with the question how these waves propagate into the extratropics and affect weather there. To assist the student in studying the material, each chapter is concluded with an abstract, listing the key concepts and summarizing the principle conclusions.

iii Weather maps, observational data and reanalysis data

There exists an immense amount of information about the weather, including observational data, processed data in the form of weather maps, forecasts made by modelling centres around the world and “reanalysis” of past observations. Since the beginning of the twentieth century the three-dimensional structure of the atmosphere has been monitored by surface observations, by radiosondes (figure 0.1), which were rare before 1950, and from the 1970’s onwards by satellite derived observations. These observations are available in raw form or in plotted form on many websites. The radiosonde network consists of about 1500 stations, which are distributed rather unevenly over the world (figure 0.2). Many of these stations are hardly in operation. During 2003, for instance, less than half of these stations were active on 80% of the days (figure 0.3). About 400 stations were not in operation at all in 2003.

The reanalyses of raw observations consists of using a specific numerical weather prediction model together with a specific data assimilation scheme in an analysis/forecast cycle with the aim to produce an internally consistent 4-dimensional analysis of the state of the atmosphere over several decades. The introductory page on the website of the European Centre for Medium Range Weather Forecasts (ECMWF) re-analysis project starts with the following explanation:

“Reanalyses of multi-decadal series of past observations have become an important and widely utilized resource for the study of atmospheric and oceanic processes and predictability. Since reanalyses are produced using fixed, modern versions of the data assimilation systems developed for numerical weather prediction, they are more suitable than operational analyses for use in studies of long-term variability in climate. Reanalysis products are used increasingly in many fields that require an observational record of the state of either the atmosphere or its underlying land and ocean surfaces. Estimation of renewable energy resources, calculation of microwave telecommunication signal losses and study of bird migration are just three examples. The first reanalysis at ECMWF was carried out in the early 1980s for the First GARP Global Experiment (FGGE) year 1979, when ECMWF operations began. Two major ECMWF reanalyses have exploited the substantial advances made since then in the forecasting system and technical infrastructure. The first project, ERA-15 (1979-1993), was completed in 1995 and the second extended reanalysis project, ERA-40 (1957-2002), in 2002. Products of ERA-15 and ERA-40 have been used extensively by the Member States and the wider user community. They are also increasingly important to many core activities at ECMWF, particularly for validating long-term model simulations, for helping develop a seasonal forecasting capability and for establishing the climate of EPS (Ensemble Prediction System) forecasts needed for construction of forecaster-aids such as the Extreme Forecast Index. ECMWF is currently producing ERA-Interim, a global reanalysis of the data-rich period since 1989. The ERA-Interim data assimilation system uses a 2006 release of the Integrated Forecasting System (IFS Cy31r2), which contains many improvements both in the forecasting model and analysis methodology relative to ERA-40. The ERA-Interim reanalysis caught up with operations in March 2009, and is now being continued in near-real time to support climate monitoring”.

---

9 see http://www.ecmwf.int/research/era/do/get/Reanalysis_ECMWF
Other reanalyses projects are being carried out by the National Center for Environmental Prediction (NCEP), under the name “NCEP/NCAR Reanalysis” (http://www.esrl.noaa.gov/psd/data/gridded/data.ncep.reanalysis.html), by the Japan Meteorological Agency under the name “Japanese reanalysis” (JRA) and NASA, under the name “MERRA” (https://gmao.gsfc.nasa.gov/merra/). The Japanese reanalysis project and the ERA-reanalysis have both yielded a fascinating atlas, which can be viewed at http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-tope.htm (JRA-25) and at http://www.ecmwf.int/research/era/ERA-40_Atlas/ (ERA-40). The most recent reanalysis project is the 20th century reanalysis of the NOAA Physical Science Division (PSD) (http://www.esrl.noaa.gov/psd/data/20thC_Rean/).

**Figure 0.2:** Locations of all stations in IGRA (Integrated Global Radiosonde Archive) (http://www.ncdc.noaa.gov/oa/climate/igra/).

**Figure 0.3:** Locations of stations in IGRA which were active during 2003.
Reanalysis data can be downloaded from ECMWF (http://apps.ecmwf.int/datasets/), or from NOAA (http://www.esrl.noaa.gov/psd/data/gridded/reanalysis/). The data is in netCDF format. The dataviewer, PANOPLY, can be used to view the data. This program can be downloaded freely from http://www.giss.nasa.gov/tools/panoply/. Information about the netCDF format can be found on http://www.unidata.ucar.edu/software/netcdf/. You can also read and process netCDF-data with the programming language, Python. For this, I recommend a book written by Johnny Lin, entitled “A Hands-On Introduction to Using Python in the Atmospheric and Oceanic Sciences” (http://www.johnny-lin.com/pyintro/).

Archived weather maps can be viewed on http://www.wetter3.de/Archiv/. Radiosonde observations since the 1970’s can be retrieved from http://weather.uwyo.edu/. A longer list of websites providing weather information can be found at http://www.staff.science.uu.nl/~delde102/WeatherDiscussions.htm.

PROBLEM 0.1. Non-linear mathematical model
The atmosphere, the ocean and climate are governed by a system of non-linear differential equations. As an example of such a system (presented first by Edward Lorenz in 1984), we study the following very simple model.

\[ X(t+1) = aX(t) - X(t)^2. \]

This is a non-linear recurrence relation, which is quite similar in structure to a “finite difference” numerical approximation of a differential equation that governs the behaviour of a fluid, such as the atmosphere or the ocean. Think of \( t \) as representing the global average surface temperature and \( a \) as representing the CO\(_2\) concentration. What is the equilibrium value of \( X \)? Write a program or script (e.g. in Python) which calculates and plots the evolution of \( X \) in time \( t \) for the following 4 values of \( a \): \( a=1 \), \( a=2 \), \( a=3.25 \) or \( a=3.75 \) (take \( X(0)=0.5 \)). Discuss the results. Define the “weather” and the “climate” of this model. For which values of \( a \), is the model-“weather” “predictable”? For which values of \( a \), is the model-“climate” “predictable”?

PROBLEM 0.2. Index of chapter 1
Chapter 1 provides a very broad overview of the concepts and models that are encountered in the study of weather and climate. While studying chapter 1, the student may be overwhelmed by the myriad manifestations of Earth’s weather. To aid the learning process, words printed in underlined bold face red letters are used in the text to elucidate important concepts. It is suggested that the student make an index of these and other important terms and concepts.

iv Acknowledgements
My first acquaintance with the subject of these notes was from the first and second edition of the, now classic, book by Jim Holton (see the list of books), who unfortunately died suddenly on March 3 2004 at the age of 65, and from lectures given at Utrecht University by Cor Schuurmans in 1979-1980, by Erland Kallen (now at ECMWF) in 1982 and by Hans Oerlemans in the 1980’s. For the chapters on quasi-geostrophic theory I relied heavily on the book by Jim Holton. For very useful comments, criticism and discussions I wish to thank the
following colleagues: Hans Oerlemans (Utrecht), Theo Opsteegh (Utrecht), Hylke de Vries (De Bilt), Rianne Giesen (Utrecht), Carleen Tijm-Reijmer (Utrecht), Yvonne Hinssen (Wageningen), Jan Lennaerts (Utrecht), Wim Verkley (De Bilt), Sander Tijm (De Bilt), Ab Maas (De Bilt), Bert Holtslag (Wageningen), Dale Durran (Seattle), Dieter Etling (Hannover, Germany), Helmut Kraus (Bonn), Peter Bannon (University Park, Pennsylvania), Christoph Schär (Zurich) and many other (anonymous) reviewers of my papers. Furthermore, I thank Sander Tijm (now at KNMI, De Bilt) and Bruce Denby (now in Norway) for helping to devise the software which I used to make weather and contour maps of data and model output, Koen Manders, Niels Zweers and Roos de Wit for help in developing and debugging the numerical model of the atmosphere (now called “PeN-model”; see chapter 10 and http://www.staff.science.uu.nl/~delde102/PeN-Model.htm), Marcel Portanger for advice and help on computer problems, Sheila McNab for advice on the English language, Seijo Kruizinga (retired from KNMI) for helping me obtain data from the ECMWF in Reading in the “pre-ERA-era” and Jaco Berbenhenegouwen, Izaak Santoe and Fred Trappenburg for drawing many figures. I have also taken many figures from the internet and thank the authors for making them available. I am also grateful to the following institutions: the Dutch weather service (KNMI) in De Bilt for giving the University of Utrecht access to the GTS and other weather data, the University of Dundee and the UK Met-Office for the satellite images and the European Centre for Medium Range Weather Forecasts (ECMWF) in Reading for making available the very valuable reanalysis data. Last but not least, I am greatful to many generations of students and all colleagues at IMAU for the stimulating interaction.

Groenekan, 30 September 2013

The clouds lose tufts of whiteness as the breeze dishevels them.
If that blue could stay for ever;
if that hole could remain for ever;
if this moment could stay for ever

Virginia Woolf (1931), The Waves.
1 Introduction to the atmosphere

On mathematics in physics
A little more mathematics isn’t going to bail you out of your troubles. Newton knew less mathematics than many a present day undergraduate, but I haven’t noticed any Principias coming out of our colleges lately. What mathematics can do is to help you formulate and express your problems in such a way that you can easily locate the critical points at which your creative physical thinking must begin. And this is no mean service to you. ... But don't expect that mathematics will give you an easy answer to any physical problem. If you find that it does, that is evidence that you picked a poor problem to begin with.

1.1 Introduction
1.2 Temperature and heating of Earth’s atmosphere
1.3 Gravity and buoyancy
1.4 Wind and Turbulence
1.5 Pressure and pressure gradient force
1.6 Inertial “force”
1.7 Momentum-, mass- and energy-budget equations
1.8 Equation of state and atmospheric composition
1.9 Clausius-Clapeyron equation
1.10 Water vapour distribution
1.11 Sources and sinks of water
1.12 Ozone distribution
1.13 Potential temperature
1.14 Hydrostatic balance
1.15 Static Stability
1.16 Potential instability and equivalent potential temperature
1.17 Convective available potential energy
1.18 Thermodynamic diagram
1.19 Geostrophic balance
1.20 Stability of geostrophic balance
1.21 The thermal wind
1.22 Stability of thermal wind balance
1.23 Non-linear balance and the Rossby number
1.24 Circulation and vorticity
1.25 Potential vorticity
1.26 Isentropic view of the atmosphere
1.27 Scales imposed by the fundamental properties of the system
1.28 Forcing and response: daily and seasonal cycle
1.29 Internally generated large-scale modes of variability
1.30 Balance and imbalance
1.31 Thermal wind and temperature advection
1.32 Fronts and mid-latitude cyclones
1.33 Frontogenesis and frontolysis: the Q-vector
1.34 The ageostrophic wind
1.35 Jetstreaks
1.36 Effect of turbulent friction on geostrophic flow
1.37 Planetary (Rossby) waves
1.38 Meridional energy transport
1.39 Zonal average, time average meridional transport by eddies or waves

Abstract of chapter 1 and further reading
List of problems

Boxes of chapter 1

Box 1.1 Pressure as a vertical coordinate
Box 1.2 Simplification of the equation of motion by scale analysis
Box 1.3 Spectral distribution of radiation: Planck’s law
Box 1.4 The origin of oxygen in the atmosphere
Box 1.5 Dew point temperature and lifted condensation level
Box 1.6 Moist adiabatic lapse rate
Box 1.7 Baroclinically unstable motions and mid-latitude cyclones
Box 1.8 Tropical Cyclones
Box 1.9 Conservation of potential vorticity: Ertel’s theorem
Box 1.10 Spectral or Fourier analysis and the mesoscale range of scales
Box 1.11 Continuity equation and the zonal mean mass-streamfunction
Box 1.12 Principal component analysis
Box 1.13 Waves: phase velocity and group velocity
Box 1.14 Heat capacity of the ocean

1.1 Introduction

Chapter 1 gives an overview of the concepts and laws that characterize and govern the physical and chemical state and evolution of the atmosphere. Many of the topics covered in this first chapter will be discussed in much more detail in later chapters.

The puzzling discrepancy between the distribution of non-adiabatic (“diabatic”) heating and cooling in the atmosphere and the corresponding response, i.e. the distribution of temperature, forms the starting point (section 1.2). One goal of these lecture notes is to resolve this puzzle. This chapter does the grounding work to accomplish this goal. Sections 1.3 to 1.6 introduce the forces that determine the acceleration of air parcels in the atmosphere. Sections 1.7 to 1.9 give an overview of the fundamental equations that govern the dynamics of a rotating fluid. The derivation of these equations should be known from a previous course on Geophysical Fluid Dynamics. Sections 1.9-1.12 are concerned with the composition of the atmosphere. Section 1.13 introduces potential temperature. Sections 1.14-1.23 treat the different types of balanced states. In Sections 1.24, 1.25 and 1.26 the measures of rotation in the atmosphere are introduced. The remaining sections in chapter 1 give an overview, albeit somewhat superficial, of the wide range of observed circulation systems and their role in determining the weather and climate of Earth’s atmosphere. A more detailed treatment of these phenomena follows in subsequent chapters.
1.2 Temperature and heating of Earth’s atmosphere

The temperature distribution prevailing in the Earth's atmosphere (figure 1.1) is controlled by Solar irradiance just outside the Earth's atmosphere (about 1366 W m$^{-2}$) and the composition of the atmosphere. The Earth's atmosphere consists mainly of nitrogen (N$_2$) (78%) and oxygen (O$_2$) (21%). These two gases are distributed in uniform proportions up to a height of about 90 km above sea level. Other constituents are water in all three phases (vapour, liquid and solid), ozone (O$_3$) and carbon dioxide (CO$_2$). Water, carbon dioxide and ozone are there in very small quantities but are nevertheless important because they absorb and emit radiation. Ozone and water vapour absorb Solar radiation. This is part of the reason for the strong heating of the equatorial atmosphere (figure 1.2), where insolation is strongest and where most water vapour is found. About 80% of the net absorbed radiation at the Earth’s surface, is used to evaporate water. Water vapour in the atmosphere represents a considerable reservoir of "latent" energy that is released as soon as saturation is reached and condensation occurs within clouds (figure 1.2). Most of the latent heat is released in the tropical belt in the layer between 2000 m (800 hPa) and 10000 m (250 hPa) above sea level.

The atmospheric circulation is driven mainly by the imbalance between absorption and emission of radiation as well as by latent heat-release. This imbalance is itself determined at least partly by the atmospheric circulation. The peculiar temperature distribution that can be seen in figure 1.1, with the tropical temperature minimum at a pressure, $p$=100 hPa (16 km above sea level), has been a longstanding puzzle. Only slowly are we beginning to understand how the interaction between the different adiabatic and non-adiabatic (diabatic) processes determines atmospheric temperature.

**FIGURE 1.1:** The zonal mean (averaged along latitude circles) and time mean (over the period 1979-2001) temperature as a function of pressure in hectopascals (hPa) (1 hPa=100 Pa; 1 Pa=1 N m$^{-2}$) and latitude. Labels are in K. The pressure can be related to the physical height via the hydrostatic equation (**Box 1.1**): 1000 hPa corresponds approximately to the Earth’s surface; 500 hPa corresponds approximately to 5 km; 250 hPa corresponds approximately to 10 km; 50 hPa corresponds approximately to 20 km; 300 hPa corresponds approximately to 10 km; 1 hPa corresponds approximately to 50 km. Note that the highest temperatures are found at the Earth’s surface near the equator, while the lowest temperatures are observed also near the equator at a height of about 15 km (100 hPa). Above this level the temperature increases due to absorption Solar ultraviolet radiation by ozone. (Source: [http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html](http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html)).
17

FIGURE 1.2: The zonal mean (averaged along latitude circles; see eq. 8 of Box 1.11 or section 1.39) and time mean (over the period 1979-2001) net non-adiabatic (diabatic) heating as a function of pressure and latitude. Labels are in K/day. See figure 1.1 for further information (Source: http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html). The net diabatic heating/cooling is a sum of the heating/cooling due to absorption or emission of radiation, due to latent heat release/consumption associated with condensation or evaporation of water and due to heat transfer to and from the Earth’s surface. This figure demonstrates that the atmosphere, above the lowest kilometre, is heated in the tropics and cooled in the extra-tropics.

Box 1.1. Pressure as a vertical coordinate

Pressure is often used as a vertical coordinate in meteorology. This is only possible if it is assumed that pressure is determined by the weight of the air above:

\[ p(z) = \rho(z) g \int_{z}^{\infty} \rho(z) dz, \]  

(1)

where \( z \) is height above sea-level, \( p \) is pressure (Pa=N m\(^{-2}\)), \( \rho \) is density (kg m\(^{-3}\)), \( g \) is the acceleration due to gravity (m s\(^{-2}\)) and \( \rho_0 \) is density at sea level (\( z=0 \)). Let us assume that density decreases exponentially with height as

\[ \rho(z) = \rho_0 \exp\left(-\frac{z}{H_\rho}\right), \]  

(2)

where \( H_\rho \) is referred to as the “density scale-height”. Substituting (2) into (1) yields

\[ p(z) = \rho_0 g \int_{z}^{\infty} \exp\left(-\frac{z}{H_\rho}\right) dz = g \rho_0 H_\rho \exp\left(-\frac{z}{H_\rho}\right). \]  

(3)

Therefore,
\[
\ln(p(z)) = \ln(g \rho_0 H) - \frac{z}{H}.
\]  
(4)

Assuming that (2) is a good approximation to the vertical density profile, we conclude that there is a linear relation between \( \ln(p) \) and \( z \), which is the reason why the vertical scale in figures 1.1 and 1.2 is logarithmic. With knowledge of the surface pressure (101300 Pa), we can even deduce the value of \( H \). Putting \( z=0 \) in (4) and with \( g \approx 9.81 \text{ m s}^{-2} \) and \( \rho_0 \approx 1.22 \text{ kg m}^{-3} \), we find that \( H = 8464 \text{ m} \). From a linear regression of an analysis of measurements of height as a function of pressure (see the figure below) we obtain \( H = 6827 \text{ m} \).

*Figure Box 1.1.* Pressure as function of height at 45°N in January according to the International Reference Atmosphere (see table 1.1). The straight line in the left diagram represents a linear fit to the data points. In the left diagram 1 in 4 data points is shown in order to reveal two straight line.

### 1.3 Gravity and buoyancy

The following four sections introduce the various types of forces acting on an *air parcel* in the atmosphere. The most dominant force is due to gravity. Every object in the universe attracts every other object with a force, which is proportional to the mass of each body (say \( m_1 \) and \( m_2 \), respectively) and varies inversely as the square of the distance, \( r \), between them. This statement can be expressed mathematically by the equation

\[
F = G \frac{m_1 m_2}{r^2}.
\]

Here, \( G = 6.673 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \). The gravitational force per unit mass exerted on a parcel of air at sea level by the Earth with mass \( m_1 = 5.988 \times 10^{24} \text{ kg} \) is

\[
g_a = G \frac{m_1}{a^2}.
\]
Here, **a is the radius of the Earth**. The average value of the radius of the Earth is 6371 km. This implies that $g_a = 9.844 \text{ N kg}^{-1}$. It is convenient to combine the gravitational attractive force of the Earth (gravitation) and the **centrifugal force** due to the Earth's rotation into an **effective gravitational force**. The magnitude of this effective gravitational force per unit mass (or acceleration) varies by less than 1% between equator and poles and between sea level and a height of 10 km. The centrifugal force is at its maximum at the equator, where it is directed exactly opposite to the gravitational force. Since the rotational speed $U$ (due only to the Earth’s rotation) at the equator is 465 m s$^{-1}$ (which is greater than the phase speed of sound waves!), the centrifugal force per unit mass is $3.4 \times 10^{-2} \text{ m s}^{-2}$. This, nevertheless, is about 300 times weaker than the gravitational force. At sea level we have$^{10}$$\[g = g_0 = 9.80616\left(1 - 0.002637\cos 2\phi + 0.0000059\cos^2 2\phi\right) \text{ m s}^{-2},\]

where $\phi$ is the latitude. Above sea level we have

$$g(z, \phi) = \frac{g_0}{(1+z/a)^2},$$

where $z$ is the vertical coordinate (perpendicular to earth's surface).

**Archimedes Principle**$^{11}$ states that an element or object immersed in a fluid at rest experiences an upward thrust, due to the pressure gradient force (see section 1.5), which is equal to the weight of the fluid displaced. If $\rho_0$ is the density of the fluid and $V_1$ is the volume of the object, this upward thrust is therefore equal to $g \rho_0 V_1$. Since the object experiences a downward force equal to its own weight, the net upward force, $F$, on the object is equal to $(g \rho_0 V_1 - g \rho_1 V_1)$, where $\rho_1$ is the density of the object. The force, $F$, is called the **buoyancy force**. Assuming now that the object immersed is an **air parcel** and that the fluid is air having a different temperature, and assuming that air is an **ideal gas** (section 1.5), we can express the net force, using the **ideal gas law**, which in meteorology is usually written as

$$p = R \rho T,$$ \hspace{1cm} (1.1)

where $p$ is pressure, $R$ is the **specific gas constant** and $T$ is temperature, in terms of temperatures as follows,

$$F = -g \left(\frac{p_1}{RT_1} - \frac{p_0}{RT_0}\right) V_1.$$

If we assume that the pressure, $p_1$, within the parcel is equal to the pressure, $p_0$, in the environment we get

$$F = -\frac{g \rho_1 V_1}{RT_1 T_0} (T_0 - T_1) = -g \rho_1 V_1 \frac{(T_0 - T_1)}{T_0} = -mg \frac{(T_0 - T_1)}{T_0},$$

---


in which \( m \) is the mass of the air parcel. Thus, the vertical component of the acceleration due to buoyancy is

\[
\frac{d^2 z}{dt^2} = g \left( \frac{T_1 - T_0}{T_0} \right) = g',
\]

(1.2)

in which \( z \) is the vertical coordinate and \( g' \), the buoyancy force per unit mass, is sometimes referred to as "reduced gravity". Eq. 1.2 demonstrates that gravity is dynamically important only if there are temperature differences or, equivalently, density differences. A typical value of \((T_1 - T_0)\) in the atmosphere is 3 K and therefore of \( g' \) is \( 10^{-1} \text{ m s}^{-2} \).

PROBLEM 1.1. Buoyancy

This problem is concerned with the concept of "buoyancy" and Archimedes principle. In the figure we see two identical rectangular beakers. The water level is exactly the same in both beakers. However, a piece of wood is floating in the right beaker.

a. Which of the two systems (beaker+water+piece of wood) is heavier, or do both systems have exactly the same weight?

b. Which fraction of the total volume of the piece of wood is below the water-level. Assume that water has a density of 1000 kg m\(^{-3}\), wood has a density of 900 kg m\(^{-3}\) and that air has a density of 1 kg m\(^{-3}\).

\[\text{FIGURE with problem 1.1.} \quad \text{On the two pans of a scale there are a container filled with water and another, identical container filled with water and a floating piece of wood. In both containers the water reaches the edges. Which one of the two is heavier? (see Fontana, F., and R Di Capua, 2005: Role of hydrostatic paradoxes towards the formation of the scientific thought of students at academic level, Eur.J.Physics., 26, 1017-1030).}\]

1.4 Wind and turbulence

Figure 1.3 shows an instrumental record of the windspeed at 10 m height during one of the most severe windstorms of the twentieth century in the Netherlands. Due to the presence of obstacles, such as trees and buildings, upstream from the measuring device, the wind speed is highly variable. In fact, the response time of the measuring instrument is probably insufficient to capture all the variations in the wind speed. Due to this fundamental measuring problem, there is an agreement that the wind speeds in standard weather
reports issued by the World Meteorological Organization (WMO), are an average value over a specified time. The agreement is that it should be a 10 minute average. Therefore, in order to compare theory and measurement, we must use equations that deal with time averages. This has some unexpected and important theoretical consequences, as will be explained in the following.

If \( u(t,x,y,z) \) is the \( x \) (west-east)-component of the velocity vector of an air parcel, where \( t \) is time and \( x, y \) and \( z \) are the three spatial coordinates. The exact definition of the coordinate system relative to the Earth will be given in section 1.7. The instantaneous acceleration, \( a \), in the \( x \)-direction is simply

\[
a = \frac{du}{dt} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right).
\]

Here, \( v \) and \( w \) are the \( y \)- and \( z \)-component of the velocity, respectively, i.e. the wind vector, \( \vec{v} \), is defined as \( \vec{v} = (u,v,w) \). In the absence of sinks or sources of mass within the atmosphere, the time rate of change of density, \( \rho \), at a fixed point in space is balanced by a divergence of the mass flux, \( \rho u \). In mathematical form this principle of mass conservation becomes
FIGURE 1.4: Surface pressure (reduced to sea level) (labeled in hPa) at 0900 UTC on February 16, 1962 over the Northern part of England. Land higher than 1000 feet above sea level is shaded. The “average” wind (about 40 m/s!) is from the north west.

\[ \frac{\partial p}{\partial t} = -\nabla \cdot \rho \mathbf{v} \]

The r.h.s. represents the convergence of the mass flux per unit volume. In a Cartesian coordinate system,

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]

The local derivative, \( \partial/\partial t \), is related to the total derivative, \( d/dt \), by

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{v} = \frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \]

With this we can express the principle of mass conservation as

\[ \frac{d\rho}{dt} = -\rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \]
The left hand side represents the time rate of change of the density of an individual air parcel. For simplicity, we assume that the air parcel is incompressible (i.e. its density does not change). Therefore the acceleration becomes

\[ a = \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}. \]  

(1.4a)

We now express the velocity components as time averages plus deviations as follows.

\[ u = \overline{u} + u'; \quad v = \overline{v} + v'; \quad w = \overline{w} + w', \]

where

\[ \overline{u} = \frac{1}{\tau} \int_{t-\tau}^{t} u \, dt. \]

The bar denotes an average over a time interval, \( \tau \). If we substitute this into the equation for \( a \) and take the time average (over a time interval, \( \tau \)) of the resulting equation, we get

\[ \overline{a} = \frac{\partial \overline{u}}{\partial t} + \frac{\partial u'\overline{u}}{\partial x} + \frac{\partial u'\overline{v}}{\partial y} + \frac{\partial u'\overline{w}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}. \]

(1.4b)

If we compare (1.4a) with (1.4b) we arrive at the unexpected and surprising conclusion that the time average acceleration depends on the velocity fluctuations. This fact was first pointed out in the nineteenth century by Osborne Reynolds. Newton’s second law relates acceleration, \( a \), to force, \( F \), as follows: \( F = ma \). Therefore, the last three terms in (1.4b) can be interpreted as a force, stress or drag “exerted” by the turbulent eddies or waves “on the average state”. This is due to the transport of momentum by the quasi-random motions associated with eddies and waves, analogous to the effect of random molecular motion giving rise to the effect called “molecular viscosity”.

Since the deviations from the time average are unknown and are usually not measured, eq 1.4b cannot be solved. To solve this problem and with the molecular analogy in mind it is frequently assumed, following Prandtl’s “mixing length theory”, that the transport of momentum by fluctuations (sometimes referred to as turbulent eddy transport) can be expressed in terms of the time average quantities as

\[ \overline{u'u'} = -K_x \frac{\partial \overline{u}}{\partial x}; \quad \overline{u'v'} = -K_y \frac{\partial \overline{u}}{\partial y}; \quad \overline{u'w'} = -K_z \frac{\partial \overline{u}}{\partial z}. \]

(1.5)

Here \( K_x, K_y, \) and \( K_z \) are eddy diffusion coefficients, which usually are much larger than their molecular counterparts. In the boundary layer of the atmosphere, \( K_z \) is in the order of 10 m\(^2\) s\(^{-1}\), while the molecular momentum diffusion coefficient (kinematic viscosity) for air

\[ ^{12} \text{An eddy is a current of air, moving contrary to the direction of the main current, especially in a circular motion.} \]
in the lower troposphere is in the order of $10^{-5}$ m$^2$ s$^{-1}$. The horizontal velocity in the atmospheric boundary layer typically varies by about 10 m s$^{-1}$ over 1 km height. This gives a typical acceleration of

$$\vec{a} = -K_z \frac{\partial^2 \vec{u}}{\partial z^2} = -\frac{10 \times 10^{-10}}{10^6} = -10^{-4} \text{ m s}^{-2}$$

This is comparable to the acceleration due to the Coriolis force when the air parcel has a velocity in order of 1 m s$^{-1}$, as we will see in section 1.6. Above the atmospheric boundary layer (above a height of about 1 km) the effect of the turbulent eddies quickly becomes smaller and in many cases is in fact negligible. Therefore, we’ll frequently neglect the effects of both molecular viscosity and turbulent viscosity. In the following we’ll assume that all variables (e.g. wind, temperature and pressure) are time averages and, for convenience we will drop the bar.$^{14}$

The principal cause of turbulence, as observed in figure 1.3, is the irregular character of the Earth’s surface, which may be rough, such as in a forest or in a city, or smooth such as over water or ice. On a larger scale the roughness of the Earth’s surface depends on the presence and dimensions of mountains or hills. Mountains exert a “form drag” on the atmosphere, which is related to the dynamic pressure differences induced by the deceleration of air as it encounters the obstacle (figure 1.4). Form drag is fundamentally different from the drag associated with turbulent vertical momentum transport (chapters 3 and 6).

### 1.5 Pressure and the pressure gradient force

The buoyancy force is the net effect of two forces, namely the effective gravitational force (section 1.2) and the pressure gradient force. We know that a gas exerts a pressure. We assume that the reader understands what this is due to. Nevertheless, we may refresh our minds on the concept of pressure with the following words from the *Feynman* Lectures on Physics (chapter 39, vol 1).

If our ears were a few times more sensitive, we would hear a perpetual rushing noise. Evolution has not developed the ear to that point because it would be useless if it were so much more sensitive - we would hear a perpetual racket. The reason is that the eardrum is in contact with the air, and air is a lot of molecules in perpetual motion and these molecules bang against the eardrums. In banging against the eardrums they make an irregular tattoo-boom, boom, boom, which we do not hear because the atoms are so small, and the sensitivity of the ear is not quite enough to notice it. The result of this perpetual bombardment is to push the drum away, but of course there is an equal perpetual bombardment of atoms on the other side of the eardrum, so the net force on it is zero.

Air pressure is ultimately determined by the average kinetic energy of the air molecules and the number of molecules per unit volume, $n$. In the kinetic theory of gases a quantity called “temperature” is introduced as a measure of the average kinetic energy of the

---

$^{13}$ Because the viscosity (in N s m$^{-2}$) is nearly constant below a height of 80 km, the kinematic molecular viscosity increases exponentially with height, to reach an estimated value of nearly 2 m$^2$s$^{-1}$ at a height of 86 km (Gill, 1982, p. 295).

molecules in an “ideal gas”. The relation between temperature and pressure is expressed as follows.

\[ p = k n T. \]  

Here, \( k \) is Boltzman’s constant \((=1.381 \times 10^{-23} \text{ J K}^{-1})\). We assume in these lecture notes that air behaves as an ideal gas. A more common version of this equation in meteorology is eq. 1.1.

If the pressure exerted on an air parcel varies from one side to the other the air parcel will experience a net force from high pressure to low pressure. This force is referred to as the pressure gradient force. Vertical pressure gradients in the lower atmosphere are of the order of 100 hPa per 1 km, whereas horizontal pressure gradient forces are of the order of 1 hPa per 100 km \((1 \text{ hPa}=1 \text{ mb}=10^2 \text{ N/m}^2)\).

### 1.6 Inertial “force”

The path of an object, which is a straight line to an observer in space will appear curved to an observer on the rotating Earth. By the latter observer it appears as if a force is pulling the object into a curved path. This apparent force is referred to as the centrifugal force in section 1.4 and is counteracted by two forces: the force associated with gravity and the pressure gradient force.

![Figure 1.5](image)

**FIGURE 1.5:** The curved coordinate system used to describe atmospheric motions relative to the spherical planet (taken from I.N.James, 1994: *Introduction to Circulating Atmospheres*. Cambridge University Press, 422 pp.

The centrifugal force has an unexpected manifestation if the object is moving relative to the earth’s surface. If we define the \( x \)-axis in the rotating frame of reference such that it points in easterly direction, the \( y \)-axis such that it points in northerly direction (figure 1.5) and we have an object which is moving in the \( x \)-\( y \) plane with a speed, \( u \), in the \( x \)-direction (relative to the rotating frame of reference), then the net centrifugal force per unit mass on
this object, according to an observer fixed in space, is

$$\frac{(u + \Omega r)^2}{r} = \frac{u^2}{r} + \Omega^2 r + 2 \Omega \dot{u},$$

where $\Omega$ is the component of angular velocity of the Earth perpendicular to the x-y plane and $r$ is the distance to the axis of rotation. If we apply Newton's second law to the motion relative to the Earth, while not being aware of its rotation, we would miss the effects represented by the last two terms on the r.h.s. of this expression. The first effect is centrifugal force resulting from the Earth’s rotation only. This term was combined with the gravitational force into an effective gravitational force (see section 1.3). The second effect is the “Coriolis effect”, which comes into play if the object is moving relative to rotating frame of reference. The Coriolis force is proportional to the absolute value of the angular velocity of the Earth, $|\Omega|$. Since $|\Omega|=7.292\times10^{-5}$ s$^{-1}$, the time scale, $\tau$, associated with the Coriolis force is of the order of $10^4$ s (i.e. $1/|\Omega|$), i.e. two orders of magnitude greater than the timescale associated with buoyancy (see section 1.3). For an air parcel moving with a velocity of 1 m s$^{-1}$, the acceleration due to the Coriolis force is in the order of $10^{-4}$ m s$^{-2}$.

The expression for the absolute acceleration of an object in terms of the velocity components relative to a frame of reference fixed to a point on the earth’s surface is rather complicated due to the twisting of this frame of reference (due to the rotation of the earth), and due to the curvature of the x and y axes as well as due to the tilting of the z axis as the object moves about over the earth.$^{15}$

Since the deflection due to the centrifugal force is always at right angles to the motion, and has no component along the movement, it can only deflect the motion, not increase or decrease the velocity and thereby change the kinetic energy.

Many of the phenomena of interest in these notes (e.g. cyclones and fronts) are the result of the fact that, because the atmosphere is a fluid and its mass must be conserved, it reacts on inertial motion by generating pressure gradients and associated forces, which eventually come into balance with the inertial forces. The most simplified version of this balance between inertial force and pressure gradient force is referred to as “geostrophic balance” (sections 1.19 and 1.20).

### 1.7 Momentum-, mass- and energy-budget equations

The structure and motion of the atmosphere is constrained by the laws of conservation of momentum, of energy and of mass. The momentum-budget equation (Newton's second law) relates the rate of change of the absolute momentum following the motion of an air parcel in an inertial reference frame to the sum of forces acting on the air parcel. The energy-budget equation states that the internal energy gained by an air parcel is equal to the heat gained by this air parcel plus the mechanical work performed on the air parcel by compression. The conservation law for mass states that the divergence of the mass flux must be balanced by local compression or expansion of the air. Here we assume that the reader is familiar with the fundamental equations that follow from these conservation laws


and which govern the dynamical and thermal state of a rotating stratified fluid. Therefore, the derivations of these equations from first principles are not presented here. Only the specific approximations that are made to these equations when they are applied to the atmosphere of the Earth are discussed here.

In a reference frame that is fixed to the Earth’s surface (figure 1.5) the momentum-, mass- and energy-budget equations can be expressed mathematically as,

\[
\begin{align*}
\frac{d\vec{v}}{dt} &= -\alpha \vec{v} \cdot \nabla p - g \hat{k} - 2\tilde{\Omega} \times \vec{v} + \vec{F}_f, \\
\frac{dp}{dt} &= -\rho \vec{v} \cdot \nabla \rho, \\
J dt &= c_v dT + p d\alpha.
\end{align*}
\]  

(Holton, 2004; Cushman-Roisin and Beckers, 2010). In these equations \(t\) is time, \(\vec{v}\) the air-velocity, \(\alpha\) the specific volume, \(p\) the pressure, \(\tilde{g}\) the acceleration due to gravity, \(\tilde{\Omega}\) the angular velocity of the earth, \(\vec{F}_f\) the friction force per unit mass (due to e.g. turbulent stress; see section 1.4), \(\rho\) the density (=1/\(\alpha\)), \(T\) the temperature, \(J\) the heating per unit mass, per unit time, \(c_v\) the specific heat at constant volume and \(\hat{k}\) is the unit vertical vector (perpendicular to the Earth’s surface). The vertical coordinate (\(z\)) is always directed perpendicular to the earth’s surface. The derivation of Newton’s second law relative to a rotating reference frame can be found in most introductory books on classical mechanics. The best reference for the derivations of all three equations is Holton (2004).

Neglecting the effects of curvature of the Earth’s surface 16, the gradient operator, \(\nabla\), is defined as

\[
\nabla = \left( \hat{i} \frac{\partial}{\partial x}, \hat{j} \frac{\partial}{\partial y}, \hat{k} \frac{\partial}{\partial z} \right),
\]

where \(\hat{i}\) and \(\hat{j}\) are the unit vectors in, respectively, the \(x\)-direction (west-east), the \(y\)-direction (south-north) (figure 1.5). The material derivative with respect to time of the vector \(\vec{v}\) relative to the curved coordinate system (figure 1.5) is (Holton, 2004)

\[
\frac{d\vec{v}}{dt} = \left( \frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} \right) \hat{i} + \left( \frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} \right) \hat{j} + \left( \frac{dw}{dt} - \frac{u^2 + v^2}{a} \right) \hat{k}
\]

(1.8)

where \(u, v\) and \(w\) are the \(x\)-, \(y\)- and \(z\)-component of the velocity, respectively. The material derivative, \(d/dt\), of a scalar, which is a function of \(x, y, z\) and \(t\), is defined in eq. 1.3. It is sometimes written as \(D/Dt\) (instead of as \(d/dt\)) if it is the material derivative of a function of more than one variable. We will, however, not make this distinction here because presumably it is always clear what kind of function we are dealing with. The latter three terms on the r.h.s. of (1.3) are referred to as advection terms. Advection significantly complicates the equations, because it makes them nonlinear.

The Coriolis term in (1.7a) can be expanded into the following sum of components.

---

16 The subtle effects of the curved coordinate system will be neglected except if explicitly stated. In the equation of motion we must take care that this approximation preserves kinetic energy conservation (Box 1.2).
In the “traditional approximation”\textsuperscript{17}, the vertical component of the Coriolis term is negligible because it is several orders of magnitude smaller than the term involving gravity (second term on the r.h.s. of 1.7a). For large scale, slowly evolving, motion systems, the vertical component of the velocity is usually much smaller than the horizontal component. Therefore, the term involving $w$ is also neglected. For small scale, fast evolving motion systems the Coriolis effect can be neglected completely (Holton, 2004). In Box 1.2 the three components of the equation of motion are simplified with scale analysis.

**PROBLEM 1.2. The law of adiabatic expansion**

Show that for an air parcel with volume $V$ in dry adiabatic circumstances, $pV' = \text{constant}$, where $\gamma = c_p / c_v$.

**Box 1.2. Simplification of the equation of motion by scale analysis**

The three components of the momentum budget are,

\[
\begin{align*}
\frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} &= - \frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi - 2\Omega w \cos \phi + F_{rx}, \quad (1) \\
\frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} &= - \frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi + F_{ry}, \quad (2) \\
\frac{dw}{dt} - \frac{u^2 + v^2}{a} &= - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2\Omega u \cos \phi + F_{rz}. \quad (3)
\end{align*}
\]

We can simplify these equations by “scale–analysis”. We define the following characteristic scales for the variables occurring in the above equations:

\[
\begin{align*}
&u = U = 10 \text{ m s}^{-1}; \quad v = U = 10 \text{ m s}^{-1}; \quad w = W = 10 \text{ cm s}^{-1}; \\
&\partial x = \partial y = L = 10^6 \text{ m}; \quad \partial z = H = 10^4 \text{ m}; \quad 2\Omega \sin \phi = 2\Omega \cos \phi = 10^{-4} \text{ s}^{-1}.
\end{align*}
\]

These values are typical for so-called “synoptic scale” weather systems in mid-latitudes, such as cyclones and anticyclones. With these scales and with $a = 6.6 \times 10^6$ m we compare the magnitudes of the curvature- and Coriolis-terms in eqs. (1) and (2):

\[
\begin{align*}
2\Omega v \sin \phi &= 2\Omega u \sin \phi = 10^{-3} \text{ m s}^{-2}; \quad 2\Omega w \cos \phi = 10^{-5} \text{ m s}^{-2}; \\
\frac{uv \tan \phi}{a} &= \frac{u^2 \tan \phi}{a} = 10^{-5} \text{ m s}^{-2}; \quad \frac{uw}{a} = \frac{vw}{a} = 10^{-8} \text{ m s}^{-2}.
\end{align*}
\]

\textsuperscript{17} Gerkema, T., J.T.F. Zimmerman, L.R.M. Maas and H. van Haren, 2008: Geophysical and astrophysical fluid dynamics beyond the traditional approximation. Reviews of Geophysics, 46, 1-33.
Retaining only the largest terms, this leads to

\[
\frac{du}{dt} = \frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi + F_{rx}, \tag{3}
\]

\[
\frac{dv}{dt} = \frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \phi + F_{ry}, \tag{4}
\]

Similarly it can be deduced that, because \( g \approx 10 \text{m s}^{-2} \gg (u^2 + v^2)/a \), the vertical component of the momentum budget reduces to

\[
\frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + F_{rz}. \tag{5}
\]

For high latitudes the terms involving \( \tan \phi \) must be retained. Therefore, the horizontal components of the equation of motion for high latitudes become:

\[
\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{uv \tan \phi}{a} + 2\Omega v \sin \phi + F_{rx}, \tag{6}
\]

\[
\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{u^2 \tan \phi}{a} - 2\Omega u \sin \phi + F_{ry}. \tag{7}
\]

**PROBLEM BOX 1.2: Energy conservation**

Demonstrate that this system of approximate equations is kinetic energy conserving in the absence of gravity, friction and pressure gradients.

---

### 1.8 Equation of state and atmospheric composition

There are four unknown variables, i.e. \( \vec{v}, \rho, T \) and \( p \) and three equations, (1.7a,b,c). Hence, we need to supplement this system of equations with one equation. This equation is the equation of state for an ideal gas, which is already employed in section 1.3 (eq.1.1) and again introduced in section 1.5 (eq. 1.6). The latter version is formulated in terms of the molecular number density, \( n \) (in numbers per m\(^3\)). The former version (eq. 1.1), containing density, \( \rho \), or specific volume, \( \alpha=1/\rho \), is encountered more frequently in atmospheric science, i.e.:

\[ p\alpha = RT. \]

If air is dry, \( R \) is the specific gas constant for dry air (=287 J K\(^{-1}\)kg\(^{-1}\)). If air is a mixture of dry air and water vapour, \( R \) is the specific gas "constant" for this mixture (we may assume that water vapour also satisfies the ideal gas law). However, since the water vapour content of air varies strongly in space and time, the specific gas constant for the mixture will vary accordingly. To circumvent this problem an alternative variable for the temperature is introduced called the **virtual temperature**, \( T_v \). The virtual temperature is the temperature at which dry air would have to be in order to have the same density as a sample of moist air,
assuming both have the same pressure. Thus, when the virtual temperature is used instead of the temperature, the equation of state for dry air may be used for moist air also. The equation for the virtual temperature is

\[ T_v = T \frac{1 + (r_v / \varepsilon)}{1 + r_v} \]

Here, \( r_v \) is the mixing ratio of water vapour, i.e. the ratio of the mass of water vapour present to the mass of dry air containing the vapour (eq. 1.10), and \( \varepsilon = 0.622 \) is the ratio of the specific gas constant of dry air to the specific gas constant of water vapour, or alternatively, the ratio of the molecular weight of water vapour to the molecular weight of dry air.

\[ \text{FIGURE 1.6:} \quad \text{Annual and global average concentration of various constituents in the atmosphere of Earth, as function of height above the Earth’s surface. The concentration is expressed as a fraction of the total molecule number density. This fraction is proportional to the mixing ratio by volume. F11 and F12 denote the chlorinated fluorocarbons Freon-11 and Freon-12. Note that the concentration of carbon dioxide is constant up to a height of 100 km, while the concentration of water vapour decreases by several orders of magnitude in the lowest 20 km. Source: Bohren, C.F. and E.E. Clothiaux, 2006: Fundamentals of Atmospheric Radiation. Wiley-VCH. 472 pp.} \]

Air is a mixture of many constituents, principally nitrogen (N\(_2\)) (780840 ppmv\(^{18}\)), oxygen (O\(_2\)) (209460 ppmv), argon (Ar) (940 ppmv), water vapour (up to 40000 ppmv) and carbon dioxide (CO\(_2\)) (390 ppmv). Trace constituents that play an important role in the radiation balance of the atmosphere are ozone (O\(_3\)) (0-100 ppbv), atomic oxygen (O) (0-10\(^9\) m\(^{-3}\)), hydrogen (H\(_2\)) (560 ppbv), nitrous oxide (N\(_2\)O) (310 ppbv) and methane (CH\(_4\)) (1.7 ppmv) (figure 1.6).

If the particular constituent has a long residence time in the atmosphere, its volume mixing ratio is relatively constant in the lowest 80 km of the atmosphere (the homosphere), where the effect of mixing due to macroscopic fluid motions dominates over molecular

---

\(^{18}\) From eq. 1.6 it can be deduced that the volumes occupied by different gases at the same temperature and pressure are proportional to the numbers of molecules of the gases. Mixing ratio can be expressed as a (molecule) number fraction in units of parts per million by volume, i.e. 1 ppmv=1 unit of volume per 10\(^6\) units.
diffusion. Nitrogen (N\textsubscript{2}) and oxygen (O\textsubscript{2}) are very well mixed because their residence times are enormous: respectively $10^7$ years and $10^4$ years. The residence time of carbon dioxide (CO\textsubscript{2}) is at least 5 years, which is long enough to be relatively well mixed. On the other hand, the radiatively active constituents, ozone and water vapour are not well mixed, since their residence times (respectively, about 100 days and about 10 days) are of the same order of magnitude or smaller than the typical time-scales associated with the variability of the circulations that are responsible for the mixing.

Both ozone and water vapour have strong sources at the height of maximum mixing ratio. In the case of ozone this is due to photochemical reactions involving ultraviolet Solar radiation and oxygen taking place at great heights in the atmosphere, while in the case of water vapour this is due to evaporation of liquid water at the Earth’s surface and removal of water vapour by condensation and precipitation in clouds. The inhomogeneity of the water vapour distribution is also connected to the strong temperature dependence of the water vapour equilibrium partial pressure (the Clausius-Clapeyron relation, discussed in section 1.9). We see in figure 1.6 that the mixing ratio of ozone actually increases with height above the Earth’s surface reaching a maximum value at a height of about 30 km, while the mixing ratio of water vapour has a maximum value at the Earth's surface and decreases by at least a factor of 10 in the lowest 5 km above the Earth’s surface (figure 1.7).

Atmospheric water vapour concentration is expressed in terms of either the fraction of the total number of molecules (figure 1.6), or as the fraction of the total mass density of air (the specific humidity),

![Figure 1.7](image_url)

**Figure 1.7.** Monthly average specific humidity, measured by radiosondes, as a function of pressure (500 hPa corresponds approximately to 5-6 km above sea level; 300 hPa corresponds to about 9-10 km above sea level) at locations possessing radically different climates, from wet tropical to midlatitude summer and winter, to
subtropical dry. The averages are for the 12-year period running from 1997 until 2008. Source of the data: http://weather.uwyo.edu/.

\[ q = q_v = \frac{\rho_v}{\rho} , \quad (1.9) \]

where \( \rho_v \) is the density of water vapour and \( \rho \) is the density of air including water vapour, or as the fraction of the mass density of "dry" air (the mixing ratio by mass),

\[ r = r_v = \frac{\rho_v}{\rho_d} , \quad (1.10) \]

where \( \rho_d \) is the density of dry air (excluding the water vapour). Figure 1.7 gives an impression of the variations in specific humidity as a function of pressure at different locations and in different seasons. An example of an extremely dry place is the city of Tamanrasset, which is located in the Sahara desert (at 23°N) at a height of 1364 m. The average total precipitation in January in Tamanrasset is less than 5 kg m\(^{-2}\) (5 mm). An example of a very wet place is Delhi at the height of the rainy season (the monsoon) in July. The average total precipitation in July in Delhi is 240 kg m\(^{-2}\). The capital of India (at 27°N) has a relatively short rainy season of about 3 months and is nearly as dry as Tamanrasset during the remaining 9 months. The climate of Singapore (at 1°N) is genuinely tropical, implying that precipitation is abundant throughout the whole year (170 kg m\(^{-2}\) in July and 2370 kg m\(^{-2}\) annually). The climate of De Bilt (at 52°N) and of Milano (45.5°N) is typical of the mid-latitudes. Despite the relatively low specific humidity in January, the total average precipitation of January in De Bilt is about the same as the total average precipitation of July (about 75 kg m\(^{-2}\)). This indicates that specific humidity is not the principal determining factor in explaining precipitation intensity. Rather, precipitation is determined by the equilibrium water vapour pressure, frequently also referred to as the saturation vapour pressure, which is a function of temperature, as we shall see in the following section.

Eqs. 1.7, and 1.1 form a closed set of equations governing the dynamics of the atmosphere. According to John Dutton\(^{19}\), the discovery of these laws is one of man's great intellectual achievements: “they summarise with but a handful of symbols very diverse and complex processes”. Analysis of the linearised versions of these equations indicates that the solution contains many kinds of waves or oscillations, such as sound waves, buoyancy oscillations/waves (section 1.15 and chapter 3), inertial oscillations/waves (section 1.20 and chapter 5) and Rossby waves (section 1.37 and chapter 9).

**PROBLEM 1.3. Mixing ratio**

(a) Write down the definition of “mixing ratio by volume”.

(b) Show that both the mixing ratio and the specific concentration of any constituent in an air parcel is conserved under advection in the absence of phase transitions and sources/sinks of this constituent.

(c) If the mixing ratio of ozone at 30 km height (\( p=10 \) hPa) is 100 ppbv (parts per billion by volume), how many ozone molecules are there in a cubic meter of air at that particular level, assuming a temperature of 200 K? The molecular weight of ozone is 48 g mol\(^{-1}\). The universal gas constant is equal to 8.3 J K\(^{-1}\) mol\(^{-1}\).

1.9 Clausius-Clapeyron equation

Water vapour, which enters the atmosphere by evaporation at the Earth's surface, is transported upwards by currents. This leads to the formation of liquid water droplets (i.e. clouds), which remain suspended in the air until they become heavy enough to fall to the ground as rain. The formation of clouds is associated with a very strong constraint on the water vapour mixing ratio, which is related to the fact that the partial pressure of the vapour phase, which is in equilibrium with the liquid phase, is an exponential function of temperature. This fact was put on a firm theoretical footing first by Emile Clapeyron and Rudolf Clausius in the nineteenth century. These two scientists derived an equation of state for a heterogeneous system, bearing their name. A heterogeneous system consists of a substance in more than one phase, such as vapour and liquid. The so-called Clausius-Clapeyron equation for the vapour pressure, \( p_e \), which is in equilibrium with the liquid phase takes the following approximate form

\[
\frac{dp_e}{dT} = \frac{p_e L_v}{R_v T^2}. \tag{1.11}
\]

The parameters \( L_v \) (which depends weakly on \( T \)) and \( R_v \) are, respectively, the specific latent heat of evaporation and the gas constant for water vapour (461.5 J K\(^{-1}\) kg\(^{-1}\)). In most applications the parameter \( L_v \) is assumed to be a constant (≈2.5×10\(^6\) J kg\(^{-1}\)).

![Equilibrium water vapour pressure](image)

**FIGURE 1.8.** Equilibrium water vapour pressure, \( e_v \), as a function of temperature, according to eq. 4, assuming \( L_v \) is constant (≈2.5×10\(^6\) J K\(^{-1}\)).

In deriving the Clausius-Clapeyron equation it has been assumed that water vapour is an ideal gas and, therefore, that the state of water vapour obeys the ideal gas law:

\[
p_v = \rho_v R_v T. \tag{1.12}
\]

Here \( p_v \) is the (partial) pressure exerted by the water vapour molecules and \( \rho_v \) is the density of water vapour in the atmosphere. The symbol \( e \) is frequently used instead of \( p_v \). The
symbol $e_s$ is used, instead of $p_e$, to designate the equilibrium value of $e$ in a mixture of liquid and gas. The subscript ‘s’ stands for “saturated”, indicating that this is the maximum value of the vapour pressure at a particular temperature. If the water vapour concentration is such that the associated partial pressure exceeds $e_s$, condensation of water vapour usually occurs, leading to the formation of clouds and ultimately precipitation (rain, hail or snow).

Eq. 1.11 can be written as follows.

$$\frac{\partial \ln e_s}{\partial T} = \frac{L_v}{R_v T^2}.$$  \hspace{1cm} (1.13)

Assuming that $L_v$ is constant (in reality $L_v$ is a weak function of temperature) this equation can be integrated to give

$$\ln e_s - \ln e_{s,0} = \frac{L_v}{R_v} \left( \frac{1}{T_0} - \frac{1}{T} \right),$$  \hspace{1cm} (1.14)

where $e_{s,0}$ is the equilibrium or “saturated” vapour pressure at $T_0=0^\circ$C. From experiment it is found that $e_{s,0}=610.78$ Pa.

We can now plot $e_s$ as a function of temperature according to eq. 1.14 (figure 1.8). Between $0^\circ$C and $36^\circ$C the value of $e_s$ increases by a factor of 10!

Applying eq. 1.14 to sub-freezing temperatures is not unusual to a meteorologist since sub-cooled water droplets in the atmosphere are the rule rather than the exception at temperatures between -40°C and 0°C.

**1.10 Water vapour distribution**

The Clausius-Clapeyron equation represents a strong constraint on the vertical distribution of the density of water vapour in the atmosphere. This is of interest, because water vapour is a potent greenhouse gas, and because its density determines the degree of absorption and emission of long wave radiation by slab of air.

Eq. (1.12) is rewritten as

$$\rho_v = \frac{e}{R_v T}.$$  \hspace{1cm} (1.16)

Let us assume that the relative humidity, $RH$, defined as

$$RH = \frac{e}{e_s},$$  \hspace{1cm} (1.17)

is constant with height, and that the temperature in the atmosphere decreases with increasing height according to
**Figure 1.9.** Monthly mean precipitable water in kg m$^{-2}$ as a function of the monthly mean mass density of water vapour at the Earth’s surface according to radiosonde measurements made at the midlatitude site of De Bilt (52°N, 5°E) (upper panel), at the subtropical site of Delhi (29°N, 77°E) and at the tropical site of Singapore (1.4°N, 104°E) (lower panel). For De Bilt, radiosonde observations from the months of January, April, July and October of the years 1993-2008 are used. Until 2001 a radiosonde was launched at De Bilt 4 times every day (at 00, 06, 12 and 18 UTC). After 2001 the frequency was reduced to 2 times every day (at 00 and 12 UTC). The linear fit to the observations in the upper panel yields a scale height, $H_v$, of 2160 m (the correlation coefficient is 0.99). The monthly mean values shown by markers in the lower panel are for the 12-year period running from 1997 until 2008. The linear fit to the observations of Singapore and Delhi in the lower panel yields a scale height, $H_v$, of 3117 m (the correlation coefficient is 0.8). Source of the data: [http://weather.uwyo.edu/](http://weather.uwyo.edu/)
\[ T(z) = T_g - \Gamma z \]  

where \( \Gamma \) is referred to as the "lapse rate"\(^{20} \) and \( T_g \) is the temperature at \( z=0 \) (the ground). Substituting (1.16) and (1.18) into (1.14) yields

\[ \rho_v = \frac{RH e_s}{R_v T} = \frac{RH e_{s,0}}{R_v (T_g - \Gamma z)} \exp \left\{ \frac{L_v}{R_v} \left( \frac{1}{T_0} - \frac{1}{T_g - \Gamma z} \right) \right\} . \]  

We now use the binomial theorem to approximate

\[ (T_g - \Gamma z)^{-1} = T_g^{-1} \left( 1 - \frac{\Gamma}{T_g} z \right)^{-1} = T_g^{-1} \left( 1 + \frac{\Gamma}{T_g} z \right) \]  

This approximation is valid as long as

---

\(^{20}\) The **lapse rate** is defined as the negative of the rate of change in an atmospheric variable, usually temperature.
Figure 1.11. Monthly mean precipitable water in kg m$^{-2}$ as a function of the monthly mean mass density of water vapour at the Earth’s surface according to radiosonde measurements made at the following different locations over the world: Tamanrasset (23°N, 5°E; continental dessert), De Bilt (52°N, 5°E; mid-latitude, west coast), St Helena (16°S, 6°W; ocean desert), Delhi (29°N, 77°E; subtropical, dry in January, transition between dry and wet season in June and wet month of July), Port Elizabeth (34°S, 26°E, subtropical, wet), Milano (45°N, 9°E; Mediterranean), and Singapore (1.4°N, 104°E; tropical, wet). The monthly mean values shown by markers are for the 12-year period running from 1997 until 2008. Source of the data: http://weather.uwyo.edu/.

The lapse rate, \( \Gamma \), in the atmosphere is equal to or less than the "dry adiabatic lapse rate", \( g/c_p \approx 10^{-2} \) K m$^{-1}$ (section 1.14). Assuming that \( T_g \approx 300 \) K, we conclude that \( \Gamma/T_g \leq (1/3) \times 10^{-4} \) m$^{-1}$. Therefore, we are allowed to make the following approximation:

\[
\left( 1 + \frac{\Gamma}{T_g} \right) z \approx 1 ,
\] (1.23)

Now (1.19) becomes

\[
\rho_v = \frac{R H e_{v,0}}{R_v T_g} \left( 1 + \frac{\Gamma z}{T_g} \right) \exp \left( \frac{L_v}{R_v T_0} \right) \exp \left[ - \frac{L_v}{R_v T_g} \left( 1 + \frac{\Gamma z}{T_g} \right) \right] .
\] (1.22)
if $z \ll 30000$ m. This implies that the vertical dependence of the density of water vapour in the lowest part of the atmosphere, where most of the water vapour is found is governed by

$$\rho_v \propto \exp \left\{ -\frac{L_v \Gamma}{R_v T_g^2} z \right\}. \quad (1.24)$$

From this we identify a **scale height**, $H_v$, that characterizes the exponential decrease of water vapour density, i.e.

$$H_v = \frac{R_v T_g^2}{L_v \Gamma}. \quad (1.25)$$

With $T_g \approx 300$ K, $\Gamma=g/c_p \approx 10^{-2}$ K m$^{-1}$, $L_v \approx 2.5 \times 10^6$ J kg$^{-1}$ and $R_v \approx 461.5$ J K$^{-1}$ kg$^{-1}$, we find that $H_v \approx 1700$ m. Since $\Gamma$ in the lower troposphere is in general smaller than the dry adiabatic lapse rate, this represents an approximate lower bound on the water vapour scale height.

With a more realistic value for $\Gamma \approx 0.65 \times 10^{-2}$ K m$^{-1}$ and $T_g = 288$ K (the global average temperature), $H_v \approx 2355$ m$^{21}$.

Roughly speaking, it may be concluded that the water vapour mass density decreases by a factor of $e^{-1}$ over the lowest two kilometres of the atmosphere. The vertical dependence of water vapour density in the lowest parts of the atmosphere can, thus, be expressed simply as

$$\rho_v = \rho_{v,g} \exp \left\{ -\frac{z}{H_v} \right\}, \quad (1.26)$$

where $\rho_{v,g}$ is the water vapour mass density at $z=0$. Integration of (1.26) from the $z=0$ to infinity (the top of the atmosphere) yields the total atmospheric water vapour content per square metre. This quantity is generally referred to as **“precipitable water”** or, in short, $PW$. The equation for $PW$ derived from (1.26) is remarkably simple, i.e.

$$PW = \rho_{v,g} H_v. \quad (1.27)$$

This equation indicates that there should be a linear relation between total precipitable water and water vapour mass density at the Earth’s surface ($z=0$).

**Figure 1.9** (upper panel) demonstrates that this is indeed true for monthly mean values of $PW$ and of $\rho_{v,g}$ over De Bilt (the Netherlands) all year round. From the upper panel in **figure 1.9** we can estimate the monthly mean value of $H_v$. A linear regression with a very high correlation coefficient of 0.99 yields

$$H_v = 2160 \text{ m}. \quad (1.28)$$

---

21 The water vapour scale height, $H_v$, therefore, is much smaller than the pressure scale height, $H_p \approx 8$ km (eq. 1.63). The importance of this fact was stressed by C.P. Weaver and V. Ramanathan in 1995 (Deductions from a simple climate model: factors governing surface temperature and atmospheric thermal structure. *J. Geophys. Res. Atmospheres*, 100(D6): 11585-11592).
Figure 1.12. The average meridional circulation in the tropics, called the Hadley circulation, is thought to be driven by latent heat release in large convective clouds in the ITCZ (sections 1.11 and 1.25). Water vapour is forced upwards over the ITCZ. Compensating subsidence in the subtropics leads to warming of the air and a concomitant reduction of the relative humidity. Source: Webster, P.J., 1994: The role of hydrological processes in Ocean-atmosphere interactions. Rev. Geophysics, 32, 427-476.

Figure 1.13. Monthly or annual mean relative humidity as a function of pressure according to radiosonde measurements for different locations. The location of Singapore (1.4°N, 104°E) is representative for the ITCZ. In fact, Singapore is near to or within the ITCZ all year round. The location of De Bilt (52°N, 5°E) is representative for the midlatitudes. Tamanrasset (23°N, 5°E) is located at an altitude of 1364 m in the middle of the Sahara desert, where evaporation is negligible. The island of St Helena (16°S, 6°W) is representative for the subtropical ocean “desert”, where evaporation of water vapour from the ocean makes the lower boundary layer very moist, but large scale downward motion (sometimes referred to as subsidence) reduces the relative humidity at higher levels. The data for De Bilt for the months of January and July as well as the data for Singapore (annual mean, using observations from the months of January, April, July and October) comprise the 16-year period running from 1993 until 2008. The data used to calculate the average relative humidity at Tamanrasset and at St. Helena is for the period 1997-2008. Source of the data: http://weather.uwyo.edu/
This agrees reasonably well with the theoretically predicted value of $H_v$.

**Figure 1.10** demonstrates very clearly that the mass density profile of water vapour at De Bilt can be represented very well by eq. 1.26, if we confine ourselves to monthly average values. The theoretical curves shown in **figure 1.10** are derived from eqs. (1.26) and (1.27) using the observed monthly average value of $\rho_{v,g}$ and the observed monthly average value of $PW$. The scale height $H_v$ hardly varies between the summer and the winter, even though the precipitable water content of the atmosphere in summer is more than twice the precipitable water content in winter.

The linear correlation between $PW$ and $\rho_{v,g}$ is quite robust, even on the global scale (**figure 1.11**). Nevertheless, it appears that the scale height, $H_v=PW/\rho_{v,g}$, is significantly larger in the tropics during the rainy season, than in the mid-latitudes. A linear fit to the observations of Delhi in June and July and of Singapore in January (when the monthly average $PW$ at these locations is greater than 40 kg m$^{-2}$) (**figure 1.9**, lower panel) yields $H_v=3117$ m. Presumably, this relatively high value of $H_v$ is the due to the sustained upward transport of water vapour in the Inter-tropical Convergence Zone (ITCZ) (**figure 1.12**).

$PW$ and $\rho_{v,g}$ do not correlate very well in sub-tropical regions, such as in the Sahara desert (Tamanrasset) and in the subtropical south Atlantic Ocean (St. Helena). An estimate of $H_v$ for St Helena in July is obtained by taking the average over 12 years (1997-2008) of $(PW/\rho_{v,g})$ which yields $H_v=1245$ m. This, in fact, corresponds to the approximate height of the relatively very moist boundary layer in this ocean area. The turbulent boundary layer is usually capped at a height of 1 to 2 km by a relatively dry layer, which is formed by large scale subsidence in the downward branch of the Hadley circulation (**figure 1.12**). The dry layer is observed in **figure 1.13**, which shows a graph of the monthly and/or annual average relative humidity, $RH$, as a function of pressure for different locations. Relative humidity is defined in eq. 1.17. The very moist lower boundary layer at St Helena is seen clearly in **figure 1.13**. At Tamanrasset, on the other hand, relative humidity is extremely low at all levels, because there is negligible evaporation at the ground. Relative humidity at upper levels increases in summer, presumably due to transport of moist air in the upper level outflow of moist air from the inter-tropical convergence zone (ITCZ) to the south (**figure 1.12**).

### 1.11. Sources and sinks of water in the atmosphere

This section is a short introduction into the water cycle of the atmosphere, which consists of evaporation at the Earth's surface, condensation in the atmosphere and associated cloud formation, precipitation and large-scale transport of water vapour in the atmosphere. The source of water in the atmosphere at a specified location on Earth is either evaporation at the Earth’s surface or moisture flux convergence (especially at low levels), while the sink of moisture is precipitation from clouds at higher levels (between 900 and 300 hPa). Because water is removed from the atmosphere at higher levels the monthly mean relative humidity decreases with height in most places (**figure 1.13**).

The atmospheric water cycle is connected to the water reservoirs of the oceans, rivers, lakes, soil, deeper ground and cryosphere (**figure 1.14**). The numbers given in **figure 1.14** are not very certain, especially precipitation, both over land and over the ocean. Ice volumes include glaciers, ice caps and ice sheets. In particular, the volume of Antarctica has often been overestimated. For permafrost or ground ice, values are also very uncertain. Permafrost occupies about 24% of the land surface in the northern hemisphere! Furthermore,
evapotranspiration, which is notoriously difficult to measure, has been computed as a residual of precipitation and runoff.

Because of climate change, the numbers shown in figure 1.14 are not constant in time. There is also a strong annual cycle in for instance the ocean precipitation with peak values of 381 units in October (1 unit is $10^9 \text{ m}^3 \text{ yr}^{-1}$; 1 m$^3$ of water weighs about 1000 kg), a minimum value of 365 units in April and an annual mean of 373 units. Over land the cycle in precipitation peaks in July at 128 units and has a minimum in February at 105 units.

Moisture converges at low levels in the ITCZ (figures 1.14 and 1.15) and to a lesser extent over the mid-latitude ocean-areas and the adjacent coastal areas (figure 1.15a). Strong moisture flux convergence in the ITCZ (usually more than 10 kg m$^{-2}$ day$^{-1}$) leads to extremely high precipitable water contents, sometimes in excess of 80 kg m$^{-2}$. Annually averaged, about 10 kg m$^{-2}$ of water vapour converges into the relatively narrow inter-tropical convergence zone over the Pacific Ocean and the Atlantic Ocean from the subtropics every day. The source of this moisture is evaporation from the sub-tropical ocean, where the annual average moisture flux divergence is about 5 kg m$^{-2}$ day$^{-1}$.

Figure 1.16 shows the global distribution of PW in respectively January and July. Obviously, the water vapour content in the atmosphere is dominated by the tropics. Here, most of the internal heating of the atmosphere is due to latent heat release as a result of condensation of water vapour and subsequent loss of this water due to precipitation (figure 1.17). Average precipitation in the ITCZ is in the order of 10 kg m$^{-2}$ day$^{-1}$ (figure 1.18). Since the vertically integrated moisture flux convergence in the ITCZ is of the same order of
magnitude we conclude that most of the precipitation in the ITCZ consists of water that has been evaporated elsewhere, i.e. from the subtropical ocean surface. Local evaporation is of minor importance as a source of moisture for rainfall in the tropics.

**PROBLEM 1.4. Precipitable water and surface humidity**

Investigate the relation between total column water vapour ($PW$) and the density of water vapour near the Earth’s surface (eq. 1.27) in the ERA-Interim reanalysis (http://apps.ecmwf.int/datasets/). Use the dataviewer, **PANOPLY** (http://www.giss.nasa.gov/tools/panoply/; see section iii of the Preface).

a. January

![Column integrated water vapor flux with their convergence](image)

b. July

![Column integrated water vapor flux with their convergence](image)

**Figure 1.15.** Column integrated water vapour fluxes (arrows) with their convergence, in mm per day (1 mm=1 kg m$^{-2}$) in January (a) and in July (b). Average for the 26-year period 1979 to 2004. Source: [http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-topo.htm](http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-topo.htm).
a. January

**Figure 1.16.** Column integrated water vapour (referred to in the text as precipitable water, PW) in January (a) and in July (b). Labels in units of kg m\(^{-2}\). Average for the 26-year period 1979 to 2004. Source: http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-tope.htm.

*Figures 1.15 and 1.16 also indicate that more moisture converges into the ITCZ if the ITCZ is located over the ocean than if the ITCZ is located over land, such as over Africa or South America. Over continents moisture flux convergence into the ITCZ (figure 1.15) depends on availability of moisture in the adjacent regions. This, in turn depends on the "wetness" of the soil (figure 1.18). Over Africa the availability of water is significantly reduced due to the presence of deserts in the north (the Sahara and Saudi Arabia) and in the south (Kalahari). Therefore, precipitation in the ITCZ over the African continent is reduced and, moreover, it exhibits and strong long-term (yearly and decadal) variation, which is related both to time variation in soil wetness, determined among other by vegetation, as well*
as to slow variations in sea surface temperature in both the tropical Atlantic and the Indian Oceans\textsuperscript{22}.

**a. January**

![Precipitation Map for January](image1)

**b. July**

![Precipitation Map for July](image2)

\textbf{FIGURE 1.17.} Precipitation in January (a) and in July (b). Labels in units of kg m\textsuperscript{2} day\textsuperscript{-1} or mm day\textsuperscript{-1}. Average for the 26-year period 1979 to 2004. By far the most precipitation (upto 15 mm/day, or about 5,000 mm per year) is observed over the ITCZ! In the south west Pacific we observe a separate region with high precipitation, also associated with low level moisture convergence. This convergence zone is referred to as the “South Pacific convergence zone” (SPCZ). Yearly precipitation in midlatitudes can sometimes also reach several thousands of millimeters, especially along the mountainous mid-latitude west coasts of south America, Canada, Alaska, Scotland and Norway. Source: [http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-tope.htm](http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-tope.htm).

**a. January**

![Map of soil water content (shallow) in January](image)

**b. July**

![Map of soil water content (shallow) in July](image)

**FIGURE 1.18.** Surface soil wetness (availability of water at the surface over land areas) in January (a) and in July (b). Labels in units of kg m$^{-2}$. Average for the 26-year period 1979 to 2004. Source: [http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-tope.htm](http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-tope.htm).

Over the tropical Pacific availability of moisture for evaporation is essentially unlimited. Here the limiting factor to evaporation is the ability of the atmosphere to mix water vapour vertically. This ability is linked to the degree of **static instability** (section 1.15), i.e. to the intensity of convection, which obviously provides the main mechanism for bringing water vapour upward in the tropical troposphere.

The hydrological cycle plays an important role in the energy balance of the atmosphere. Because energy is required to evaporate water vapour from the surface, while this energy is released upon condensation in clouds, large **latent fluxes of energy** are associated with the
hydrological cycle. Furthermore, the distribution of water vapour and clouds is crucial in determining the “strength” of the greenhouse effect (chapter 2).

1.12 Ozone distribution

Like water vapour, ozone is inhomogeneously distributed in the atmosphere. Both water vapour and ozone interact strongly with radiation and thus determine the temperature distribution in the atmosphere. Here we explain the reasons for the existence of ozone in the atmosphere and its peculiar distribution, particularly the formation of ozone in the layer between 10 and 50 above sea level (figure 1.6). Ozone and oxygen absorb Solar ultraviolet radiation. The associated heating in the stratosphere (figure 1.2) leads to the temperature maximum that is observed at approximately 50 km above sea level (figure 1.19). It should be stressed directly that an explanation of the exact distribution of ozone requires knowledge of the large scale circulation, which is in fact partly induced by the heating due absorption of solar radiation by ozone. Therefore, we will be concerned in this section with a small part of the interesting non-linear problem involving the interaction of the chemistry, radiation and fluid dynamics of the atmosphere.

**FIGURE 1.19.** Pressure (solid), density (dashed), and temperature (dotted), as functions of altitude according to the U.S. standard atmosphere (Table 1.1) (Salby, M.L., 1996: Fundamentals of Atmospheric Physics. Academic Press, San Diego, 627 pp). See also Box 1.1. The temperature profile is typical for the extra-tropics.
Figure 1.20. Spectral distribution of Solar radiation reaching the top of the top of the atmosphere (yellow) and reaching sea level (red) (1 nm=10⁻⁹ m). Also shown is the spectrum of a black body with a temperature of 5250°C (eq. 3-Box 1.3), which is approximately the Sun's surface temperature. Ozone and water vapour are the principal constituents that absorb Solar radiation. Adapted from the following source: http://www.globalwarmingart.com/.

The basic mechanism that gives rise to the ozone layer was put forward by Sidney Chapman in 1930\textsuperscript{23}. It involves the following (photo-) chemical reactions.

\begin{align}
O_2 + h\nu_1 & \rightarrow O + O; \\
O + O_2 + M & \rightarrow O_3 + M; \\
O_3 + h\nu_2 & \rightarrow O + O_2; \\
O + O_3 & \rightarrow 2O_2;
\end{align}

where $h$ is Planck’s constant (Box 1.3) and $\nu$ is the frequency of the incident radiation that is absorbed in the particular photochemical reaction and $M$ is any other air molecule needed for the energy balance of the reaction. The wavelength bands involved in the two photochemical reactions are (roughly)

\begin{align}
\nu_1 &= 0.19 - 0.24 \text{ \mu m (dissociation of } O_2) \quad (1.30a) \\
\nu_2 &= 0.2 - 0.3 \text{ \mu m (dissociation of } O_3) \quad (1.30b)
\end{align}

These wavelengths correspond to the extreme low wavelength end of the Solar spectrum.

(Figure 1.20). The molecule number concentrations (per cubic meter) of $O$, $O_2$ and $O_3$ are denoted by, respectively, $n_1$, $n_2$ and $n_3$. The number concentration of $O_2$-molecules is assumed to be constant fraction of the total number of molecules, which can be computed from the equation of state (1.6). Assuming that 21% of the air molecules are oxygen molecules, we find

$$n_2 = 0.21 \frac{p}{kT}. \tag{1.31}$$

The rates of change of the number concentrations of $O$ and of $O_3$, as a result of the four reactions (1.29a-d) involved in what is now referred to as the Chapman cycle are

$$\frac{dn_1}{dt} = 2j_a n_2 - k_b n_1 n_2 n + j_c n_3 - k_d n_1 n_3; \tag{1.32}$$

$$\frac{dn_3}{dt} = k_b n_1 n_2 n - j_c n_3 - k_d n_1 n_3. \tag{1.33}$$

In these equations $k_b$ and $k_d$ are constants determining the rate of, respectively, reactions (1.29b) and (1.29d). The exact values of these so-called rate constants are determined in laboratory experiments\textsuperscript{24}. The parameters $j_a$ and $j_c$ are first order photolysis rate coefficients.

The magnitudes of $j_a$ and $j_c$ depend on the probability that a photon will be absorbed by, respectively, an oxygen molecule or an ozone molecule (expressed in terms of an absorption cross-section), the probability that (if absorption occurs) the oxygen or ozone molecule will dissociate (the “quantum yield”), and on the radiation flux, $I$. The values of $j_a$ and $j_c$ are thus calculated from

$$j_a = \int_{\lambda_1}^{\lambda_2} \sigma_{O_2}(\lambda,T) \phi_{O_2}(\lambda,T) I(\lambda) d\lambda, \tag{1.34}$$

$$j_c = \int_{\lambda_1}^{\lambda_2} \sigma_{O_3}(\lambda,T) \phi_{O_3}(\lambda,T) I(\lambda) d\lambda. \tag{1.35}$$

The parameters $\sigma_{O_2}$ and $\sigma_{O_3}$ are the wavelength- and temperature-dependent absorption cross-sections (in m$^2$) (see chapter 2); $\phi_{O_2}$ and $\phi_{O_3}$ are the quantum yields for, respectively, reaction (1.29a) and reaction (1.29c). The units of $I$ are now photons per square meter per second per meter of wavelength.

Let us estimate the values of $j_a$ and $j_c$. We first simplify matters strongly by assuming that the quantum yields corresponding to the two photochemical reactions in the Chapman cycle are equal to 1. Reaction (1.29c), for instance, is sensitive to radiation in a wavelength band between 0.2 and 0.3 µm. The global average and yearly average Solar flux \textit{at the top of the atmosphere} within this wavelength band is about 10 W m$^{-2}$. Let us assume that absorption cross-section corresponding to this photochemical reaction is constant and equal to $5 \times 10^{-23}$ m$^2$ molecule$^{-1}$. In that case

\textsuperscript{24} see http://jpldataeval.jpl.nasa.gov/
\[ j_c = 5 \times 10^{-23} \times 10 \times \frac{\overline{\lambda}}{hc} = 6 \times 10^{-3} \ \text{s}^{-1} \] (1.36)

where \( \overline{\lambda}/hc \) is the average energy of one photon within the spectral interval (1.30b) of interest, where \( c \) is the speed of light (\( h=6.63 \times 10^{-34} \ \text{J s} \) and \( c=3 \times 10^8 \ \text{m s}^{-1} \)). In the case of reaction (1.29c), the average wavelength is \( \overline{\lambda}=0.25 \ \text{µm} \).

Reaction (1.29a) is sensitive to radiation in a wavelength band between 0.19 and 0.24 µm. The global average and yearly average Solar flux within this wavelength band at the top of the atmosphere is about 2.15 W m\(^{-2} \). Assuming that the absorption cross-section corresponding to photochemical reaction (1.29a) is constant and equal to \( 3 \times 10^{-28} \ \text{m}^2 \) molecule\(^{-1} \)

\[ j_a = 3 \times 10^{-28} \times 2.15 \times \frac{\overline{\lambda}}{hc} = 7 \times 10^{-10} \ \text{s}^{-1} \] (1.37)

(\( \overline{\lambda}=0.215 \ \text{µm} \)). It is important to remember that the above estimates of \( j_a \) and \( j_c \) apply to the top of the atmosphere. Obviously, since the radiation is depleted quickly as one goes downward into the atmosphere, the value of the photolysis coefficient will also decrease strongly with decreasing height. This is illustrated in Figure 1.21. In any case \( j_a<<j_c \) at all levels. Therefore, we conclude that reaction (1.29a) is relatively very slow compared to reaction (1.29c). We may therefore assume that reaction (1.29a) is “slaved” to reaction (1.29c), i.e.

\[
\frac{dn_1}{dt} = 0. \] (1.38)

**Figure 1.21.** Photolysis rates as a function of height for reactions (1.29a) and (1.29c). The symbols \( k_1 \) and \( k_3 \) are used here instead of \( j_a \) and \( j_c \), respectively. (Source: Daniel Jacob, 1999: *An Introduction to Atmospheric Chemistry*. Princeton University Press).
Box 1.3. Spectral distribution of radiation: Planck’s law

Matter emits radiation. The energy of this radiation is distributed over a spectrum of waves according to Planck’s law:

\[ B_\nu(T) = \frac{2 h \nu^3}{c^2} \frac{1}{\exp\left(\frac{h \nu}{kT}\right) - 1} \quad [\text{J s}^{-1} \text{m}^{-2} \text{sr}^{-1} \text{Hz}^{-1}] \]  

(1)

Here \( B_\nu(T) \) is the energy per unit time per unit surface area per unit solid angle per unit frequency. In the above equation \( \hbar \) is Planck’s constant \((=6.63 \times 10^{-34} \text{ J s})\), \( \nu \) is the frequency, \( c \) is the phase speed, \( k \) is Boltzmann’s constant \((=1.38 \times 10^{-23} \text{ J K}^{-1})\) and \( T \) is the temperature of the object that is emitting the radiation. The energy emitted by a black body can also be expressed in terms of wavelength instead of frequency. We then have the following equation.

\[ B_\lambda(T) = \frac{2 \pi h c^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1} \quad [\text{J s}^{-1} \text{m}^{-2} \text{sr}^{-1} \text{m}^{-1}] \]  

(2)

This radiant energy is emitted in all directions. We would like to have an expression for the component of the radiant energy emitted by a flat surface perpendicular to the surface. This expression is found by integrating the radiant energy component perpendicular to the emitting surface over a full hemisphere. This yields

\[ B_\lambda(T) = \frac{2 \pi h c^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1} \quad [\text{J s}^{-1} \text{m}^{-2} \text{m}^{-1}] \]  

(3)

An expression for the total radiant intensity emitted by black body in a particular direction is found by integrating (3) over all wavelengths, i.e.,

\[ B(T) = \int_0^\infty B_\lambda(T) d\lambda. \]  

(4)

This yields the Stefan-Boltzmann law:

\[ B(T) = \sigma T^4. \]  

(5)

Here, \( \sigma \) is the Stefan-Boltzmann constant \((=5.67 \cdot 10^{-8} \text{ W m}^2 \text{K}^{-4})\). The radiation emitted by a non-black body is usually approximated by

\[ B(T) = \varepsilon(\lambda) \sigma T^4. \]  

(5)
Here, $\varepsilon(\lambda) (\leq 1)$ is the wavelength dependent emission coefficient. Simplified theories of emission of radiation assume that $\varepsilon$ is wavelength-independence.

**Reference to BOX 1.3**


**Figure 1.22.** Time average (based on measurements made between 1980 and 1991), global average ozone molecule number concentration ($n_3$) and ozone mixing ratio ($n_3/n$) ($n$ is the total molecule number concentration of air). Source: J.P.F Fortuin and H. Kelder, 1998: An ozone climatology based on ozonesonde and satellite measurements. *J. Geoph. Res.*, 103, 31709-31734.

Therefore, from (1.32) we obtain,

$$n_1 = \frac{2j_an_2 + j_en_3}{k_dn_3 + k_bn_2n}.$$  

(1.39)

Substituting this into (1.33) gives

$$\frac{dn_3}{dt} = 2j_an_2 - \frac{2n_3(2j_an_2 + j_en_3)}{n_3 + n_2n(k_b/k_d)}.$$  

(1.40)

The first term on the r.h.s. of (1.40) represents the ozone-formation rate, while the second term on the r.h.s. of (1.40) represents the ozone-destruction rate. If these two processes are in equilibrium we have

$$\frac{dn_3}{dt} = 0.$$  

(1.41)
From this we obtain the following expression for the steady state value \([n_3]_0\) of the ozone concentration:

\[
[n_3]_0 = \frac{j_a n_2}{2 j_c} \left\{ -1 + \left( 1 + \frac{4 j_c k_b n}{j_a k_d} \right)^{1/2} \right\} \text{[molecules per m}^3]\ (1.42)
\]

The values of the reaction coefficients \(k_b\) and \(k_d\) can be found in the NASA-publication given in a footnote earlier in this section. We find that

\[
k_b = 6 \times 10^{-46} \text{ m}^6 \text{s}^{-1} \text{ and } k_d = 8 \times 10^{-18} \exp\left( \frac{-2060}{T} \right) \text{ m}^3 \text{s}^{-1}. \ (1.43)
\]

With (1.6) and (1.31) and the estimates of the values of the photolysis coefficients in (1.36) and (1.37), we obtain a rough estimate of the steady state ozone concentration as a function of pressure and temperature. For instance, at a height of 20 km, where \(p=50 \text{ hPa}, T=217 \text{ K} \) (figure 1.19), \(j_a=10^{-13} \text{ s}^{-1}, j_c=5 \times 10^{-4} \text{ s}^{-1}\) and \(k_b\) and \(k_d\) given by (1.43) we obtain an ozone molecule concentration, \(n_3\), of about \(20 \times 10^{18}\) molecules m\(^{-3}\). According to the observations (figure 1.22) the global average ozone concentration at 50 hPa (20 km) is, approximately, \(3.5 \times 10^{18}\) molecules m\(^{-3}\). The value found from (1.42) and (1.43), incidentally, is sensitive to the temperature.
The Chapman theory systematically over-estimates the ozone concentrations, but nevertheless, qualitatively it does very well in explaining why there is a maximum in the ozone concentration at a certain height in the stratosphere. Basically, this is due to the fact that there are very few oxygen molecules at very high levels to form enough oxygen atoms, while there is insufficient ultraviolet radiation at low levels to form ozone from oxygen atoms. The ozone concentration in the stratosphere does not only depend on the reactions (1.29a-d). There are other important reactions that promote the destruction of ozone. Also, transport of ozone by the circulation is an important factor, which explains the fact that most ozone is found over the poles (figure 1.23), while most ozone is produced in the tropics.

An analysis of the stability of the chemical equilibrium (1.42) is of great interest, because it provides us with a criterion for the existence of the equilibrium as well as a timescale of adjustment to this equilibrium. Suppose we perturb the state of chemical equilibrium (1.42) slightly. Mathematically this can be expressed as follows:

\[ n_3 = \left[n_3\right]_0 + n_3', \]  
\[ \text{(1.44)} \]

where \( n_3' \) is the perturbation ozone molecule number concentration. For convenience, equation (1.40) is written as

\[ \frac{dn_3}{dt} = an_3^2 + bn_3 + c, \]  
\[ \text{(1.45)} \]

where, because

\[ n_3 \ll n_2 n \frac{k_b}{k_d}, \]

\[ a = -\frac{2jckd}{n_2nk_b}; \quad b = -\frac{4ja^2k_b}{nk_b}; \quad c = 2j_an_2. \]  
\[ \text{(1.46)} \]

Substituting (1.44) into (1.45) we obtain

\[ \frac{dn_3'}{dt} = a(n_3')^2 + 2an_3'[n_3]_0 + bn_3'. \]  
\[ \text{(1.47)} \]

If \( n_3' \ll [n_3]_0 \) (i.e. the perturbation is small), we are left with the following simple equation governing the time-evolution of the perturbation.

\[ \frac{dn_3'}{dt} = (2a[n_3]_0 + b)n_3'. \]  
\[ \text{(1.48)} \]

The solution of this equation is of the form

\[ n_3' \approx \exp(\lambda t), \]  
\[ \text{(1.49)} \]

with
\[ \lambda = \left( 2a[n_3]_0 + b \right) < 0, \] (1.50)

implying that the equilibrium is a stable equilibrium. The time scale associated with (re)adjustment to photochemical equilibrium is

\[ \tau = \frac{-1}{2a[n_3]_0 + b} \approx \frac{n_2 nk_b}{4k_d \left( j_a n_2 + j_c \right) [n_3]_0}. \] (1.51)

We see that, since \( n \) decreases with height while \( j_a \) and \( j_c \) increase with height, \( \tau \) is smaller (adjustment is faster) at upper levels than at lower levels.

The estimates of \( j_a \) and \( j_c \), given in, respectively (1.37) and (1.36) are most appropriate for the upper atmosphere, say at 70 km, because the insolation intensity, assumed to obtain these estimates, is representative for the top of the atmosphere. Using the average values of \( p=0.05 \) hPa and \( T=220 \) K for \( z=70 \) km, (1.51) gives

\[ \tau(70 \text{ km}) \approx 1 \text{ hour}. \] (1.52a)

The coefficients \( j_a \) and \( j_c \) decrease very quickly with decreasing height, because Solar ultraviolet radiation with wavelengths smaller than 300 nm is depleted nearly completely above the tropopause. Therefore, the adjustment time of the ozone layer below 70 km height is significantly greater than 1 day. In fact in the lower stratosphere the adjustment time is in order of several months to one year! For instance, at a height of 20 km, where \( p=55 \) hPa, \( T=217 \) K, \( j_a=10^{-13} \text{ s}^{-1} \), \( j_c=5\times10^{-4} \text{ s}^{-1} \) and \( k_b \) and \( k_d \) given by (1.43) we get

\[ \tau(20 \text{ km}) \approx 210 \text{ days}. \] (1.52b)

Because transport by the circulation can alter the ozone distribution significantly on a time scale of several weeks, we expect that the ozone-distribution in the lower stratosphere is not in photochemical equilibrium. This explains a remarkable feature seen in figure 1.23, namely that the ozone molecule number concentration is higher over the spring pole than over the equator, despite the fact that, due to lack of insolation, ozone has hardly been produced (or destroyed) over the spring pole during the previous months. The stratospheric meridional circulation that brings ozone rich air from the tropics to the winter pole is called the Brewer-Dobson circulation (figure 1.66). The reason for the existence of this circulation is explained in section 1.29 (figure 1.65) and chapters 11 and 12.

A more precise impression of the ozone concentration as a function of height, according to the Chapman theory, is obtained when equation (1.40) is coupled to a radiation model such as the one that is described in section (2.7). The radiation model provides the temperature and radiation-fluxes that are needed to calculate the photolysis coefficients.

**PROBLEM 1.5. Ozone concentration profile according to Chapman’s theory**

Write a computer program to evaluate the ozone molecule number concentration as a function of height in an isothermal atmosphere (section 1.14), assuming that the temperature is 300 K or 200 K and in the US-1976 Standard Atmosphere (table 1.1). In all these three cases the surface pressure is 1013 hPa. Plot the result as a function of pressure. Compare the result with figure 1.22.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>231.29</td>
<td>231.58</td>
</tr>
<tr>
<td>30</td>
<td>218.25</td>
<td>222.74</td>
</tr>
<tr>
<td>50</td>
<td>213.67</td>
<td>218.09</td>
</tr>
<tr>
<td>70</td>
<td>210.23</td>
<td>217.00</td>
</tr>
<tr>
<td>90</td>
<td>207.88</td>
<td>217.00</td>
</tr>
<tr>
<td>110</td>
<td>207.25</td>
<td>217.00</td>
</tr>
<tr>
<td>130</td>
<td>208.98</td>
<td>217.42</td>
</tr>
<tr>
<td>150</td>
<td>211.57</td>
<td>218.25</td>
</tr>
<tr>
<td>170</td>
<td>214.63</td>
<td>219.08</td>
</tr>
<tr>
<td>190</td>
<td>217.78</td>
<td>219.92</td>
</tr>
<tr>
<td>210</td>
<td>220.97</td>
<td>220.75</td>
</tr>
<tr>
<td>230</td>
<td>224.11</td>
<td>221.58</td>
</tr>
<tr>
<td>250</td>
<td>227.07</td>
<td>222.42</td>
</tr>
<tr>
<td>270</td>
<td>230.04</td>
<td>223.98</td>
</tr>
<tr>
<td>290</td>
<td>233.00</td>
<td>227.23</td>
</tr>
<tr>
<td>310</td>
<td>235.82</td>
<td>230.37</td>
</tr>
<tr>
<td>330</td>
<td>238.65</td>
<td>232.82</td>
</tr>
<tr>
<td>350</td>
<td>241.47</td>
<td>235.27</td>
</tr>
<tr>
<td>370</td>
<td>244.30</td>
<td>237.78</td>
</tr>
<tr>
<td>390</td>
<td>246.71</td>
<td>240.33</td>
</tr>
<tr>
<td>410</td>
<td>249.06</td>
<td>242.87</td>
</tr>
<tr>
<td>430</td>
<td>251.41</td>
<td>244.87</td>
</tr>
<tr>
<td>450</td>
<td>253.76</td>
<td>246.84</td>
</tr>
<tr>
<td>470</td>
<td>256.11</td>
<td>248.80</td>
</tr>
<tr>
<td>490</td>
<td>258.10</td>
<td>250.85</td>
</tr>
<tr>
<td>510</td>
<td>259.83</td>
<td>252.91</td>
</tr>
<tr>
<td>530</td>
<td>261.56</td>
<td>254.97</td>
</tr>
<tr>
<td>550</td>
<td>263.29</td>
<td>256.79</td>
</tr>
<tr>
<td>570</td>
<td>265.02</td>
<td>258.37</td>
</tr>
<tr>
<td>590</td>
<td>266.75</td>
<td>259.95</td>
</tr>
<tr>
<td>610</td>
<td>268.48</td>
<td>261.53</td>
</tr>
<tr>
<td>630</td>
<td>269.82</td>
<td>263.15</td>
</tr>
<tr>
<td>650</td>
<td>271.04</td>
<td>264.80</td>
</tr>
<tr>
<td>670</td>
<td>272.27</td>
<td>266.45</td>
</tr>
<tr>
<td>690</td>
<td>273.49</td>
<td>268.09</td>
</tr>
<tr>
<td>710</td>
<td>274.72</td>
<td>269.57</td>
</tr>
<tr>
<td>730</td>
<td>275.94</td>
<td>270.85</td>
</tr>
<tr>
<td>750</td>
<td>277.17</td>
<td>272.13</td>
</tr>
<tr>
<td>770</td>
<td>278.39</td>
<td>273.40</td>
</tr>
<tr>
<td>790</td>
<td>279.61</td>
<td>274.68</td>
</tr>
<tr>
<td>810</td>
<td>280.63</td>
<td>276.01</td>
</tr>
<tr>
<td>830</td>
<td>281.65</td>
<td>277.36</td>
</tr>
<tr>
<td>850</td>
<td>282.68</td>
<td>278.70</td>
</tr>
<tr>
<td>870</td>
<td>283.70</td>
<td>280.05</td>
</tr>
<tr>
<td>890</td>
<td>284.72</td>
<td>281.39</td>
</tr>
<tr>
<td>910</td>
<td>285.74</td>
<td>282.58</td>
</tr>
<tr>
<td>930</td>
<td>286.76</td>
<td>283.63</td>
</tr>
<tr>
<td>950</td>
<td>287.79</td>
<td>284.68</td>
</tr>
<tr>
<td>970</td>
<td>288.81</td>
<td>285.74</td>
</tr>
<tr>
<td>990</td>
<td>289.83</td>
<td>286.79</td>
</tr>
<tr>
<td>1013</td>
<td>291.01</td>
<td>288.00</td>
</tr>
</tbody>
</table>

**Table 1.1.** The annual average temperature in the northern hemisphere (average for 0°-80°) according to the COSPAR International Reference Atmosphere (Committee on Space Research\(^{26}\)) and the so-called "US-1976 Standard Atmosphere" at specified pressure levels between 1013 (global average sea level pressure) and 10 hPa (about 30 km above sea level).

\(^{26}\) The COSPAR International Reference Atmosphere (CIRA) provides empirical models of atmospheric temperature and densities as recommended by the Committee on Space Research (COSPAR). Since the early 1960’s different editions of CIRA have been published. The CIRA Working Group meets bi-annual during the COSPAR General Assemblies. CIRA-86 is described in *Advances in Space Research*, Volume 8, Numbers 5-6, 1988. See also [http://badc.nerc.ac.uk/data/cira/](http://badc.nerc.ac.uk/data/cira/).
Box 1.4. The origin of oxygen in the atmosphere

About 21% of all molecules in the atmosphere are oxygen molecules. This has not always been the case. The composition of the early atmosphere, more than 4 billion years ago, was determined by the products of emissions of volcanic eruptions, which were more frequent then, than now. These products are nitrogen, water vapour, carbon dioxide and sulfur gases. Water vapour condensed to form the oceans. Carbon dioxide dissolved in the oceans, leaving nitrogen as the dominant gas in the atmosphere. The original source of oxygen in the atmosphere was photolysis of water. Ultraviolet rays split water to form hydrogen and oxygen. The hydrogen gas is light enough to escape Earth’s gravitational field. Most of the oxygen that was left in the atmosphere reacted with iron in the rocks. This process is called weathering. Weathering is also an important mechanism for loss of atmospheric carbon dioxide (rocks are eroded by carbon dioxide dissolved in rain drops). The most important source of oxygen is associated with the growth of plants and algae. In a process called photosynthesis these organisms grow by absorbing light and using carbon dioxide from the atmosphere and water from the soil. The waste product of photosynthesis is oxygen ($O_2$).

![Graph](image.png)

**FIGURE BOX 1.4**: Changes in atmospheric concentration of oxygen and carbon dioxide over the past 600 million years, based on models of Robert Berner. Oxygen levels reached a peak of 35% in the late Carboniferous and early Permian, before falling to 15% in the late Permian. Oxygen levels peaked again in the Cretaceous (K) before falling to present levels in the Tertiary. Carbon dioxide levels fell from 0.5% in the Silurian to around 0.03% at the end of the Carboniferous. Source: Nick Lane, 2002: *Oxygen: the molecule that made the world*. Oxford University Press, 374. pp. (p. 83).

Most of this oxygen is consumed by respiration by organisms, such as bacteria, which consume dead plant material. Because the waste product of respiration is carbon dioxide, the biochemical cycle of oxygen and carbon dioxide is closed. However, about 0.1% of the dead plant material escapes consumption because it ends up in the deep ocean or is buried under sediments. Therefore, the complete re-uptake of oxygen by consumers is prevented and oxygen slowly accumulates in the atmosphere. A 3 billion year mismatch between oxygen
generated by photosynthesis and oxygen consumed by respiration represents the present oxygen content in the atmosphere. During the past 600 million years the atmospheric oxygen content has varied between 13 % and 35 %. The rise to 35% during the carboniferous era, about 300 million years before present, is thought to be associated with the appearance of bark bearing trees, such as the lepidodendron, containing lignin, which is needed to keep the trees upright. Bacteria had difficulty in digesting lignin. Therefore more dead plant material was buried, escaping decomposition by bacteria, and so forming the coal, oil and gas reserves that we are mining and burning now.

1.13 Potential temperature

After this intermezzo on the composition of the atmosphere, we return to the elementary aspects of the fluid dynamics of the atmosphere. We begin by defining the Exner function\(^{(27)}\), \(\Pi\), and the potential temperature, \(\theta\), as substitutes of the pressure, \(p\), and the density, \(\rho\), as variables in the equations. The Exner function is defined as,

\[
\Pi = c_p \left( \frac{p}{p_{\text{ref}}} \right)^\kappa ,
\]

(1.53)

where \(p_{\text{ref}}\) is a reference pressure (if not stated otherwise \(p_{\text{ref}}=1000\) hPa = 10\(^5\) Pa) and \(\kappa\) is the ratio \(R/c_p = (c_p-c_v)/c_p\). The factor \(c_p\) is sometimes omitted in (1.53). The potential temperature is defined as,

\[
\theta = T \left( \frac{p_{\text{ref}}}{p} \right)^\kappa = \frac{c_p T}{\Pi}.
\]

(1.54)

If the air contains water vapour, a virtual potential temperature, \(\theta_v\), can be defined by substituting \(T_v\) for \(T\) and \(\theta_v\) for \(\theta\) in eq. 1.54. From eqs. 1.7c, 1.1 and 1.54 it can be deduced that

\[
\frac{d\theta}{dt} = \frac{J}{\Pi}.
\]

(1.55)

In the absence of heat sources (\(J=0\)), this equation reduces to the statement that potential temperature is conserved by fluid elements. This is a fundamental constraint on the dry adiabatic motion in the atmosphere.

Eq. 1.7a, in terms of \(\vec{v}, \theta\) and \(\Pi\), becomes

\[
\frac{d\vec{v}}{dt} = -\theta \vec{\nabla} \Pi - g \hat{k} - 2\vec{\Omega} \times \vec{v} + \vec{F}_r.
\]

(1.56)

We can eliminate the density from the continuity equation (1.7b) with the help of (1.1), (1.53) and (1.54) to obtain the new form of the continuity equation (problem 1.6),

---

\[
\frac{d}{dt} \left( \ln \theta + \left( 1 - \frac{1}{\kappa} \right) \ln \Pi \right) = \nabla \cdot \vec{v} .
\] (1.57)

The system of equations (1.55-1.57) forms a closed set of three prognostic equations with three unknown variables, \( \vec{v} \), \( \theta \) and \( \Pi \). We do not need to supplement this system with a diagnostic equation, as was the case with the original set (1.7a-c).

Equation (1.57) can be written in an alternative form using eq. 1.55:

\[
\begin{align*}
\frac{c_v}{R} \frac{d \Pi}{dt} + \nabla \cdot \vec{v} &= -\frac{J}{\theta \Pi} .
\end{align*}
\] (1.58)

Here we have used

\[
1 - \frac{1}{\kappa} = 1 - \frac{c_p}{R} = -\frac{c_v}{R} .
\] (1.59)

Eq. 1.58 states that the changes in time of the pressure of an air parcel are either due to heating or due to compression or expansion.

**PROBLEM 1.6. Continuity equation in terms of the Exner function**

Show that for an ideal gas (such as the atmosphere) the continuity equation:

\[
\frac{d \rho}{dt} = -\rho \nabla \cdot \vec{v}
\]

can also be written as:

\[
\frac{d}{dt} \left( \ln \theta + \left( 1 - \frac{1}{\kappa} \right) \ln \Pi \right) = \nabla \cdot \vec{v} .
\]

**1.14 Hydrostatic balance and thermal stratification**

The force of gravity, acting downwards, is nearly always approximately in balance with the vertical component of the pressure gradient force. This state is referred to as hydrostatic balance. A mathematical expression for this balanced state naturally follows from the vertical component of (1.56):²⁸

\[
\frac{\partial \Pi}{\partial z} = -\frac{g}{\theta} .
\] (1.60)

In this balanced state the pressure decreases with increasing height. An explicit expression for the pressure as a function of height within an isothermal atmosphere can be derived from (1.60) using (1.54) to give

²⁸ The vertical component of the Coriolis force can be neglected compared to the term involving acceleration due to gravity.
\[ \Pi = \Pi_s \exp \left( - \frac{gz}{c_p T} \right) \text{ or } p = p_s \exp \left( - \frac{gz}{RT} \right), \]  

(1.61)

where \( \Pi_s \) and \( p_s \) are, respectively, the Exner function and the pressure at \( z=0 \). The so-called scale height characterizing the exponential decrease of the Exner function in an isothermal atmosphere is

\[ H_{\Pi} = \frac{c_p T}{g}. \]  

(1.62)

The scale height characterizing the exponential decrease of the pressure (and density) in an isothermal atmosphere is

\[ H_p = \frac{RT}{g}. \]  

(1.63)

With \( c_p=1005 \text{ J kg}^{-1} \text{ K}^{-1}, R=287 \text{ J kg}^{-1} \text{ K}^{-1}, g=9.8 \text{ m s}^{-2} \) and \( T=300 \text{ K} \) we have \( H_{\Pi}=30 \text{ km} \) and \( H_p=9 \text{ km} \).

An air parcel, which ascends in an environment in hydrostatic balance, will expand and do work on the environment at the cost of its internal energy, which implies a temperature decrease with increasing height. This temperature decrease can be computed from law of energy conservation (1.7c) and the equation state (1.1). Assuming dry adiabatic conditions \((J=0)\) we find that

\[ \frac{dT}{dp} = \frac{\alpha}{c_p}. \]  

(1.64)

The temperature decrease apparently depends only on the pressure decrease. If the pressure within the air parcel adjusts instantaneously to the hydrostatic pressure in the environment we have

\[ \frac{dp}{dz} = -\rho g \text{ or } c_d p = -g dz, \]  

(1.65)

which is the common way of expressing hydrostatic balance (Box 1.1).

Combining (1.64) and (1.65) yields

\[ \frac{dT}{dz} = -\frac{g}{c_p}. \]  

(1.66)

This constant temperature gradient is referred to as the dry adiabatic lapse rate. With (1.54) and (1.65) it can easily be demonstrated that the dry adiabatic lapse rate is equivalent to \( \frac{d\theta}{dz}=0 \). Sometimes the heating rate, \( J \), is expressed in terms of a change in entropy, \( dS \), such that \( J dt = T dS \). The change of the entropy of an air parcel is (eq. 1.55), \( dS = c_v d(\ln \theta) \). Therefore, the dry adiabatic lapse rate is also referred to as the isentropic lapse rate. A detailed discussion of entropy is found in chapters 2 and 3 of Ambaum (2010), (see the list of books at the end of this chapter).
FIGURE 1.24: Vertical profile of the thermal equilibrium temperature under permanent equinox conditions, assuming an effective mean zenith angle of 60°, according to calculations by S. Manabe and R. Strickler (*Thermal equilibrium of the atmosphere with convective adjustment*, *J. Atmos. Sci.*, 21 (1964), p. 371) for the case of a pure H₂O, H₂O+CO₂, and H₂O+CO₂+O₃ atmosphere. Note that the relatively warm stratosphere is caused by absorption of Solar radiation by ozone (figure 1.2). However, the characteristic stable thermal stratification of the stratosphere itself does not require ozone as an explanation. The calculated temperature profile, including the effect of ozone, is very similar to the US-standard atmosphere (second column in table 1.1), but clearly different to the northern hemisphere average profile (first column in table 1.1). The northern hemisphere average profile is biased towards the tropics and is thus characterized by a sharp temperature minimum at about 100 hPa, “the cold point tropopause” (section 1.15). For more details, see chapter 2.

PROBLEM 1.7. Reducing pressure to sea-level
A weather station, situated 200 m above sea level, observes a pressure of 985 hPa and a temperature of 15°C.
(a) The relation, \( \frac{\partial p}{\partial z} = -\frac{pg}{RT} \), is usually employed to estimate the sea level pressure. On which assumptions is this relation based?
(b) Assuming a constant dry adiabatic lapse rate of about 10 K km\(^{-1}\) in the hypothetical atmosphere between the Earth’s surface and sea level, estimate the sea level pressure, using this relation.
(c) Would the estimate be significantly different if this hypothetical atmosphere were assumed to be isothermal?

An atmosphere, which is completely mixed is said to be in *convective equilibrium*. Such an atmosphere is characterized by equality of entropy throughout, implying an "isentropic" lapse rate and, therefore a finite vertical extent with height

\[
H = \frac{c_p T_0}{g},
\]  

(1.67)

where \( T_0 \) is the temperature at the Earth's surface. With \( T_0 = 300 \) K we find that \( H = 30 \) km. At
this height the temperature and pressure would be equal to zero. However, we know that the atmosphere does not end at \( z \approx 30 \) km. In fact, the average temperature at \( z \approx 30 \) km is approximately 220 K (figure 1.19). Indeed, one of the most conspicuous properties of the basic (time-averaged) state of the atmosphere is the stratification of the atmosphere in the vertical direction into layers possessing radically different thermal properties. In 1926 the famous English meteorologist, Sir Napier Shaw, wrote about the discovery of these thermal properties of the atmosphere in the following terms:

The use of these balloons [ballon-sondes] has resulted in the most surprising discovery in the whole history of meteorology. Contrary to all expectation the thermal structure of the atmosphere in the upper regions is found to be such that isothermal surfaces are vertical surfaces succeeding one another with diminishing temperature from the pole outwards towards the equator, while underneath this remarkable structure, which shows the lowest temperatures of the atmosphere to be at very high levels over the equator, lies the structure with which we are ordinarily familiar consisting of approximately horizontal layers warmer in the lower latitudes and colder in the higher and, with some interesting exceptions, diminishing in temperature with height until the upper layer of vertical columns is suddenly reached. The lower region of the atmosphere in which temperature is arranged in horizontal layers was called by Teisserenc de Bort the troposphere and the upper region of the so-called "isothermal columns" the stratosphere. The name tropopause has been coined to indicate the level at which the troposphere terminates.

(Shaw, 1926, page 225-226)\(^29\). Somewhere else Shaw states that

The end of the nineteenth century is a suitable epoch in the history of dynamical meteorology because the vital division of the atmosphere into troposphere and stratosphere dates from about that time. Before that it may be said to have been usual for the purpose of theory to regard the atmosphere either as isothermal or as isentropic. The systematic investigation of the upper air has put an end to that kind of simplification.

The stratosphere (the isothermal layer) was discovered around 1900 by Léon Teisserenc de Bort and Richard Assman\(^30\). The striking layering of the atmosphere is due principally to the interaction of radiation with ozone, water vapour and clouds. Because the distribution of these constituents is strongly dependent on the flow patterns in the atmosphere, the explanation of why there is such a sharply defined tropopause and why it is found at a height of about 10 km in midlatitudes and at a height of about 15 km in the tropics involves both radiation transfer theory and the theory of the geophysical fluid dynamics of the atmosphere.

The upper limit of the stratosphere is called the stratopause. Here, at about 50 km above sea level, the temperature reaches a maximum (about 0°C) due to absorption of solar radiation by ozone, the importance of which is illustrated in figure 1.24, which shows temperature profiles calculated with a radiation model of the atmosphere, with and without ozone.

At the stratopause the pressure is about 1 hPa, implying that about 99.9% of the total mass of the atmosphere is located below this level. Events in the atmospheric layers above this level (i.e. in the mesosphere and the thermosphere) will therefore have little immediate effect on the structure and motion of the troposphere and, thus, on the dynamics of weather

---


systems which are observed principally in the troposphere. The troposphere contains about 75% of the total mass of the atmosphere. At the tropopause the density is about 30% of the density at the earth's surface.

1.15 Static stability

This section studies the stability of hydrostatic balance. We continue the argument of section 1.3. Formulating the r.h.s. of eq. 1.2 in terms of potential temperature, we have

$$\frac{d^2 z}{dt^2} = g \frac{(T_i - T_0)}{T_0} = g \frac{(\theta_i - \theta_0)}{\theta_0}. \quad (1.68)$$

According to this equation, the vertical acceleration of an air parcel is determined by the buoyancy force. Suppose that this air parcel has a potential temperature, $\theta_i=\theta^*$, identical to the potential temperature of the air in the “environment” of this air parcel at the same height. In adiabatic conditions (no heat sources or mixing with the environment) the potential temperature of the air parcel will remain constant. Suppose now that, for some unspecified reason, the parcel is brought to a different height, a distance $\delta z$ above or below the original height. If the environment is stratified, the potential temperature in the environment ($\theta_0$) at the new height is different from the potential temperature of the air parcel ($\theta^*$). Therefore, a buoyancy force is exerted on the air parcel, which is a manifestation of the pressure gradient force, which, we assume, is not influenced by the displacement of the air parcel.

We approximate the vertical profile of the environmental potential temperature profile with the following formula (a truncated Taylor series approximation).

$$\theta_0 = \theta^* + \frac{d\theta_0}{dz} \delta z. \quad (1.69)$$

Eq. 1.68 now becomes

$$\frac{d^2 \delta z}{dt^2} = -g \frac{d\theta_0}{dz} \delta z = -g \frac{\theta^*}{\theta^*} \frac{d\theta_0}{dz} \delta z. \quad (1.70)$$

This equation has a solution of the form

$$\delta z = \text{Re}\left[A \exp(\pm iNt)\right], \quad (1.71)$$

where Re[…] stands for “the real part of […]”. $A$ is an imaginary amplitude and

$$N = \sqrt{\frac{g}{\theta^*}} \frac{d\theta_0}{dz} \quad (1.72)$$

is the so-called Brunt-Väisälä frequency. If $d\theta_0/dz > 0$, $N$ is a real number and the solution (1.71) represents a so-called buoyancy-oscillation with a frequency equal to $N$. With $\theta^*=300$ K and $d\theta_0/dz=4$ K km$^{-1}$ (a typical tropospheric value), we have $N=1.14 \times 10^{-2}$ s$^{-1}$. The
associated period is $2\pi/N = 551$ s (about 10 min). If $d\theta_0/dz < 0$, $N$ is an imaginary number and the solution (1.71) represents an exponential function of time. The air parcel does not return to its original balanced position, implying that the state of hydrostatic balance is unstable to perturbations. This will lead to vertical motion of air parcels, a process which is referred to as convection. Usually, this situation arises only near the surface of the Earth, due to heating of the surface layer of the atmosphere by molecular and turbulent transfer of “sensible heat” from the Earth’s surface, which has been heated due to absorption of radiation. The atmosphere above the surface layer is nearly always stably stratified, i.e. $d\theta_0/dz > 0$. This is mainly due to the influence of radiation. In fact, the radiatively determined state of the atmosphere is such that potential temperature increases with height and from the poles to the equator. This led Sir Napier Shaw to construct a simplified picture of the potential temperature distribution in which isentropic surfaces ("isentropes", i.e. iso-surfaces of potential temperature) form “caps” over the pole (figure 1.25). The cylindrical form, which passes tangentially from the Earth’s surface at the equator, separates the atmosphere into an "Overworld" and an "Underworld". The isentropes within the Underworld cut the Earth in rings of latitude (figure 1.26). If conditions are adiabatic, i.e. heat is not added to or extracted from air parcels, air below specific isentropes within the Underworld is trapped by the Earth’s surface. An isolated reservoir of potentially very cold air exists over the poles in winter.

**FIGURE 1.25**: Nineteenth century (after Helmholtz, Brillouin and Shaw) depiction of isentropes in the atmosphere encircling the Earth. Only one hemisphere is shown. PN indicates North Pole. The 300 K isentrope usually grazes the Earth’s surface in the tropics.

In the extratropics, poleward of 20-30° latitude, the isentropes between 320 K and 375 K are squeezed together vertically (figure 1.26). This is a manifestation of the extratropical tropopause. The upward bulging isentropes between 350 K and 400 K in the tropics are a manifestation of the very low temperatures in this region (figure 1.1). The tropical tropopause, equatorward of 20° latitude, coincides by definition with the temperature minimum. It is referred to as the “Cold Point Tropopause (CPT)”. The tropical cold layer used to be one of the outstanding paradoxes of meteorology until the 1960’s, and is still
subject of very active research. The tropical cold layer is referred to as the “tropical transition layer (TTL)”. Air generally ascends through this layer, whereby it loses most of its moisture, i.e. air is “freeze-dried” as it enters the tropical stratosphere. The reasons for the ascent of air through the TTL and the embedded “Cold Point Tropopause (CPT)”, into the tropical stratosphere, are understood much better now than 45 years ago. They are set forth in section 1.29 (figures 1.65 and 1.66) and in chapters 11 and 12.

![Diagram of zonal mean and monthly mean potential temperature in the “COSPAR International reference atmosphere” as a function of latitude and pressure. Labels are in K. The isentropes in the “Underworld” are blue; the isentropes in the “Overworld” are red. Cyan isentropes belong to the “Middleworld” (defined in section 1.26). The isolated mass of very cold air over the north pole grows between September and December. Source: Fleming, E. L., Chandra, S., Barnett, J. J. and Corney, M. 1990. Zonal Mean Temperature, Pressure, Zonal Wind, and Geopotential Height as Functions of Latitude. Advances in Space Research, 10, No. 12, 11-59.]

**FIGURE 1.26**: The zonal mean and monthly mean potential temperature in the “COSPAR International reference atmosphere” as a function of latitude and pressure. Labels are in K. The isentropes in the “Underworld” are blue; the isentropes in the “Overworld” are red. Cyan isentropes belong to the “Middleworld” (defined in section 1.26). The isolated mass of very cold air over the north pole grows between September and December. Source: Fleming, E. L., Chandra, S., Barnett, J. J. and Corney, M. 1990. Zonal Mean Temperature, Pressure, Zonal Wind, and Geopotential Height as Functions of Latitude. *Advances in Space Research*, 10, No. 12, 11-59.

**PROBLEM 1.8 Parcel model of a buoyancy oscillation in a stratified environment**

Construct a numerical model, which calculates the vertical position and vertical velocity of a dry air-parcel, which is initially at rest just above the earth’s surface. Initially this air parcel is warmer than its environment. The potential temperature, $\theta$, of the air parcel is given by

$$\theta(z,t) = \theta_0(z) + \theta'(z,t),$$

where $\theta_0(z)$ is the potential temperature of the environment of this air parcel, which is a function of height, $z$.

The equations governing the vertical position above the earth’s surface, $z$, the vertical velocity, $w$, and the potential temperature, $\theta$, are

$$\frac{dz}{dt} = w, \quad \frac{dw}{dt} = g\frac{\theta'}{\theta_0}, \quad \frac{d\theta}{dt} = 0.$$

Note that potential temperature is conserved.

Define an air parcel with a potential temperature, $\theta = \theta^*$. Assume that this air parcel is at the surface of the earth, and that $\theta^* > \theta_0$ at this level. Write a program that computes the time evolution of the height and vertical velocity of this air parcel, as it moves through the environment while it neither mixes with this environment, nor disturbs the environment. The potential temperature of the environment of the air parcel is computed from

$$\theta_0(z) = \theta_0(0) + \Gamma z.$$

In other words, in this example $\theta_0$ is a simple analytical function of height. For simplicity, $\Gamma$ is assumed constant (e.g. 0.005 K/m). If the air parcel is 1°C warmer than its environment initially, it will accelerate upwards. The program computes the height of the air parcel at equally spaced points in time. Assume that $\theta(0) = 300$ K.

On the basis of analytical theory (this section and section 1.17), the following questions can be answered (approximately). What maximum upward velocity will it attain? At which height will it reach this velocity? What maximum height will it attain? Eventually the air parcel will move up and down with a constant frequency. Compute this frequency. This information will be a guide to your choice of the grid distance and time step.

Numerical approximation of differential equations is not an exact science. Finding the best numerical scheme is a matter of “educated trial and error”. These means that you should check the consistency of the analytical theory with your numerical results. The most obvious and, at first sight, best numerical scheme does not necessarily give the best results (tip: the semi-implicit Euler method is a simple and accurate numerical method). Use Python with numpy and matplotlib (see “A Hands-On Introduction to Using Python in the Atmospheric and Oceanic Sciences” by Johnny Lin (http://www.johnny-lin.com/pyintro/)). Describe the numerical method that you used to obtain these numerical results. Repeat the exercise with the following lapse rate of the environmental potential temperature distribution.

$$\Gamma = 0 \text{ if } z < 2 \text{ km}; \quad \Gamma = 40 \text{ K/km if } 2 \leq z \leq 2.25 \text{ km}; \quad \Gamma = 5 \text{ K/km if } z > 2.25 \text{ km}.$$

The environment is neutrally stratified up to 2 km above the earth’s surface. Between 2 and 2.25 km above the surface there is a very stable "temperature-inversion". Above this inversion potential temperature increases by 5 K per km.

1.16 Potential instability and equivalent potential temperature

In the previous section an analysis is given of the vertical acceleration of a dry air parcel in a stratified static atmosphere. In reality air parcels contain water vapour. Water has a
tremendous influence on buoyancy. First, water vapour influences the density and therefore buoyancy of an air parcel. Furthermore, according to the Clausius-Clapeyron equation (1.11) an air parcel will become saturated with water vapour at a certain point during its ascent. Further ascent will lead to condensation of water vapour, whereby latent heat is released. This has a profound effect on the buoyancy and further acceleration of the air parcel. The importance of the discovery of this effect in the 18th century by Jean André Deluc and Joseph Black cannot be overemphasized.32

Let us consider a saturated atmosphere in which there is rising and falling motion. Condensation and associated latent heat release will only take place in the updraught. It will be shown in the following that the latent heat released in the updraught is quite closely proportional to the upward velocity, \( w \). If we let \( r_s \) denote the mass of water vapour per unit mass of dry air in a saturated air parcel (\( r_s \) is called the saturation mixing ratio), then the rate of heating per unit mass due to condensation, assuming that the heat is absorbed by the dry air, is

\[
J = -L \frac{dr_s}{dt}, 
\]

(1.73)

where \( L (=2.5\times10^6 \text{ J kg}^{-1}) \) is the specific latent heat of condensation. Assuming that the change in \( r_s \) following the motion is primarily due to ascent, i.e.

\[
\frac{dr_s}{dt} = \frac{\partial r_s}{\partial t} + u \frac{\partial r_s}{\partial x} + v \frac{\partial r_s}{\partial y} + w \frac{\partial r_s}{\partial z} = w \frac{\partial r_s}{\partial z},
\]

we get

\[
\frac{dr_s}{dt} = w \frac{dr_s}{dz} \quad \text{for} \quad w > 0;
\]

\[
\frac{dr_s}{dt} \approx 0 \quad \text{for} \quad w \leq 0.
\]

Assuming that the potential temperature of the air parcel is

\[
\theta = \theta_0(z) + \theta',
\]

the potential temperature equation (1.55) can be written as

\[
\frac{d\theta}{dt} = \frac{d\theta'}{dt} + w \frac{d\theta_0}{dz} = \frac{J}{\Pi}.
\]

We now have \( J=0 \) if \( w \leq 0 \) and

\[
J = -L \frac{dr_s}{dt} \quad \text{if} \quad w > 0.
\]

With (1.72) we may express the potential temperature equation as follows.

---

where $N$ is the Brunt Väisälä frequency (see eq. 1.72) and, by analogy, $N_m$ is the "moist" Brunt Väisälä frequency, defined as

$$N_m^2 = N^2 + \frac{gL}{\theta^* \Pi_0} \frac{dr_s}{dz}, \quad (1.74)$$

where we have assumed that $\Pi = \Pi_0(z) + \Pi' = \Pi_0(z)$. There are more accurate definitions of $N_m$ available in the literature\(^{33}\), but the present definition is sufficiently accurate to illustrate the principle influence of moisture on atmospheric circulations.

Because $dr_s/dz<0$, $N_m^2$ may easily become negative, even if $N^2$ is positive but not too large. In these circumstances the atmosphere is statically or buoyantly unstable only with respect to saturated upward motion. This is called conditional instability. The word "conditional" refers to the condition that the air must be saturated and ascending.

By analogy with dry adiabatic motion, where potential temperature is constant following the motion, we may define a pseudo- or moist adiabatic process in which a so-called equivalent potential temperature, $\theta_e$, is constant. That is, $\theta_e$ is constant following saturated ascent. To this end, we simply define,
\[
N_m^2 = \frac{g}{\theta_0} \frac{d\theta_e}{dz},
\]
which implies, using (1.72) and (1.74), that
\[
\frac{d}{dz} (\ln \theta_e) = \frac{d}{dz} (\ln \theta_0) + \frac{L}{\theta_0 \Pi_0} \frac{dr_s}{dz},
\]
so that the relation between equivalent potential temperature and potential temperature is
\[
\theta_e = \theta \exp \left( \frac{L r_s}{\theta \Pi} \right).
\]
This is the definition of the equivalent potential temperature for a saturated air parcel. It should be noted that there are a lot of subtleties involved in the definition and derivation of \(\theta_e\) and \(N_m\), which have been bypassed here for the sake of simplicity. The criterion for static instability of a saturated updraught (independent of the motion in the environment) is
\[
\frac{\partial \theta_e}{\partial z} < 0.
\]
The definition of the equivalent potential temperature for an unsaturated air parcel is
\[
\theta_e = \theta \exp \left( \frac{L r}{\theta \Pi_{LCL}} \right) = \theta \exp \left( \frac{L r}{c_p T_{LCL}} \right),
\]
where \(r\) is the actual mixing ratio of the air parcel, \(\Pi_{LCL}\) and \(T_{LCL}\) are, respectively, the Exner function and the temperature that the parcel would have if expanded adiabatically to saturation. Usually an ascending air-parcel is not saturated until a so-called "lifting condensation level" (LCL) is reached. In reality the atmosphere is absolutely stable everywhere (i.e. the potential temperature increases with height everywhere), except sometimes near the Earth’s surface. Air-parcels rising spontaneously from the Earth’s surface due to the static instability of the thin surface layer, will nevertheless quickly lose their positive buoyancy, except if they reach the LCL before this happens or if some lifting mechanism is operating (chapter 4). If the atmosphere at the LCL is conditionally unstable (i.e. \(\theta_e\) decreases with increasing height) the ascending air parcel will retain its positive buoyancy and continue its ascent. This is also called potential instability. Sometimes latent heat release is not enough to make the buoyancy positive, implying that the ascent is arrested. This is the case when \(\theta_e\) in the environment of the ascending air parcel increases with height. In that case meteorologists usually say that the air parcel has not reached its so-called "level of free convection" (LFC). In other words, it is only when the parcel reaches its LFC that conditional instability is released. Section 1.18 will explain how the LCL and LFC are determined in practice.

When the air parcel reaches the LCL, condensation occurs and a cloud forms. The formation of puffy clouds due to condensation of water vapour in ascending positively buoyant air is called “cumulus convection”. The clouds are called cumulus clouds (figure 1.27).
1.17 Convective available potential energy

Convection currents are the result of an unstable balance of forces in the vertical direction. The deviations from this balance are governed by eq. 1.68, which can be rewritten as

$$\frac{d^2 z}{dt^2} = \frac{dw}{dt} = g \frac{\theta'}{\theta_0}.$$  

Assuming a stationary state, this equation becomes

$$w \frac{dw}{dz} = g \frac{\theta'}{\theta_0} \text{ or } w \frac{dw}{dz} = gB$$  

(1.79)

It can now easily be deduced that a parcel starting its ascent at a level $z_1$ with vertical velocity $w_1$, will have a velocity $w_2$ at a height $z_2$ given by

$$w_2^2 = w_1^2 + 2 \times CAPE,$$  

(1.80)

where the **Convective Available Potential Energy** (CAPE) is defined by

![Figure 1.28](image.png)

**Figure 1.28.** The vertical motion measured from an aircraft in (a) a fair weather cumulus cloud 1.5 km deep, over a track about 250 m below cloud top (from Telford, J.W. and J. Warner, 1962: On the measurement from an aircraft of buoyancy and vertical velocity. *J.Atmos.Sci.*, 19, 415-423), and (b) a “supercell” thunderstorm over a track about 6000 m above the ground (from Musil, D.J., A.J. Heymsfield and P.L. Smith, 1986: Microphysical characteristics of a well developed weak echo region in a high plains supercell thunderstorm. *J.Clim.Appl.Met.*, 25, 1037-1051).
Figure 1.29. Hocker’s photogrammetric analysis of the 1957 Dallas tornado showing (a) tangential and (b) vertical velocities in m s\(^{-1}\) (from Kessler, E., ed. 1985: Thunderstorm Morphology and Dynamics. Second edition. University of Oklahoma Press, Norman, 411 pp.)

\[ CAPE = g \int_{z_1}^{z_2} B dz \]  

(1.81)

In the practice of weather forecasting, CAPE is determined for a hypothetical situation of an air parcel that starts its ascent at the Earth’s surface. CAPE is calculated by taking the integral in (1.81) from the level of free convection (LFC) of this hypothetical air parcel to its equilibrium level (EL), or level of no buoyancy (LNB) (see section 1.18).

The magnitude of CAPE can be as large as 4500 m\(^2\) s\(^{-2}\). A value of 2500 m\(^2\) s\(^{-2}\) would, according to (1.80), translate into a maximum possible updraught of about 70 m s\(^{-1}\), if \(w_1=0\). Such strong updraughts are never observed because of friction and water loading. Water loading refers to the effect on buoyancy of the weight of condensed water in the atmosphere. However, sometimes updraught-intensities in thunderstorms come surprisingly close to this value. While upward velocities in ordinary non-precipitating cumulus clouds are usually of the order of 3 m s\(^{-1}\) (figure 1.28a), the upward velocities observed in “supercell” thunderstorms may reach hurricane force (figure 1.28b)! Note that these velocities are directed in the horizontal in hurricanes.

Updraughts in tornadoes are even more spectacular, especially because of the associated exceptional vertical accelerations. Figure 1.29 represents a rare observation of the distribution of vertical velocity in a tornado. Updraughts of the order of 50 m s\(^{-1}\) are apparently observed at heights of about 100 m above the Earth’s surface. Accelerations associated with these intense updraughts can be deduced from the l.h.s. of (1.79). For the cumulus cloud (figure 1.28a), the thunderstorm cloud (figure 1.28b) and the tornado (figure 1.29) we have, respectively:

\[
\text{cumulus cloud: } \frac{dw}{dt} = 0.01 \text{ m s}^{-2} \ll g; \]

cumulo-nimbus cloud (thunderstorm): \( \frac{dw}{dt} = 0.25 \text{ m s}^{-2} < g \);
tornado: \( \frac{dw}{dt} = 25 \text{ m s}^{-2} > g \).

To get a vertical acceleration of 0.01 m s\(^{-2}\), a rather small potential temperature perturbation, \( \theta' \), of about 0.3 K is required. To get a vertical acceleration of 0.25 m s\(^{-2}\), a much larger potential temperature perturbation, \( \theta' \), of about 7.5 K is required. In tornadoes, however, the accelerations (typically about 25 m s\(^{-2}\)) cannot be due to buoyancy only, because these accelerations would require a potential temperature perturbation, \( \theta' \), of about 750 K (!). Excess potential temperatures in clouds much greater than 10 K are hardly ever observed. Therefore, vertical accelerations in clouds that produce tornadoes must be due to an effect that has been neglected here until now. This has to do with the assumption that the pressure in an air parcel adjusts instantaneously to the hydrostatic pressure in the environment. While this assumption is valid for small cumulus clouds\(^{34}\), it does not hold in the environment of tornadoes. The question of how the pressure gradients are created, that produce the tremendous vertical accelerations of air in tornadoes, is addressed in chapter 4.

PROBLEM 1.9. Downdraughts
The relatively sharp downdraughts at the edge of the cumulus cloud (figure 1.28a) are a very typical feature of cumulus clouds\(^{35}\). What effect is responsible for these downdraughts?

1.18 Thermodynamic diagram

Meteorologists usually plot temperature and water vapour content in a thermodynamic diagram or tephigram, which is a graph with temperature and potential temperature as principal axes. These axes are rotated through about 45° so that lines of equal pressure are almost horizontal. The tephigram displays lines representing isobaric, isothermal, dry adiabatic and pseudo- or moist adiabatic processes, as well as lines of constant saturation mixing ratio, which is a function of pressure and temperature only.

An example of a tephigram is shown in figure 1.30. The isotherms (isopleths of temperature) slope upwards to the right of the diagram at an angle of about 45° with the x-axis and 90° with the isentropes (isopleths of potential temperature or “dry adiabats”). The moist adiabats (isopleths of equivalent potential temperature) are distinctly curved, running more parallel to the dry adiabats at low temperatures. The saturation mixing ratio isopleths are nearly straight and can be used to find the lifting condensation level and the level of free convection of an air parcel if the height, temperature and dew point temperature (Box 1.5) of an air parcel are known. The data plotted on a tephigram are temperature and dew point temperature. The tephigram is very useful in assessing the likelihood of convection, cloud formation and thunderstorm.

---


PROBLEM 1.10. What can we do with a tephigram?
Plot the data shown in table 1.2 in a tephigram. Determine the lifting condensation level (LCL) of an air parcel at the ground from the tephigram. Determine the height of the LCL from the theory described in Box 1.5. Will this air parcel reach the LCL spontaneously? What does your numerical model (problem 1.8) say about this? Once it has reached the LCL, over how large a vertical distance will it rise? Estimate this vertical distance from the tephigram and also by using the theory of Box 1.6. Verify the value of \( \theta_e \) at the surface using eq. 1.78. Estimate the value of CAPE. Repeat this for the data in table 1.3.

**FIGURE 1.30.** Thermodynamic diagram (or “tephigram”). Solid lines (red) sloping upwards to the right are isotherms, labeled in bold numbers in °C. Solid lines (brown) sloping upwards to the left are isentropes or “dry-adiabats”, labeled in bold numbers in °C. Dashed, curved lines (blue) sloping upwards to the left are “pseudo-adiabat”s (isopleths of saturation equivalent potential temperature). Dashed lines (green) sloping upwards to the right are isopleths of saturation mixing ratio. Values are indicated in g/kg at the bottom of the figure. The nearly horizontal dashed lines represent isobars (lines of equal pressure). Values are indicated in units of hPa on the right hand side of the figure. The diagram is due to Ian M. Brooks, University of Leeds, UK. The construction of a tephigram is explained in Ambaum, M.H.P., 2010: *Thermal Physics of the Atmosphere*. Wiley-Blackwell, 239 pp.
Box 1.5. Dew point temperature and lifted condensation level

The *approximate* integrated Clausius Clapeyron equation can be written as follows (see eq. 1.14):

\[
e_s = e_{s0} \exp \left\{ \frac{L_v}{R_v T_0} \left( \frac{T - T_0}{T} \right) \right\}.
\]

(1)

With \(e_{s0}=610.78\) Pa (eq.1.15), \(L_v=2.5 \times 10^6\) J kg\(^{-1}\), \(R_v=461.5\) J kg\(^{-1}\)K\(^{-1}\) and \(T_0=273.16\) K this becomes

\[
e_s = 610.78 \exp \left\{ 19.83 \left( \frac{T[^\circ C]}{T[^\circ C]+273.16} \right) \right\}
\]

(2)

with \(T[^\circ C]\) the temperature in degrees Celcius. For realistically varying \(L_v\) (with temperature) a more accurate empirical formula for the saturated vapour pressure as a function of temperature is *Tetens’s formula*:

\[
e_s = 611.2 \exp \left\{ 17.67 \left( \frac{T[^\circ C]}{T[^\circ C]+243.5} \right) \right\}
\]

(3)

Tetens published this formula in 1930, but slightly different versions of this formula have been published since 1828 (Lawrence, 2005)

Radiosondes carry a relative humidity sensor, which provides measurements of relative humidity: \(RH=e/e_s\) (eq. 1.17). Radiosondes also carry a thermometer, thus yielding enough information to calculate the water vapour pressure, \(e\), from Tetens’s formula and the definition of relative humidity. The dew point temperature is calculated from Tetens’s formula by substituting \(e\) in place of \(e_s\):

\[
T_d[^\circ C] = \frac{243.5 \ln \left( \frac{e}{611.2} \right)}{17.67 - \ln \left( \frac{e}{611.2} \right)}.
\]

(4)

An ascending air parcel that is not saturated cools by about 1°C for every 100 m ascent (eq. 1.66), assuming that it does not mix with the environment. The condensation level is reached when \(T=T_d\). But, because the vapour pressure inside the air parcel changes as it ascends, its dew point temperature also changes as it ascends.

Since the vapour pressure of an unsaturated air parcel is equal, by definition, to the saturated vapour pressure at the dew point temperature of the air parcel we are allowed to write Clusius-Clapeyron equation (1.13) as

\[
\frac{de}{dT_d} = \frac{Le}{R_v T_d^2} .
\]

(5)

With the equation of state (1.1) applied to the air and to the vapour (assuming that \(e<<p\)), the vapour pressure can also be written as
\[ e = \frac{r_v p}{\varepsilon}, \]  
where
\[ \varepsilon = \frac{R_d}{R_v}, \]  
where \( R_d \) is the specific gas constant for dry air (section 1.8). Because \( R_d = 287 \text{ J K}^{-1}\text{kg}^{-1} \) and \( R_v = 461.5 \text{ J K}^{-1}\text{kg}^{-1} \), \( \varepsilon = 0.62 \). The ratio \( e/p \) in an air parcel is constant as long as its mixing ratio is constant, which is the case if its water vapour does not condense. This leads to
\[
\frac{de}{dT} = \frac{r_v}{\varepsilon} \frac{dp}{dT} = \frac{L r_v p}{R_v \varepsilon T_d^2} \text{ or } \frac{dp}{dT} = \frac{L p}{r_v T_d^2}.
\]  
With the hydrostatic equation (1.65) we arrive at an equation for the **dew point lapse rate**:
\[
\Gamma_{\text{dew}} = -\frac{dT_d}{dz} = \frac{g T_d}{L \varepsilon},
\]  
where it has been assumed that \( T = T_d \). Furthermore, \( L = 2.5 \times 10^6 \text{ J kg}^{-1} \) and \( g = 9.81 \text{ m s}^{-2} \). These values yield a dew point lapse rate of 0.0018 K m\(^{-1}\) if \( T_d = 285 \text{ K} \). The dew point lapse rate varies from about 0.0016 K m\(^{-1}\) at \( T_d = 250 \text{ K} \) to 0.0019 K m\(^{-1}\) at \( T_d = 300 \text{ K} \).

With knowledge of the temperature and the dew point temperature at or near the Earth’s surface the meteorologist can estimate the likelihood of cloud formation by performing the following “thought experiment”. An air parcel is lifted adiabatically (section 1.14) from the Earth’s surface to the lifted condensation level, assuming that it does not mix with the environment. At the lifted condensation level \( T = T_d \). The height of the lifted condensation level above the Earth’s surface, \( z_{\text{LCL}} \), is found by solving

![Figure 1 Box 1.5](http://en.wikipedia.org/wiki/Lifted_condensation_level).

**Figure 1 Box 1.5.** How to graphically determine the Lifted Condensation Level (LCL) of an air parcel with a surface temperature, \( T \), and a surface dew point temperature, \( T_d \), that is forced to rise from the surface. An explanation of the moist adiabat is given in **Box 1.6**. Source: [http://en.wikipedia.org/wiki/Lifted_condensation_level](http://en.wikipedia.org/wiki/Lifted_condensation_level).
\[ z_{LCL} = \left( T_s - T_{ds} \right) \left( \frac{dT_d}{dz} - \frac{dT}{dz} \right)^{-1} = \left( T_s - T_{ds} \right) \left( -0.0018 + 0.01 \right)^{-1} = 122 \left( T_s - T_{ds} \right), \tag{10} \]

where \( T_s \) and \( T_{ds} \) are the temperature and the dew point temperature at the Earth’s surface. In other words, the height of the cloud base can be estimated from the surface dew point depression.

Lawrence (2005) has shown that for relative humidities, \( RH > 50 \% \), the dew point depression is related to \( RH \) by the following simple formula:

\[ RH = 100 - 5 \left( T - T_d \right) \left[ \% \right]. \tag{11} \]

This very useful rule of thumb allows a quick conversion from \( RH \) to \( (T - T_d) \) and so to \( z_{LCL} \) for moist air (\( RH > 50\% \)).

**PROBLEM BOX 1.5. Relative humidity and dew point**

At a pressure of 1000 hPa and a temperature of 20°C air has a mixing ratio of 6 g kg\(^{-1}\). Determine the corresponding relative humidity and dewpoint.

**References to BOX 1.5**

Lawrence, M.G., 2005: Relationship between relative humidity and dew point temperature in moist air. BAMS, 86, 225-233.

---

**Box 1.6. Moist adiabatic lapse rate**

For an ascending saturated air parcel the first law of thermodynamics (Eq. 1.7c) can be written as,

\[ -L_v dr_s = c_p dT - \alpha dp. \tag{1} \]

With the hydrostatic equation \( dp = -\rho g dz \) this becomes

\[ \frac{dT}{dz} = -\frac{g}{c_p} - \frac{L_v}{c_p} \frac{dr_s}{dz}. \tag{2} \]

Assuming that water vapour is an ideal gas, the saturation mixing ratio can be written as

\[ r_s = \frac{e e_s}{p - e_s} = \frac{e e_s}{p}. \tag{3} \]

Therefore, with the hydrostatic equation \( dp = -\rho g dz \), we obtain,

\[ \frac{dr_s}{dz} = \frac{e e_s dp}{p^2 dz} = \frac{\varepsilon d e_s}{p^2 dz} = \frac{\varepsilon e_s}{p^2} \rho g . \tag{4} \]

With the Clausius-Clapeyron equation (1.13) this becomes
\[
\frac{dr_s}{dz} = \frac{\varepsilon L_v e_s}{R_v p T^2} \frac{dT}{dz} + \frac{\varepsilon e_s \rho g}{p^2}.
\] (5)

Substituting this into (2) and using (3) and the equation of state yields the **moist adiabatic lapse rate** or **pseudo-adiabatic lapse rate**:

\[
\frac{dT}{dz} = -\frac{g}{c_p} \left( \frac{1 + \frac{L_v r_s}{RT}}{1 + \frac{L_v^2 r_s}{c_p R_v T^2}} \right).
\] (6)

The factor in between brackets on the r.h.s. of this equation is smaller than 1 if

\[
L_v > \frac{R_v}{R} c_p T \approx 5 \times 10^5 \text{ J kg}^{-1}.
\] (7)

Because \( L_v = 2.5 \times 10^6 \text{ J kg}^{-1} \), this is indeed the case at realistic temperatures. The moist adiabatic lapse rate decreases with temperature and pressure, as is shown in the table below.

<table>
<thead>
<tr>
<th>P [hPa]</th>
<th>1000</th>
<th>700</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>T [°C]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-30</td>
<td>9.2</td>
<td>9.0</td>
<td>8.7</td>
</tr>
<tr>
<td>-20</td>
<td>8.6</td>
<td>8.2</td>
<td>7.8</td>
</tr>
<tr>
<td>-10</td>
<td>7.7</td>
<td>7.1</td>
<td>6.4</td>
</tr>
<tr>
<td>0</td>
<td>6.5</td>
<td>5.8</td>
<td>5.1</td>
</tr>
<tr>
<td>+10</td>
<td>5.3</td>
<td>4.6</td>
<td>4.0</td>
</tr>
<tr>
<td>+20</td>
<td>4.3</td>
<td>3.7</td>
<td>3.3</td>
</tr>
</tbody>
</table>

**TABLE 1 (Box 1.6).** Values of the moist adiabatic lapse rate (K km\(^{-1}\)) for different temperatures and pressures.

**PROBLEM 1 BOX 1.6. Cloud formation and vertical motion (1)**

A parcel of air with a temperature of 15°C and a dewpoint temperature of 2°C is lifted from the ground at 1000 hPa, without mixing with the environment. Determine its LCL and temperature at that level. The air parcel is lifted a further 200 hPa above its LCL. What is its final temperature and how much liquid water per kg of air is condensed?

**PROBLEM 2 BOX 1.6. Cloud formation and vertical motion (2)**

If \( L \) does not meet the criterion (7), clouds will *not* form when air ascends, but when air descends. Give a physical interpretation of this conclusion. Can you derive this criterion also from the expression (9 of Box 1.5) for the dew point lapse rate?

**References to BOX 1.6**


<table>
<thead>
<tr>
<th>PRES</th>
<th>HGHT</th>
<th>TEMP</th>
<th>DWPT</th>
<th>RELH</th>
<th>MIXR</th>
<th>DRCT</th>
<th>SKNT</th>
<th>THTA</th>
<th>THTE</th>
<th>THTV</th>
</tr>
</thead>
<tbody>
<tr>
<td>hPa</td>
<td>m</td>
<td>C</td>
<td>C</td>
<td>%</td>
<td>g/kg</td>
<td>deg</td>
<td>knot</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
<tr>
<td>1009.0</td>
<td>61</td>
<td>23.8</td>
<td>19.1</td>
<td>75</td>
<td>13.99</td>
<td>20</td>
<td>3</td>
<td>296.2</td>
<td>336.7</td>
<td>298.7</td>
</tr>
<tr>
<td>1000.0</td>
<td>129</td>
<td>24.6</td>
<td>17.6</td>
<td>65</td>
<td>12.82</td>
<td>16</td>
<td>3</td>
<td>297.8</td>
<td>335.2</td>
<td>300.0</td>
</tr>
<tr>
<td>979.0</td>
<td>316</td>
<td>26.4</td>
<td>14.4</td>
<td>48</td>
<td>10.64</td>
<td>8</td>
<td>4</td>
<td>301.4</td>
<td>333.0</td>
<td>303.3</td>
</tr>
<tr>
<td>941.0</td>
<td>664</td>
<td>23.8</td>
<td>12.8</td>
<td>50</td>
<td>9.96</td>
<td>5</td>
<td>5</td>
<td>302.1</td>
<td>331.9</td>
<td>304.0</td>
</tr>
<tr>
<td>925.0</td>
<td>814</td>
<td>23.4</td>
<td>13.4</td>
<td>53</td>
<td>10.49</td>
<td>8</td>
<td>4</td>
<td>303.2</td>
<td>334.9</td>
<td>305.1</td>
</tr>
<tr>
<td>906.0</td>
<td>995</td>
<td>23.0</td>
<td>13.0</td>
<td>53</td>
<td>10.49</td>
<td>8</td>
<td>4</td>
<td>304.6</td>
<td>336.3</td>
<td>306.5</td>
</tr>
<tr>
<td>865.0</td>
<td>1396</td>
<td>19.4</td>
<td>10.4</td>
<td>56</td>
<td>9.23</td>
<td>6</td>
<td>6</td>
<td>304.9</td>
<td>332.9</td>
<td>306.6</td>
</tr>
<tr>
<td>850.0</td>
<td>1547</td>
<td>20.2</td>
<td>10.2</td>
<td>53</td>
<td>9.27</td>
<td>7</td>
<td>7</td>
<td>307.3</td>
<td>335.7</td>
<td>309.0</td>
</tr>
<tr>
<td>840.0</td>
<td>1649</td>
<td>20.6</td>
<td>9.6</td>
<td>49</td>
<td>9.01</td>
<td>7</td>
<td>7</td>
<td>308.8</td>
<td>336.5</td>
<td>310.4</td>
</tr>
<tr>
<td>810.0</td>
<td>1962</td>
<td>18.2</td>
<td>2.2</td>
<td>34</td>
<td>5.57</td>
<td>8</td>
<td>8</td>
<td>313.8</td>
<td>329.0</td>
<td>314.7</td>
</tr>
<tr>
<td>700.0</td>
<td>3191</td>
<td>8.8</td>
<td>-0.2</td>
<td>53</td>
<td>5.42</td>
<td>235</td>
<td>11</td>
<td>319.7</td>
<td>327.9</td>
<td>320.2</td>
</tr>
<tr>
<td>636.0</td>
<td>3978</td>
<td>2.6</td>
<td>-3.4</td>
<td>65</td>
<td>4.70</td>
<td>228</td>
<td>12</td>
<td>321.8</td>
<td>325.4</td>
<td>321.9</td>
</tr>
<tr>
<td>532.0</td>
<td>5399</td>
<td>-7.5</td>
<td>-15.5</td>
<td>53</td>
<td>2.16</td>
<td>215</td>
<td>13</td>
<td>318.1</td>
<td>325.5</td>
<td>318.6</td>
</tr>
<tr>
<td>500.0</td>
<td>5880</td>
<td>-10.9</td>
<td>-14.9</td>
<td>72</td>
<td>2.42</td>
<td>210</td>
<td>13</td>
<td>319.7</td>
<td>327.9</td>
<td>320.2</td>
</tr>
<tr>
<td>493.0</td>
<td>5988</td>
<td>-11.7</td>
<td>-15.1</td>
<td>76</td>
<td>2.41</td>
<td>209</td>
<td>13</td>
<td>320.0</td>
<td>328.2</td>
<td>320.5</td>
</tr>
<tr>
<td>457.0</td>
<td>6563</td>
<td>-15.9</td>
<td>-25.9</td>
<td>42</td>
<td>1.02</td>
<td>204</td>
<td>14</td>
<td>321.8</td>
<td>325.4</td>
<td>321.9</td>
</tr>
<tr>
<td>433.0</td>
<td>6966</td>
<td>-18.9</td>
<td>-23.6</td>
<td>66</td>
<td>1.32</td>
<td>200</td>
<td>14</td>
<td>322.9</td>
<td>327.7</td>
<td>323.2</td>
</tr>
<tr>
<td>421.0</td>
<td>7174</td>
<td>-20.3</td>
<td>-38.3</td>
<td>18</td>
<td>0.33</td>
<td>198</td>
<td>15</td>
<td>323.8</td>
<td>325.1</td>
<td>323.8</td>
</tr>
<tr>
<td>400.0</td>
<td>7550</td>
<td>-22.5</td>
<td>-40.5</td>
<td>18</td>
<td>0.28</td>
<td>195</td>
<td>15</td>
<td>325.7</td>
<td>326.8</td>
<td>325.7</td>
</tr>
<tr>
<td>360.0</td>
<td>8318</td>
<td>-27.3</td>
<td>-47.3</td>
<td>13</td>
<td>0.15</td>
<td>204</td>
<td>15</td>
<td>329.2</td>
<td>329.8</td>
<td>329.2</td>
</tr>
<tr>
<td>351.0</td>
<td>8500</td>
<td>-28.7</td>
<td>-39.7</td>
<td>34</td>
<td>0.35</td>
<td>206</td>
<td>15</td>
<td>329.7</td>
<td>331.1</td>
<td>329.8</td>
</tr>
<tr>
<td>322.0</td>
<td>9115</td>
<td>-32.7</td>
<td>-52.7</td>
<td>12</td>
<td>0.09</td>
<td>214</td>
<td>14</td>
<td>332.4</td>
<td>332.8</td>
<td>332.4</td>
</tr>
<tr>
<td>313.0</td>
<td>9314</td>
<td>-34.3</td>
<td>-44.3</td>
<td>36</td>
<td>0.24</td>
<td>216</td>
<td>14</td>
<td>332.9</td>
<td>333.8</td>
<td>332.9</td>
</tr>
<tr>
<td>300.0</td>
<td>9610</td>
<td>-36.9</td>
<td>-44.9</td>
<td>43</td>
<td>0.23</td>
<td>220</td>
<td>14</td>
<td>333.2</td>
<td>334.2</td>
<td>333.3</td>
</tr>
<tr>
<td>285.0</td>
<td>9961</td>
<td>-40.1</td>
<td>-46.1</td>
<td>53</td>
<td>0.22</td>
<td>209</td>
<td>13</td>
<td>333.6</td>
<td>334.5</td>
<td>333.6</td>
</tr>
<tr>
<td>250.0</td>
<td>10840</td>
<td>-46.1</td>
<td></td>
<td>180</td>
<td>10</td>
<td>337.4</td>
<td></td>
<td>337.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200.0</td>
<td>12290</td>
<td>-56.5</td>
<td></td>
<td>220</td>
<td>4</td>
<td>343.1</td>
<td></td>
<td>343.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>186.0</td>
<td>12744</td>
<td>-59.9</td>
<td></td>
<td>215</td>
<td>8</td>
<td>344.8</td>
<td></td>
<td>344.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>171.0</td>
<td>13264</td>
<td>-60.7</td>
<td></td>
<td>213</td>
<td>13</td>
<td>351.9</td>
<td></td>
<td>351.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>150.0</td>
<td>14080</td>
<td>-57.1</td>
<td></td>
<td>210</td>
<td>21</td>
<td>371.5</td>
<td></td>
<td>371.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0</td>
<td>16640</td>
<td>-60.1</td>
<td></td>
<td>200</td>
<td>14</td>
<td>411.3</td>
<td></td>
<td>411.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>93.0</td>
<td>17090</td>
<td>-60.9</td>
<td></td>
<td>198</td>
<td>12</td>
<td>418.4</td>
<td></td>
<td>418.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70.0</td>
<td>18860</td>
<td>-58.3</td>
<td></td>
<td>190</td>
<td>4</td>
<td>459.3</td>
<td></td>
<td>459.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50.0</td>
<td>20990</td>
<td>-55.3</td>
<td></td>
<td>100</td>
<td>8</td>
<td>512.7</td>
<td></td>
<td>512.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.0</td>
<td>24280</td>
<td>-50.7</td>
<td></td>
<td>100</td>
<td>23</td>
<td>605.8</td>
<td></td>
<td>605.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.0</td>
<td>26940</td>
<td>-47.1</td>
<td></td>
<td>70</td>
<td>17</td>
<td>691.2</td>
<td></td>
<td>691.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 1.2.** Radiosonde measurements at characteristic levels on July 20, 1992, 00 UTC at Bordeaux (France) (Data obtained from [http://weather.uwyo.edu/](http://weather.uwyo.edu/)).
<table>
<thead>
<tr>
<th>PRES</th>
<th>HGHT</th>
<th>TEMP</th>
<th>DWPT</th>
<th>RELH</th>
<th>MIXR</th>
<th>DRCT</th>
<th>SKNT</th>
<th>THTA</th>
<th>THTE</th>
<th>THTV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1005.0</td>
<td>61</td>
<td>32.2</td>
<td>22.2</td>
<td>56</td>
<td>17.09</td>
<td>160</td>
<td>6</td>
<td>304.9</td>
<td>356.3</td>
<td>308.0</td>
</tr>
<tr>
<td>1000.0</td>
<td>94</td>
<td>31.8</td>
<td>21.8</td>
<td>55</td>
<td>16.75</td>
<td>160</td>
<td>6</td>
<td>304.9</td>
<td>355.3</td>
<td>308.0</td>
</tr>
<tr>
<td>926.0</td>
<td>777</td>
<td>24.2</td>
<td>17.2</td>
<td>65</td>
<td>13.51</td>
<td>180</td>
<td>11</td>
<td>303.9</td>
<td>344.4</td>
<td>306.4</td>
</tr>
<tr>
<td>925.0</td>
<td>786</td>
<td>24.2</td>
<td>17.2</td>
<td>65</td>
<td>13.53</td>
<td>180</td>
<td>11</td>
<td>304.1</td>
<td>344.6</td>
<td>306.5</td>
</tr>
<tr>
<td>911.0</td>
<td>920</td>
<td>24.0</td>
<td>17.0</td>
<td>65</td>
<td>13.56</td>
<td>183</td>
<td>12</td>
<td>304.1</td>
<td>345.2</td>
<td>306.4</td>
</tr>
<tr>
<td>907.0</td>
<td>959</td>
<td>25.2</td>
<td>16.2</td>
<td>57</td>
<td>12.93</td>
<td>183</td>
<td>12</td>
<td>305.2</td>
<td>346.0</td>
<td>307.4</td>
</tr>
<tr>
<td>895.0</td>
<td>1076</td>
<td>25.4</td>
<td>8.4</td>
<td>34</td>
<td>7.78</td>
<td>186</td>
<td>13</td>
<td>306.8</td>
<td>346.0</td>
<td>309.5</td>
</tr>
<tr>
<td>850.0</td>
<td>1526</td>
<td>22.0</td>
<td>5.0</td>
<td>33</td>
<td>6.47</td>
<td>195</td>
<td>15</td>
<td>309.2</td>
<td>346.0</td>
<td>310.0</td>
</tr>
<tr>
<td>775.0</td>
<td>2318</td>
<td>15.6</td>
<td>-1.4</td>
<td>31</td>
<td>4.48</td>
<td>193</td>
<td>19</td>
<td>310.6</td>
<td>324.9</td>
<td>311.4</td>
</tr>
<tr>
<td>700.0</td>
<td>4718</td>
<td>-4.3</td>
<td>-7.4</td>
<td>79</td>
<td>3.81</td>
<td>193</td>
<td>19</td>
<td>311.4</td>
<td>326.9</td>
<td>315.2</td>
</tr>
<tr>
<td>578.0</td>
<td>4718</td>
<td>-4.3</td>
<td>-7.4</td>
<td>79</td>
<td>3.81</td>
<td>193</td>
<td>19</td>
<td>311.4</td>
<td>326.9</td>
<td>315.2</td>
</tr>
<tr>
<td>524.0</td>
<td>5487</td>
<td>-7.5</td>
<td>-24.5</td>
<td>24</td>
<td>1.01</td>
<td>194</td>
<td>23</td>
<td>319.5</td>
<td>323.1</td>
<td>319.7</td>
</tr>
<tr>
<td>500.0</td>
<td>5850</td>
<td>-9.9</td>
<td>-24.9</td>
<td>28</td>
<td>1.02</td>
<td>195</td>
<td>23</td>
<td>320.9</td>
<td>324.6</td>
<td>321.1</td>
</tr>
<tr>
<td>433.0</td>
<td>6942</td>
<td>-17.7</td>
<td>-28.7</td>
<td>38</td>
<td>0.83</td>
<td>201</td>
<td>23</td>
<td>324.5</td>
<td>327.5</td>
<td>324.6</td>
</tr>
<tr>
<td>424.0</td>
<td>7099</td>
<td>-18.9</td>
<td>-35.9</td>
<td>21</td>
<td>0.42</td>
<td>202</td>
<td>23</td>
<td>324.9</td>
<td>326.5</td>
<td>325.0</td>
</tr>
<tr>
<td>400.0</td>
<td>7530</td>
<td>-21.9</td>
<td>-39.9</td>
<td>18</td>
<td>0.30</td>
<td>205</td>
<td>23</td>
<td>326.4</td>
<td>327.6</td>
<td>326.5</td>
</tr>
<tr>
<td>321.0</td>
<td>9110</td>
<td>-33.7</td>
<td>-52.7</td>
<td>13</td>
<td>0.09</td>
<td>213</td>
<td>22</td>
<td>331.3</td>
<td>331.7</td>
<td>331.3</td>
</tr>
<tr>
<td>300.0</td>
<td>9580</td>
<td>-37.3</td>
<td>-48.3</td>
<td>31</td>
<td>0.16</td>
<td>215</td>
<td>22</td>
<td>332.7</td>
<td>333.4</td>
<td>332.7</td>
</tr>
<tr>
<td>290.0</td>
<td>9815</td>
<td>-39.1</td>
<td>-48.1</td>
<td>38</td>
<td>0.17</td>
<td>217</td>
<td>22</td>
<td>333.4</td>
<td>334.1</td>
<td>333.4</td>
</tr>
<tr>
<td>250.0</td>
<td>10820</td>
<td>-47.7</td>
<td>225</td>
<td>22</td>
<td>335.0</td>
<td>335.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>225.0</td>
<td>11504</td>
<td>-53.7</td>
<td>230</td>
<td>22</td>
<td>336.1</td>
<td>336.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>200.0</td>
<td>12250</td>
<td>-58.1</td>
<td>235</td>
<td>22</td>
<td>340.6</td>
<td>340.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>186.0</td>
<td>12703</td>
<td>-60.9</td>
<td>235</td>
<td>22</td>
<td>343.2</td>
<td>343.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150.0</td>
<td>14050</td>
<td>-57.3</td>
<td>220</td>
<td>22</td>
<td>371.1</td>
<td>371.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100.0</td>
<td>16610</td>
<td>-58.9</td>
<td>200</td>
<td>20</td>
<td>413.6</td>
<td>413.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>93.1</td>
<td>17058</td>
<td>-59.1</td>
<td>196</td>
<td>18</td>
<td>421.8</td>
<td>421.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70.0</td>
<td>18850</td>
<td>-57.3</td>
<td>180</td>
<td>9</td>
<td>461.4</td>
<td>461.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50.0</td>
<td>20990</td>
<td>-55.1</td>
<td>125</td>
<td>10</td>
<td>513.2</td>
<td>513.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32.6</td>
<td>23748</td>
<td>-52.3</td>
<td>108</td>
<td>18</td>
<td>587.3</td>
<td>587.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30.0</td>
<td>24290</td>
<td>-49.9</td>
<td>105</td>
<td>19</td>
<td>608.0</td>
<td>608.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29.0</td>
<td>24512</td>
<td>-49.1</td>
<td>104</td>
<td>19</td>
<td>616.1</td>
<td>616.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22.4</td>
<td>26203</td>
<td>-49.5</td>
<td>98</td>
<td>23</td>
<td>662.1</td>
<td>662.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.0</td>
<td>26950</td>
<td>-45.9</td>
<td>95</td>
<td>24</td>
<td>694.9</td>
<td>694.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 1.3.** Radiosonde measurements at characteristic levels on July 20, 1992, 12 UTC at Bordeaux (France) (Data obtained from [http://weather.uwyo.edu/](http://weather.uwyo.edu/)).

1.19 Geostrophic balance

When a fluid is heated at a different rate at different places a pressure gradient is created within the fluid. As a result, there will be a flow from high to low pressure, and a reduction of the pressure gradient. A non-rotating fluid system will evolve towards the state of rest,
assuming that friction damps possible oscillations. However, this does not happen in rotating fluid systems, such as the atmosphere or the ocean. Due to the Coriolis force, and for reasons which will become apparent in chapter 5, such fluid systems will not evolve towards the state of rest, rather they will adjust towards a different equilibrium state, referred to as "geostrophic equilibrium" or "geostrophic balance", in which the local horizontal components of the Coriolis force and of the pressure gradient force are equal and opposite, i.e.:

\[ \theta \vec{V}_h \Pi = -2\vec{\Omega} \times \vec{v}, \]  
(eq. 1.56), where

\[ \vec{V}_h \equiv \left( \hat{i} \frac{\partial}{\partial x}, \hat{j} \frac{\partial}{\partial y}, 0 \right). \]  
(1.83)

The balance of forces in the vertical direction does not involve the Coriolis force because it is "swamped" by the force of gravity (Box 1.2 and section 1.14).

Chapter 5 of these lecture notes is intended to provide an answer to the question why, when and how quickly the atmosphere evolves towards geostrophic balance, a process that is called geostrophic adjustment. In section 1.30 observational evidence will be provided of the approximate existence of geostrophic balance in the atmosphere. The geostrophic balance equation (1.82) is in fact used as a simplifying constraint in so-called "quasi-geostrophic" theories of "large-scale" circulation- or weather-systems (chapter 9). The question what exactly is "large-scale" (in space and time) is addressed in chapter 5.

The sphericity of the Earth is the cause of an additional interesting dynamical effect, which is referred to as the "\( \beta \)-effect". At the poles Earth’s rotation vector is oriented parallel to the local vertical, while at the equator Earth’s rotation vector is oriented parallel to the local horizontal. Due to this, the Coriolis force at high latitudes is oriented approximately parallel to the local horizontal, while at low latitudes it is oriented approximately parallel to the local vertical. Therefore, the horizontal component of the Coriolis force approaches zero near the equator. The meridional gradient of the orientation of \( \vec{\Omega} \) with respect to the local coordinate system gives rise to a certain type of oscillation or wave, which is called a "planetary Rossby wave" (section 1.37).

### 1.20 Stability of geostrophic balance

This section presents a simplified analysis of the stability of geostrophic balance, using the parcel method (section 1.15), yielding an additional fundamental time-scale or frequency: the "inertial frequency". Suppose that air in the atmosphere flows in west-east (zonal) direction and that, associated with this flow pattern, the meridional (\( y \)-) component of the pressure gradient force and the meridional component of the Coriolis force are in exact balance. This implies that the \( y \)-component of eq. 1.82 becomes,

\[ fu = -\theta \frac{\partial \Pi}{\partial y}, \]  
(1.84)

where the Coriolis parameter, \( f \), is defined as
\[ f = 2\Omega\sin\phi, \]  
\[ (1.85) \]

with \( \phi \) the latitude. In this hypothetical state there is no pressure gradient in the zonal (\( x \)-) direction. The two components of the equation of motion (1.56) now become

\[ \frac{du}{dt} = f \frac{dy}{dt}, \]  
\[ (1.86) \]

\[ \frac{dv}{dt} = -\theta \frac{\partial \Pi}{\partial y} - fu = f \left( u_g - u \right), \]  
\[ (1.87) \]

where

\[ u_g = -\frac{\theta}{f} \frac{\partial \Pi}{\partial y} \]  
\[ (1.88) \]

is the zonal geostrophic velocity, which we assume depends only on latitude, \( y \). The geostrophic wind at a point \( y = y_0 + \delta y \) can, therefore, be approximated by

\[ u_g(y_0 + \delta y) = u_g(y_0) + \frac{\partial u_g}{\partial y}(y_0)\delta y. \]  
\[ (1.89) \]

Let us now consider an air parcel with a fixed mass which is located at \( y = y_0 \), moving with the geostrophic velocity at that latitude. By an unspecified external force it is brought to a new meridional position: \( y = y_0 + \delta y \). In view of (1.86), which is actually equivalent to the principle of conservation of "linear momentum per unit mass",

\[ M = u - fy, \]  
\[ (1.90) \]

we can state that

\[ u(y_0 + \delta y) = u(y_0) + f\delta y. \]  
\[ (1.91) \]

Eq. (1.87) can be written as

\[ \frac{dv(y_0 + \delta y)}{dt} = f \left[ u_g(y_0 + \delta y) - u(y_0 + \delta y) \right]. \]  
\[ (1.92) \]

With (1.89) and (1.91) this becomes

\[ \frac{dv(y_0 + \delta y)}{dt} = f \left[ u_g(y_0) + \frac{\partial u_g}{\partial y}\delta y - (u(y_0) + f\delta y) \right], \]  
\[ (1.93) \]

or
The solution of this equation is

$$\delta y = \Re\{A \exp(\pm iFt)\},$$  \hspace{1cm} (1.95)

with

$$F^2 = f \left( f - \frac{\partial u_g}{\partial y} \right).$$  \hspace{1cm} (1.96)

If $f(\partial u_g/\partial y)>0$, $F$ is a real number and the solution (1.95) represents a so-called **inertial-oscillation** with a frequency equal to $F$. If $f(\partial u_g/\partial y)<0$, $F$ is an imaginary number. In that case, solution (1.95) represents an exponential function of time. The air parcel does not return to its original balanced position, implying that the state of geostrophic balance is unstable to the perturbation. The parameter, $F$, is referred to as the **inertial stability**.

It is interesting to note that the Earth’s rotation imparts to the flow an intrinsic stability. In the northern hemisphere the (inertial) oscillations that are a result of this stability have a period equal to $2\pi/f$ (the **inertial period**) which is two orders of magnitude larger than the period of oscillations, associated with oscillations around hydrostatic balance (section 1.15). When the geostrophic flow itself possesses lateral shear such that $\partial u_g/\partial y>f$, we find that this intrinsic stability vanishes. The parameter $-\partial u_g/\partial y$ is in fact equivalent to the rotation of the geostrophic wind-vector, also called **relative vorticity** (section 1.24). The parameter, $f$, represents the **planetary vorticity**, i.e. the vorticity of an air parcel due the rotation of the Earth. The parameter, $(f-\partial u_g/\partial y)$, represents the **absolute vorticity**. Thus, the stability criterion, $(f-\partial u_g/\partial y)>0$, can be interpreted for the northern hemisphere as a contraint on the absolute vorticity associated with the average geostrophic (horizontal) motion in the atmosphere.

**PROBLEM 1.11. Parcel model of an inertial oscillation**

Construct a numerical model (in e.g. Python, so that you can plot the result directly), which simulates the position and velocity of an air parcel in the longitude-latitude plane (neglect the convergence of the meridians) under influence of a stationary pressure field for which $\Pi(y)=C_0+C(y-y_0)^2$. The parameters $C_0$ and $C$ are constants, which may be specified. Let us restrict our attention to middle latitudes, close to $y_0=45^\circ$N. Neglect friction. Air is moving in the zonal direction with the geostrophic wind velocity. At a certain time, $t=0$, an air parcel, which does not interact or influence the environment, is displaced in the meridional direction from $y=y_0$ such that its zonal velocity increases by $10$ m s$^{-1}$. How large is this displacement for $f=10^{-4}$ s$^{-1}=\text{constant}$? In which way does the answer to this question depend on $C$? First write down the governing equations. For simplicity, take $f=f(y_0)=\text{constant}$. For which range of values of $C$ will the air parcel undergo an oscillation after it is displaced to a different latitude. Will it undergo an oscillation afterwards? If so, with what frequency. Verify your answers both analytically (approximately) and numerically. You must discretise the time-axis and decide how you to approximate the time derivatives numerically (tip: the **semi-implicit Euler method** is a simple and accurate numerical method). What happens when $C= -0.3 \times 10^{-10} \text{ J K}^{-1}\text{ kg}^{-1} \text{ m}^{-2}$? Assume that $\theta=300$ K.
1.21 The thermal wind

Consider again the case of a zonal flow in which there is geostrophic and hydrostatic equilibrium. The geostrophic wind is given by (1.88). Using eq. 1.60 (the equation expressing hydrostatic balance), we can eliminate the pressure, and obtain an equation for the vertical shear of the geostrophic wind, the so-called thermal wind equation:

\[
\frac{\partial u_g}{\partial z} = \frac{u_g}{\theta} \frac{\partial \theta}{\partial z} - \frac{g}{f} \frac{\partial \theta}{\partial y}.
\]  

(1.97)

The ratio of the two terms on the r.h.s. of (1.97) is

\[
\frac{\frac{g}{f} \frac{\partial \theta}{\partial y}}{\frac{u_g}{\theta} \frac{\partial \theta}{\partial z}} = \frac{(10 \text{ m s}^{-2})(5 \times 10^{-5} \text{K m}^{-1})}{(10^{-4} \text{s}^{-1})(5 \times 10^{-3} \text{K m}^{-1})} = 10^2.
\]

So the second term on the r.h.s. of (1.97) is by far dominant, implying that the shear of the geostrophic wind is proportional to the horizontal potential temperature gradient. More specifically: a northward decreasing potential temperature is associated with an increasing geostrophic wind with increasing height. As a linear approximation we may write that

![FIGURE 1.31: Isentropes (thin solid lines, labelled in Kelvin) and isotachs (isopleths of the velocity) (dashed lines, m s\(^{-1}\)) in a vertical section through a cold front. The \(y\)-coordinate is positive towards the left. Heavy lines mark the tropopause and frontal boundaries. The section extends approximately 1200 km in the horizontal direction (Palmen, E. and C.W. Newton, 1969: Atmospheric Circulation Systems. Academic Press, 603 pp).](image)
The parameter $u_T(z, \delta z)$ is referred to as the thermal wind.

The relation between the potential temperature and the wind, imposed by the thermal wind equation (1.97) is illustrated in figure 1.31. In accord with the thermal wind equation, the geostrophic wind speed increases with increasing height if $\partial \theta / \partial y < 0$ (in the troposphere), while it decreases with increasing height if $\partial \theta / \partial y > 0$ (in the stratosphere). This is the reason for the existence of a jet near the tropopause over the zone with the most intense temperature gradient, the so-called polar front, where the potential temperature near the Earth’s surface decreases in northerly direction by 10 K over a distance of about 50 km.
Figure 1.32a shows the average position of the circumpolar jet at p=250 hPa (about 10 km above sea level) in December, January and February, in the northern hemisphere (winter) and in the southern hemisphere (summer). The meridional temperature gradient is largest in the winter hemisphere. Due to thermal wind balance, this leads to the highest wind speeds in the winter hemisphere (the northern hemisphere in this case). The jet observed at a latitude of approximately 30° over Africa and Asia in the northern hemisphere winter is usually referred to as the “subtropical jet”. This jet is connected to the “Hadley circulation” (Figure 1.12). The intensity of this jet is significantly reduced over eastern Pacific Ocean and over the Atlantic Ocean. A localized jet is observed over the Eastern United States in winter. This so-called “jetstreak” (section 1.35) is associated with the strong temperature contrast near the Earth’s surface across the east coast of North America.

The near-surface temperature contrast is not the only factor that determines the intensity of jets at the top of the troposphere. This is demonstrated indirectly in Figure 1.32b, which shows the average position and the intensity of the circumpolar jet in June, July and August, in the northern hemisphere (summer) and in the southern hemisphere (winter). Again, the winter hemisphere exhibits the strongest jet, located at about 35°S. The wind speeds observed in the southern hemisphere are about as strong as the wind speeds observed in the northern hemisphere in winter, despite the omnipresence of ocean in the southern hemisphere (and thus weak surface temperature gradients) as opposed to in the northern hemisphere.

1.22 Stability of thermal wind balance

This section presents an analysis of the stability of thermal wind balance, using the parcel method (sections 1.15 and 1.20). The three components of the equation of motion are (neglecting friction) (eq. 1.56)

\[
\begin{align*}
\frac{du}{dt} - f v &= -\theta \frac{\partial \Pi}{\partial x} \\
\frac{dv}{dt} + f u &= -\theta \frac{\partial \Pi}{\partial y} \\
\frac{dw}{dt} &= -\theta \frac{\partial \Pi}{\partial z} - g
\end{align*}
\]

(1.99a)

(1.99b)

(1.99c)

Let us assume a basic state \((u_0, \theta_0, \Pi_0)\) in geostrophic- and hydrostatic balance as follows.

\[
\begin{align*}
f u_0 &= -\theta_0 \frac{\partial \Pi_0}{\partial y} \\
\theta_0 \frac{\partial \Pi_0}{\partial z} &= -g
\end{align*}
\]

(1.100)

(1.101)

The pressure does not vary in the zonal \((x-)\) direction. Differentiating (1.100) with respect to \(z\) and differentiating (1.101) with respect to \(y\), and subtracting the result gives the following expression for thermal wind balance.

\[
\frac{\partial u_0}{\partial z} = - \frac{g}{f \theta_0} \frac{\partial \theta_0}{\partial y} + \frac{u_0}{\theta_0} \frac{\partial \theta_0}{\partial z}.
\]

(1.102)
This equation is identical to equation (1.97). On the basis of the scale analysis given in the previous section, we can approximate (1.102) by

$$\frac{\partial u_0}{\partial z} \equiv -\frac{g}{f\theta_0} \frac{\partial \theta_0}{\partial y}.$$ (1.103)

Imagine that an air parcel, that moves with the average flow $u_0$ at a fixed latitude and at some point $(x_0, y_0, z_0)$, is given a small “push” in meridional- and vertical direction, such that its position becomes $(x_0, y_0 + \delta y, z_0 + \delta z)$. Since the pressure is independent of $x$, we may simplify (1.99a) by

$$\frac{d}{dt}(u - fy) \equiv \frac{dM}{dt} = 0.$$ (1.104)

The parameter $M$ is the so-called “linear momentum per unit mass” (eq. 1.90). In the example discussed here (1.104) implies

$$u(y_0 + \delta y, z_0 + \delta z) = u(y_0, z_0) + f\delta y.$$ (1.105)

We neglect perturbations in the pressure and in the potential temperature. We can write (1.99b) (using 1.100) as follows.

$$\frac{dv}{dt} = f(u_0 - u).$$ (1.106)

For small departures $(\delta y, \delta z)$ from equilibrium:

$$\frac{d}{dt}v(y_0 + \delta y, z_0 + \delta z) = f\left(u_0(y_0, z_0) + \frac{\partial u_0}{\partial y} \delta y + \frac{\partial u_0}{\partial z} \delta z - u(y_0 + \delta y, z_0 + \delta z)\right)$$ (1.107)

With (1.105), we arrive at

$$\frac{d}{dt}v(y_0 + \delta y, z_0 + \delta z) = f\left(\frac{\partial u_0}{\partial z} \frac{\delta z}{\delta y} - \left(f - \frac{\partial u_0}{\partial y}\right)\frac{\delta z}{\delta y}\right)$$ (1.108)

We assume now that the air parcel moves on an isentrope (a surface of equal potential temperature, $\theta$). On an isentrope we have

$$\delta \theta = \frac{\partial \theta}{\partial y} \delta y + \frac{\partial \theta}{\partial z} \delta z = 0,$$ (1.109)

so that (with eq. 1.103),

$$\frac{\delta z}{\delta y} = \left(\frac{\delta z}{\delta y}\right)_\theta = -\frac{\partial \theta_0 / \partial y}{\partial \theta_0 / \partial z} = f \frac{\partial u_0 / \partial z}{N^2},$$ (1.110)
where the Brunt-Väisälä, $N$, is defined according to,

$$N^2 = \frac{g}{\theta_0} \frac{\partial \theta_0}{\partial z}. \quad (1.111)$$

Substituting (1.110) into (1.108) yields

$$\frac{d^2 \delta y}{dt^2} = -f^2 \left( \frac{f - \partial u_0 / \partial y}{f} \right) - \frac{1}{N^2} \left( \frac{\partial u_0}{\partial z} \right)^2 \delta y. \quad (1.112)$$

This equation has the same form as (1.94). The answer to the question whether the parcel will return to its original equilibrium position (stability) or will be accelerated away from this equilibrium position (instability) depends on the sign of factor in front of $\delta y$ on the r.h.s. of (1.112). We thus conclude that thermal wind balance is stable if

$$\left( \frac{f - \partial u_0 / \partial y}{f} \right) - \frac{1}{N^2} \left( \frac{\partial u_0}{\partial z} \right)^2 > 0. \quad (1.113a)$$

This stability-criterion can also be expressed in terms of the linear momentum, $M_0 = u_0 f y$, as follows.

$$f \left( f - \frac{\partial u_0}{\partial y} \right) > \left( \frac{f}{N} \frac{\partial u_0}{\partial z} \right)^2. \quad (1.113b)$$

In the absence of a horizontal temperature gradient, this criterion reduces to the criterion for inertial stability. If there is no temperature gradient (i.e. $\partial \theta_0 / \partial y = 0$) thermal wind balance is always stable to small perturbations for negative values of $\partial M_0 / \partial y$. However, in the presence of a horizontal temperature gradient, thermal wind balance may be unstable to small perturbations for negative values of $\partial M_0 / \partial y$. This type of instability, which is called “baroclinic instability”, can set in even though the atmosphere is inertially stable and hydrostatically stable (see Box 1.7).

**Box 1.7 Baroclinically unstable motions and mid-latitude cyclones**

A simple theory of symmetric (two-dimensional) baroclinic instability was presented here principally to give physical insight into the complicated process of baroclinic instability. Obviously, two-dimensional baroclinic instability will in general not occur in the real atmosphere.

The two-dimensional constraint implies angular momentum conservation. This constraint is broken when variations along the basic flow direction ($x$) are allowed. The theory of baroclinic instability then becomes quite complicated. A qualitative picture of what could happen, in the case that the zonal symmetry is broken, was provided by Hoskins (see the caption of figure 1 of this box). Suppose that some air parcels at a specific longitude are displaced poleward (along isentropes) and at different longitudes other air parcels are displaced equatorward along similar trajectories. In this sloping plane the motions will look...
like the dashed vectors shown in figure 1 (this box), with air parcels curving in a clockwise direction in the northern hemisphere. This tendency for air parcels to curve can however be opposed by pressure gradient forces corresponding to a simple pressure pattern of the form shown. The meridional motion will then be in geostrophic balance.

**Figure 1 (Box 1.7).** The motion of air, as seen in the plane of the trajectories. The dashed vectors are those that would occur in the absence of pressure variation indicated by L and H, and the continuous vectors are the actual balanced motions (from Hoskins, B.J., 1990: Theory of extratropical cyclones. In: Newton, C.W., and E.O. Holopainen (editors), 1990: Extratropical Cyclones. American Meteorological Society, Boston, 262 pp, p. 63-105).

**Figure 2 (Box 1.7).** A longitude-height section showing poleward, equatorward and vertical motion of warm (W) and cold (C) air, and the longitudinal variation in pressure for a growing disturbance (Source: see the caption of figure 1 of this box).

A vertical section in longitudinal direction across the system must then look like the schematic shown figure 2 (this box). There is poleward motion to the east of the low (L) and equatorward motion to the west of the low. Because the trajectory of the meridional motion must be less steep than the slope of the isentropes, the equatorward motion is downward, while the poleward motion is upward, i.e. warm air is moving poleward and upward to the east of the low pressure region and cold air is moving equatorward and downward to the west. From hydrostatic balance, the higher density of cold air corresponds to a relatively large change (decrease) in pressure with height. This implies a tendency for the low pressure at upper levels to be shifted toward the cold air at low levels. Thus, the low pressure region (i.e. the trough) and the high pressure region (i.e. the ridge) tilt westward with height.

Let us assume a tropospheric potential temperature distribution with potential temperature increasing equatorward and upward, as is illustrated in a meridional cross section in figure 3 (see also figure 1.26). Let an air parcel at position 1 be interchanged with an air parcel at position 2. The potential temperature of parcel 1, which ascends, is higher
than the potential temperature of parcel 2, which descends. When displaced adiabatically to its new position, parcel 1 just fits into the space vacated by parcel 2, and *vice versa*. Thus, the pressure field is the same before and after the exchange of the two air parcels. If \( V_1 \) and \( V_2 \) are the respective volumes and \( p_1 \) and \( p_2 \) the pressures, the law of adiabatic expansion (problem 1.2) demands that

\[
p_1 V_1^\gamma = p_2 V_2^\gamma \quad \left( \gamma = \frac{c_p}{c_v} \right), \tag{1}\]

*Figure 3 (Box 1.7).* Height-latitude cross section of the troposphere showing potential temperature, \( \theta \), increasing equatorward and upwards (statically stable stratification). Even though the system is stable to vertical displacements, potential energy can still be converted to kinetic energy if particles are exchanged along the thick solid double arrow. Source: J.S.A. Green, 1979: Topics in dynamical meteorology: 8. Trough-ridge systems as slantwise convection (1). *Weather*, 34, 2-10.

The ratio of masses, \( M \), of the two parcels is

\[
\frac{M_2}{M_1} = \frac{\rho_2 V_2}{\rho_1 V_1} = \frac{\rho_2 p_1^{1/\gamma}}{\rho_1 p_2^{1/\gamma}} = \frac{\theta_1}{\theta_2}, \tag{2}\]

The change in potential energy, \( \Delta PE \), is given by final value minus the initial value:

\[
\Delta PE = \left( M_1 g z_2 + M_2 g z_1 \right) - \left( M_1 g z_1 + M_2 g z_2 \right) = M_1 g (z_1 - z_2) \left( \frac{M_2}{M_1} - 1 \right), \tag{3}\]

where \( z_1 \) and \( z_2 \) are, respectively, the heights of the positions of parcel 1 and parcel 2 before the exchange. With (2), (3) becomes

\[
\Delta PE = M_1 g (z_1 - z_2) \left( \frac{\theta_1 - \theta_2}{\theta_2} \right) < 0. \tag{4}\]

Thus, because \( z_1 < z_2 \) and \( \theta_1 > \theta_2 \), potential energy is reduced during the interchange. This
energy must go into kinetic energy.

We can now calculate the maximum possible decrease of the potential energy by first expressing $\Delta PE$ as a function of the slope of the straight trajectory, which connects position 1 with position 2, and the distance, $L$, between the initial position and the final position of the air parcels. The slope of the trajectory is represented by the angle $E$ in figure 3. We can now write:

$$\theta_2 - \theta_1 = (y_2 - y_1) \frac{\partial \theta}{\partial y} + (z_2 - z_1) \frac{\partial \theta}{\partial z} = \left( \frac{\partial \theta}{\partial y} \cos E + \frac{\partial \theta}{\partial z} \sin E \right) L,$$

(5)

which implies, with (4), that

$$\Delta PE = \frac{M_1 g L^2}{\theta_2} \sin E \left( \frac{\partial \theta}{\partial y} \cos E + \frac{\partial \theta}{\partial z} \sin E \right).$$

(6)

Minimising this expression with respect to variations in $E$ implies

$$\frac{d}{dE} \left[ \sin E \left( \frac{\partial \theta}{\partial y} \cos E + \frac{\partial \theta}{\partial z} \sin E \right) \right] = 0,$$

(7)

which yields

$$\tan 2E = -\frac{\partial \theta / \partial y}{\partial \theta / \partial z}$$

(8)

as the “optimum” orientation of the parcel trajectory. The r.h.s. of (8) represents the slope of the isentropes. Therefore, the “optimum” or “preferred” slope of parcel trajectories, in which there is a maximum conversion of potential energy into kinetic energy, is half the slope of the isentropic surfaces.

**PROBLEM Box 1.7. Conversion of potential energy into kinetic energy**

Assume that all the potential energy is converted into kinetic energy. Derive an approximate expression for the final velocity of each air parcel after the exchange. What is this velocity (approximately) if $L=200$ km? Assume that

$$\frac{\partial \theta}{\partial y} \approx -5 \times 10^{-5} \text{ K m}^{-1} \text{ and } \frac{\partial \theta}{\partial z} \approx 5 \times 10^{-3} \text{ K m}^{-1}.$$

Careful analysis of surface observations, by the members of the “Bergen School” in the period 1910-1920, and of observations of the upper air by many other groups in the last century, taking advantage of the evolving radiosonde network, have led to a three-dimensional view of the morphology of baroclinically unstable motions in mid-latitudes as illustrated in figure 4 of this box in which we see warm air ascending and cold air descending at very small angles to the horizontal. The ascending warm current in the warm sector produces layered clouds, such as cirrus and (nimbo-)stratus, and persistent precipitation.
Typical flow pattern between two isobaric surfaces (1000 hPa is near the Earth’s surface; 500 hPa is at about 5 km above sea level) in a mid-latitude baroclinically unstable disturbance in the northern hemisphere. Such a disturbance is also known as a **middle latitude cyclone**. The thin dashed lines are lines of equal height of the 1000 hPa surface, which can also be interpreted as streamlines with the direction of the flow indicated by the arrows. The thicker solid lines represent lines of equal height of the 500 hPa surface (also streamlines). The open arrows represent trajectories of air parcels. The boundaries between cold and warm air masses are indicated by a line with closed half-circles (**warm front**) and a line with closed triangles (**cold front**). Air within the warm sector (between the cold front and the warm front) is rising slowly, while the air within the cold sector is descending. Both the ascent and the descent is occurring at an angle to the horizontal that is smaller than the slope of the isentropes, as is required for baroclinic instability (section 1.22). The centre of the cyclonic circulation at 1000 hPa is indicated by “L” (“Low”). The centre of the anti-cyclonic circulation at 1000 hPa is indicated by “H” (“High”). The “T” stands for “trough”, whiel the “R” stands for “Ridge”. Note that the trough is and the ridge are out of phase with, respectively the low and the high. In other words, the wave tilts towards the west with increasing height as is illustrated also in figure 2 (this box). Note also that the instantaneous wind-direction at a specific point in the warm sector turns clockwise with increasing height, while the opposite is the case in the cold sector. This is a property of thermal wind balance as will become clear in section 1.31. This figure is based on a figure in: Palmen, E. and C.W. Newton, 1969: *Atmospheric Circulation Systems*. Academic Press, 603 pp.

**Reference to Box 1.7**

1.23 Non-linear balance and the Rossby number

It is important to remark that the atmosphere is usually not in a stationary state when it is in geostrophic balance. Strictly speaking, in the stationary state the horizontal components of the equation of motion (see e.g. 1.99a,b) are

\[
\frac{\partial u}{\partial x} + \frac{v}{\partial y} - f v = -\theta \frac{\partial \Pi}{\partial x} \tag{1.114a}
\]

and

\[
\frac{\partial v}{\partial x} + \frac{v}{\partial y} + f u = -\theta \frac{\partial \Pi}{\partial y} \tag{1.114b}
\]

The wind obtained from the solution of eqs. 1.114a,b is referred to as gradient wind. If the flow is curved, the gradient wind may deviate strongly from the geostrophic wind (see problem 1.23 in section 1.30).

A traditional argument given in many textbooks, as a justification for neglecting the advection terms in the above equations, is based on a rough scale-analysis, which reveals that the Coriolis-term dominates the advection term if the so-called “Rossby number”,

\[
\text{Ro} \equiv \frac{U}{fL} \ll 1. \tag{1.115}
\]

In this expression \(U\) and \(L\) are so-called “characteristic” velocity- and length-scales, respectively. It is usually stated that \textbf{for Rossby numbers in the order of 1 or larger, the balanced state is non-linear.}

It is, however, difficult to specify the value of \(L\). \(L\) is a typical distance over which the velocity varies (see e.g. figure 1.32). If the wind strength varies only in the direction perpendicular to the wind vector, such as in the case of a straight homogeneous jet stream, e.g. oriented in east-west direction, so that \(u\) is only a function of \(y\) and \(v=0\), the Rossby number may exceed a value of 1, even though the advection terms in the nonlinear balance equations are zero, so that geostrophic balance is a perfect representation of the dynamic state of the atmosphere!

1.24 Circulation and vorticity

\textbf{Vorticity} can be understood by considering first its relation to \textbf{circulation}, \(\Gamma\). Circulation is the anticlockwise integral of the fluid velocity around an arbitrary closed curve, \(C\), i.e.

\[
\Gamma \equiv \oint_C \mathbf{v} \cdot d\mathbf{l} \tag{1.116}
\]

With Stokes theorem, this becomes

\[
\Gamma \equiv \oint_C \mathbf{v} \cdot d\mathbf{l} = \oint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \equiv \oint_S \mathbf{\omega} \cdot d\mathbf{S} \tag{1.116}
\]
The area, $S$, is the area enclosed by the closed curve, $C$. **Vorticity**, $\vec{\omega}$, is defined as the curl of the velocity vector:

$$\vec{\omega} = \nabla \times \vec{v}.$$  (1.117)

**The vorticity averaged over the area, S, is equal to the circulation around S divided by the area enclosed by S.**

Circulation and vorticity have been recognized as fundamental concepts in meteorology since Bjerknes formulated his circulation theorem in 1902. An equation for the vorticity can be deduced from the eq. of motion (1.7a). Using the vector identity,

$$\nabla \times \vec{v} = \vec{\omega} \times \vec{v} = \nabla \cdot \vec{v} - \frac{1}{2} \nabla|\vec{v}|^2,$$

we can write (1.7a), neglecting friction, as follows:

$$\frac{\partial \vec{v}}{\partial t} + (2\vec{\Omega} + \vec{\omega}) \times \vec{v} = -\frac{1}{\rho} \vec{\nabla} p - gk - \frac{1}{2} \nabla|\vec{v}|^2.$$  (1.118)

To find the vorticity equation we now take the curl of (1.118) and find

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times \{(2\vec{\Omega} + \vec{\omega}) \times \vec{v}\} = \frac{\vec{\nabla} \rho \times \vec{\nabla} p}{\rho^2}.$$  (1.119)

Using the vector identity,

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A} \vec{\nabla} \cdot \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{A} - \vec{B} \vec{\nabla} \cdot \vec{A} - (\vec{A} \cdot \vec{\nabla}) \vec{B},$$

we get

$$\vec{\nabla} \times \{(2\vec{\Omega} + \vec{\omega}) \times \vec{v}\} = (2\vec{\Omega} + \vec{\omega}) \vec{\nabla} \cdot \vec{v} + (\vec{v} \cdot \vec{\nabla})(2\vec{\Omega} + \vec{\omega}) - \{(2\vec{\Omega} + \vec{\omega}) \cdot \vec{\nabla}\} \vec{v},$$  (1.120)

while keeping in mind that $\vec{\nabla} \cdot (2\vec{\Omega} + \vec{\omega}) = 0$. The result is

$$\frac{\partial \vec{\omega}}{\partial t} = -(2\vec{\Omega} + \vec{\omega}) \vec{\nabla} \cdot \vec{v} - (\vec{v} \cdot \vec{\nabla})(2\vec{\Omega} + \vec{\omega}) + \{(2\vec{\Omega} + \vec{\omega}) \cdot \vec{\nabla}\} \vec{v} + \frac{\vec{\nabla} \rho \times \vec{\nabla} p}{\rho^2}.$$  (1.121)

This is the **vorticity equation**. If we assume incompressibility, the first term on the right hand side (r.h.s.) of (1.121) vanishes.

The baroclinic term (last term on the r.h.s. of 1.121) can also be written in terms of potential temperature and Exner function as

$$-\vec{\nabla} \theta \times \vec{\nabla} \Pi.$$  

The vertical component of the vorticity represents rotation or shear in a horizontal plane. Its definition is
$\omega \cdot k = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ \hfill (1.122)

For large-scale circulations (horizontal scale much larger than vertical scale) this is the most important component of the vorticity. An equation for $\zeta$ can be deduced from eq. 1.56 by differentiating its $y$ component with respect to $x$ and differentiating its $x$-component with respect to $y$ and subtracting the results, giving (note that this is not a trivial derivation, due to the non-linearity of the advection terms)

$$\frac{d\zeta}{dt} = -(f + \zeta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left( \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) + \left( \frac{\partial \theta}{\partial y} \frac{\partial \Pi}{\partial x} - \frac{\partial \theta}{\partial x} \frac{\partial \Pi}{\partial y} \right) - v \frac{df}{dy}. \hfill (1.123)$$

Apparently, there are four effects that influence material change of the vertical component of the vorticity: horizontal divergence of the wind (vortex stretching), tilting of horizontal vortex tubes into the vertical (second term on the r.h.s. of 1.123), baroclinicity (third term on the r.h.s. of 1.123) and advection of planetary vorticity (the $\beta$-effect) (fourth term on the r.h.s. of 1.123).

**Planetary vorticity is defined as “cyclonic”**. Because $f>0$ in the northern hemisphere, positive (negative) values of the relative vorticity, $\zeta$, in the northern hemisphere are referred to as cyclonic (anticyclonic) relative vorticity. Likewise, because planetary vorticity is negative in the southern hemisphere, this terminology is reversed for the southern hemisphere.

**Figure 1.33.** The time mean axisymmetric swirling (azimuthal or tangential) wind (solid line) and the gradient wind (sections 1.23) (crosses) at the 850 hPa pressure level, measured in tropical cyclone (hurricane) Alicia (28°N, 94°W) in the time interval 11:59 to 18:18 UTC, August 17, 1983. From H.E. Willoughby, 1990: Gradient balance in tropical cyclones. *J.Atmos.Sci.*, 47, 265-274. More about Alicia can be found in the following paper: H.E. Willoughby, 1990: Temporal changes of the primary circulation in tropical cyclones. *J.Atmos.Sci.*, 47, 242-264.
Box 1.8 Tropical cyclones

Tropical cyclones, which are formed over the tropical oceans, are not accompanied by frontal cloud bands, but by spiral bands of cumulus clouds (figure 1, this box). The centre of a tropical cyclone is usually cloud free. This feature is referred to as the "eye" of the tropical cyclone. Clouds surrounding the eye, in the so-called eye-wall, produce enormous amounts of precipitation and coincide with the highest wind speeds. In a mature tropical cyclone temperatures in the eye are in the order of 10 K higher than in the environment of the cyclone. In other words, a tropical cyclone has a warm core. This warm core is principally due to compression of air associated with descent in the eye. In the lower part of the cyclone warm and moist air spirals cyclonically into the eye wall, while in the upper part of the cyclone colder air spirals outwards anticyclonically (see the lower right panel in figure 1 of this box). Whereas a middle latitude baroclinic cyclone (figure 4 of Box 1.7) tilts westward with height, a mature tropical cyclone stands practically upright and is highly axisymmetric.

Tropical cyclones usually move westward with the average upper level wind in the tropics, and also slightly poleward (figure 2, this box). They reach their maximum intensity at subtropical latitudes. When a tropical cyclone makes landfall, it quickly looses its intensity. This also happens when it reaches relatively cold waters at higher latitudes. This clearly indicates that moisture is required to maintain a tropical cyclone. The most intense tropical cyclones, which are classified as category 5 cyclones according to the Saffir-Simpson scale, coloured bright red in figure 2, occur most frequently over the warm waters of the Western Pacific (figure 1.47), near the Philippines. These cyclones are called...
“Typhoons”. Most tropical cyclones in the Caribbean and the Gulf of Mexico, called “Hurricanes”, are category 2 to 4 cyclones. Blue trajectories in figure 2 reveal storms that are not classified as tropical cyclones, but as weaker tropical depressions or tropical storms. Most of these tracks reveal storms in their initial stages of development, or at the end of their life-cycle, when the make landfall or reach cooler regions.

**FIGURE 2 (Box 1.8).** Tracks of tropical cyclones over nearly 150 years. The tracks are coloured according to the Saffir-Simpson tropical cyclone intensity scale: category 5: windspeed at 10 m height> 135 knots; category 4: windspeed: 114-135 knots; category 3: windspeed: 96-113 knots; category 2: windspeed: 84-95 knots; category 1: windspeed: 65-83 knots; tropical storm (TS): 34-64 knots; tropical depression (TS): <34 knots (1 knot=0.5 m/s). Source: [http://earthobservatory.nasa.gov/IOTD/view.php?id=7079](http://earthobservatory.nasa.gov/IOTD/view.php?id=7079).

**PROBLEM 1.12. Vorticity in, wind in, and formation of a tropical cyclone**

An approximate model of the horizontal distribution of velocity in a tropical cyclone (Box 1.8) is the so-called "Rankine vortex". This is an axisymmetric circular vortex with an azimuthal velocity, $v_\theta$, which is a function of the radius, $r$ (the distance from the centre of the vortex), as follows.

$$v_\theta = \frac{v_0 r}{R} \quad \text{for } r \leq R$$

$$v_\theta = \frac{v_0 R}{r} \quad \text{for } r > R$$

Here, $v_0$ is the maximum wind velocity and $R$ is the radius of maximum wind velocity.

(a) Calculate and plot the vorticity as a function of $r$ for a Rankine vortex with $v_0=40$ m s$^{-1}$ and for hurricane Alicia (at 850 hPa) (figure 1.33), assuming axisymmetry.

(b) How would you define the Rossby number, $Ro$, in this context (section 1.23).

(c) Where at 850 hPa in Alicia does the Rossby number exceed a value of 1?

(d) Estimate the inertial period “inside” the radius of maximum wind at 850 hPa in Alicia.

(e) Why do tropical cyclones not form over the equator, over the Eastern South Pacific Ocean and over the South Atlantic Ocean (see figure 2, Box 1.8)?
1.25 Potential vorticity

A beautiful theorem, concerning vorticity in a stratified fluid, was derived by Ertel (1942) (Box 1.9). It states that, if (eq. 1.55)

\[ \frac{d\theta}{dt} = \frac{J}{\Pi} = 0, \]

then

\[ \frac{d}{dt} \left( \frac{\tilde{\omega}_a \cdot \tilde{\nabla} \theta}{\rho} \right) = 0. \] (1.124)

where the absolute vorticity

\[ \tilde{\omega}_a = 2\tilde{\Omega} + \tilde{\omega}. \] (1.125)

The derivation of this theorem from the vorticity equation (1.121), the continuity equation (1.7b), the potential temperature equation (1.55) and the definition of potential temperature (1.54) is given in Box 1.9.

The left hand side of eq. (1.124) represents the material time derivative of a quantity, \( \Omega_{\text{pot}} \), called Ertel's potential vorticity, i.e.

\[ \Omega_{\text{pot}} = \frac{\tilde{\omega}_a \cdot \tilde{\nabla} \theta}{\rho} . \] (1.126)

Therefore, in adiabatic conditions fluid elements conserve Ertel's potential vorticity, i.e.

\[ \frac{d\Omega_{\text{pot}}}{dt} = 0. \] (1.127)

This is a very remarkable constraint on the motion of the atmosphere. The theorem is derived from the equations of conservation of mass and momentum without making approximations. We’ll see in chapters 5, 7 and 12 that potential vorticity plays a central role in the dynamics of a rotating stratified fluid, such as the atmosphere. An approximate version of this conservation law, for the case of hydrostatically balanced flow, is derived schematically below and more accurately in chapter 7.

Averaged over a horizontal scale of 100 km or more the vertical component of the vorticity dominates over the horizontal vorticity components. Therefore, for these scales we can approximate (1.126) by (using the hydrostatic eq.1.65)

\[ \Omega_{\text{pot}} = \frac{\left( \zeta + f \right)}{\rho} \frac{\partial \theta}{\partial z} = -g \left( \zeta + f \right) \frac{\partial \theta}{\partial p} = \frac{\left( \zeta_{\theta} + f \right)}{\sigma} \equiv Z_{\theta}, \] (1.128a)

where \( \zeta_{\theta} \) is the relative vorticity on a constant potential temperature (isentropic) surface:
FIGURE 1.34. Monthly mean (upper panel: January; lower panel: July), zonal mean of isentropic potential vorticity (eq. 1.128a) (green lines, labeled in PVU, where 1 PVU=10^{-6} \text{K m}^2 \text{kg}^{-1} \text{sec}^{-1}), of pressure (dashed lines, labeled in hPa) and of zonal wind (red lines, labeled in m/s, interval: 5 m/s) as a function of potential temperature and latitude, derived from the COSPAR international Reference Atmosphere (Fleming, E. L., Chandra, S., Barnett, J. J. and Corney, M., 1990: Zonal Mean Temperature, Pressure, Zonal Wind, and Geopotential Height as Functions of Latitude. Advances in Space Research, 10, No. 12, 11-59). The thick black line indicates the Earth’s surface. The monthly average latitude of the overhead sun is indicated in red.
\[ \zeta_\theta = \left( \frac{\partial v}{\partial x} \right)_\theta - \left( \frac{\partial u}{\partial y} \right)_\theta, \]  

(1.128b)

The subscript, \( \theta \), indicates differentiation at constant potential temperature. \( Z_\theta \) is potential vorticity in the isentropic coordinate system, which employs potential temperature, \( \theta \), as a vertical coordinate, which is introduced in detail in chapter 7. This requires that potential temperature be a monotic function of height, or pressure. This is indeed the case in the greatest part of the atmosphere (figure 1.26). The symbol, \( \sigma \), defined as

\[ \sigma \equiv \frac{-1}{g} \frac{\partial p}{\partial \theta}, \]  

(1.128c)

is called isentropic density. It’s units are kg m\(^{-2}\) K\(^{-1}\). This, indeed, represents mass per unit volume in the coordinate system with potential temperature as the vertical coordinate.

Figure 1.34 shows the monthly mean and longitudinal (zonal) mean distributions of potential vorticity \( (Z_\theta) \), pressure and zonal wind as a function of potential temperature and latitude, for January and July. The subtropical jet is located at a latitude of 30° in winter and 45° in summer, close to \( \theta = 350 \) K, or \( p = 200 \) hPa. It is much stronger in winter than in summer. A strong eastward jet is observed in the winter stratosphere (above 200 hPa), while westward winds are observed in the tropics and in the summer stratosphere.

The border between the troposphere and the stratosphere, i.e. the tropopause, is defined by the World Meteorological Organization (WMO) as the lowest level at which the temperature lapse rate decreases to 2 K/km or less, provided that the average lapse rate (eq. 1.18) between this level and all higher levels within 2 km does not exceed 2 K/km. An alternative definition of the tropopause is the so-called dynamic definition of the tropopause, with potential vorticity instead of temperature lapse rate as the defining variable. The threshold value usually chosen is 2 Potential Vorticity Units (PVU). A PVU is equivalent to \( 10^{-6} \) K m\(^2\) kg\(^{-1}\) s\(^{-1}\). The troposphere is characterized by low values of potential vorticity, while the stratosphere is characterized by high values of potential vorticity. With the dynamical definition, the tropopause is a material surface under adiabatic conditions. With the WMO definition of the tropopause, the stratosphere and the troposphere do not represent separate air masses, which is not as convenient conceptually.

Note (figure 1.34) that the isobars between 100 hPa and 500 hPa are packed much closer together in the tropics than in the extratropics. This implies that this layer is characterized by high values of the isentropic density in the tropics and low values of the isentropic density in the extra-tropics. Chapter 7 discusses the dynamical implications of these remarkable characteristics of the thermal stratification of the Upper Troposphere and Lower Stratosphere (“UTLS”), which, until the 1960’s were still left unexplained. Chapter 12 reveals the underlying mechanisms.

**Box 1.9 Conservation of potential vorticity: Ertel’s theorem**

With the definition of absolute vorticity (eq. 1.125), the vorticity equation (1.121) can be written as

\[ \frac{\partial \bar{\omega}_a}{\partial t} = -\bar{\omega}_a \bar{\nabla} \cdot \bar{v} - (\bar{v} \cdot \bar{\nabla}) \bar{\omega}_a + \left( \bar{\omega}_a \cdot \bar{\nabla} \right) \bar{v} + \frac{\bar{v} \times \bar{\nabla} p}{\rho^2}, \]  

(1)

which, since,
\[
\frac{\partial \tilde{\omega}_a}{\partial t} + (\tilde{\nabla} \cdot \tilde{v})\tilde{\omega}_a = \frac{d\tilde{\omega}_a}{dt}
\]  
(2)

is also be expressed as

\[
\frac{d\tilde{\omega}_a}{dt} = -\tilde{\omega}_a \tilde{\nabla} \cdot \tilde{v} + (\tilde{\omega}_a \cdot \tilde{\nabla})\tilde{v} + \frac{\tilde{\nabla} \rho \times \tilde{\nabla} p}{\rho^2}.
\]  
(3)

The continuity equation,

\[
\frac{d\rho}{dt} = -\rho \tilde{\nabla} \cdot \tilde{v},
\]  
(4)

(eq. 1.7b) is now used simplify the vorticity equation (3):

\[
\frac{d\tilde{\omega}_a}{dt} = \frac{\tilde{\omega}_a}{\rho} \frac{d\rho}{dt} + (\tilde{\omega}_a \cdot \tilde{\nabla})\tilde{v} + \frac{\tilde{\nabla} \rho \times \tilde{\nabla} p}{\rho^2},
\]  
(5)

which, because (product rule),

\[
\frac{d}{dt}\left(\frac{\tilde{\omega}_a}{\rho}\right) = \frac{1}{\rho} \frac{d\tilde{\omega}_a}{dt} - \frac{\tilde{\omega}_a}{\rho^2} \frac{d\rho}{dt}
\]  
(6)

further simplifies into

\[
\frac{d}{dt}\left(\frac{\tilde{\omega}_a}{\rho}\right) = \left(\frac{\tilde{\omega}_a}{\rho} \cdot \tilde{\nabla}\right)\tilde{v} + \frac{\tilde{\nabla} \rho \times \tilde{\nabla} p}{\rho^3}.
\]  
(7)

We now combine eq. 7 with an equation describing the material time rate of change of the gradient of an arbitrary scalar quantity, \(\lambda\). This quantity will be associated with a physical quantity later. We start by writing,

\[
\tilde{\omega}_a \cdot \frac{d}{dt} \tilde{\nabla} \lambda = \tilde{\omega}_a_i \frac{d}{dt} \frac{\partial \lambda}{\partial x_i} = \tilde{\omega}_a_i \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) \frac{\partial \lambda}{\partial x_i}
\]

\[
= \tilde{\omega}_a_i \frac{\partial}{\partial x_i} \frac{\partial \lambda}{\partial t} + \tilde{\omega}_a_i \frac{\partial}{\partial x_i} \left( u_j \frac{\partial \lambda}{\partial x_j} \right) - \tilde{\omega}_a_i \frac{\partial u_j}{\partial x_i} \frac{\partial \lambda}{\partial x_j}
\]

\[
= \tilde{\omega}_a_i \frac{\partial}{\partial x_i} \frac{d\lambda}{dt} - \tilde{\omega}_a_i \frac{\partial u_j}{\partial x_i} \frac{\partial \lambda}{\partial x_j}
\]

which is written in vector form as

\[
\tilde{\omega}_a \cdot \frac{d}{dt} \tilde{\nabla} \lambda = (\tilde{\omega}_a \cdot \tilde{\nabla}) \frac{d\lambda}{dt} - (\tilde{\omega}_a \cdot \tilde{\nabla}) \tilde{v} \cdot \tilde{\nabla} \lambda.
\]  
(8)

Dividing eq. (8) by \(\rho\) and combining it with (7), we get
\[
\frac{d}{dt} \left( \frac{\vec{\omega}_a}{\rho} \cdot \vec{\nabla} \lambda \right) = \vec{\nabla} \lambda \cdot \left( \frac{\vec{\omega}_a}{\rho} \cdot \vec{\nabla} \right) \vec{v} + \vec{\nabla} \lambda \cdot \vec{\nabla} \rho \times \vec{\nabla} p + \frac{1}{\rho} \left( \vec{\omega}_a \cdot \vec{\nabla} \right) \frac{d\lambda}{dt} - \frac{1}{\rho} \left( \vec{\omega}_a \cdot \vec{\nabla} \right) \vec{v} \cdot \vec{\nabla} \lambda .
\]

which simplifies into

\[
\frac{d}{dt} \left( \frac{\vec{\omega}_a}{\rho} \cdot \vec{\nabla} \lambda \right) = \vec{\nabla} \lambda \cdot \vec{\nabla} \rho \times \vec{\nabla} p + \frac{1}{\rho} \left( \vec{\omega}_a \cdot \vec{\nabla} \right) \frac{d\lambda}{dt} .
\]

This is Ertel's theorem. If \( \lambda \) is materially conserved, i.e.

\[
\frac{d\lambda}{dt} = 0
\]

and \( \lambda \) is a function only of \( \rho \) and \( p \), then

\[
\frac{d}{dt} \left( \frac{\vec{\omega}_a}{\rho} \cdot \vec{\nabla} \lambda \right) = 0.
\]

That is, the quantity between brackets on the left hand side of (11), referred to as “potential vorticity”,

\[
\frac{\vec{\omega}_a}{\rho} \cdot \vec{\nabla} \lambda
\]

is materially conserved. If we intend to apply this theorem to the atmosphere we must identify a materially conserved atmospheric scalar quantity that is a function only of \( \rho \) and \( p \). Potential temperature, \( \theta \), conforms to these requirements. Thus, the usual definition of potential vorticity for Earth’s atmosphere is

\[
\Omega_{pot} = \frac{\vec{\omega}_a}{\rho} \cdot \vec{\nabla} \theta,
\]

with

\[
\frac{d\Omega_{pot}}{dt} = 0
\]

in adiabatic circumstances.

**References to Box 1.9**


1.26 Isentropic view of the atmosphere

**Figure 1.35** shows a schematic view of the zonal average, time average vertical structure of the atmosphere as a function of latitude and pressure in terms of potential temperature and potential vorticity. The isentropes between, approximately, 310 K and 380 K intersect the dynamical tropopause. These isentropes belong to the “Middleworld”, a term which was proposed by Hoskins in 1991\(^{36}\), who was inspired by the terms, “Underworld” for the isentropes cutting the Earth's surface and “Overworld” for the isentropes spanning the globe, which were both proposed by Napier Shaw in the 1920’s (figure 1.25).

![Schematic view of the atmosphere](image)

**Figure 1.35.** Schematic view of the atmosphere, as a function of latitude (pole-equator) and height (surface-30 hPa). The dynamical tropopause is marked by a thick line, and isentropes every 30 K from 270-390 K by thin lines. The arrows indicate some diabatic cross-isentropic and adiabatic isentropic transports (from Hoskins, B.J., 1991. *Tellus*, 43AB, 27-35).

Middleworld isentropes probe both the stratosphere and the troposphere. This is illustrated in **Figure 1.36**, which shows the evolution of the potential vorticity field at 24 hour intervals on the 350 K isentropic surface between 00 UTC on 27 January 2007 and 00 UTC on 29 January 2007. The transition from the troposphere to the stratosphere is marked by a strong isentropic potential vorticity gradient at sub-tropical latitudes (about 30°). Both hemispheres are characterized by two regions of approximately uniform potential vorticity at 350 K: low tropospheric values (<1 PVU) in the tropics, equatorward of 30° latitude, and high stratospheric values (> 4 PVU) in the extra-tropics. The relatively narrow transition region is referred to as the isentropic tropopause\(^{37}\). Meanders in the isentropic tropopause reveal the presence of large scale planetary waves, which are referred to also as Rossby waves, in honour of Carl Gustav Rossby, who was the first to present a theory of the origin and propagation of these waves (section 1.37).

Diabatic effects at 350 K are weak enough that they do not significantly affect the evolution of potential vorticity over periods of several days. In such nearly adiabatic conditions, air parcels do not depart from an isentropic surface, i.e. **an isentrope is a**

---


material surface. Since potential vorticity (PV or $Z_{\theta}$, defined in eq. 1.128a) is also materially conserved, changes in the PV-distribution are determined by two-dimensional isentropic advection on an isentropic surface. Observing the evolution of $Z_{\theta}$ on an isentrope, therefore, offers insight into the turbulent adiabatic dynamics of the atmosphere.

The consequence of adiabatic potential vorticity mixing “across” the zonal mean isentropic tropopause is observed over the eastern Atlantic Ocean in the Northern Hemisphere (figure 1.36). A “pocket” of low potential vorticity tropical air, referred to as a negative potential vorticity (PV) anomaly, penetrates into the middle latitudes. This process is referred to as “Rossby wave breaking”. If the isentropic PV-anomaly is sufficiently large horizontally, it is associated with, or “induces”, an “anticyclone” over the full depth of the troposphere (figure 1.93). The association of a negative PV-anomaly with an anticyclone and a positive PV-anomaly with a cyclone is a manifestation of thermal wind balance and is explained in chapter 7.

Figure 1.37 demonstrates that the strength of the subtropical jet is positively correlated with the meridional potential vorticity gradient in the subtropics, i.e. high (low) potential vorticity gradients are connected to strong (weak) zonal flow. This link is also a manifestation of thermal wind balance. Isentropic mixing of PV across the zonal mean isentropic tropopause, due to “breaking” Rossby waves, which reduces the zonal mean meridional PV gradient, thus, governs the strength of the subtropical jet.

Over periods much longer than one week, diabatic effects determine the distribution of potential vorticity. We shall see (chapter 12) that the almost piecewise uniform potential vorticity distribution on isentropic surfaces in the Middleworld is in fact a result of the peculiar distribution of diabatic heating, illustrated in figure 1.2, which is characterized by heating in the tropics and cooling in the extra-tropics. Because potential temperature in the Middleworld increases with height, diabatic heating is perceived as “upward motion” on an isentropic surface, while diabatic cooling is perceived as “downward motion” on an isentropic surface (figure 1.35). Since potential vorticity increases with height, diabatic heating usually leads to a local decrease of isentropic potential vorticity, while diabatic cooling leads to a local increase of isentropic potential vorticity, $Z_{\theta}$. At 350 K, in other words, diabatic heating in the tropics represents a permanent “sink” of potential vorticity, while diabatic cooling in the extratropics represents a permanent “source” of potential vorticity.

Identifying the processes that determine the large-scale distribution of diabatic heating (figure 1.2), and so the sources and sinks of potential vorticity on isentropic surfaces, is an important task that will keep us busy in the latter part of these notes. We’ll see (especially in chapters 7 and 12) that the potential vorticity distribution can be understood only by studying the complex non-linear interaction between dynamics and the diabatic processes that are associated with radiation and phase changes of water.

**PROBLEM 1.13. Isentropic mixing across the isentropic tropopause**

Use the dataviewer PANOPLY, (see section iii of the Preface) to make an animation of the evolution of potential vorticity on the 350 K isentrope in the northern hemisphere in the winters of 2006-2007 and 2009-2010, according to the ERA-Interim reanalysis (http://apps.ecmwf.int/datasets/). In which of these two winters is cross-tropopause isentropic mixing strongest? In which way did these winters differ, as far as circulation over the Atlantic Ocean and the weather in Europe is concerned?

---

FIGURE 1.36. Potential vorticity ($Z_\theta$) on the 350 K isentropic surface, according to the ERA-Interim reanalysis at 24 hour intervals, i.e. at (a) 00 UTC January 27, 2007, (b) 00 UTC January 28, 2007, and (c) 00 UTC January 29, 2007. Contour correspond to ±1.6, ±4.8 and ±8 PVU (dashed contours: negative values). Data from http://apps.ecmwf.int/datasets/.
FIGURE 1.37: Zonal (longitudinal) mean zonal velocity (upper panel) and zonal mean potential vorticity (lower panel) at $\theta$=350 K in a so-called “Hovmöller diagram” for the winter of 2006-2007 (1 Dec. 2006 to 28 Feb. 2007). Based on ERA-Interim reanalysis. The original Hovmöller diagram, defined as a contour plot with time along one axis and a spatial coordinate along the other axis, is shown in figure 1.91 (section 1.37). Contour interval is 10 m/s and 1 PVU, respectively. Labels are in units of m/s and $10^{-6}$ PVU, respectively. The zonal mean is defined mathematically in Box 1.11 (eq. 8). Data from http://apps.ecmwf.int/datasets/.

1.27 Scales imposed by the fundamental properties of the climate system

The fundamental quantities that shape the atmospheric circulation in space and time are (1) the mass of the atmosphere, $p$/g (kg m$^{-2}$) (problem 1.14), (2) the specific heat capacity at constant pressure of air, $c_p$ (1004 J kg$^{-1}$K$^{-1}$), (3) Earth’s angular velocity, $\Omega$ ($7.292 \times 10^{-5}$ s$^{-1}$), (4) the Solar energy flux at the top of the atmosphere, $Q$ (the annual and global average value of $Q$ is 342 W m$^{-2}$) and (5) the fraction of the Solar beam that is reflected back to
space, the so-called “planetary albedo”, \( \alpha_p \). Planetary albedo is determined significantly by the composition of the atmosphere and by the condition of Earth’s surface (land, ocean, ice, vegetation).

**PROBLEM 1.14. Mass of the atmosphere**

Why can the total mass of the atmosphere per unit surface area be written as \( m = p / g \)?

In radiative equilibrium, the radiation that is emitted by both the Earth’s surface and the atmosphere, which escapes to space at the top of the atmosphere (TOA), is equal to the absorbed Solar radiation. The heat absorbed by a vertical column of air in the atmospheric (in J m\(^{-2}\)) is approximately by \( mc_p T_E \), where \( m \) is the mass of the atmosphere per unit surface area (= \( p / g \); problem 1.14) and \( T_E \) is the average temperature of the atmosphere. The power (in J m\(^{-2}\) s\(^{-1}\)) absorbed by the planet is \((1-\alpha_p)Q\), This yields a radiative equilibrium timescale, \( \tau_E \), which is determined by the following equation.

\[
\tau_E = \frac{p c_p T_E}{g(1-\alpha_p)Q}.
\]  

(1.129)

With realistic values of the parameters involved (for example: \( T_E = 250 \) K, \( p = 10^5 \) Pa, \( c_p = 1005 \) J kg\(^{-1}\) K\(^{-1}\) and \( \alpha_p = 0.3 \)), this yields \( \tau_E = 10^7 \) s or about 120 days (4 months). Of course, never did this situation occur in reality. The closest analogue of this situation is the Northern winter, in which there is no insolation for several months. The atmosphere over the poles cools by emission of radiation to space for up to 6 months during the polar night. Thus, radiative equilibrium is not possible in this region during winter. During the summer the atmosphere over the pole has time to absorb heat from the Sun and readjust to radiative equilibrium. This topic is the subject of chapter 2.

The above argument assumes that Solar power is ultimately absorbed completely by the atmosphere. This assumption holds only as long as the Earth itself (soil or water) does not retain the Solar energy after absorption, i.e. as long as the heat capacity of the Earth’s surface is negligible. Over land this is a reasonable assumption. However, over appreciable water bodies, such as the oceans, we must take account of the absorption and retention of Solar heat by water. Solar radiation can easily penetrate into the upper 25 m of the ocean. The heat capacity of a layer of water of depth, \( d_w = 25 \) m, is \( \rho_c c_w d_w = 10^3 \times 4218 \times 25 \approx 10^5 \) J K\(^{-1}\) m\(^{-2}\). The heat capacity of the entire atmosphere is \( c_p p / g = 10^5 \times 10^5 / 10 \approx 10^7 \) J K\(^{-1}\) m\(^{-2}\), i.e. a factor 10 less! In other words, the thermal capacity of the entire atmosphere is equal to the thermal capacity of less than 2.5 m of water. Absorption of Solar radiation by the ocean must, therefore, be an important factor in determining the timescale of temperature variations in the lower atmosphere over the oceans and coastal land areas (Box 1.14).

Frequencies of oscillations around the fundamental equilibria of hydrostatic balance and geostrophic balance, i.e. the Brunt-Väisälä frequency, \( N \) (s\(^{-1}\)) (section 1.15) and the inertial frequency, \( F \) (s\(^{-1}\)), (section 1.20) bring forth two fundamental time scales: \( \tau_1 = 1/N \) and \( \tau_2 = 1/F = 1/f \). The parameter, \( \tau_1 \), multiplied by \( 2\pi \), is called the buoyancy period. In the atmosphere its magnitude is usually about \( 2\pi \times 10^2 \) s (about 10 minutes). The parameter, \( \tau_2 \), multiplied by \( 2\pi \), is called the inertial period. Its magnitude, if the relative vorticity is small compared to the planetary vorticity, is \( 2\pi / (2\Omega \sin\phi) \) (12 hours at the poles; 24 hours at 30° latitude, increasing towards the equator).

For the three time scales, that we have just identified, we may write...
These three time scales, i.e. $\tau_1$, $\tau_2$, and $\tau_E$, reflect the classical division of the subject of Atmospheric Dynamics into, respectively, the subdisciplines, Boundary Layer Meteorology, Dynamical Meteorology and Climate Dynamics.

As far as length scales are concerned, it is not straightforward to come up with one that is externally imposed. A vertical length scale, $H$, may vary according to the location (height) of temperature-inversions, which act as a ceiling on atmospheric circulations. One such ceiling, obviously, is the tropopause at a height of about 10 km (figure 1.34). Therefore, we introduce the (vertical) lengthscale, $\lambda_1 = H = 10$ km.

It is then possible to construct two additional length-scales using the parameters $H$, $N$ and $f \approx f$, i.e.

$$
\lambda_2 = \frac{NH}{f}; \lambda_3 = \frac{fH}{N}.
$$

We’ll see that $\lambda_2$ and $\lambda_3$ appear naturally in the theory of atmospheric dynamics. The length-scale, $\lambda_2$, is usually called the Rossby radius of deformation after Rossby and takes on a value of about $10^5$-$10^6$ m (100-1000 km). The Rossby radius of deformation governs the horizontal scale of the process of adjustment of the velocity field to the mass field (or vice-versa) towards geostrophic balance (chapter 5). The length-scale, $\lambda_3$, is referred as the Rossby height. The Rossby height governs the vertical scale of the process of adjustment towards thermal wind balance (chapter 7). The ratio of these length scales is the dynamically relevant quantity, i.e. the aspect ratio (the horizontal scale divided by the vertical scale), $N/f \approx 100$. Circulation systems that are in thermal wind balance, typically have an aspect ratio in order of 100, i.e. they are very flat, which is accord with the typically observed aspect ratio of cyclones and anticyclones in the atmosphere (chapter 7).

---

**Box 1.10 Spectral or Fourier analysis and the mesoscale range of scales**

A series of $N$ consecutive measurements of a variable, $U_n$ ($n=0$ to $n=N-1$), regardless of how complex in form (except that it must be continuous or without breaks, and there can only be one value of $U$ for each value of $t$), can be represented as the real part of the sum of a series of complex exponential functions, or sine/cosine time ($t$) dependent wave-forms, as follows.

$$
U(t) = \text{Re} \left[ \sum_{k=0}^{N/2} C_k \exp\{i(2\pi kt/T)\} \right].
$$

The amplitudes or spectral (“Fourier”) coefficients, $C_k$, are complex, i.e.:

$$
C_k = \alpha_k + i\beta_k
$$
$T$ is the total length in time of the record or series of measurements. The parameter $k$ is the wave number, or frequency. The corresponding period is

$$T_k = \frac{T}{k}.$$

The amplitudes for each wave number, $k$, i.e. the spectral coefficients, $\alpha_k$ and $\beta_k$, for wave numbers $k=0$ to $k=N/2$, can be found by evaluating

$$\alpha_k = \frac{2}{N} \sum_{n=0}^{N-1} U_n \cos\left(\frac{-2\pi kn}{N}\right), \quad \beta_k = \frac{2}{N} \sum_{n=0}^{N-1} U_n \sin\left(\frac{-2\pi nk}{N}\right) \quad \text{and} \quad \alpha_0 = \frac{1}{N} \sum_{n=0}^{N-1} U_n.$$

$\alpha_0$ represents the average. Note that $\beta_0 = 0$. Hence we have $N+1$ spectral (Fourier) coefficients. The amplitude of the $k^{th}$ harmonic is

$$A_k = \sqrt{\alpha_k^2 + \beta_k^2}.$$

A summary of spectral analysis of several meteorological time series is shown in figure 1 (this box). The maximum in the spectrum at periods of several days is identified with the synoptic scale (Rossby waves, cyclones, anticyclones, etc.), while the maximum at periods of several minutes is identified with the micro-scale (e.g. turbulence). A minimum in the variance spectrum, i.e. a spectral gap, at periods ranging from half an hour to several hours is identified with the meso-scale.

![Figure 1](https://example.com/figure1.png)

**FIGURE 1 (Box 1.10):** Spectrum of the kinetic energy associated with the horizontal wind measured at different locations. Each curve is from a different study. The dominant peak in the wavelength range from 3000 to 6000 km represents the troughs and ridges in the large scale pressure field at mid-latitudes. The small bump near 500 km represents the fronts, while the peak at about 1 km represents the energy associated with cumulus convection. Source: J.S.A. Green, 1979: Topics in dynamical meteorology: 8. Trough-ridge systems as slantwise convection (1). *Weather*, 34, 2-10.
The existence of a spectral gap is not an expression of the fact that individual meso-scale circulation systems contain little energy and that they are therefore unimportant, rather it is an indication that these circulations are highly intermittent\textsuperscript{40}. They occur only if certain conditions are fulfilled. For example, a sea breeze circulation (section 1.28) will occur only when the temperature difference between the sea and the land is large enough, and even then, other conditions must be met. Thunderstorms (section 1.29), hurricanes (problem 1.12), lee-troughs (figure 1.4) and sea-breeze circulations (section 1.28) all of which are examples of meso-scale circulations, which are characterized significant departures form hydrostatic- and/or geostrophic-balance, occur only in certain areas and in certain seasons. This is in contrast to the ever-present small scale turbulent eddies in the boundary layer as well as the continuous meandering of the isentropic tropopause (figure 1.36).

What makes the range of time-scales between 1 hour and 1 day "intermediate" (mesoscale) and, therefore, defines the boundary between "slow" and "fast"? These lecture notes intend to provide physical and mathematical insight into the answer to this question. This insight is gained by studying the problem of adjustment to balance. The atmosphere is continuously brought out of balance by differential absorption of Solar radiation or by other non-adiabatic (i.e. diabatic) effects. Much of the motion that is observed in the atmosphere is a result of (re-)adjustment to thermal wind balance. The principal chapters in these lecture notes are thus concerned with the stability properties of, and the adjustment to, hydrostatic balance (chapters 3 and 4), geostrophic balance (chapter 5) and thermal wind balance (chapters 7, 8 and 9).

\textbf{PROBLEM 1 (Box 1.10). Spectral analysis}

Perform a spectral analysis of a series of hourly pressure-, temperature-, and wind-measurements near the Earth’s surface at a weather station in the Netherlands (e.g. Schiphol). Hourly measurements are available from 1951 until today. Plot the spectra, i.e. the amplitude as a function of the frequency or period. Interpret the spectra. The data can be downloaded from the following website:

\url{http://www.knmi.nl/klimatologie/uurgegevens/selectie.cgi}

1.28 \textbf{Forcing and response: daily and seasonal cycle}

This section discusses examples of the response of the atmosphere to heating. The first example is the atmospheric tide. Atmospheric tides are global scale daily oscillations (in pressure, temperature and wind), which are primarily forced by diurnal variations of the heating due to absorption of Solar radiation by atmospheric water vapour and ozone. The Solar and lunar gravitational forcing of tides is much less important for the atmosphere than for the ocean. We can distinguish migrating tides and nonmigrating tides. Migrating tides move with the overhead Sun, while nonmigrating tides are associated with, for example, geographically fixed tropospheric heat sources, associated with the distribution of land and sea. Most of the Solar radiation reaches the Earth’s surface and heats the surface of the Earth. The response to this heating in terms of temperature is much stronger over land than over sea. This is because Solar radiation penetrates into the water over a depth in the order of 10 m, while Solar radiation heats only a very thin layer (order 1 cm) of soil. Moreover, a much larger portion of the Solar radiative energy is “used” to evaporate water over the oceans than

---

\textsuperscript{40}Ishida, H., 1990: Seasonal variations of spectra of wind speed and temperature in the mesoscale frequency range. \textit{Boundary-Layer Meteorol.}, 52, 335-348.
over land. If insolation is strong enough during day light hours this state of affairs leads to high temperatures over land and low temperatures over sea, which, in turn, leads to an expansion of the air column over land relative to the air column over sea. Details about how exactly this expansion is communicated to upper levels by acoustic (sound) waves are given in chapter 3. Because pressure at upper levels over land is higher than at the same level over sea, air masses at upper levels move from land to sea. This, in turn, leads to relatively low sea level pressure over land and relatively high sea level pressure over sea (figure 1.38). The compensating low-level air flow from sea to land represents the well known sea breeze, which is frequently experienced at the beach on a summer afternoon.

![Figure 1.38](image)

**Figure 1.38.** Pattern of isobars (lines of equal pressure) at two constant height levels (constant with respect to sea level) in the atmosphere, associated with heating of the atmosphere between these two levels. In this example the heating is due to release of latent heat in clouds. Air motion is upward over land and downward over sea. The Coriolis effect is not taken into account in this picture. So, this picture illustrates the sea breeze circulation in the tropics. Source: figure 10.9 of J.M.Wallace and P.V. Hobbs, 2006: *Atmospheric Science: An Introductory Survey*. Academis Press/Elsevier, 483 pp.

**Figure 1.39** shows the annual mean diurnal variation of the pressure near the Earth's surface over the island of Java (Indonesia) over land and over the adjacent sea. The double diurnal tidal period of the pressure and the single diurnal period of the pressure-difference between land and sea is clearly seen. The double diurnal variation is associated with a migrating tide, while the weaker single diurnal variation, which leads to the sa breeze circulation, is non-migrating.
The annual mean diurnal variation of the pressure in Java over land and sea. Dashed line: Djakarta; full line: open sea 100 km off the coast. Between 11 hour local time and 19 hours local time (indicated by the vertical arrows) pressure over sea is higher than over land (Source: Kraus H., 1987: Specific surface climates. In Landolt-Börnstein, New Series, vol 4c, part 1, p. 29-92.).

A graph of the typical daily variation of the solar heating, $J_{\text{solar}}$ (heavy curve). The horizontal dashed line represents the diurnal (or zonal) average of $J_{\text{solar}}$, which is assumed to be balance by the diurnal (or zonal) average infrared cooling, so that the net heating is zero (from Andrews et al., 1987, see list in section ii).

The time evolution of Solar heating, which forces the tide, might look something like that shown in figure 1.40. Fourier analysis (Box 1.10) of this curve will include a steady component, a diurnal component (with a 24 hr period), a somewhat smaller semidiurnal component (with a 12 hr period), and so on. The response of the atmosphere to this heating can likewise be decomposed into a steady part, and diurnal, semidiurnal and higher frequency oscillations. A puzzle, raised by Lord Kelvin in 1882, is to explain why the semidiurnal surface pressure oscillation (about 1 hPa in amplitude at the Earth's surface) is larger and more regular than the diurnal surface pressure oscillation.
Figure 1.41. The “classical” analysis by van Bemmelen. Isolines of the speed (in m/s) of the north-south component of the wind velocity above Batavia (now Djakarta, Java) are obtained from hourly mean values evaluated from pibal observations from May to November 1909 to 1915. Wind from the sea (sea breeze) is hatched. There is a pronounced return current with a maximum shortly before sunset and in height of 2 km (Van Bemmelen, W., 1922: Land- und seebrise in Batavia. Beitr. Phys. frei. Atmos., 10, 169-177).

The diurnal surface pressure oscillation is manifested principally in the pressure difference between land and sea (figure 1.39). These pressure differences are responsible for the existence of the sea breeze circulation. Figure 1.41 shows an analysis by the Dutch meteorologist, Willem van Bemmelen, of observations of the sea breeze over Java.

The influence of the Coriolis force appears in the gradual turning of the wind from a direction perpendicular to the coast at the beginning of the day to a direction parallel to the coast in the course of the afternoon. This manifestation of the process of adjustment to geostrophic balance in which the pressure gradient force comes into balance with the Coriolis force, is not noticed over Java, since the Coriolis parameter, $f$, tends to zero at the equator. It is, however, noticed strongly in the extra-tropics. Along the western coast of the Netherlands the adjusted state would imply a wind coming approximately from the north to north east (parallel to the coast), as is observed in figure 1.42. In fact, in the example shown in figure 1.42, the wind turning is observed as far inland as Deelen (station 275) (at a distance of about 100 km from the coast).

The influence of heating can also be recognized on a global scale (figure 1.43). In general, the sea-level pressure over the continents is lower in summer than in winter, while the reverse is the case over the oceans. This implies that the so-called "sea-breeze-effect" penetrates inland over thousands of kilometres! In fact diurnal pressure oscillations at the Earth's surface are observed very far inland over, for instance, the European continent in the summer (section 3.10).
FIGURE 1.42. Upper left: Map of the Netherlands and the vicinity. The positions of measuring sites are shown by white circles (seabreeze case) and black circles (thunderstorm case, see next section) (thanks to Peter Duynkerke). Lower left: Temperature as a function of time on May 8, 1976, at three stations in the Netherlands. Station 225 lies at the coast while stations 260 (De Bilt) and 375 lie 60 km and more than 100 km inland, respectively. Right: An illustration of the influence of the Coriolis force on the wind-direction. Shown are hourly wind vectors at 10 m above the Earth’s surface, as a function of time on May 8, 1976, at six stations in the Netherlands. The station on the left (225) is closest to the coast, while the station on the right (290) is furthest from the coast.

The anticyclonic turning of the wind in time, observed in figure 1.42, is an example of an inertial oscillation (section 1.20). The low-level jet, which forms around sunset when the Earth’s surface cools and a stable stratification develops (\( \partial \theta / \partial z \gg 0 \)) in the lowest 10 to 100 metres above the Earth’s surface also exhibits an inertial oscillation. The stable stratification just above the Earth’s surface suppresses turbulence and eddy viscosity (section 1.4). The layer above this thin stable layer becomes decoupled from the Earth’s surface. The rather abrupt reduction of the drag force, due to eddy viscosity, at sunset disturbs the daytime balance in the surface layer between the drag force, the pressure gradient force and the Coriolis force. This accelerates the horizontal wind above the stable layer, thereby producing a wind maximum (i.e. a low-level jet) at heights between 10 and several 100 m. The direction of the low level jet rotates anticyclonically with a period corresponding to the inertial period, \( 2 \pi / \omega \), as is observed in figure 1.44.
Figure 1.43. Mean sea-level pressure in December, January and February (upper panel) and in June, July and August (labeled in hPa) (average for the period 1979-2004). The subtropical high-pressure systems are most intense in the winter over the oceans. The Asian winter high-pressure system is due to the presence of a relatively thin layer of cold continental air. Sea level pressure is in general relatively low at a latitude of about 60° in both hemispheres. The low surface pressure over the south Asian in summer (June-August) is caused by intense (latent) heating. At upper levels an anticyclone is observed in this region (figure 1.50). Source JRA-25 Atlas. (see http://ds.data.jma.go.jp/gmd/jra/atlas/eng/atlas-top.htm).

Problem 1.15. Probability density function
Compute and plot the probability density function of the wind direction at Schiphol Airport for the month of May and for the month of September. Repeat this analysis for the wind direction at 10 UTC and for 15 UTC. Discuss the results, i.e. the differences between the two times and two months.
(http://en.wikipedia.org/wiki/Probability_density_function)
Box 1.11 Continuity equation and the zonal mean mass-streamfunction

Consider an element of air that is initially rectangular with sides $\delta x$, $\delta y$ and $\delta z$. A short time later, this element will be slightly distorted. To first order in $\delta x$, $\delta y$ and $\delta z$, the volume changes only because of small changes in the lengths of the sides. Slight rotations of the edges are not associated with significant volume changes to this order of accuracy. The mass of this element of air, $\delta m$, is equal to $\rho \delta x \delta y \delta z$. With (1.65) and $\delta x = a \cos \phi \delta \lambda$, $\delta m$ becomes,

$$\delta m = -\frac{a \cos \phi \delta \lambda \delta y \delta p}{g}.$$

(1)

Following the motion, the mass $\delta m$ is conserved, i.e.

$$\frac{d}{dt}(\delta m) = \frac{d}{dt}\left(-\frac{a \cos \phi \delta \lambda \delta y \delta p}{g}\right) = 0.$$

(2)

Dividing by $\delta m$ and carrying out the differentiation gives

$$-\frac{1}{\cos \phi \delta \lambda} \frac{d}{dt}(\delta m) + \frac{1}{\delta y} \frac{d}{dt}(\delta y) + \frac{1}{\delta p} \frac{d}{dt}(\delta p) = -\frac{\tan \phi \delta y}{a} \frac{d}{dt}(\delta \lambda) + \frac{1}{\delta \lambda} \frac{d}{dt}(\delta \lambda) + \frac{1}{\delta y} \frac{d}{dt}(\delta y) + \frac{1}{\delta p} \frac{d}{dt}(\delta p) = 0.$$

(3)

The third term in this equation may be rewritten as follows

$$\frac{1}{\delta y} \frac{d}{dt}(\delta y) = \frac{d}{dt}\left(y(B) - y(A)\right) = \frac{1}{\delta y} \left(v(B) - v(A)\right) = \frac{\delta(v)}{\delta y} = \frac{\partial v}{\partial y},$$

(4)

where $(vB) - v(A)$ is the velocity difference across the element in the $y$-direction, i.e.
between two points, A and B, a distance $\delta y$ apart in the y-direction. If the other two terms are rewritten similarly, (3) becomes

$$-\frac{v \tan \phi}{a} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \omega}{\partial p} = 0,$$

(5)

or

$$\left(\frac{\partial u}{\partial x} + \frac{1}{a \cos \phi} \frac{\partial (v \cos \phi)}{\partial \phi}\right) + \frac{\partial \omega}{\partial p} = 0,$$

(6)

where $\omega$ is referred as the *"vertical velocity" on an isobaric surface*, i.e.

$$\omega = \frac{dp}{dt}.$$

(7)

The subscript $p$ in (5) and (6) indicates that the partial derivatives are taken with $p$ is constant. We now define the zonal average of any quantity, for instance the zonal average of the vertical velocity ($\langle \omega \rangle$), as follows:

$$[\omega] = \frac{1}{2\pi} \int_0^{2\pi} \omega \sin \lambda d\lambda.$$

(8)

Let us now take the zonal average of (6) around a full latitude circle. This yields,

$$\frac{1}{a \cos \phi} \left(\frac{\partial [v \cos \phi]}{\partial \phi}\right)_p + \frac{\partial [\omega]}{\partial p} = 0.$$

(9)

This can also be expressed as

$$\left(\frac{\partial [v \cos \phi]}{\partial y}\right)_p + \frac{\partial [\omega \cos \phi]}{\partial p} = 0.$$

(10)

The zonal average northward mass-flux through a vertical section of the atmosphere at latitude circle at latitude $\phi$, above a particular pressure level $p$, is expressed in terms of a mass streamfunction:

$$[\Psi_m] = \frac{2\pi a \cos \phi}{g} \int_0^p [v] dp \ [\text{kg s}^{-1}].$$

(11)

This quantity is plotted in figure 1.4.5. Equations (10) and (11) require that

$$[v] = -\frac{g}{2\pi a \cos \phi} \frac{\partial [\Psi_m]}{\partial p}; \ [\omega] = -\frac{g}{2\pi a \cos \phi} \frac{\partial [\Psi_m]}{\partial y}.$$

(12)
The Hadley circulation is the longitudinally average (or “zonal average”; see eq. 8 of Box 1.11) meridional circulation in the tropics. It represents the zonal average dynamical response to excess heating in the tropics. This excess heating is due to absorption of radiation as well as due to latent heat release in large clouds over the ITCZ (Intertropical Convergence Zone) (figure 1.45). On average, the layer between 500 hPa and 10 hPa is heated in the tropics and cooled elsewhere (figure 1.2). The reasons for the cooling of the upper troposphere and lower stratosphe in the extra-tropics are interesting and far from
trivial to understand. The downward branch of the Hadley circulation in the subtropics and the resulting divergence at the Earth’s surface leads to the formation of the subtropical anticyclones, such as the Azores anticyclone (figure 1.43).

The “Ferrel cell”, observed in midlatitudes, is an “indirect” circulation, meaning that cold air rises on the poleward side and warm air descends on the equatorward side. Therefore, the Ferrel cell cannot be the response to heating. The explanation of the origin of the Ferrel circulation is the subject of chapter 11.

At the intertropical convergence zone (ITCZ) (figure 1.45, lower panel), where the trade winds from both hemispheres converge, air must rise. This is the area where large clusters of precipitating clouds are formed. The associated maximum in the precipitation is clearly observed in figure 1.17.

![Figure 1.46](image)

**Figure 1.46.** Average (extreme) position of the ITCZ (intertropical convergence zone) in July and January, respectively. Source: Bridgeman, H.A., and J.E. Oliver: 2006: The global climate system. Cambridge University Press. 331 pp.

The ITCZ is not always located in the summer hemisphere, as should be expected (figure 1.46). In the northern hemisphere the ITCZ reaches the Tropic of Cancer, at 23.5°, only over the central Sahara Desert and over South Asia, where it even lies north of the Tropic of Cancer in July. Surprisingly, despite the moisture convergence that is associated with the ITCZ in July, and in contrast to Southern Asia, the Central Sahara receives almost no monsoonal rains. According to paleoclimatic data analysis, this has been the case only for the past 5500 years. Before that the Sahara was green and teeming with life. Model investigations have revealed that the interaction between the atmosphere and vegetation in this region is highly non-linear. A slight change in the orbital parameters of the Earth (chapter 2) can have a huge impact on climate in this region due to this nonlinear interaction.

Over the Eastern Pacific Ocean and over the Eastern Atlantic Ocean, the ITCZ remains in the northern hemisphere nearly permanently. This is probably due to the relatively low sea surface temperatures at and just south of the equator, due to upwelling of cold water. A double-ITCZ is also observed in some regions, such as in the Pacific, i.e. a precipitation maximum is observed on both sides of the equator. In the western Pacific in the southern

---

41 The reasons for the upper tropospheric and lower stratospheric cooling in the extra-tropics and its interesting dynamical consequences are discussed in section 7.14 and in chapter 12.

hemisphere the “South Pacific Convergence Zone” (SPCZ) is a prominent weather feature. The SPCZ is manifest as a band of high cloudiness and high precipitation (figure 1.17) extending from New Guinea southeastward into the mid-latitudes, appearing to merge with the high precipitation-band associated with the mid-latitude storm track. The SPCZ is collocated with a zone of high sea surface temperatures in the equatorial southern hemisphere west Pacific (figure 1.47). A similar feature is observed in the western Atlantic. Here a zone of high average precipitation is observed extending from southern Brazil southeastwards over the Atlantic Ocean. This zone and the SPCZ are not shown in figure 1.46.

a. January

b. July

The sea surface temperature distribution is quite variable in time, especially in the equatorial Pacific. The variations in the Pacific are associated with an important climatological phenomenon that determines climate variations over a large part of the globe. This phenomenon is referred to by the acronym, “ENSO”, which stands for “El Niño-Southern Oscillation”. El Niño is the oceanic manifestation of ENSO. Every few years the temperature-distribution in the equatorial Pacific changes radically: the cold surface waters in the east become warmer, while the warm surface waters in the equatorial western Pacific become colder (figure 1.48). The precipitation shifts towards the east, bringing floods in Peru and drought in Indonesia and Australia. The El Niño of 1997-1998 was one of the most intense El Niño’s on record. High evaporation rates over the central and eastern equatorial...
Pacific during this period, raised the global and yearly average precipitable water content of the atmosphere in 1998 by about 1%. Globally, the year 1998 is one of the warmest years on record. The enhanced greenhouse effect due to water vapour may have had an important contribution here.

Most of the moisture that is precipitated over the ITCZ comes from evaporation of sea water over the subtropical oceans (figure 1.49). Dry land areas, such as the Sahara and Middle East, are of course not a source for moisture in the atmosphere.

In the tropics and sub-tropics the SST distribution is relatively insensitive to seasonal changes of insolation. The SST distribution is determined more strongly by ocean circulation. In particular, in regions where ocean water rises towards the surface (“upwelling”) SST’s are low. Normally this is observed on the east side of the oceans, in particular in the sub-tropics and lower mid-latitudes (figure 1.47). Over the oceans the ITCZ, therefore, does not follow the overhead position of the Sun. The continents, however, respond almost directly to insolation changes. This, of course, is especially important in the northern hemisphere. Over Asia and Africa the ITCZ moves far northward in summer. Air in the trade winds crosses the equator from the southern (winter) hemisphere to the northern hemisphere and is deviated towards the east by the Coriolis force (figure 1.50, upper panel). This makes the low-level winds blow from the south-west over the northern Indian Ocean and deposit evaporated moisture as rain over southern Asia. This phenomenon is called the Asian Monsoon. The winds at 850 hPa are particularly strong in the so-called “Somali jet” just south of the Arabian Peninsula, presumably due to the large temperature difference between the land and sea in that area and the tendency of the atmosphere to adjust to thermal wind balance. Inertial instability may also be playing a role here (section 1.20). Most of the moist air over the Indian Ocean is transported towards western India, leading to intense rainfall events. Persistent heating, mainly due to latent heat release in intense precipitation systems, is observed over southern Asia (figure 1.51).

**Figure 1.49.** The annual mean (1979-2002) total evaporation minus annual mean total precipitation [mm/day]. Source: ERA-40 Atlas. ERA-40 project report series number 19 (see http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html).

---

43 For example on 25 July 2005, 650 mm fell within 6 hours in Santacruz (Mumbai, India).
FIGURE 1.50. Mean wind at 200 hPa (12 km above sea level) (upper panel) and at 850 hPa (1.5 km above sea level) (lower panel) in June-August (lower panel) (1979-2002). Note the different flow patterns over Southern Asia and the Indian Ocean at these two levels. Source: ERA-40 Atlas. ERA-40 project report series number 19 (see http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html).

The Tibetan plateau, which is the world’s largest highland, with an area of more than 2.5 million square kilometres and an average height of more than 4.5 km, with an elliptical shape with the major axis extending from 75°E to 103°E and the minor axis extending from 27°N to 38°N (figure 1.97, right panel), acts as an elevated heat source in summer. This leads to the formation of the Asian summertime anticyclone (clockwise flow) over southern Asia (figure 1.46), lower panel). This anticyclone is most intense over the Tibetan plateau and Northern India, but tends to extend westwards towards the Middle East and even to Northern Africa. It is observed at levels between 500 hPa (5 km above sea level) and 50 hPa (20 km above sea level). It is most intense at levels between the 200- and 100 hPa isobaric levels (about 12 to 16 km above sea level).
The zonal asymmetry in the vertically integrated heating across the Pacific Ocean (i.e. between the east and the west) is the reason for the existence of an equatorial circulation, normally with upward motion over the western tropical Pacific and Indonesia and downward motion over the Eastern tropical Pacific. This so-called **Walker circulation**, named after the British Meteorologist, Sir Gilbert Walker, is illustrated in **figure 1.52**. The Walker circulation is a key feature in understanding the atmospheric circulation in the tropical Pacific.
Circulation in the atmosphere is coupled to a similar circulation in the Pacific Ocean. Wind stress associated with the westward equatorial trade winds over the surface of the tropical Pacific Ocean drives the surface ocean water towards the west. A compensating current exists at greater depth in the ocean, leading to upwelling of cold water in the east and downwelling of warmer water in the west, which explains the sea surface temperature distribution that is shown in figure 1.47. The coupled system of two circulations is illustrated in figure 1.52. The atmospheric Walker circulation drives the oceanic “El Niño/La Niña” circulation, which determines the sea surface temperature difference across the Pacific Ocean, which, in turn, determines the sea level longitudinal pressure difference across the ocean. The strength of the Walker circulation is determined by this pressure difference and thus indirectly by the associated sea surface temperature difference.

Figure 1.52. Schematic diagram of the atmospheric and oceanic circulations over and in the Equatorial Pacific Ocean between Indonesia and South America. The Walker circulation is seen at the surface as easterly trade winds which move water and air warmed by the sun towards the west. The ocean is some 60 cm higher in the western Pacific as the result of this motion. An El Niño episode is characterised by a reversal or at least severe weakening of the Walker circulation. Source: http://en.wikipedia.org/wiki/Walker_circulation.

As was stated before in this section, the coupled ocean-atmosphere system undergoes an interesting oscillation with an irregular period of several years, which is known as ENSO, which stands for “El Niño-Southern Oscillation”. The amplitude and the phase of ENSO are measured by the Southern Oscillation Index (SOI). The SOI is proportional to the anomaly in the sea level pressure difference between Tahiti and Darwin (figures 1.52 and 1.53).
high sea level pressure anomaly in Tahiti relative to the sea level pressure anomaly in Darwin indicates a strengthening of the Walker circulation and an increase in rainfall on the western side of the tropical Pacific, including Indonesia and Australia. This phase of ENSO is called “La Niña”. The opposite phase of ENSO (“El Niño”), which brings severe drought in Australia, is associated with sustained SOI-values below -8. In the past 30 years this occurred two times: between June 1982 and May 1983 and between April 1997 and May 1998 (figures 1.53). These two El Niño events, which had a strong impact on the weather world wide, started and ended rather abruptly. An understanding of these abrupt and irregular changes will open the possibility for medium range weather forecasting in countries surrounding the Pacific Ocean.

![SOI graph](http://www.bom.gov.au/climate/current/soihtm1.shtml)


In the mid-latitudes the circulation is dominated by the circumpolar flow (figure 1.32), which is westerly in the troposphere ($u>0$), i.e winds blow from the west to the east (figure 1.34). In the northern hemisphere winter (December, January and February) winds are westerly near the earth’s surface at latitudes between 30° and 70°, while winds are easterly near the earth’s surface in the subtropics. These (sub)tropical easterly winds were called the “trade winds” by sailors many centuries ago. The trade winds are related to the relatively high sea level pressure at about 35° latitude (i.e. the presence of sub-tropical high pressure areas) and the tendency, even near the earth’s surface, of the atmosphere to adjust to geostrophic balance.

The zonal mean subtropical jet undergoes meridional shifts with the seasons from a latitude of about 30° in winter to a latitude of about 40-50° in summer, especially in the southern hemisphere (figures 1.50 and 1.68). Also, the intensity of the subtropical jet varies...
from a 3-monthly mean of about 40 m s\(^{-1}\) in winter to a 3-monthly mean of about 25 m s\(^{-1}\) in summer. The latitudinal position of the subtropical jet is connected to the position of the overhead sun, which reaches its maximum (23.5°) on the “solstice” dates of 21 June and 21 December. In the ideal case, the Hadley circulation at solstice would have its upward motion branch at this latitude and equally strong downward motion branches on either side of this latitude. However, in reality, the downward motion branch is most intense over the winter hemisphere (figure 1.57), i.e. the most intense Hadley cell has its upward motion branch in the summer hemisphere and its downward motion branch in the winter hemisphere.

Sea level pressure is usually low near the Arctic Circle at 66.6°N, because it lies within the latitude belt where most mid-latitude cyclones, which are characterized by low central pressure at sea level, reach their maximum intensity. Anticyclones, which are characterized by a high central pressure at sea level, on the other hand, are most frequently observed in the sub-tropics or lower middle latitudes (30°-40° latitude) in connection with the downward branch of the Hadley circulation cell.

**Figure 1.54.** Annual march of the zonal mean and monthly mean sea level pressure at 36°N and at 66°N, averaged over the years 1979-2012. The phases are discussed in the text. Based on ERA-Interim reanalysis (http://data-portal.ecmwf.int/data/d/interim_moda/).

The Arctic Circle (e.g. 66°N) and the subtropical latitudes (e.g. 36°N) are each part of a so-called “annular belt of action”. The average annual march of sea-level pressure in these annular belts is shown in a phase diagram in figure 1.54. It can be divided into 3 phases. In phase 1 (August-December) the atmosphere over the northern hemisphere cools, while the atmosphere over the southern hemisphere heats. This leads to a shift of mass from the southern hemisphere to the northern hemisphere, leading to a sea-level pressure rise at both 36° and 66° latitude. Between December and April (phase 2) sea-level pressure at high
northern latitudes continues to increase, especially in January and February, while sea-level pressure at lower mid-latitudes decreases. Between May and July/August (phase 3) sea-level pressure decreases in both annular belts of action, especially in May and June. The physical mechanisms that are associated with the pressure changes in phase 1 are better understood than the physical mechanism behind the more complex sea-level pressure changes in phases 2 and 3, but it is certain that the so-called Brewer-Dobson circulation is involved in all phases (section 1.29 and figure 1.66). This circulation consists of a poleward flux of mass in the upper troposphere and stratosphere and an equatorward mass flux near the earth’s surface, by cold air outbreaks (figure 11.10).

\[ y = -0.0020384 \cdot 1.3355x \quad R^2 = 0.63871 \]

**Figure 1.55.** Zonal mean and monthly mean sea level pressure (mslp)-anomalies at 36°N and at 66°N in December, January, February and March for the years 1979-2012 (136 points), illustrating the negative correlation in mslp-anomalies between the two annular belts of action. The square of the correlation coefficient (the coefficient of determination) is 0.41, which means that 41% of the variance in the data is explained by the red line. Monthly mean anomalies are defined with respect to the monthly mean climatology, which is shown in figure 1.54. Positive anomalies at low latitudes appear to be strongly associated with negative anomalies at high latitudes and vice versa. These regimes are referred to, respectively, as “high zonal index” and “low zonal index” (see the text). Based on ERA-Interim reanalysis (http://data-portal.ecmwf.int/data/d/interim_moda/).

1.29 Internally generated large-scale modes of variability

To a large extent zonal mean sea level pressure is determined by irregular shifts of mass within one hemisphere, principally from one annular belt of action to the other. Irregular deviations of sea level pressure from the long term average seasonal cycle, which are called “anomalies”, occur in particular in phase 2 (figure 1.54). They determine whether a winter
is colder or warmer than normal, or whether a summer is drier or wetter than normal, as we will see in this section. The zonal mean and monthly mean sea level pressure anomaly at 66°N may be as large as 10 hPa (figure 1.55), which is much larger than the amplitude of the long term average seasonal cycle in zonal mean and monthly mean sea level pressure (±2.5 hPa) (figure 1.54). The anomalous mass shifts which cause these sea level pressure anomalies are internally generated in the atmosphere, probably in connection with variations in the number and intensity of middle latitude cyclones and/or the characteristics of large scale planetary waves (section 1.37). These deviations from zonal symmetry take care of a large proportion of the total meridional transport of mass (chapter 11).

![Figure 1.56](image-url). Zonal mean sea level pressure (monthly means of daily means) as a function of time and latitude from January 1979 to December 2012 according to the ERA-Interim reanalysis (http://data-portal.ecmwf.int/data/d/interim_moda/). The double arrows indicate consecutive winters with relatively high zonal mean sea level pressure in the subtropics.

Figure 1.56 shows the monthly mean zonal mean sea level pressure as a function of latitude and time. This is an example of a “Hovmöller diagram” (figure 1.91). Largest sea level pressure variability is observed in the two annular belts of action between 30°N and 40°N and poleward of 60 °N. We see that the subtropical high between 30°N and 40°N is most intense in the winter. This is because subsidence (downward motion), which is associated with the winter cell of the Hadley circulation, is much more intense than subsidence, which is associated with the “summer Hadley cell” (figure 1.57). The zonal average intensity of winter subtropical highs varies strongly from year to year. In fact, northern hemisphere winters, which are characterized by anomalously intense subtropical highs, seem to come together in groups of two or three, indicated by the double arrows in figure 1.56. The northern hemisphere winters of 1988-1989, 1989-1990, 1992-1993, 1993-1994, 1998-1999, 1999-2000, 2006-2007, 2007-2008, 2008-2009 were characterized by higher than average zonal mean, monthly mean sea level pressure in the subtropics. Most of these winters were also characterized by lower than average mean sea level pressure at the Arctic Circle and mild winters in Western Europe.
FIGURE 1.57. Seasonal average (December, January, February) zonal average streamlines in the vertical plane perpendicular to the equator. Flow converges near the equator. The winter Hadley cell is much more intense than the summer Hadley cell. Compare this picture with the picture in the lower panel of figure 1.45. From the ERA-40 Atlas. ERA-40 project report series number 19 (see http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html). Streamlines are iso-lines of the mass-streamfunction (Box 1.11).

FIGURE 1.58. Correlation between zonal-mean and monthly mean sea level pressure anomalies in the northern hemisphere (1958-2000). The contour interval is 0.3 for positive values and 0.15 for negative values. Negative values lower than -0.3 are shaded. Note that the map is symmetric about the diagonal, as expected. Source: J. Li and J.X.L. Wang, 2003: A modified zonal index and its physical sense. Geophys.Res.Lett., 30, 1632.
Indeed, as was noticed first by Carl Gustav Rossby in 1939\textsuperscript{44}, zonal mean sea-level pressure anomalies in the annular belts of action are anti-correlated. Figure 1.58, published by Li and Wang in 2003\textsuperscript{45}, shows cross correlations between the zonally averaged sea level pressure anomalies at different latitudes in the northern hemisphere for the years 1958-2000. The monthly anomaly at a specific latitude is defined with respect to the 1958-2000 monthly mean at that latitude. The outstanding feature in this figure is the relatively large negative correlation between the sea level pressure anomalies in the latitude band between 20°N and 40°N and the sea level pressure anomalies in the latitude band between 60°N and 70°N, in agreement with figure 1.55. To quantify this effect, Li and Wang proposed a Zonal Index, defined as the normalized\textsuperscript{46} difference between 35°N and 65°N in the zonal mean sea level pressure anomalies. The monthly average value of this index, since 1873, is plotted in figure 1.59. The zonal index clearly varies strongly from month to month, but decadal trends are also apparent in figure 1.59. High values of the zonal index were observed more frequently in the 1980’s and 1990’s, while low values of the Zonal Index were observed more frequently in the 1940’s to 1960’s. Positive and negative extreme values are only observed between October and April. Very low values were observed in recent winters, especially in


\textsuperscript{46} “Normalized” means that the anomalies are divided by the standard deviation.
December 2009, January 2010, February 2010 and December 2010, which were also relatively cold and snowy in Western Europe.

The sea surface temperature in the North Atlantic Ocean has undergone an oscillation with a similar relatively decadal time scale (figure 1.60). The acronym for this oscillation is **Atlantic Multidecadal Oscillation (AMO)**. The AMO also shows up in the North Pacific Ocean, i.e. sea surface temperatures in the North Pacific are positively correlated with the sea surface temperature of the North Atlantic. The reasons for this large scale and long-term variability are still actively debated.

**Figure 1.60.** (A) Index of the Atlantic multidecadal Oscillation (AMO), 1871 to 2003. The index was calculated by averaging annual mean SST observations (29) over the region 0°N to 60°N, 75°W to 7.5°W. The resulting time series was low-pass filtered with a 37-point Henderson filter and then detrended, also removing the long-term mean. The units on the vertical axis are °C. This index explains 53% of the variance in the detrended unfiltered index. (B) The spatial pattern of SST variations associated with the AMO index shown in (A). Shown are the regression coefficients (°C per SD) obtained by regressing the detrended SST data on a normalized (unit variance) version of the index. Source: Sutton, R.T., Hodson, D.L.R., 2005: Atlantic Ocean Forcing of North American and European Summer Climate. Science, 309, 115-118.

For a description of the winter weather in these months see the issues of *Weather* of February 2010, of March 2010, of April 2010, of February 2011 and of December 2011.
The Zonal Index is correlated relatively highly with the winter temperature in Western Europe. This can be seen in figure 1.61 (see also Box 1.12), which shows a scatter plot of monthly mean temperatures in De Bilt in December, January and February in the years 1979 to 2011 as a function of the corresponding monthly mean zonal index. High winter temperatures are associated with high Zonal Index while low winter temperatures are associated with low Zonal Index. A linear regression yields a correlation coefficient of 0.65. The correlation coefficient would be much higher if it were not for several extremely cold months. These months are cold, apparently not only because of the low Zonal Index and, thus, because of the higher frequency of easterly winds, bringing cold continental air towards western Europe, but also for another reason, probably related to the extent and amount of *accumulated* snow cover over Europe and Asia in the previous month (usually December).

The irregular variations in the Zonal Index reflect an oscillation, which is called the "**Northern Annular Mode**" (NAM). The NAM reflects variations in meridional mass transfer, not only in the troposphere, but especially also in the lower stratosphere, where there is a systematic transport of mass from the tropics to the polar caps, in particular to the winter polar cap. This poleward mass transfer was identified first by Alan Brewer in 194948 and Dobson in 1956, who investigated the distributions of water vapour and of ozone in the stratosphere (see section 1.12 and figure 1.66). Extreme values of the Zonal Index (figure 1.59), both low and high, are observed exclusively between October and April, while the variability of the Zonal Index in summer is relatively weak.

**Figure 1.61.** Scatter plot of the monthly mean northern hemisphere Zonal (Annular Mode) Index (NAM-index) as a function of monthly mean temperature in De Bilt in December, January and February in the 33-year period from 1979 to 2011. The correlation coefficient between the data points and the linear approximation (the red line) is 0.65, implying that this linear approximation explains 42% of the covariance in the data.

PROBLEM 1.16. Cross-correlation- or covariance-matrix

Figure 1.58 is a graphical representation of a cross-correlation- or covariance-matrix (Box 1.12). Reproduce this figure using monthly mean zonal mean sea level pressure analyses based on the ERA-Interim reanalysis for the 34-year period from 1979 to 2012 (http://apps.ecmwf.int/datasets/).

PROBLEM 1.17. Time filtering

The curve in the upper panel of figure 1.60 is highly smoothed. Smoothing or (time) filtering of a (time)series, $Y_n$, involves replacing the n-th member of the series with a suitably weighted average of the neighbouring members:

$$Y_n^F = \sum_{i=-j}^{j} w_i Y_{n+i}$$

A simple choice for the filter, which gives a “running mean” of the series, is

$$w_i = \text{constant} = \frac{1}{2j+1}$$

Apply this time filter to the time series of the Zonal (NAM) Index (figure 1.59). Plot the result for different choices of $j$. Perform a spectral analysis (see Box 1.9) of the running mean and of the associated deviation of this running mean. The Zonal (NAM) Index can be downloaded from the following website:

http://ljp.lasg.ac.cn/dct/page/65544.

You can also apply this analysis to the Southern Oscillation Index (SOI), which can be downloaded from the following website:

file:///Users/Shared/Climate&WeatherIndex/MonthlySouthernOscillationIndex.webarchive.

Identify dominant time scales in both time series with physical processes.

In winter and spring strong positive anomalies of temperature in the polar stratosphere occur occasionally, especially in the northern hemisphere. Temperatures in the polar stratosphere increase by more than 25 K within a week while the zonal mean zonal winds in the stratosphere reverse from being eastward to being westward. These events are referred to as Sudden Stratospheric Warmings (SSW’s).

Let us look at an example of a Sudden stratospheric Warming. Figure 1.62 shows the march of the zonal mean temperature and zonal mean zonal wind at 10 hPa (about 30 km above sea level) as a function of latitude during the winter of 2008-2009. In early January 2009, the axis of the polar night jet at 10 hPa is located around 65°N. The zonal mean temperature over the polar cap at 10 hPa is about 200 K. After 15 January the zonal wind speed in the jet decreases from about 80 m/s to about 40 m/s on 20 January. At this point in time a sharp temperature increase is observed poleward of 60°N. Within only a few days a temperature increase of about 50 K takes place at 10 hPa, which is accompanied by a reversal of the zonal mean zonal wind, in which the cyclonic polar vortex is replaced by an anticyclonic polar vortex (figure 1.63), which is the normal circulation in the summer stratosphere. The SSW of January 2009 is an example of a “major SSW”, in which the cyclonic polar vortex splits into two separate vortices (figure 1.63). A “minor SSW” is associated with a displacement of the cyclonic polar vortex from the pole.
The SSW of January 2009 was followed by a slow cooling of the polar cap stratosphere and a recovery of the cyclonic polar vortex (figure 1.64). This cooling is induced by emission of radiation to space by greenhouse gases in the stratosphere (e.g. carbon dioxide, ozone, methane and water vapour). The highest levels cool fastest because emitted radiation can escape most easily to space. As a result of this, a downward propagation of the temperature minimum and an associated downward expansion of the cyclonic polar vortex is observed. However, the polar cyclonic vortex never attains the same intensity of the vortex prior to the SSW in January, probably because the lower stratosphere remains relatively warm for months after the SSW and, moreover, absorption of Solar radiation by ozone gradually takes effect over the polar cap as the date approaches equinox (21 March).

The occurrence of Sudden Stratospheric Warming is correlated with upward (from the troposphere) and poleward propagation of planetary waves (section 1.37). The energy of these waves is proportional to the mass density and the amplitude of the wave. Because density decreases exponentially with height and wave propagation is energy conserving, the amplitudes of planetary waves increase exponentially as they propagate upward into the stratosphere (this applies to all kinds of upward propagating waves), eventually leading to “overturning” of streamlines, analogous to the fate of ocean waves along the coastal beaches. In winter and early spring the wave flux from the troposphere into the stratosphere is sometimes so large that the stratospheric “planetary wave breaking” is manifest as an SSW.
Figure 1.64. Time–height cross sections of (a) zonal-mean north polar cap temperatures averaged over 75°–90°N and (b) zonal-mean zonal winds at 60°N during the winter of 2008/09. Contour intervals are 10 K for the temperature and 10 m s\(^{-1}\) for the zonal wind, respectively. Shading shows deviations from the climatological means in units of K and m s\(^{-1}\), respectively. Climatological means are calculated for the period from 1979 to 2004. Source: Harada, Y., A. Goto, H. Hasegawa, N. Fujikawa, Hiroaki Naoe, Toshihiko Hirooka, 2010: A Major Stratospheric Sudden Warming Event in January 2009. J. Atmos. Sci., 67, 2052–2069.

Figure 1.36 gives an impression of what wave-breaking means for the isentropic potential vorticity distribution, in this case near the tropopause. During this process “cut-off lows” and “blocking highs” are formed. The continuous “breaking” of planetary waves in the so-called stratospheric “surf zone” is accompanied by turbulent velocity fluctuations that act to transport mass and vorticity in the meridional direction along isentropic surfaces. This effect acts to disturb thermal wind balance. According to a theory of the zonal mean state of the atmosphere, which is described in chapter 11, the maintenance of thermal wind balance by the atmosphere appears as a drag force on the zonal average flow. This so-called “wave drag” reduces the zonal average zonal wind velocity and hence also the zonal average Coriolis force, which is therefore out of balance with the zonal average pressure gradient force (figure 1.65), which then leads to a poleward drift of air parcels and downward motion over the pole, two branches of the Brewer-Dobson circulation. This circulation is drawn schematically in figure 1.66.
Because planetary waves cannot propagate upwards from the troposphere to the stratosphere when the wind is westward, i.e. in the summer stratosphere (figure 1.34), wave drag is not active in the middle stratosphere in summer. Therefore, ozone is transported from the tropics, where it is formed (section 1.12), preferably towards the winter pole. Because ozone is not destroyed photochemically during the polar night, relatively high ozone concentrations are observed over the pole at the end of winter and in early spring (figure 1.23). These important properties of the atmosphere are discussed further in chapters 11 and 12.
**Figure 1.67.** Zonal mean-monthly mean zonal wind at 10 hPa (about 30 km above sea level) (upper panel) and at 50 hPa (about 20 km above sea level) (lower panel) from January 1979 until December 2011, according to the ERA-Interim reanalysis. Contour interval is 20 m s\(^{-1}\).


The Hovmöller diagrams of the zonal mean monthly mean zonal winds at different levels in the stratosphere (figure 1.67) and the troposphere (figure 1.68) for the period 1979-2011 reveal interesting zonal wind oscillations, with time scales of seasons to several years. In the stratosphere (figure 1.67) we observe the yearly cycle of mid-latitude eastward winds (\(u>0\)) in the winter and mid-latitude westward winds (\(u<0\)) in the summer. The spring reversal of the zonal mean zonal wind is always rather abrupt, usually because it is associated with a “final sudden stratospheric warming”. In the northern hemisphere it occurs in March
Figure 1.68. Zonal mean-monthly mean-zonal wind at 300 hPa (about 9 km above sea level) (upper panel) and at 850 hPa (about 1.5 km above sea level) (lower panel) from January 1979 until December 2011, according to the ERA-Interim reanalysis. Contour interval is 10 m s$^{-1}$ in the upper panel and 5 m s$^{-1}$ in the lower panel. The high wind speeds observed in the southern are called “the Roaring Forties”. Source: http://data-portal.ecmwf.int/data/d/interim_moda/.

or April, while it occurs much later in spring (at the end of November or beginning of December) in the southern hemisphere. The period of summer westward winds around the pole in the stratosphere is very short in the southern hemisphere (only 2 or 3 months at 10 hPa and less or non-existent at 50 hPa). The eastward winds in the winter are weaker in the
northern hemisphere than in the southern hemisphere, reflecting more intense wave activity and stronger wave drag in the northern hemisphere than in the southern hemisphere.

In the tropical stratosphere we observe (figure 1.67) an oscillation of the zonal mean zonal wind, between westward and eastward winds, with a period of more than two years, which is referred to as the **Quasi-Biennial Oscillation (QBO)**. The QBO seems to originate at the stratopause, at about 1 hPa, and appears to propagate downward to the tropical tropopause (figure 1.69). Unexpected eastward winds at the equator (why unexpected?) occur roughly every 28 months.

![Figure 1.69](image)

**Figure 1.69.** Monthly mean - zonal mean - equatorial zonal wind in m s\(^{-1}\) between 1961 and 1999, according to the ERA-40 reanalysis. Source: [http://www.ecmwf.int/research/era/ERA-40_Atlas/](http://www.ecmwf.int/research/era/ERA-40_Atlas/).

Because of the relatively long period of the QBO, we expect that the equatorial atmosphere is in thermal wind balance. This means that (eq. 1.97)

\[
2\Omega \sin \phi \frac{\partial v_g}{\partial z} = 2\Omega \phi \frac{\partial v_g}{\partial z} = \frac{2\Omega y}{a} \frac{\partial v_g}{\partial z} = -\frac{g}{\theta} \frac{\partial \theta}{\partial y}.
\]

Assuming equatorial symmetry we have \(\partial \theta / \partial y = 0\) at \(y = 0\) (the equator). Now, with L'Hospital’s rule we obtain the following relation between the vertical wind shear and the temperature at the equator:

\[
\frac{\partial u_g}{\partial z} = -\frac{ag}{2\Omega \theta} \frac{\partial^2 \theta}{\partial y^2}.
\]

(1.133)

Therefore, shear zones at the equator are associated with temperature anomalies. Positive shear is associated with a warm anomaly (figure 1.70), while negative shear is associated with a cold anomaly. Maintenance of these temperature anomalies against “radiative damping” requires “adiabatic heating/cooling”, which occurs when air is moving downwards/upwards and being compressed/expanded. Monthly mean temperatures at the equator in the stratosphere are indeed highly anticorrelated with monthly mean vertical velocities (figure 1.71). Continuity then implies the existence of a “**residual meridional**
circulation” with pole-ward motion below and equator-ward motion above a positive temperature anomaly, as shown in figure 1.70.

We are left with many questions about the mechanism that creates and sustains the QBO. What creates the equatorial shear zone in the first place. How and why does the shear zone propagate downward. How are eastward winds at the equator, implying “super-rotation”, possible? These questions will be tackled in chapter 13.

**Figure 1.70.** Schematic latitude-height section showing the zonal mean meridional circulation associated with a positive zonal mean equatorial temperature anomaly. Isopleths of zonal mean temperature are drawn as solid contours; isopleths of zonal mean zonal wind are drawn as dashed contours. Source: Plumb, R.A., and R.C. Bell, 2001: A model of the Quasi-biennial oscillation on an equatorial beta-plane. Quart.J.R.Met.Soc., 108, 335-352.

**Figure 1.68** shows the monthly mean zonal mean zonal wind at two levels in the troposphere: at 300 hPa (9 km above sea level) and at 850 hPa (1.5 km above sea level). At 300 hPa we observe **permanent eastward winds in the subtropics and mid-latitudes** and nearly permanent westward winds in the tropics. Signs of the QBO are very weak at this level. The subtropical jet at about 30° exhibits a seasonal cycle, being most intense in the winter in both hemispheres.

At 850 hPa we observe the easterly (westward) **Trade winds** in the tropical belt between 30°S and 30°N. In the northern hemisphere, between 5°N and 10°N, the Trade winds are interrupted in summer by a spell of weak westerly (eastward) winds. This is a manifestation of cross-equatorial flow from the southern hemisphere to the northern hemisphere, which is part of the intense “winter” Hadley cell, with upward motion in the summer and downward in the winter (**figure 1.57**). When air crosses the equator from the southern hemisphere to the northern hemisphere, it is accelerated towards the east by the Coriolis force. This happens especially over the Indian Ocean, where the upward branch of the Hadley cell is forced far north over the Asia continent, resulting in the **Somali Jet** (upper panel of **figure 1.50**).

The atmospheric modes of variability, which have been the subject of discussion in this section, i.e. the Southern Oscillation, the Index Cycle in the strength of the westerlies, Sudden Stratospheric Warmings and the Quasi-Biennial Oscillation, are all the result of complex non-linear interactions between different components of the atmosphere-ocean
system, working on the very different time- and space scales. Present day theories of these phenomena are still in their infancy. Simulations with free-running numerical models still yield imprecise results, especially with regard to ENSO, the Index Cycle (Annular Modes) and Sudden Stratospheric Warmings 49.

**PROBLEM 1.18. Equatorial temperature anomalies associated with the QBO**

What is the order of magnitude of the temperature anomaly if \( \frac{\partial u}{\partial z} = 4 \times 10^{-3} \text{ s}^{-1} \) and the meridional scale of the QBO is approximately 1200 km? Compare your answer with your estimate based on figure 1.71.

**PROBLEM 1.19. Principal Component Analysis (PCA)**

Apply Principal Component Analysis (PCA) (explained in Box 1.12) to a combination of three (or more) variables that according to your hypothesis, may be “tele-connected”. Examples of interesting variables are zonal mean wind at 50 hPa at the equator (as a measure of the QBO), zonal mean wind at 10 hPa at 60°N (as measure of the strength of the stratospheric vortex), the NAM-index (fig. 1.59), the Southern Oscillation Index (SOI-index), the precipitation in the ITCZ (as a measure of the amplitude of the subtropical highs). You may combine any other set of three or more time series that you hypothesize are teleconnected. First formulate your hypothesis, then collect the necessary data, perform the PC-analysis and discuss the results in the light of the hypothesis. Perform the PCA for anomalies with respect to the mean, as in Box 1.12.

You must present your results orally (in 15 minutes) and in a written report of no more than 1500 words and 5 figures. You may do this exercise in couples.

Data can be found on the internet, e.g. at [http://apps.ecmwf.int/datasets/](http://apps.ecmwf.int/datasets/) (ERA-Interim reanalysis data), [http://climexp.knmi.nl/](http://climexp.knmi.nl/) (site giving links to many data-sites), [http://eca.knmi.nl/](http://eca.knmi.nl/) (Daily observations at many stations in especially Europe), [http://www.knmi.nl/klimatologie/](http://www.knmi.nl/klimatologie/) (Dutch weather observations).

**PROBLEM 1.20. Influence of filtering out the trends in Principal Component Analysis (PCA) and the influence of the ocean (AMO-index) and of CO₂**

Repeat the PCA for anomalies with respect to the best linear fit to each times series (the red lines in figures 1 to 6 in Box 1.12 represent the best linear fit)? Interpret your results (see section 6.2 -on linear regression- of the book by Daniel S. Wilks (2005), which is listed at the end of Box 1.12). Extend the PCA in Box 1.12 (using an online matrix calculator to find the eigenvalues and eigenvectors) with the time series of yearly mean CO₂ concentrations, which may be represented by the following quadratic function:

\[
pCO_2 = 0.013t^2 + 0.518t + 310.44 \text{ [ppmv]},
\]

where \( t \) is time measured in years since 1950. Extend the PCA by including the AMO-index.

---

49 [http://www.atmos.colostate.edu/ao/index.html](http://www.atmos.colostate.edu/ao/index.html).

FIGURE 1.71. Graphs, illustrating that the relation between monthly mean zonal mean temperature (red, solid line), monthly mean zonal mean vertical wind shear, defined as the difference in zonal mean zonal wind speed between 10 hPa and 50 hPa (blue, dashed line in upper panel) and monthly mean zonal mean vertical velocity (blue, dashed line in lower panel) at the equator at 30 hPa is approximately in accord with the theoretical predictions. The lower panel demonstrates that the movement of air at 30 hPa is mostly upward (reflecting the upward branch of the Brewer-Dobson circulation) except for short episodes when the vertical shear is positive. These episodes are characterized by high temperatures, downward motion and positive shear. The time axis runs from January 1979 to December 2003. Eleven positive shear zones propagate downward through the 30 hPa level in 25 years, which indeed implies a period which slightly exceeds 2 years. Source: [http://data-portal.ecmwf.int/data/d/interim_moda/levtype=pl/](http://data-portal.ecmwf.int/data/d/interim_moda/levtype=pl/).
Box 1.12. Principal component analysis

Principal component analysis is a statistical technique that is used by meteorologists (and others) to discover hidden relations or correlations between more than two different variables. The technique is useful in attributing variations and trends in e.g. the temperature to specific physical mechanisms. For example, the positive trend (s) in the mean temperature in De Bilt (Netherlands) in January (figure 1) over a period of 62 years (s=0.034°C per year) can be attributed to the enhanced greenhouse effect. Due to the steady increase in the CO₂ concentration of about 1.4 ppmv per year the temperature should be increasing. But other factors, such as gradual atmospheric circulation changes, could perhaps also explain part of the observed trend, while the yearly variability of the temperature is very likely linked to the variability of the circulation.

A frequently used measure of the large-scale middle latitude circulation is the “Northern Annular Mode Index” (NAM-Index), also termed the “Zonal Index” by Rossby in 1939. Li and Wang (2003) defined a Zonal Index for the northern hemisphere as the normalized difference in zonal average sea-level pressure anomalies between 35°N and 65°N. The NAM-index is related to another often used index: the North Atlantic Oscillation index (NAO-index), which is defined as the normalized difference in sea level pressure between Lisbon (Portugal) and Iceland (Reykjavik). Figure 2 shows the variability and trend in the monthly mean NAM-Index in January over the same period of 62 years. An upward trend in both the temperature in De Bilt and the NAM-index with a large variability superposed on both trends is observed. The large “natural” variability of the circulation makes it very difficult to be certain about a systematic upward trend in temperature, let alone to attribute this trend to a specific physical mechanism. The natural variability could also possess timescales larger than 62 years. The records shown in figures 1 and 2 may well be part of one large natural upward swing in both the temperature and the NAM-index, which will, at some time in the future, be followed by a natural downward swing. However, figure 3 demonstrates that the variability in temperature is very likely physically linked to the variability in the NAM-index, because the NAM-index correlates positively with the temperature. The correlation coefficient is 0.57. This positive correlation is expected, because a positive NAM-index is related to a lower than normal pressure over the polar cap and a higher than normal pressure over the subtropics, leading to stronger than normal westerly (eastward) winds in the mid-latitudes. Therefore, because De Bilt (at 52°N) is located on the east side of the Atlantic Ocean, a positive NAM-index in the winter is associated with relatively warm air advection from the ocean. The opposite correlation between NAM-index and temperature is expected on the other side of the Atlantic Ocean. This effect is indeed seen when we relate the NAM-index to the January temperature for the same period of time at Goose (Eastern Canada), which is located at nearly the same latitude (53°N) as De Bilt. Figure 4 demonstrates that the temperature at Goose is indeed negatively correlated with the NAM-index.

It has been known for a long time (at least since the 18th century) that the weather in winter in Greenland is just the reverse to that in Europe (van Loon and Rogers, 1978). Temperatures in Eastern Canada exhibit an identical relation with temperatures in Europe (figure 5). Because of the average upward trend in the NAM-index in January over the period 1950-2011 (figure 2), the temperature in January at Goose has not risen as strongly as the temperature at De Bilt. In fact, there is no measurable trend in the temperature at Goose in January during this period (figure 6). Are the greenhouse effect and the circulation effect on the temperature at Goose canceling each other? A clue to the answer to this question can be obtained from principal component analysis.
Let us apply principal component analysis to the time series of, respectively, the January mean temperature at De Bilt, the January mean temperature at Goose and the January mean NAM-index from 1950 to 2011. We start by calculating the average, the variance and the standard deviation of each of the three time series (Table 1). We then normalize the three time series with their respective standard deviations. The variance of each normalized time series should be equal to one. Next, we compute the covariances and write down the covariance matrix. The covariance of the normalized NAM-index and the normalized temperature at De Bilt is 0.5669 (Figure 3). The covariance of the normalized NAM-index and the normalized temperature at Goose is -0.5036 (Figure 4). The covariance of normalized temperature at Goose and the normalized temperature at De Bilt is -0.3217 (Figure 5). Therefore, the covariance matrix becomes

$$M = \begin{bmatrix}
1.000 & 0.5669 & -0.5036 \\
0.5669 & 1.000 & -0.3217 \\
-0.5036 & -0.3217 & 1.000
\end{bmatrix}$$

Figure 1.58 is, in fact, a graphical representation of a covariance matrix. Finding the eigenvalues of $M$ reduces to solving a cubic equation of the form,

$$Ay^3 + By^2 + Cy + D = 0$$

The coefficients of this cubic eigenvalue equation are
There is no significant trend in Goose.

\[ A = 1.000; B = -3.000; C = 2.321; D = -0.505 \]

This yields 3 eigenvalues:

\[ \lambda_1 = 1.935; \lambda_2 = 0.682; \lambda_3 = 0.383 \]

The sum of the eigenvalues (total variance) is 3. These eigenvalues represent the contribution to the total variance of each “principal component”. The relative contributions to the total variance of each component are, respectively, 64.5%, 22.7%, 12.8%.

The three associated eigenvectors, normalized (i.e. dividing each component of the eigenvector by the absolute value of the eigenvector) such that they become so-called empirical orthogonal functions (“EOF’s”), are
where the first component of the eigenvector represents the NAM-index, the second component of the eigenvector represents the temperature in De Bilt, and the third component of the eigenvector represents the temperature in Goose.

The first eigenvector, or first principal component, explaining 64.5% of the original variance, is a linear function of the NAM-index, the temperature at De Bilt and the temperature at Goose such that, if the original data is “projected” onto this linear function, explains the maximum possible variance. How the original data is “projected” onto this linear function is explained shortly in this box. The first principal component represents the NAM-induced anti-correlation between the temperature in De Bilt and the temperature in Goose with lower than average temperatures in De Bilt correlating with higher than average temperatures in Goose and negative NAM-index and vice-versa.

The second eigenvector or principal component is the linear function with the maximum possible variance subject to being uncorrelated with the first principal component. This principal component explains 22.7% of the variance. It hardly projects on to the NAM-index. Hence it may be related to the increase of the CO$_2$ concentration from about 310 ppmv in 1950 to about 390 ppmv in 2011, which is expected to lead to temperature anomalies with an identical sign at both Goose and De Bilt. The upward trend in CO$_2$ concentrations might also have affected the NAM-index. Moreover, it is very likely that ocean variability is correlated in the same way with temperatures at both Goose and De Bilt (figure 1.60). These questions can be investigated by extending the PCA with the time series of monthly mean CO$_2$ concentrations and/or AMO-index (problem 1.20).

The projection of the original observational data onto the three eigenvectors (i.e. the computation of the values of each data point in units of the principal components) reduces to computing

$$Y = XE$$

where $E$ are the eigenvectors, written in matrix form as

$$E = \begin{bmatrix} -0.630 & 0.077 & 0.773 \\ -0.565 & 0.638 & -0.524 \\ 0.533 & 0.766 & 0.358 \end{bmatrix}$$

and $X$ are the original observational data, written as a matrix with 3 columns (representing the three variables) and 62 rows (representing the 62 years from 1950 to 2011) as

$$X = \begin{bmatrix} \text{NAM}[1950] & \text{TDB}[1950] & \text{TG}[1950] \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \text{NAM}[2011] & \text{TDB}[2011] & \text{TG}[2011] \end{bmatrix}$$
where $NAM$ is the deviation of the NAM-index from the 62-year average divided by the standard deviation, $TDB$ is the deviation of the January average temperature from the 62-year average divided by the standard deviation in De Bilt and $TG$ is the deviation of the January average temperature from the 62-year average divided by the standard deviation in Goose.

Figure 7 shows a scatter plot of the amplitude of the first principal component for the month of January as a function of the year. The 7 years with an average January temperature in De Bilt below 0°C are indicated in the colour blue, while the seven warmest January months are shown in the colour red. Clearly, low (high) temperatures in the Bilt correspond to a positive (negative) amplitude of the first principal component.

Note that 5 of the 7 coldest months and 6 of the 7 warmest months occurred in the second half of the 62-year period under investigation, implying that the frequency of cold extremes as well as the frequency of warm extremes has increased. However, we have to be careful with these conclusions. This is fortuitous. With the 1940’s included in the dataset, only 3 of the 7 coldest months and 5 of the seven warmest months occurred in the second half of the 72-year period under investigation.

**Figure 7**

References


1.30 Balance and imbalance

Jule Charney, one of the greatest theoreticians in Atmospheric Dynamics, states in his paper, “On the Scale of Atmospheric Motions”, that “…the motion of large-scale atmospheric disturbances is governed by the laws of conservation of potential temperature and absolute potential vorticity, and by the conditions that the horizontal velocity be quasi-geostrophic and the pressure quasi-hydrostatic”. Chapters 3, 5, 7 and 9 of these lecture notes hope to make clear why, according to Norman Phillips, Charney’s well known contemporary, these 35 words should be considered one of the most effective meteorological statements of the twentieth century. To appreciate the importance of Charney’s words, we need to go into the details of the theories of adjustment to hydrostatic balance (chapter 3) and adjustment to geostrophic balance (chapter 5). The reader will then understand why the atmosphere is usually in “quasi-hydrostatic” balance, and why the “Rossby radius of deformation” (eq. 1.131) represents the lower limit of the “large scale”.

*Figure 1.72* gives an overview of the typical time- and horizontal length scales of weather systems in relation to some basic parameter values and fundamental time and length scales, which were defined in section 1.27. Note that the scales are plotted logarithmically. The upper left corner of the figure contains slow “motion systems” that are dominated by molecular diffusion. The associated Reynolds number, \( Re = L^2 / \nu \), where \( \nu \) is the kinematic molecular viscosity coefficient [units: m\(^2\) s\(^{-1}\)], is smaller than 1. The lower right corner of the figure contains “motion systems” that are dominated by large accelerations (>g), such as sound waves, which are associated with the process of adjustment to hydrostatic balance (chapter 3). The vertical blue lines define the borders of the meso-length-scale, while the horizontal red lines define the borders of the meso-time-scale. Motion systems with a horizontal scale smaller than about 1 km, which is representative for the depth of the turbulent boundary layer, are defined as “small”. The “Rossby radius of deformation” separates “large scale” from “mesoscale”. Large scale, slow motion systems (upper right corner), such as extra-tropical cyclones and planetary waves are in “quasi-geostrophic” and “quasi-hydrostatic balance”.

The degree of balance at specific instants in time and points in space, and within specific weather systems may vary considerably, especially in the case of geostrophic balance. This is illustrated here with two examples: the circulation associated with thunderstorms, which is strongly out of balance and the circulation associated with a planetary wave, which is very close to geostrophic balance.

Let us begin by considering the case of two thunderstorm-showers, which travelled over the area shown in the lower left panel of *figure 1.42*, on July 11, 1984. *Figure 1.73* displays two satellite pictures of the thunderstorms. Both thunderstorms are travelling from southwest to northeast at a relatively high speed of about 26 m s\(^{-1}\). The first thunderstorm is dissipating and is already very weak at the time of the second photograph. The horizontal scale of the circulation that is associated with a thunderstorm is similar to the horizontal scale of a sea breeze circulation. The similarity in scale is however not apparent from the pressure measurements at a particular station, as can be seen in *figure 1.74*. The thunderstorms are manifest in the pressure records as humps with amplitudes of about 5-10 hPa and a period of about 1 hour. The second thunderstorm is the most marked. Zierikzee (325) in the south-west of the Netherlands (figure 1.42) records a relatively quiet pressure

---

rise of 6 hPa in 35 minutes, followed by a pressure fall of 9 hPa in the next 45 minutes. De Bilt (260) records a pressure-fall of about 5 hPa in less than three minutes. From these pressure records one would of course not conclude that the thunderstorm is short-lived, but rather that it travels fast. In fact, the thunderstorms formed during the night over the Bay of Biscay and intensified strongly over France. They had existed for at least 12 hours when they were crossing the Netherlands in the morning and early afternoon.

FIGURE 1.72: Spatial and temporal scales of atmospheric dynamics. The region in the lower right corner of the figure, is associated with phenomena exhibiting accelerations greater than the gravitational acceleration, $g$. The region in the upper left corner is associated with phenomena dominated by molecular diffusion, i.e. low Reynolds number phenomena. The upper horizontal red line separates the upper domain, in which Earth’s rotation is important to circulation systems, from the lower domain in which Earth’s rotation is not important. The domain below the lower red line is associated with significant vertical accelerations and deviations from hydrostatic balance. The weather systems falling into the lower domain appear to lie on the dashed line on which accelerations scale with $L$ to the power (-1/3), which characterizes three-dimensional turbulence. The right vertical blue line corresponds to the Rossby radius of deformation, defined in section 1.27 (eq. 1.31). GC stands for “General Circulation”. QBO stands for “Quasi-Biennial Oscillation”. This figure is inspired by figure 1 in Smagorinsky, J., 1979: Topics in dynamical meteorology: 10. Epilogue – a perspective of dynamical meteorology. Weather, 34, 126-135.

FIGURE 1.73. Two photographs of Western Europe made by a polar orbiting satellite on July 11, 1984; (a) at 7:48 UT; (b) at 14:29 UT. Courtesy of the University of Dundee, Scotland.

FIGURE 1.74. Surface pressure as a function of time at four stations in the Netherlands along the path of the thunderstorms. See figures 1.42 and 1.75 for the location of the stations.
Figure 1.75. Schematic representation of the precipitation features in the thunderstorms derived from the radar equipment based at De Bilt (DB) on July 11, 1984, 12:32 UT (black: relatively intense precipitation). The other stations indicated are Zierikzee (Z) (325 in figure 1.42), Lelystad (L) (268) and Eelde (E) (280). The complex of thunderstorms is moving in northeasterly direction. Also indicated are the maximum heights of the echos at several points.

This case serves to illustrate that we must make a distinction between so-called "Eulerian" time-scales, so-called "Lagrangian" time-scales and the life-time of the circulation system. The Eulerian time-scale of a circulation system is the time-scale we measure at a fixed point in space, i.e. the length of the humps in the pressure-records in figure 1.72, which gives a time-scale of about one hour for the individual thunderstorms on 11 July 1984. The Lagrangian time-scale, on the other hand, is the time it takes an air parcel to move once through the entire system, or, if we are dealing with wavelike oscillations, to cover one wavelength.

The second thunderstorm on July 11, 1984 had a vertical extent of approximately 14 km, whereas the horizontal extent was about 50 km (figure 1.75). A representative air parcel velocity through the disturbance is about 10 m s\(^{-1}\). Such an air parcel takes less than 4 hours to travel through the disturbance.

Yet another relevant time-scale is the total lifetime of a weather system. The sea breeze circulation has a lifetime of about 12 hours, after which it becomes a "land-breeze", or vanishes. The lifetime of a typical fair-weather cumulus cloud is about 1 hour, whereas thunderstorms may persist for more than 24 hours. Tropical cyclones (problem 1.12) may easily persist for more than a week. One of the most persistent Atlantic tropical cyclones on record is "Ginger". It lasted about 1 month (5 September-5 October 1972).

By dividing the total lifetime by the Lagrangian time-scale, we obtain a dimensionless time-scale, \(T\), which gives an impression of the coherence in time of the particular circulation system. For an arbitrary eddy in the surface layer of the atmosphere, \(T\) will be close to or less than 1. This implies that, after a time equal to the Lagrangian time-scale, the
Eddy has lost its identity. This is a characteristic of incoherence in time and unpredictability, i.e. turbulence. In the case of the fair weather cumulus cloud, $T$ lies between 1 and 2. For the thunderstorm and the sea breeze, $T$ is of the order of 5. A tropical cyclone is one of the most "time-coherent" weather systems. Its Lagrangian time-scale may be about 12 hours, while its total lifetime is typically 5 to 15 days, which makes $T \sim 10-30$.

Figure 1.76 shows the conditions at the Earth's surface in terms of wind and sea-level pressure over Europe on July 11, 1984. It can be seen that the wind direction is very variable from one place to the other and does not seem to bear any relation to the pressure pattern. In some areas the wind blows from high to low pressure, as would be expected in the absence of forces other than the pressure gradient force. But, in other areas, such as in the vicinity of the thunderstorm-high (located over western Belgium at 12 UTC), the wind blows from low to high pressure.

**Figure 1.76.** Surface weather maps of July 11, 1984, 1200 UTC. Isobars are shown by solid lines with values indicated in hPa. The letters “H” and “L” denote surface pressure maxima and minima, respectively. A dot marks the position of a weather station as well as the end point of a wind vector. The bars attached to the wind vector indicate the 10-minute average wind speed, $ff$, measured at a height of 10 m above the ground: no barb corresponds to $ff<2.5$ m s$^{-1}$; one barb corresponds to $2.5 \leq ff < 5$ m s$^{-1}$; one barb is added for every 2.5 m s$^{-1}$. Dashed lines mark zones ("lines") of surface confluence and dotted-dashed lines mark zones of surface diffluence. (Note: the analysis of the pressure field and the confluence/diffluence lines has been performed subjectively on the basis of an observation network with more than double the density of the observations shown on these maps). The pre-low, thunderstorm high and wake low, which propagate in north-easterly direction, can be identified also in the pressure records shown in figure 1.74. The dynamical origin of the surface pressure pattern around a thunderstorm is discussed in chapter 4.
**Figure 1.77.** Upper panel: 500 hPa plotted weather maps 3 March 1995, 12 UTC. The position of a measuring station is indicated by a square. Also indicated are the temperature (°C) (upper left), the dew point depression (°C) (lower left) and geopotential height (dm) (upper right). Barbs, attached to the wind vector, which points towards the station position, indicate the windspeed, f f. No barb corresponds to ff<2.5 knots (1 knot is 0.5 m/s); each whole (half) barb corresponds to an extra 10 (5) knots. If windspeed is equal or greater than 25 m/s, this is indicated by a triangle attached to the wind vector, while the value of the wind speed is indicated in bold numbers in m/s to the right of the plot. Lower panels: Analysis of the observed wind at 500 hPa (left) (see problem 1.21) and analysis of 500 hPa isobaric surface height (contour interval is 25 m; thick contour corresponds to 5300 m), together with the geostrophic wind vector computed from the analysed isobaric height (right).
**Figure 1.78.** Plotted weather maps of 3 March 1995, 00 UTC for two isobaric levels: 500 hPa (panel on this page) and 850 hPa (panel on next page). The position of a radiosonde station is indicated by a square. Also indicated are the temperature (°C) (upper left), the dew point depression (°C) (lower left) and geopotential height (dm) (upper right). Barbs, attached to the wind vector, which points towards the station position, indicate the windspeed. No barb corresponds to \( \leq 2.5 \) knots (1 knot is 0.5 m/s); each whole (half) barb corresponds to an extra 10 (5) knots. If the windspeed is equal or greater than 25 m/s, this is indicated by a triangle attached to the wind vector, while the value of the wind speed (in m/s) is indicated to the right of the plot in bold numbers.

*Figure 1.76* should be compared with *figure 1.77*, which shows an example of (nearly) balanced flow at 500 hPa. In this case, there is a clear relation between the **height of a surface of constant pressure** (500 hPa in this case) and the wind. The wind direction is approximately **perpendicular** to the height gradient (lower right panel).

Assuming hydrostatic equilibrium, \( \partial p/\partial z = -\rho g \) (section 1.14), any scalar, \( A(t,x,y,z) \), can be written as \( A(t,x,y,z(t,x,y,p)) \). Taking the partial derivative of \( A(t,x,y,z(t,x,y,p)) \) with respect to \( x \) and \( y \), we get, respectively

\[
\left( \frac{\partial A}{\partial x} \right)_{y,p,t} = \left( \frac{\partial A}{\partial x} \right)_{y,z,t} + \left( \frac{\partial A}{\partial z} \right)_{x,y,t} \left( \frac{\partial z}{\partial x} \right)_{y,p,t}
\]
and

\[
\left( \frac{\partial A}{\partial y} \right)_{x,p,J} = \left( \frac{\partial A}{\partial y} \right)_{x,z,J} + \left( \frac{\partial A}{\partial z} \right)_{x,y,J} \left( \frac{\partial z}{\partial y} \right)_{x,p,J} \]

Applying this to the case \( A=p \) yields

\[
\left( \frac{\partial p}{\partial x} \right)_z = \rho g \left( \frac{\partial z}{\partial x} \right)_p \quad \text{and} \quad \left( \frac{\partial p}{\partial y} \right)_z = \rho g \left( \frac{\partial z}{\partial y} \right)_p
\]

We can now relate the \textbf{geostrophic wind} to the gradient of the \textbf{isobaric height} as follows:

\[
v_g = \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_p ; \quad u_g = -\frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_p
\]

which, if \( g \) is constant, can also written in terms of the \textbf{geopotential height}.

\[
\Phi \equiv gz
\]
The subscript \( g \) stands for “geostrophic” and \( z \) is the height (in m) of the isobaric surface. The observed wind at 500 hPa on 3 March 1995, 12 UTC, shown in the lower left panel of figure 1.77, and the geostrophic wind, computed from the height gradient (problem 1.21) (lower right panel of figure 1.77) are very similar, indicating that the atmosphere at this level is approximately in geostrophic balance so that the isopleths of the height of an isobaric surface can also be interpreted as streamlines.

A weather system, or atmospheric circulation system, can be classified as being “balanced” or as being “unbalanced”. The cluster of thunderstorms (figures 1.73-1.76) is an example of an “unbalanced weather system”, i.e. deviations from hydrostatic- and geostrophic balance are relatively large. The trough in the jet, which is illustrated in figure 1.77, is an example of a “balanced weather system”, i.e. deviations from hydrostatic- and geostrophic balance are small.

Atmospheric circulation systems possessing horizontal scales of several hundred km or larger are usually observed to be in geostrophic- and hydrostatic-balance. This information is used to properly initialize a numerical weather prediction model. Wind measurements, obtained from radiosondes, are relatively inaccurate, principally because of their lack of representativeness for an area in the order of 25 by 25 km, which is the approximate horizontal resolution of an average numerical weather prediction model in the year 2013. Temperature measurements do not suffer as much from this problem. The height of a pressure surface is calculated from the hydrostatic balance equation, using the observed surface pressure as a boundary condition. From this the geostrophic wind is calculated (eq. 1.136; problem 1.22). By initializing a numerical weather prediction model with the geostrophic wind, instead of with the observed wind, false oscillations in the model, introduced by imbalances due to errors in the wind observations, are avoided. Interestingly, it appears that not capturing the real imbalances at initial time does not adversely affect the predictability of the weather on horizontal scales of several hundred kilometers or larger. Chapter 5 explains why this is the case.

For an atmosphere in hydrostatic balance, any variable \( s \) (such as velocity or temperature) is a function of \( t, x, y \) and \( p \). Therefore, with the chain rule we can write:

\[
\frac{ds}{dt} = \frac{\partial s}{\partial t} dt + \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy + \frac{\partial s}{\partial p} dp
\]

Dividing this equation by \( dt \) yields

\[
\frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} \frac{dx}{dt} + \frac{\partial s}{\partial y} \frac{dy}{dt} + \frac{\partial s}{\partial p} \frac{dp}{dt} = \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + \omega \frac{\partial s}{\partial p},
\]

where the (partial) \( x \)-derivative is performed with \( t, y \) and \( p \) kept constant and the \( y \)-derivative is performed with \( t, x \) and \( p \) kept constant. The parameter, \( \omega \),

\[
\omega = \frac{dp}{dt}.
\]
is the “vertical velocity” in the isobaric coordinate system (Box 1.11, eq.7).

**PROBLEM 1.21. Analysis of observations**

Observations are performed at randomly spaced points (see figure 1.78), while theoretical analysis and/or numerical models usually require data on a regular grid of points. This requires interpolation of the observations. The most simple interpolation method is called **piecewise linear interpolation**. Suppose that the gridpoint to which we want to interpolate the observations has the horizontal coordinates, \((x_0, y_0)\). The horizontal coordinates of three measuring points in the vicinity of this gridpoint are given by \((x_i, y_i)\), with \(i=1, 2, 3\). The value of a variable (for instance the height of a pressure surface) at one of these points can be expressed as follows.

\[
z_i = z_0 + (x_i - x_0) \frac{\partial z}{\partial x} + (y_i - y_0) \frac{\partial z}{\partial y},
\]

The desired value of the height at the gridpoint \((x_0, y_0)\) is \(z_0\). Applying this formula to the three measuring points yields three equations with three unknowns, \(z_0\), \(\frac{\partial z}{\partial x}\) and \(\frac{\partial z}{\partial y}\), which can be written concisely as

\[
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix} =
\begin{pmatrix}
1 & (x_1 - x_0) & (y_1 - y_0) \\
1 & (x_2 - x_0) & (y_2 - y_0) \\
1 & (x_3 - x_0) & (y_3 - y_0)
\end{pmatrix}
\begin{pmatrix}
z_0 \\
\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial y}
\end{pmatrix}
\]

which can be written shortly as

\[
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix} = M
\begin{pmatrix}
z_0 \\
\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial y}
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
z_0 \\
\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial y}
\end{pmatrix} = M^{-1}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix}
\]

so that we have to invert the matrix,

\[
M =
\begin{pmatrix}
1 & (x_1 - x_0) & (y_1 - y_0) \\
1 & (x_2 - x_0) & (y_2 - y_0) \\
1 & (x_3 - x_0) & (y_3 - y_0)
\end{pmatrix}
\]

Note that, since the locations of stations are given in degrees latitude and longitude in table 1.4, it may be quicker to work in these units. An important action in this procedure, which
precedes the actual interpolation, is the search of suitable measurements (i.e. which three measurements are used for this interpolation?).

Compute the 500 hPa isobaric height \((z)\) and wind \((u, v)\) at the points \((x_0, y_0) = (5.17^\circ E, 52.09^\circ N)\) (De Bilt) and \((x_0, y_0) = (10.38^\circ E, 50.56^\circ N)\) (Meiningen) on 3 March 1995, 00 UTC (figure 1.78). Also compute the isobaric height- and wind-gradients. We can use this information to compute the geostrophic wind at 500 hPa (problem 1.22) and the gradient wind (problem 1.23). Compare the interpolated 500 hPa isobaric height and wind with the measured height and wind. The observations of the measuring stations can be obtained from table 1.4 (see also figure 1.78). Take account of the convergence of the meridians (lines of equal longitude).

PROBLEM 1.22. Geostrophic wind
Estimate the geostrophic wind \((u_g, v_g)\) on 3 March 1995, 00 UTC at the points \((x_0, y_0) = (5.17^\circ E, 52.09^\circ N)\) (De Bilt) and \((x_0, y_0) = (10.38^\circ E, 50.56^\circ N)\) (Meiningen) from the gradient of the isobaric height (problem 1.21). Compare the geostrophic wind with the interpolated measured winds at the same point (problem 1.21) (table 1.4). Explain the differences.

The nonlinear balance equations (section 1.23) in the isobaric coordinate system are

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= f\nu_{grad} - g \left(\frac{\partial z}{\partial x}\right)_p = f\left(v_{grad} - v_g\right); \\
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -fu_{grad} - g \left(\frac{\partial z}{\partial y}\right)_p = -f\left(u_{grad} - u_g\right).
\end{align*}
\tag{1.140}
\tag{1.141}
\]

Here, the subscript “\(\text{grad}\)” stands for “gradient wind”. The left hand side of these equations represents the “stationary” part of the ageostrophic wind (section 1.34).

PROBLEM 1.23. Non-linear balance and gradient wind
From eqs. (1.140) and (1.141) calculate the gradient wind at the points \((x_0, y_0) = (5.17^\circ E, 52.09^\circ N)\) (De Bilt) and \((x_0, y_0) = (10.38^\circ E, 50.56^\circ N)\) (Meiningen) on the 500 hPa level on 3 March 1995, 00 UTC with the velocity- and height gradients calculated in problems 1.21 and 1.22. Compare the gradient wind with the geostrophic wind (problem 1.22). Explain the differences.

The vertical component of the relative vorticity on an isobaric surface is

\[
\zeta = \left(\frac{\partial v}{\partial x}\right)_{t,p,y} - \left(\frac{\partial u}{\partial y}\right)_{t,p,x}.
\tag{1.142a}
\]

The horizontal divergence on an isobaric surface is

\[
\delta = \left(\frac{\partial u}{\partial x}\right)_{t,p,y} + \left(\frac{\partial v}{\partial y}\right)_{t,p,x}.
\tag{1.142b}
\]
<table>
<thead>
<tr>
<th>PRES</th>
<th>HGHT</th>
<th>TEMP</th>
<th>DWPT</th>
<th>RELH</th>
<th>MIXR</th>
<th>DRCT</th>
<th>SKNT</th>
<th>THTA</th>
<th>THTE</th>
<th>THTV</th>
</tr>
</thead>
<tbody>
<tr>
<td>hPa</td>
<td>m</td>
<td>°C</td>
<td>°C</td>
<td>%</td>
<td>g/kg</td>
<td>deg</td>
<td>knot</td>
<td>K</td>
<td>K</td>
<td>K</td>
</tr>
</tbody>
</table>

**10393 Lindenberg (Berlin)** Station latitude: 52.20 Station longitude: 14.10

| 700.0 | 2858 | −15.1 | −19.9 | 67   | 1.13 | 235   | 48    | 285.7 | 289.3 | 285.9 |
| 500.0 | 5330 | −31.1 | −41.1 | 37   | 0.21 | 265   | 70    | 295.1 | 295.8 | 295.1 |

**10548 Meiningen** Station latitude: 50.56 Station longitude: 10.38

| 700.0 | 2881 | −12.1 | −27.1 | 28   | 0.60 | 245   | 43    | 289.1 | 291.0 | 289.2 |
| 500.0 | 5370 | −29.1 | −34.0 | 63   | 0.43 | 260   | 59    | 297.5 | 299.0 | 297.6 |

**10739 Stuttgart** Station latitude: 48.83 Station longitude: 9.19

| 700.0 | 2920 | −8.7  | −24.7 | 26   | 0.74 | 292.8 | 295.3 | 292.9 |
| 500.0 | 5440 | −27.1 | −32.1 | 62   | 0.52 | 299.9 | 301.7 | 300.0 |

**10618 ETGI Idar-Oberstein** Station latitude: 49.70 Station longitude: 7.32

| 700.0 | 2869 | −9.5  | −10.7 | 91   | 2.43 | 245   | 65    | 291.9 | 299.4 | 292.4 |
| 500.0 | 5390 | −27.3 | −29.9 | 78   | 0.64 | 255   | 59    | 299.7 | 301.9 | 299.8 |

**10006 Nuernberg-Oberschleissheim** Station latitude: 50.25 Station longitude: 11.15

| 700.0 | 2943 | −8.9  | −35.9 | 9    | 0.26 | 265   | 24    | 292.6 | 293.5 | 292.6 |
| 500.0 | 5470 | −26.1 | −43.1 | 19   | 0.17 | 270   | 37    | 301.2 | 301.8 | 301.2 |

**10235 Praha** Station latitude: 50.00 Station longitude: 14.13

| 700.0 | 2920 | −8.5  | −55.5 | 1    | 0.03 | 260   | 31    | 293.0 | 293.2 | 293.1 |
| 500.0 | 5440 | −27.1 | −45.1 | 16   | 0.14 | 270   | 49    | 299.9 | 300.4 | 300.0 |

**10771 ETGK Kuehmersbruck** Station latitude: 49.43 Station longitude: 10.89

| 700.0 | 2926 | −9.3  | −37.3 | 8    | 0.22 | 250   | 35    | 292.2 | 292.9 | 292.2 |
| 500.0 | 5440 | −26.9 | −38.9 | 31   | 0.26 | 265   | 51    | 300.2 | 301.1 | 300.2 |

**10486 Dresden** Station latitude: 51.12 Station longitude: 13.68

| 700.0 | 2891 | −13.9 | −22.9 | 47   | 0.87 | 250   | 38    | 287.1 | 289.8 | 287.2 |
| 500.0 | 5380 | −28.9 | −39.9 | 34   | 0.24 | 270   | 63    | 297.7 | 298.6 | 297.8 |

**10410 EDZE Essen** Station latitude: 51.40 Station longitude: 6.96

| 700.0 | 2817 | −12.5 | −14.6 | 84   | 1.77 | 225   | 50    | 288.6 | 294.0 | 288.9 |
| 500.0 | 5300 | −31.8 | −43.3 | 32   | 0.46 | 245   | 52    | 296.8 | 298.4 | 296.9 |

**06260 De Bilt** Station latitude: 52.09 Station longitude: 5.17

| 700.0 | 2769 | −11.9 | −12.9 | 92   | 2.03 | 225   | 47    | 289.3 | 295.5 | 289.6 |
| 500.0 | 5250 | −30.9 | −41.9 | 19   | 0.19 | 230   | 58    | 295.3 | 294.0 | 295.3 |

**06447 Uccle** Station latitude: 50.79 Station longitude: 3.34

| 700.0 | 2789 | −11.1 | −37.1 | 10   | 0.23 | 235   | 56    | 290.2 | 291.0 | 290.2 |
| 500.0 | 5290 | −27.9 | −49.9 | 10   | 0.08 | 245   | 67    | 299.0 | 299.3 | 299.0 |

**03496 Hemsby** Station latitude: 52.68 Station longitude: 1.67

| 700.0 | 2656 | −11.5 | −16.5 | 66   | 1.51 | 170   | 8     | 289.7 | 294.4 | 290.0 |
| 500.0 | 5140 | −31.5 | −43.5 | 30   | 0.16 | 205   | 20    | 294.6 | 295.2 | 294.6 |

**10200 Emden-Flugplatz** Station latitude: 53.38 Station longitude: 7.23

| 700.0 | 2793 | −14.1 | −27.7 | 12   | 0.22 | 230   | 22    | 286.4 | 287.2 | 286.4 |
| 500.0 | 5240 | −33.9 | −38.4 | 64   | 0.28 | 245   | 48    | 291.6 | 292.6 | 291.7 |

**Table 1.4.** Radiosonde observations of 3 March 1995, 00 UTC at different station in Europe, including station code, latitude(°N), longitude (°E). Wind direction is defined as follows: 0° or 360° corresponds to wind blowing from north; 90° corresponds to wind blowing from east; 180° corresponds to wind blowing from south; 270° corresponds to wind blowing from west. Source: [http://weather.uwyo.edu/upperair/sounding.html](http://weather.uwyo.edu/upperair/sounding.html).
PROBLEM 1.24 Relative vorticity and horizontal divergence

Estimate the relative vorticity and the horizontal divergence on the 500 hPa isobaric level on 3 March 1995, 12 UTC at the \((x_0, y_0) = (5.17^\circ E, 52.09^\circ N)\) (De Bilt) and \((x_0, y_0) = (10.38^\circ E, 50.56^\circ N)\) (Meiningen) with information about the velocity gradients calculated in problem 1.21. Compare the relative vorticity with the planetary vorticity. Also give an estimate of the Rossby number (section 1.23).

1.31 Thermal wind and temperature advection

The equation expressing hydrostatic equilibrium (section 1.14) can be rewritten, using the equation state, \(p = \rho RT\) (eq. 1.1), as

\[
\frac{\partial z}{\partial \ln p} = \frac{RT}{g}.
\]  

(1.143)

Now, differentiating (1.134) with respect to \(\ln(p)\) yields

\[
\frac{\partial v_g}{\partial \ln p} = \frac{g}{f} \frac{\partial^2 z}{\partial x \partial \ln p}, \quad \frac{\partial u_g}{\partial \ln p} = -\frac{g}{f} \frac{\partial^2 z}{\partial y \partial \ln p}.
\]  

(1.144)

Combining (1.143) and (1.144) yields the two components of the thermal wind equation:

\[
\frac{\partial v_g}{\partial \ln p} = -\frac{R}{f} \frac{\partial T}{\partial x}, \quad \frac{\partial u_g}{\partial \ln p} = \frac{R}{f} \frac{\partial T}{\partial y}.
\]  

(1.145)

**FIGURE 1.79.** Three-dimensional portrayal of the relationship between turning of the geostrophic wind and temperature advection: (a) cold advection and backing of the wind with increasing height; (b) warm advection and veering of the wind with increasing height (from Visconti, G., 2001: Fundamentals of Physics and Chemistry of the Atmosphere. Springer Verlag, Berlin, 593pp).
or, in vectorial form,

$$\frac{\partial \tilde{v}_g}{\partial \ln p} = -\frac{R}{f} \tilde{k} \times \tilde{\nabla}_p T. \tag{1.146}$$

The symbol “×” stands for “cross product”.

The **thermal wind vector**, $\tilde{v}_T$, is defined as the vector difference between the

**FIGURE 1.80.** NOAA image in channel 4 (IR) made on (a) March 2, 1995, at 1343 UTC; (b) on March 3, 1995, at 0157 UTC; (c) on March 3, 1995, at 1133 UTC.
geostrophic winds at two levels. Integrating (1.146) from \( p=p_0 \) to \( p=p_1 \) (with \( p_1<p_0 \)) yields

\[
\vec{v}_T = \vec{v}_g(p_1) - \vec{v}_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} \left( \hat{k} \times \vec{v}_g \right) d\ln p .
\] (1.147)

The \( x \)- and \( y \)-component of this equation are

\[
u_T = -\frac{R}{f} \left[ \frac{\partial(T)}{\partial y} \right]_{p} \ln \left( \frac{p_0}{p_1} \right) \quad \text{and} \quad v_T = \frac{R}{f} \left[ \frac{\partial(T)}{\partial x} \right]_{p} \ln \left( \frac{p_0}{p_1} \right) ,
\]

where the \textbf{mean temperature in the layer}, \((T)\), is defined by

\[
\langle T \rangle = \frac{1}{\int_{p_0}^{p_1} d\ln p} \left[ \int_{p_0}^{p_1} T d\ln p \right]^{-1} .
\](1.149)

The thermal wind equation can be used to estimate the mean horizontal temperature advection in a layer. It is clear from (1.147) that the \textbf{thermal wind vector and the isotherms on an isobaric surface are parallel}, with the warm air to the right looking in the direction of the thermal wind in the northern hemisphere. Thus, as is illustrated in figure 1.79\textit{a}, a geostrophic wind that turns counterclockwise (backs) with increasing height is associated with cold air advection. Conversely, as shown in figure 1.79\textit{b}, clockwise turning (veering) of the geostrophic wind with increasing height implies warm air advection by the geostrophic wind.

**PROBLEM 1.25. Identifying areas of cold and warm air advection**

Figures 1.78 shows the wind vectors at, respectively, 500 hPa (upper panel) and 850 hPa (lower panel) on 3 March 1995, 00 UTC. Locate areas of warm advection and areas of cold advection in the layer 850-500 hPa, using the theory described in this section. Indicate the relationship between these areas and the observed cloud pattern (figure 1.80\textit{b}).

**PROBLEM 1.26. Testing the thermal wind equation (1.148)**

Test the validity of eq. 1.148 between 700 hPa and 500 at \((x_0, y_0)=(5.17^\circ E, 52.09^\circ N)\) (De Bilt) and \((x_0, y_0)=(10.38^\circ E, 50.56^\circ N)\) (Meiningen) on 3 March 1995, 00 UTC. This means that you should compute the two components of the thermal wind for the layer between 700 hPa and 500 hPa from the temperature distribution and compare it with the observed thermal wind. For radiosonde data see \texttt{http://weather.uwyo.edu/} and Table 1.4.

### 1.32 Fronts and mid-latitude cyclones

**Figure 1.81** shows the temperature distribution, as well as the geopotential height, at 850 hPa over central-western Europe on 25 May 2010. A zone exhibiting a rather strong meridional temperature-gradient is observed between England and Poland. In meteorological terms, it is said that a \textbf{front} is lying over this region. Fronts are in fact omnipresent in the atmosphere, especially in mid-latitudes. In the 1920’s fronts were conceived by members of the “\textit{Bergen school}” (V. Bjerknes, J. Bjerknes, T. Bergeron and H. Solberg) as \textbf{interfaces between warm and humid and cold and dry air masses}. Fronts are usually recognized on
FIGURE 1.81. A stationary temperature front over central Europe at 850 hPa on 25 May 2010, 12 UTC (upper panel) and 15 UTC (lower panel). Colours and black contours represent temperature at 850 hPa, labeled in °C; white contours represent height of the 850 hPa isobaric surface, labeled in dm. Source: http://www.wetterzentrale.de/

satellite images as cloud bands, which usually produce precipitation. This fact suggests that motion is upward along the front. Usually, the upward motion occurs on the warm side of the front, while the compensating downward motion occurs on the cold side of the front. Why this is the case, is explained most clearly chapter 8.

The Bergen school identified four types of fronts: cold, warm, stationary and occluded.
All these frontal types, except for the stationary front, are typically observed within a mid-latitude cyclone (figure 1.82). The warm front separates the receding cold air mass from the advancing warm air, which is most frequently observed as a warm moist plume of air, moving polewards. This plume is sometimes called the warm sector of the mid-latitude cyclone or, alternatively, the warm conveyor belt, when the warm travels poleward in a narrow stream just in advance of the cold front. The cold front separates the cold air mass, which undercuts the warm conveyor belt, from the warm air mass. The occluded front is formed when a cold front overtakes a warm front. The advancing cold air then forms a new boundary with the receding cold air ahead of the warm front, while the warm sector is “forced” upwards, so that it is not observed at the surface anymore.

Because radiosondes were not launched routinely at weather stations until the 1940’s, the investigators of the Bergen school could only use surface measurements. It is thus remarkable how well the conceptual model of the structure and life cycle of a mid-latitude cyclone, which was devised by the Bergen school, still holds. Figure 1.83 gives a schematic impression of the life cycle of a mid-latitude depression according to the Norwegian (Bergen) school as well as a more modern (1990) “American” view of this life cycle, due to M.A. Shapiro and D. Keyser. The main difference between the two conceptual models is the fracture of the cold front, in the case of the Shapiro-Keyser model, allowing the cold front to progress through the warm sector perpendicular to the warm front. Also, the “occlusion” in the old Norwegian model is not defined as such in the modern view. Instead, a new type of front is introduced, which is called “back-bent front”.

**Figure 1.82.** Conceptual model (due to the Bergen school) of the positions of fronts at or near the Earth’s surface within a full-grown midlatitude cyclone. Black lines indicate streamlines. Green shading indicates regions of 100%-cloud cover. A midlatitude cyclone forms due to the baroclinic instability of a quasi-stationary front. This instability is illustrated in figure 1.80 (the details will be treated in chapter 9). An occluded front is formed during the process of cyclogenesis when a cold front overtakes a warm front. When this occurs, the warm air is separated (occluded) from the cyclone centre at the Earth’s surface. (Based on an image taken from Wikimedia Commons).
FIGURE 1.83. The life cycle of an extratropical cyclone according to, respectively, the Norwegian (Bergen) school (left) and according to M.A. Shapiro and D. Keyser (1990) (right). The life cycle according to Shapiro and Keyser is characterized by four stages: (I) incipient frontal cyclone; (II) frontal fracture; (III) bent-back warm front and frontal T-bone; (IV) warm-core frontal seclusion. Upper: sea level isobars and fronts. Lower panel: surface isotherms. (Shapiro, M.A., D. Keyser, 1990: Fronts, jet streams and the tropopause. Chapter 10 of Extratropical Cyclones: The Erik Palmen Memorial Volume. American Meteorological Society, 262 pp.)

PROBLEM 1.27. Identifying the location of fronts.
Draw the fronts (cold, warm and occluded/back-bent) in the lower panel (850 hPa) of figure 1.78. The conceptual models shown in figure 1.82 and 1.83, as well as the satellite images shown in figure 1.80 will be of help. To which of the two conceptual models of the life cycle of a mid-latitude cyclone (figure 1.83) does this case conform best?

1.33 Frontogenesis and frontolysis: the Q-vector

The formation or strengthening of a front in the atmosphere is referred to as “frontogenesis”. The vanishing or weakening of a front in the atmosphere is referred to as “frontolysis”.

Miller (1948)\(^{52}\) defined “frontal strength” as the absolute value of the gradient of potential temperature, \(|\vec{V} \cdot \theta|\). Nowadays the definition of frontal strength is based on the magnitude of the horizontal temperature gradient:

\[
|\vec{V}_h \theta| = \sqrt{\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2}.
\]

The “frontogenesis function” quantifies the time rate of change in potential horizontal

\(^{52}\text{Miller, J.E., 1948: On the concept of frontogenesis. J. Meteor. 5, 169-171.}\)
temperature gradient following the motion of an air-parcel. Its definition is\(^{53}\):

\[
\text{Frontogenesis function} = \frac{d(\vec{V}_h \theta)^2}{dt} = 2\vec{V}_h \theta \cdot \frac{d\vec{V}_h \theta}{dt} = 2\vec{V}_h \theta \cdot \vec{Q}.
\]

(1.150)

The vector \(\vec{Q}\) is known as the \(\text{“Q-vector”}\). A non-zero frontogenesis function implies changing temperature gradients. From eq. 1.150 we see that \textbf{frontogenesis} (absolute temperature gradient intensifies) occurs if the Q-vector points up the potential temperature gradient, and that \textbf{frontolysis} (absolute temperature gradient weakens) occurs if the Q-vector points down the potential temperature gradient. If the Q-vector points in the direction perpendicular to the temperature gradient (i.e parallel to the isotherms), the absolute value of the temperature gradient is invariant. Nevertheless, this case is still of interest because the vector \(\vec{V}_h \theta\) changes direction.

A change in the absolute temperature gradient and a change in the direction of the temperature gradient have similar dynamical consequences, i.e. both represent a perturbation to thermal wind balance, necessitating a dynamical response in the atmosphere. The theory of this dynamical response, i.e. the adjustment to thermal wind balance, is central in Dynamical Meteorology and is the main theme of chapters 8 and 9.

Let us first derive an expression for the Q-vector from the potential temperature equation (1.55). which repeated here:

\[
\frac{d\theta}{dt} = \frac{J}{\Pi}.
\]

From this equation it is easily deduced that the Q-vector can be written as

\[
\vec{Q} = (Q_1, Q_2),
\]

(1.151)

with

\[
Q_1 = \frac{d}{dt} \left( \frac{\partial \theta}{\partial x} \right) = -\frac{\partial u}{\partial x} \frac{\partial \theta}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial \theta}{\partial y} - \frac{\partial w}{\partial x} \frac{\partial \theta}{\partial z} + \frac{\partial J}{\partial x} \frac{1}{\Pi}
\]

(1.152a)

and

\[
Q_2 = \frac{d}{dt} \left( \frac{\partial \theta}{\partial y} \right) = -\frac{\partial u}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial v}{\partial y} \frac{\partial \theta}{\partial y} - \frac{\partial w}{\partial y} \frac{\partial \theta}{\partial z} + \frac{\partial J}{\partial y} \frac{1}{\Pi}
\]

(1.152b)

According to eqs. (1.152a,b), there are four mechanisms for changing the horizontal temperature gradient.

(i) \textbf{Horizontal shear} of the along front wind in combination with an along front temperature gradient (second term on the r.h.s. of 1.152a and first term on the r.h.s. of 1.152b).

(ii) \textbf{Convergence or confluence of the cross front wind} (first term on the r.h.s. of 1.153a and second term on the r.h.s. of 1.152b).

(iii) **Differential vertical motion or tilting** of the vertical potential temperature gradient onto the horizontal (third term on the r.h.s. of 1.152a,b). This term vanishes in the geostrophic approximation.

(iv) **Differential diabatic heating** (fourth term on the r.h.s. of 1.152a,b).

The first two mechanisms are most important in surface frontogenesis, while the tilting term (mechanism (iii)) often dominates the process of frontogenesis in the upper troposphere. The fourth frontogenetic mechanism is involved in producing basic equator to pole temperature gradient. **Figure 1.84** visualizes two kinematic mechanisms responsible for frontogenesis.

It is frequently convenient to use the geostrophic wind in the above equations so that the \( Q \)-vector can be calculated from the temperature- and geopotential fields. The expressions for the \( x \)- and \( y \)-component of this “geostrophic \( Q \)-vector”, \( \mathbf{Q}_g \), are, respectively:

\[
Q_{g1} = \frac{d}{dt} \left( \frac{\partial \theta}{\partial x} \right) = -\frac{\partial u_g}{\partial x} \frac{\partial \theta}{\partial x} - \frac{\partial v_g}{\partial x} \frac{\partial \theta}{\partial y} + \frac{\partial}{\partial x} \left( \frac{J}{\Pi} \right) \tag{1.153a}
\]

and

\[
Q_{g2} = \frac{d}{dt} \left( \frac{\partial \theta}{\partial y} \right) = -\frac{\partial u_g}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial v_g}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial}{\partial y} \left( \frac{J}{\Pi} \right) \tag{1.153b}
\]

The superpositioning of \( Q \)-vectors on a temperature analysis can be used to identify regions of significant frontogenesis or frontolysis. In these regions the atmosphere is **undergoing a process of adjustment where potential temperature and momentum are redistributed in such a way that thermal wind balance is maintained**. This process is in general accompanied by forced lifting and cloud formation and precipitation (chapters 8-10).
Figure 1.85. Potential temperature, isobaric height and geostrophic $Q_g$-vector 30 hours apart at the 864 hPa-isobaric level in a simulation of the life-cycle of a mid-latitude cyclone. Thin solid dark blue lines are isopleths of isobaric height (contour interval: 50 m, thick line: 1250 m). Thick light blue lines are isopleths of potential temperature (contour interval: 5 K, thick line: 0°C). Geostrophic $Q_g$-vectors (arrows) are shown only if the absolute value exceeds $10^{-11}$ K m$^{-1}$ s$^{-1}$. The vertical axis corresponds to latitude. The horizontal axis corresponds to longitude. Boundary conditions at the western and eastern boundary are periodic. The $Q_g$-vector in the cold front ($cf$) (lower panel: lower left corner) is directed towards the warm air, indicating frontogenesis. This is also the case in the warm front ($wf$) to north of the warm sector. In the back-bent front ($bbf$) the $Q_g$-vectors are directed approximately parallel to the isotherms, indicating that isotherms in the backbent front are rotating. We’ll see (chapters 9) that this has important dynamical consequences. More information on this simulation is given in chapter 10, and at http://www.staff.science.uu.nl/~delde102/BaroclinicLifeCycle.htm.
FIGURE 1.86. The flow field (wind vectors) and the absolute value of the temperature gradient (blue solid lines) 30 hours apart at the 864 hPa-isobaric level in a simulation of the life-cycle of a mid-latitude cyclone with a numerical model of the atmospheric circulation, which neglects frictional and diabatic effects. Contour interval is $0.4 \times 10^{-5}$ K m$^{-1}$, labels are in units of $10^{-5}$ K m$^{-1}$. See chapter 10 for further information about the parameter values. Note the similarity between the simulated frontal morphology in the lower panel and the frontal morphology in stage 3 of the conceptual model of Keyser and Shapiro (Figure 1.83). Note also the increase of the temperature gradients across the fronts (see also figure 1.85).
**Figure 1.85** shows an example of an analysis of the Q-vectors superposed on the isotherms, based on a simulation of the life-cycle of a mid-latitude cyclone. In particular, in the more mature stage of the cyclone (lower panel) the Q-vectors point across the isotherms from the cold to the warm air mass, which indicates that the front is intensifying (undergoing frontogenesis). This is confirmed in **figure 1.86**, which shows an analysis of the absolute value of the temperature gradient, demonstrating that both the cold front in the lower left corner (the south-west) and the warm front in the north have nearly “doubled” their intensity within 30 hours. The intensifying temperature gradients in a *freely evolving rotating stratified fluid flow* are truly remarkable!

### 1.34 The ageostrophic wind

Suppose that we break up the horizontal wind into a geostrophic part \((\vec{v}_g)\) and an *ageostrophic part* \((\vec{v}_a)\), so that

\[
\vec{v} = \vec{v}_g + \vec{v}_a.
\]  
(1.154)

The horizontal components of the equation of motion in isobaric coordinates are (section 1.31)

\[
\begin{align*}
\frac{du}{dt} & = fv - g \left( \frac{\partial z}{\partial x} \right)_p, \\
\frac{dv}{dt} & = -fu - g \left( \frac{\partial z}{\partial y} \right)_p.
\end{align*}
\]  
(1.155a, 1.155b)

With (1.154) and (1.134) these equations become

\[
\begin{align*}
\frac{du}{dt} & = f\vec{v}_a, \\
\frac{dv}{dt} & = -f\vec{v}_a.
\end{align*}
\]  
(1.156a, 1.156b)

In vector notation these equations translate into:

\[
\vec{v}_a = -\frac{1}{f} \hat{k} \times \frac{d\vec{v}}{dt} - \frac{1}{f} \hat{k} \times \frac{d\vec{v}_g}{dt} + \frac{1}{f} \hat{k} \times \frac{d\vec{v}_a}{dt}.
\]  
(1.157)

The ageostrophic wind vector always points perpendicular and to the left of the parcel acceleration in the northern hemisphere. The geostrophic wind (1.134) can be written in vector notation as

\[
\vec{v}_g = \frac{g}{f} \hat{k} \times \nabla p\hat{z}.
\]  
(1.158)
FIGURE 1.87. Caption on following page.
FIGURE 1.87. (a) Sea level isobars (thick grey lines, labeled in hPa) and sea level isallobars (thin solid lines, labeled in hPa per 3 hours), May 28, 2000, 06 UTC; (b) isallobaric wind (bold arrows) at sea level between 3 and 6 UTC on May 28 2000 and geostrophic wind at 6 UTC; (c) observed 10-min mean wind at 6 UTC at a height of 10 m above the Earth’s surface. Barbs, attached to the wind vector, which points towards the station position, indicate the windspeed, ff. No barb corresponds to ff<2.5 knots (1 knot is 0.5 m/s); each whole barb corresponds to an extra 10 knots; each half a barb corresponds to an extra 5 knots. The analysis is based on an interpolation of the observations to a "lat-lon"-grid of 24 by 25 points with an interval 0.75° in the zonal direction and 0.5° in the meridional direction. The boundaries of the domain shown are at 8°W, 9.25°E, 56.5°N and 45.5°N. Stormy conditions occur if the isallobaric wind and the geostrophic wind point in the same direction. This was the case on the south-east side of the low (marked by the letter "L" in (a)).

where

$$\vec{V}_p = \left( \begin{pmatrix} \frac{\partial}{\partial x} \rho \ e^{\frac{\partial}{\partial y}} \rho \ e^{0} \end{pmatrix} \right). \quad (1.159)$$
Substituting the geostrophic wind eq. 1.158 into eq. 1.157, yields

$$\vec{v}_a = \frac{1}{f} \hat{k} \times \left[ \frac{g}{f} \left( \hat{k} \times \hat{\nabla} p \frac{\partial z}{\partial t} \right) + \text{other terms} \right].$$

(1.160)

The $x$- and $y$- components of this equation are, respectively, (including the “other terms”)

$$u_a = -\frac{1}{f} \left( \frac{g}{f} \frac{\partial}{\partial x} + \frac{\partial v_a}{\partial t} + u \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} \right),$$

(1.161a)

$$v_a = \frac{1}{f} \left( \frac{g}{f} \frac{\partial}{\partial y} + \frac{\partial u_a}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} \right).$$

(1.161b)

When height is used as a vertical coordinate, instead of pressure, these expressions become:

$$u_a = -\frac{1}{\rho f^2} \frac{\partial}{\partial x} + \text{"other terms";}$$

(1.162a)

$$v_a = -\frac{1}{\rho f^2} \frac{\partial}{\partial y} + \text{"other terms".}$$

(1.162b)

The first term on the r.h.s. of (1.162a,b) represents the component of the ageostrophic wind that is due to the gradient of the pressure- or height-tendency. This component of the ageostrophic wind, which is called the isallobaric wind, is directed down the gradient of height- or pressure-tendencies. The isallobaric wind can be substantial in the presence of a rapidly deepening cyclone or a rapidly building anticyclone, or fast moving pronounced cyclones or anticyclones, or around mountain ranges such as the Alps. Storms are frequently associated with areas where the isallobaric wind vector points in the same direction as the geostrophic wind. An example is given in figure 1.87. On May 28 2000 a cyclone travelled over the English Channel and the North Sea along the coast of Northern France, Belgium and the Netherlands. Wind gusts of more than 30 m s$^{-1}$ occurred to the south-east of the centre of the cyclone, while comparatively calm conditions occurred to the north-west. This can be explained by noting that the geostrophic wind and the isallobaric wind re-enforce each other in the south-east, while they counteract each other in the north-west.

Due to the effect of friction, the actual wind speed near the Earth’s surface is usually much smaller than the sum of the geostrophic wind speed and the isallobaric wind speed (figure 1.87c).

### 1.35 Jet streaks

When the cross-front temperature gradient is not uniform along a front, thermal wind balance dictates a local wind maximum over the most intense part of the front. This local wind maximum is referred to as a jet streak. The accelerations and deceleration of air parcels into the jetstreak or out of the jetstreak lead to significant departures from geostrophic balance.

**Figure 1.88** shows a schematic picture of a jet streak in a straight jet aligned along the $x$-axis. The jetstreak travels very slowly with respect to the Earth’s surface, so that parcels of air travelling through the upper level jet core accelerate into the jet streak.
The acceleration is given by (eq. 1.156a)

$$\frac{du}{dt} = f\nu_a;$$

This equation shows that an ageostrophic flow, $\nu_a$, is necessary to realize the acceleration or deceleration.

If $du/dt>0$, as in the entrance region of the jetstreak, $\nu_a>0$, i.e. the ageostrophic motion is directed towards lower geopotential height. After passing through the region of maximum winds, the parcels of air decelerate ($du/dt<0$) through ageostrophic motion towards higher pressure ($\nu_a<0$). In this way secondary circulations perpendicular to the jet axis are set up at the entrance and exit regions of the jet streak. The secondary circulation is thermally direct in the entrance or confluence region and thermally indirect in the exit region of the jet streak. Moreover, in the exit region, the secondary circulation is frontogenetic because the downward motion on the warm side of the front causes a adiabatic temperature increase (assuming that the potential temperature increases with height), while the upward motion on the cool side of the front causes a temperature decrease, i.e. there is frontogenesis by differential vertical motion which tilts the vertical potential temperature gradient onto the horizontal (the third mechanism in eq. 1.152). In the entrance region the situation is reversed, i.e. the secondary circulation is frontolytic (provided again that the temperature stratification is stable). Hence, the jet streak progresses forward, along the current. Simultaneously, due to the secondary circulation, relatively cool air is advected horizontally into the exit region of jet streak at upper levels while relatively warm air is
advected horizontally into the exit region of the jet streak at lower levels. Hence, the stratification becomes less stable. Again, the opposite is the case in the entrance region of the jet streak. Clearly, the left exit region of the jet streak is the most favourable area for the formation of precipitation systems that depend upon upward motion and destabilization of the atmosphere.

The vertical motion that is associated with the secondary circulation in a stably stratified atmosphere is strongly slanted. As the air parcel travels from one side of the jet or front to the other, while slowly descending or ascending, it also moves parallel to the front or jet with the relatively strong average flow velocity. This kind of vertical motion, therefore, is of a totally different nature than the vertical motion that is the result of (conditional) hydrostatic instability (section 1.16). The slanted upward motion around fronts or jets gives rise to the characteristic “layered” clouds, such as cirrus and stratus. Conditional hydrostatic instability, on the other hand, gives rise to the characteristic “puffy” clouds, that are referred to as cumulus.

Semi-permanent jetstreaks, embedded in the subtropical jet, are observed in winter (figure 1.89) over the north-western Pacific and the north-western Atlantic. The left exit regions of these jetstreaks are preferred areas of cyclogenesis.

**Figure 1.89.** The mean wind at 200 hPa (12 km above sea level) in the northern hemisphere winter (December, January and February) (1979-2002). Note the semi-permanent jetstreaks over the western Pacific and the western Atlantic. Source: ERA-40 Atlas. ERA-40 project report series number 19 (see http://www.ecmwf.int/research/era/ERA-40_Atlas/docs/index.html).

### 1.36 Effect of turbulent friction on geostrophic flow

So far we have neglected friction and dissipation (except in section 1.4), not because it is unimportant, but more because a physically sound theory on the nature of friction and dissipation in the atmosphere is lacking, even though a large amount of literature exists on this topic.

Broadly speaking, there are two types of sources of friction or dissipation to the smoothed circulation systems we have attempted to describe in theory in previous sections. There is friction due to the direct interaction of the circulation-system with the Earth’s
surface and there is the smoothing or mixing effect induced by circulations of smaller scale (i.e. turbulent eddies), alluded to by Edward Lorenz in his famous book on the general circulation of the atmosphere:\(^{54}\):

It is utterly impracticable to describe every gust of wind or even every cumulus cloud occurring at a particular time (...). It is therefore customary in problems of global scale to define the circulation as a smoothed circulation, from which motion systems of thunderstorm size or less have been subtracted. Meanwhile the effects of these systems cannot be disregarded. Ordinarily it is postulated that the statistical properties of the small-scale motions can be described in terms of the smoothed circulation, although really suitable formulae that accomplish this have yet to be established. The simplest way to represent these properties is through the use of coefficients of turbulent viscosity and conductivity, which may exceed the corresponding molecular coefficients by a factor of 10\(^5\) or more. Qualitatively, this idealization treats the atmosphere as a highly viscous, highly thermally conductive fluid.

Circulations of small scale are most prominent in the lowest 1 or 2 kilometres of the atmosphere (the so-called **boundary layer**). Instead of geostrophic balance we then have the following balance of forces.

\[-fv_g + fv + Fr_x = 0\]  \hspace{1cm} (1.63a)
\[+fu_g - fu + Fr_y = 0\]  \hspace{1cm} (1.63b)

where \((u_g, v_g)\) is the geostrophic wind. The frictional force can be expressed as an eddy flux of momentum in terms of "K-theory", as follows (section 1.4)

\[Fr_x = K \frac{\partial^2 u}{\partial z^2}, \quad Fr_y = K \frac{\partial^2 v}{\partial z^2}.\]  \hspace{1cm} (1.64)

If it is assumed that the geostrophic wind is constant, that \(u=v=0\) at \(z=0\) (the earth's surface), that \(u \to u_g\) and \(v \to v_g\) as \(z \to \infty\), and that the axes are oriented such that \(v_g=0\), eqs. (1.63.b) with (1.64) have the solutions\(^{55}\)

\[u = u_g \left[1 - \exp(-\gamma_E z) \cos(\gamma_E z)\right]\]  \hspace{1cm} (1.65)
\[v = u_g \exp(-\gamma_E z) \sin(\gamma_E z)\]

where

\[\gamma_E = \sqrt{\frac{f}{2K}}.\]  \hspace{1cm} (1.66)

When \(z=\pi/\gamma_E\), the wind is parallel to and nearly equal to the geostrophic value. It is conventional to designate this level as the top of the "**Ekman boundary layer**" and to define the Ekman boundary layer depth as \(D_e=\pi/\gamma_E\).

Solution (1.65), which represents the so-called "**Ekman spiral**", after the Swedish oceanographer, V.W. Ekman, indicates that the horizontal wind in the boundary layer has a

---


component directed towards lower pressure. This implies mass convergence in a cyclonic vortex (low pressure area) and mass divergence in an anticyclonic vortex (high pressure area). By mass continuity this, in turn, requires vertical motion, which is upward in a cyclone and downward in an anticyclone (figure 1.90). Obviously, this promotes clear skies in anticyclones and cloudy skies in cyclones.

**Figure 1.90.** Schematic surface wind pattern (arrows) associated with high (H)- and low (L)-pressure centres in the northern hemisphere. Isobars are shown by solid lines. (Courtesy of R. Stull).

The frictionally forced upward motion in a cyclone will very likely lead to cloud formation and latent heat release in the middle troposphere, which in turn could lead to further intensification of the cyclonic circulation. This effect is, however, counteracted by the effect of mass convergence in the boundary layer, which causes a “spin-down” of the vortex. This is the essence of an old and controversial theory for the growth of a tropical depression (hurricane or typhoon) known under the acronym “CISK” (Conditional Instability of the Second Kind) \(^{56}\). The central question is whether the effect of latent heat release on the intensification of the vortex can overcome the effect of the spin down due to boundary layer frictional convergence on the decay of the vortex. This is still an open question, principally because the “ideal” Ekman layer (solution 1.65) is never observed in the atmospheric boundary layer. The turbulent momentum fluxes are not simply proportional to the gradient of the mean momentum (section 1.4). Even if this were a correct assumption, it would still not be proper to assume a constant eddy viscosity coefficient. Furthermore, we have not demonstrated that the solution (1.65) is indeed a stable steady state solution.\(^{57}\)

---


1.37 Planetary (Rossby) waves

The jet stream in middle and subtropical latitudes exhibits meanders (figure 1.36). These meanders are a response of the jet to blocking action of large mountain ranges and to inhomogeneous surface conditions (land-sea contrasts). The meanders seem to propagate in eastward direction. The trough shown in figure 1.77 is part of such a meander. This section presents a simple theory that demonstrates that these meanders can be interpreted as waves that owe their existence to the variation of the Coriolis parameter with latitude.

Based on the considerations in section 1.31, the two horizontal components of the equation of motion can be expressed in pressure coordinates as follows.

\[
\frac{\partial u}{\partial t} + u \left( \frac{\partial u}{\partial x} \right)_p + v \left( \frac{\partial u}{\partial y} \right)_p + \omega \frac{\partial u}{\partial p} = fv - g \left( \frac{\partial z}{\partial x} \right)_p ,
\]

\[
\frac{\partial v}{\partial t} + u \left( \frac{\partial v}{\partial x} \right)_p + v \left( \frac{\partial v}{\partial y} \right)_p + \omega \frac{\partial v}{\partial p} = -fu - g \left( \frac{\partial z}{\partial y} \right)_p .
\]

The continuity equation in pressure coordinates, neglecting the curvature term (Box 1.11), becomes

\[
\left( \frac{\partial u}{\partial x} \right)_p + \left( \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0 .
\]

Let us assume that the eastward flow can be represented by

\[
u(x,y,p,t) = U + u'(x,y,p,t) ,
\]

where \( U \) represents a “basic state” constant zonal mean wind velocity, while \( u'(x,y,p,t) \) represents a perturbation to this basic state zonal current. In other words

\[
u'(x,y,p,t) << U .
\]

The basic state wind is purely zonal. Furthermore, the basic state is in geostrophic balance according to

\[
U = -\frac{g}{f} \left( \frac{\partial \bar{Z}}{\partial y} \right)_p ,
\]

where

\[
z(x,y,p,t) = \bar{Z}(y) + z'(x,y,p,t) \text{ with } |\varepsilon| << |\bar{Z}| .
\]

Substituting these assumptions into the equations of motion (1.163a,b) and neglecting the products of perturbations yields the following linearized equations.
Figure 1.90 Maps showing isopleths of 500 hPa geopotential (black solid lines, labeled in m of contour interval of 4 m or 40 m) and isopleths of 500 hPa temperature (colours and white lines, labeled in °C). The time interval is 12 hours. The letters “R” and “T” indicate the positions of a trough (T) and a ridge (R) corresponding to a Rossby wave. This trough-ridge system moves eastward with a speed of about 500 km per day with little dispersion. Source: http://www.wetter3.de/Archiv/.

\[
\frac{\partial u'}{\partial t} + U \left( \frac{\partial u'}{\partial x} \right)_p = f v' - g \left( \frac{\partial z'}{\partial x} \right)_p, \tag{1.169a}
\]

\[
\frac{\partial v'}{\partial t} + U \left( \frac{\partial v'}{\partial x} \right)_p = -f u' - g \left( \frac{\partial z'}{\partial y} \right)_p. \tag{1.169b}
\]

The relative vorticity perturbation is

\[
\zeta' = \left( \frac{\partial v'}{\partial x} \right)_p - \left( \frac{\partial u'}{\partial y} \right)_p. \tag{1.170}
\]

It is now easy to derive an equation for the time evolution of the relative vorticity from (1.169a) and (1.169b)). This yields,
\[
\frac{\partial \zeta'}{\partial t} + U \left( \frac{\partial \zeta'}{\partial x} \right)_p = -f \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right)_p - v' \frac{df}{dy}.
\]

(1.171)

For simplicity we assume that the motion is “barotropic”, i.e. purely horizontal, i.e. \( \omega = 0 \). This implies that

\[
\left( \frac{\partial u}{\partial x} \right)_p + \left( \frac{\partial v}{\partial y} \right)_p = 0.
\]

(1.172)

With this we may define a streamfunction, \( \psi' \), so that

\[
\begin{align*}
\psi' &\equiv -\frac{\partial \psi'}{\partial y}, & v' &\equiv \frac{\partial \psi'}{\partial x} \\
\zeta' &\equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi' = \nabla^2 \psi'.
\end{align*}
\]

(1.173)

We now introduce the “beta-parameter”:

\[
\beta \equiv \frac{df}{dy} = \frac{2\Omega \cos \phi}{a},
\]

(1.174)

\( a \) being the mean radius of the Earth, \( \Omega \) Earth’s angular velocity and \( \phi \) the latitude. With this, the linearized vorticity equation is

\[
\frac{\partial \nabla^2 \psi'}{\partial t} + U \left( \frac{\partial \nabla^2 \psi'}{\partial x} \right)_p + \beta \frac{\partial \psi'}{\partial x} = 0.
\]

(1.175)

The value of \( \beta \) decreases from \( 1.983 \times 10^{-11} \) m\(^{-1}\) s\(^{-1}\) at 30\(^\circ\)N to \( 1.145 \times 10^{-11} \) m\(^{-1}\) s\(^{-1}\) at 60\(^\circ\)N. Nevertheless, when the above equation is applied to mid-latitudes it is usually assumed that \( \beta \) is constant. We refer to this approximation as the “beta-plane” approximation.

On an infinite domain (beta-plane) we may assume wave-like solutions of the form

\[
\psi' = \text{Re} \left\{ \Psi \exp \left[ il (lx + my - \omega t) \right] \right\}.
\]

(1.176)

By substituting this into eq. (1.175) we find that (1.176) is a solution of (1.175) if

\[
\omega = Ul - \frac{\beta l}{l^2 + m^2}.
\]

(1.177)

This relation between the frequency, \( \omega \), of the wave and its wavelength, represented by the wavenumbers, \( l \) and \( m \), is called a “dispersion relation”. This particular dispersion relation applies to so-called “barotropic planetary waves”, also named “Rossby waves” after Carl-Gustav Rossby\(^{58}\). It is clear that the existence of these waves depends on the beta-effect, i.e.


on the condition $\beta \neq 0$.

The zonal component of the phase speed of the wave is $c_x = \omega / l$:

$$c_x = U - \frac{\beta}{l^2 + m^2}.$$  \hfill (1.178)

Since $\beta \geq 0$, Rossby waves propagate towards the west with respect to the average basic state flow. This relative westward propagation is strongest for long waves.

**Figure 1.91.** The 500 hPa geopotential (given in dm) as a function of time and longitude in November 1945. The values are average values of geopotentials between 35°N and 60°N. Ridges are shown by horizontal hatching; troughs are shown by vertical hatching. The slanted straight lines indicate a succession of maximum development of troughs and ridges. Note that most ridges and troughs propagate in easterly direction. This means that they are embedded in a strong zonal average easterly flow. Source of this figure: Hovemöller, E., 1949: The Trough-and-Ridge diagram. *Tellus*, 1, 62-66.
Rossby waves can be identified with the troughs and ridges that are characteristic of upper air charts of the geopotential height at, for instance 500 hPa (figure 1.90). A compact method to visualize the propagation in the zonal direction as well as the amplitude of these waves was introduced by Ernest Hovemöller\(^{59}\). This method consists of constructing a contour map of the mean height between fixed latitudes of (e.g.) the 500 hPa level as a function of longitude and time. Figure 1.91 shows an example (the original figure published by Hovemöller) corresponding to the month of November 1945. It shows at a glance that an almost stationary ridge (high values of the geopotential height) prevailed near longitude 0° throughout this month. To the west of this ridge, several troughs and ridges are moving eastward during the month. Some of the troughs seem to penetrate into the stationary ridge, leading to temporary weakening of this ridge, but it is always “rebuilt” relatively quickly. Therefore, the ridge at 0° longitude is sometimes referred to as a “blocking high”. The phase velocity of the Rossby waves is given by the orientation of the "peak line" of the ridges. For the interval between 180°W and the stationary ridge, the phase velocity varies between 5 and 15 ° longitude per day (at 45°N this would correspond to approximately 400-1200 km per day).

The following dominant trough/ridge systems can be identified in figure 1.91. Between

---

90°E and 180°E (eastern Asia to the middle Pacific) we predominantly observe troughs. Between 30°W and 60°E (Eastern Atlantic and Europe) we predominantly observe ridges. Between 180°E and 30°W (Eastern Pacific to Western Atlantic via North America) we observe more variability, but there a preference for ridges in the Eastern Pacific. The Eastern Pacific and Eastern Atlantic sectors (including western Europe) are indeed preferred longitudinal locations of blocking highs or anticyclones in the northern hemisphere between 40°N and 80°N throughout the whole year. They are also the locations where Rossby wave wave usually reach their maximum amplitude and “break”.

Figure 1.92 illustrates the two types of Rossby wave breaking: anticyclonic wave breaking and cyclonic wave breaking. Both types of wave breaking usually lead to the formation of an anticyclone poleward of a cyclone. At upper tropospheric levels (above 500 hPa) the anticyclone possesses a warm core, while the cyclones possess a cold core. In the case of the cyclone over Russia, the cold core is already apparent at 850 hPa. The cold core cyclones are referred to also as “cut-off cyclones”.

Figure 1.93 shows a beautiful example of a “blocking” pressure distribution, consisting of an anticyclone over Scandinavia and a cyclone over the Mediterranean Sea. In between these pressure systems an easterly wind brings very cold air from Russia towards Western Europe. Because this blocking persisted for many weeks, the month that followed (February 1956) was the coldest of the 20th century over the major part of Western Europe, including Spain.

Looking more closely at the conditions of the western hemisphere in November 1945 (figure 1.91), we note a remarkable rhythm in the intensity of troughs and ridges: an intensity-maximum of a trough is in several cases followed after one to three days by a maximum of the ridge nearest ahead (to the east) of the trough. Good examples of this phenomenon are found, above all, during the first ten days of the month of November 1945. This is essentially in agreement with the theory of dispersion of Rossby waves. The group velocity follows from eq. 1.177 and eq. 1.178:

\[ c_{gx} = \frac{\partial \omega}{\partial l} = c_x + \frac{2 \beta l^2}{(l^2 + m^2)^2}. \] (1.179)

In other words, the group velocity is greater than the phase velocity. A group of Rossby waves travels faster in eastward direction than the individual troughs and ridges.

Rossby waves are often excited when air flows eastward over a mountain range, such as the Rocky mountains. Because the eastward group speed of Rossby waves is greater than the eastward phase speed, we then observe “downstream development” of a series of troughs and ridges in the lee of the mountain.

PROBLEM 1.28. The basic state zonal flow
Estimate the zonally averaged, time-mean value of \( \overline{U} \) in the western hemisphere for the month of November 1945, using figure 1.91.

---

61 The linear theory presented in this section is only applicable to waves with small amplitudes. Therefore it cannot be used to understand the breaking of Rossby waves.
**Figure 1.93.** A blocking high over Scandinavia and a cut-off low over Central/Southern Europe at 0600 UTC, on 1 February 1956, marking the beginning of the coldest 20th century winter month in Western Europe. Upper panel shows the 500 hPa geopotential height (colours), the sea level pressure (white lines) and the thickness of the layer between 1000 hPa and 500 hPa, which is proportional to the average temperature of this layer (eq. 1.143). Lower panel shows the 850 hPa geopotential height and the 850 hPa temperature (colours), labeled in °C. Source of the figure: [http://www.wetter3.de/Archiv/](http://www.wetter3.de/Archiv/).
Box 1.13. Waves: phase velocity and group velocity

Consider a function, \( \xi = f(x) \), drawn as the solid curve in the figure above. If we replace \( x \) by \( (x-a) \) we get \( \xi = f(x-a) \). The shape of the curve does not change, but the same values of \( \xi \) are now found for \( x \)-values that are \( a \) greater. If \( a > 0 \) the curve is displaced towards the right. If we now assume that \( x = ct \), with \( t \) the time, we obtain a “propagating” curve, \( \xi = f(x-ct) \), that propagates towards the right with a phase speed, \( c \).

The wave-like solution (eq. 1.176) of eq. 1.175 (neglecting the \( y \)-dependence for simplicity),

\[
\xi = \exp\{i(\alpha x - \omega t)\},
\]

(\( \alpha \) is the wave number) can be written as

\[
\xi = \exp\left\{i\alpha \left( x - \frac{\omega}{\alpha} t \right) \right\} = \exp\{i\alpha(x - ct)\}
\]

with

\[
c = \frac{\omega}{\alpha}.
\]

The parameter \( \omega \) is called the angular frequency. The period is

\[
T = \frac{2\pi}{\omega}.
\]

The frequency, \( 1/T \), is measured in cycles per second (Hz). The wave-number, \( \alpha \), has a special meaning, which can be seen when we observe that

\[
\xi\left(x + \frac{2\pi}{\alpha}, t\right) = \xi(x,t).
\]

This implies that the curve repeats itself after a distance, \( \lambda = 2\pi/\alpha \). This distance is the wavelength of the signal.
Superposition of two waves propagating in the $x$-direction with slightly different wavelength and frequency,

$$\xi(x,t) = \xi_0 \exp[i((\alpha + \delta \alpha)x - (\omega + \delta \omega)t)] + \xi_0 \exp[i((\alpha - \delta \alpha)x - (\omega - \delta \omega)t)],$$

yields

$$\xi(x,t) = \xi_0 \exp[i(\delta \alpha x - \delta \omega t)] + \exp[-i((\delta \alpha x - \delta \omega t))],$$

which is the same as

$$\xi(x,t) = \xi_0 \{2\cos(\delta \alpha x - \delta \omega t)\} \exp[i(\alpha x - \omega t)].$$

The factor $\{2\cos(\delta \alpha x - \delta \omega t)\}$ is referred to as the “envelope wave”, which modulates the amplitude of the “carrier wave”, $\exp[i(\alpha x - \omega t)]$. The envelope wave travels with a speed,

$$c_g = \frac{\delta \omega}{\delta \alpha} = \frac{\partial \omega}{\partial \alpha}.$$

This is referred to as the “group velocity”. The relation between group velocity and phase velocity is

$$c_g = c + \alpha \frac{\partial c}{\partial \alpha}.$$

### 1.38 Meridional energy transport

For the Earth as a whole and averaged over many years the absorbed Solar radiation and the outgoing long wave radiation at the top of the atmosphere are approximately in balance. However, at a specific latitude this is obviously not the case. This is illustrated in figure 1.94. More radiation is absorbed than is lost to space at the equator, while at the poles the reverse is the case. Since the emitted longwave radiation is proportional to the temperature, this implies that the poles are warmer than would be expected from radiation balance, while the equator is colder than would be expected on the grounds of radiation balance. The physical process that is responsible for the deviation from radiation balance is transport of energy due to the circulation in both the ocean and the atmosphere.

The energy budget at the top of the atmosphere (TOA), shown in figure 1.94, is brought into balance at each latitude by energy transport from the tropics to the poles by the atmosphere and the oceans. The total energy, $E$, in the atmosphere per unit mass (in J kg$^{-1}$) is the sum of kinetic energy ($K=(u^2+v^2+w^2)/2$), internal energy ($I=cT$), potential energy

---

62 The units of energy are Joules=N m (force×distance), which is equivalent to kg m$^2$ s$^{-2}$ (mass×acceleration×distance).
(PE=gz) and latent energy \((LH=Lq)\) \(^{63}\). Here, \(c_v\) is the specific heat capacity of air at constant volume \((c_v=717 \text{ J K}^{-1}\text{kg}^{-1})\), and \(L\) is the specific latent heat of condensation or evaporation, which are the dominant phase transitions of water in the atmosphere \((L=2.501 \times 10^6 \text{ J kg}^{-1})\). Thus,

\[
E = K + L + PE + LH.
\]  

\[(1.180)\]

For the oceans we can obtain similar expressions for the forms of energy with the exception, of course, of latent energy.

![Figure 1.94](image)

**Figure 1.94.** Annual mean, zonal mean absorbed Solar radiation and outgoing long wave radiation at the top of the atmosphere. There is a surplus of about 50 W m\(^{-2}\) at the equator and a deficit of nearly 100 W m\(^{-2}\) over the polar cap. Source: W. Ruddiman, 2001: *Earth's Climate. Past and Future*. W.H. Freeman. 465 pp. (figure 2-14a).

Most of the energy in the atmosphere is in the form of internal energy and potential energy. Assuming that the total mass of the atmosphere is about \(10^4\) kg m\(^{-2}\), and that the average temperature is 255 K, the global average internal energy in a column of the atmosphere is about \(1.8 \times 10^9\) J m\(^{-2}\). Globally averaged the atmosphere contains 25 kg m\(^{-2}\) of water vapour. This represents about \(0.06\times10^9\) J m\(^{-2}\), which is about 3\% of the internal energy. This may seem a small portion of the internal energy, but, because water cycles through the atmosphere on average in about 10 days, this means that the equivalent of the total internal energy of the atmosphere is released in clouds upon condensation of water vapour during a period of about 300 days! By far most of the latent heat is released in clouds in the ITCZ. This heat comes from evaporation of water at the Earth’s surface, principally over the subtropical oceans. Over the oceans about 90 \% of the net radiation (absorbed Solar radiation+net absorbed long wave radiation) at the surface is used to evaporate water. A large fraction of the evaporated water is transported to the ITCZ by the Trade winds. In other words, in the (sub) tropics latent heat is transported up the existing temperature!

---

Determining the partitioning of energy transport between the ocean and the atmosphere has been a subject of research for decades. A few decades ago it was thought that ocean transport and atmospheric transport are approximately equal. However, the most recent estimates from reanalysis of observations (figure 1.95) show that significantly more energy is transported pole-ward through the atmosphere than through the ocean, especially at latitudes pole-ward of 30°, even though the atmosphere contains only a fraction of the energy in the ocean. Atmospheric meridional energy transport, which consists of sensible heat transport and latent heat (water vapour) transport, is mediated mainly by large scale eddies and planetary (Rossby) waves in mid-latitudes, and mainly by the Hadley circulation in the tropics. An energy flux of 6 PW (figure 1.95) represents about $2 \times 10^{23}$ J per year, which is twice the amplitude of the seasonal swings in energy content of the ocean (Box 1.14).

**Figure 1.95.** Meridional energy transport in Peta-Watts (PW=10$^{15}$ Watts). Upper left: total; upper right: atmospheric transport; lower left: ocean transport; lower right: annual-zonal mean energy transport by ocean, atmosphere and total (source: Fasullo, John T., Kevin E. Trenberth, 2008: The Annual Cycle of the Energy Budget. Part II: Meridional Structures and Poleward Transports. *J. Climate*, 21, 2313–2325.

**Problem 1.29. Is the energy budget closed?**

According to the lower-right panel of figure 1.95, energy converges poleward of 40°N. Estimate the average energy convergence due to the circulation in the ocean and the atmosphere in W. Estimate the average energy divergence at the top of the atmosphere over the area poleward of 40°N. Discuss your findings.

A much more accurate estimate of the net radiation fluxes at the top of the atmosphere
(TOA), including their inter-yearly variability can be obtained from ERA-Interim reanalysis data ([http://data-portal.ecmwf.int/data/d/interim_mnth/](http://data-portal.ecmwf.int/data/d/interim_mnth/)). Reconstruct figure 1.94 using ERA-Interim data of the “Top Solar Radiation” and the “Top Thermal Radiation” and investigate and interpret the inter-yearly variability of these fields. Restrict your analysis to the northern hemisphere and the tropics, north of 30°S.

Evaluate the importance of the northward transport water vapour in the energy budget of the area north of 40°N with the help of the reanalysis of the “vertical integral of the northward transport of water vapour flux”.

**Box 1.14 Heat capacity of the ocean**

The seasonal swings in the ocean heat content are huge. With an average depth of 4000 m, a global coverage of about 70% and a heat capacity of 4218 J kg⁻¹ K⁻¹ the oceans thermal capacity is a factor of more than a thousand larger than that of the atmosphere. However, yearly temperature variations below 100 m depth are very small, i.e. only the upper 100 m of the ocean shows a significant seasonal cycle in temperature (figure 1). Nevertheless, this upper mixing layer still represents a thermal capacity, which is nearly 30 times greater than the thermal capacity of the atmosphere.

![Figure 1](image)

**FIGURE 1** (Box 1.14). Seasonal cycle of temperature as a function of depth at 50°N, 145°W (eastern north Pacific). Contours are labeled in units of °C. Based on figure 4.8 (p. 79) of L.D. Talley et al., 2011: *Descriptive Physical Oceanography*. 6th edition, Elsevier.

In the extra-tropics the upper 100 m of the ocean gains heat principally by absorption of Solar radiation in the spring and early summer and looses this heat to the atmosphere and to space in between late summer and late winter mainly by emission of long wave radiation, by sensible heat transfer and by latent heat transfer (evaporation) (figure 2.26).

Most of the global ocean coverage is in the southern hemisphere. Globally averaged, the ocean looses (gains) about \(10^{23}\) J of heat between April (September) and August (March). If this heat were to be absorbed by the atmosphere, this would lead to a global average temperature increase of nearly 20 K. This effect reduces the amplitude of the annual atmospheric temperature cycle. However, at the end of the winter and in early spring a significant portion of this heat is used to melt snow and ice. Of course, heat gained by the atmosphere is also lost to space by emission of radiation.

There are strong indications that, on a time scale of decades, the upper layer of the ocean is slowly warming due to the anthropogenic greenhouse effect (chapter 2). Measurements from about 3000 Argo floats, that provide temperature profiles over the upper 2 km of the ocean, indicate a warming of about 0.8 W m⁻² for the entire ocean between 2003 and 2008.
This represents about 0.9×10^{22} J per year, which is nearly 10% of the amplitude of the annual cycle of the ocean heat content. Therefore, the ocean is also damping the temperature-response to the greenhouse effect.

References to box 1.14


1.39 Zonal average, time average meridional transport by eddies or waves

When diagnosing the meridional transport of a quantity, \( Q \) (this quantity may represent energy or mass or momentum), we usually integrate over a certain period of time, such as several years or several seasons (an ensemble average). We must then take account of the fluctuations of the transport during this time period. Thus, following the method introduced in section 1.4, we assume that

\[
Q = \overline{Q} + Q',
\]

where

\[
\overline{Q} = \frac{1}{\tau} \int_{0}^{\tau} Q d\tau
\]

is the time average of \( Q \) over a period of time of length \( \tau \). For the perturbation, \( Q' \), we have

\[
\overline{Q}' = 0.
\]

Popular averaging periods are 30 years, which is “working definition” of “climate”, 1 year, 1 season (i.e. December, January and February (DJF) or June, July and August (JJA)), or several seasons in an ensemble mean.

The most popular spatial average is the zonal average, which, for \( Q \), is defined as

\[
[Q] = \frac{1}{2\pi} \int_{0}^{2\pi} Q d\lambda,
\]

where \( \lambda \) represents longitude in radians. The deviation of \( Q \) from the zonal average is written as \( Q^* \). Thus

\[
Q = [Q] + Q^*.
\]

with

\[
[Q^*] = 0 \text{ and } \left[ \frac{\partial Q}{\partial x} \right] = 0.
\]
**Figure 1.96.** Zonal mean zonal wind as a function of latitude and pressure for the seasons DJF (upper panel) and JJA (lower panel). Note that the zonal wind is eastward at nearly all levels in the winter hemisphere, but not in the summer hemisphere. In the summer hemisphere the wind is westward. This is a very important property in relation to eddy or wave activity. Source: [http://www.ecmwf.int/research/era/ERA-40_Atlas/](http://www.ecmwf.int/research/era/ERA-40_Atlas/).

Most of the distributions that are shown in the ERA40-reanalysis atlas ([http://www.ecmwf.int/research/era/ERA-40_Atlas/](http://www.ecmwf.int/research/era/ERA-40_Atlas/)) represent the zonal average of the time average of a particular quantity, such as the zonal wind (**figure 1.96**).

The zonal average of the time average of the meridional (northward) flux of $Q$ is

\[
\bar{v}Q = \left[ \bar{v} + v' \right] (\bar{Q} + Q') = \left[ \bar{v}\bar{Q} + v'\bar{Q} + Q'\bar{v} + v'Q' \right] = \left[ \bar{v}\bar{Q} + v'Q' \right].
\]  

(1.187)
Figure 1.97. The terrain height that was adopted by Manabe and Terpstra (1974) in their numerical model to obtain the results that are shown in Figure 1.98. The red double arrow indicates the latitude band of the westerlies, between 30°N and 60°N. Note that this latitude band is practically free of mountains in the southern hemisphere.

Figure 1.98. Observed stationary (time mean) planetary waves at 45° latitude in the northern hemisphere, in terms of the meridional component of the velocity (v) (contours labeled in m/s), induced by mountains and other irregularities in the Earth’s surface. Source: Manabe and T.B. Terpstra, 1974: The effects of mountains on the general circulation of the atmosphere as identified by numerical experiments. J.Atmos.Sci., 31, 3-42.

The time average can be written as

$$\bar{v} = \bar{\bar{v}} + \bar{v}^*$$

and

$$\bar{Q} = \bar{\bar{Q}} + \bar{Q}^*,$$

so that

$$[\bar{v}\bar{Q}] = [\bar{\bar{v}}\bar{Q} + \bar{v}^*\bar{Q}^*] = [\bar{\bar{v}}\bar{Q}] + [\bar{v}^*\bar{Q}^* + \bar{v}^*\bar{Q} + \bar{\bar{v}}\bar{Q}^*] + [\bar{v}^*\bar{Q}^*],$$

or

$$[\bar{v}\bar{Q}] = [\bar{\bar{v}}\bar{Q}] + [\bar{v}^*\bar{Q}^* + \bar{v}^*\bar{Q} + \bar{\bar{v}}\bar{Q}^*] + [\bar{v}^*\bar{Q}^*].$$

\[ \bar{v}'Q = \left[ \bar{v}' \bar{Q} \right] + \left[ \bar{v}' \bar{Q}' \right] + \left[ \bar{v}' \bar{Q}'' \right]. \]  \hspace{1cm} (1.190)

According to eq. 1.190, the zonal average of the time average of the meridional flux of \( Q \) is determined by the zonal average circulation (first term on the r.h.s.) the “stationary eddies” (second term on the r.h.s.) and the “transient eddies” (third term on the r.h.s.).

Stationary eddies or waves are the time-averaged, or time-invariant, component of the eddy- or wave field. These waves are associated with orography or land sea contrasts (figure 1.97). Figure 1.98 illustrates the stationary wave-like response of westerly flow in the northern hemisphere mid-latitudes (40°N-60°N) to large-scale mountain ranges, such as the Rockies and the Himalaya’s. At levels below about 500 hPa the troughs of these waves are usually located on the lee side of mountain ridges. However, the waves appear to tilt towards the west with increasing height. This indicates that the stationary wave is transporting energy upwards and that the wave is transporting heat from the equator to the pole. This can be understood as follows.

Imagine an idealized wave at midlatitudes, in geostrophic and hydrostatic balance. The geometry of this wave in a longitudinal plane is shown in figure 1.99. Three surfaces of equal pressure are shown. The geostrophic wind is related to the height of a pressure surface, \( Z \), by

\[ u_g = -\frac{g}{f} \frac{\partial Z}{\partial y}, \quad v_g = \frac{g}{f} \frac{\partial Z}{\partial x}. \]  \hspace{1cm} (1.191)

The temperature on a pressure surface is related to the thickness of a layer by the hydrostatic relation:

\[ \frac{\partial Z}{\partial p} = -\frac{1}{\rho g} \frac{RT}{pg}, \]  \hspace{1cm} (1.192)
so that the temperature at level zero is

$$T_0 = \frac{\rho_0 g}{R} \frac{Z_1 - Z_{-1}}{2\Delta p}.$$  \hspace{1cm} (1.193)

Suppose that the geopotential height varies sinusoidally in $x$, such that

$$Z_i = Z_{Rj} + A_i \sin(kx + i\delta), \ i = -1, 0, 1,$$  \hspace{1cm} (1.194)
where $A_i$ is the amplitude of the wave and $Z_{R_i}$ is a function of height. Positive $\delta$ means the wave is tilting to the west with increasing height. Suppose, for simplicity, that the amplitude, $A_i$, is constant with height ($=A$). Then the temperature at level zero is

$$T_0 = \frac{p_0 g (Z_{R,1} - Z_{R,-1})}{2\Delta p R} + \frac{p_0 g A}{\Delta p R} \cos(kx)\sin(\delta).$$  \hfill (1.195)

The temperature-perturbation (deviation from the zonal mean) is:

$$T_0^* = \frac{p_0 g A}{\Delta p R} \cos(kx)\sin(\delta).$$  \hfill (1.196)
No fluctuation of temperature with $x$ occurs unless there is a phase tilt of geopotential height with height. The poleward wind perturbation is

$$v_0^* = \frac{g}{f} \frac{\partial Z}{\partial x} = \frac{kgA}{f} \cos(kx).$$

(In the zonal mean state $v_0^* = 0$). Integrating the heat flux over one wavelength, we obtain

$$\left[ v_0^* T_0^* \right] = \frac{1}{2} \frac{p_0 kg^2 A^2}{f R \Delta \rho} \sin(\delta).$$

**Figure 1.102.** Pressure level climatology (latitude-pressure projections) of the zonal mean of the total stationary (upper panel) and total transient (lower panel) northward flux of westerly wind for December-February (JJA) for the years 1979-2002. $Q=u$ in eq. 1.190. Source: [http://www.ecmwf.int/research/era/ERA-40_Atlas](http://www.ecmwf.int/research/era/ERA-40_Atlas).
**Figure 1.103.** Pressure level climatology (latitude-pressure projections) of the zonal mean of the total stationary (upper panel) and total transient (lower panel) northward flux of westerly wind for June-August (JJA) for the years 1979-2002. \( \dot{Q} = u \) in eq. 1.190. Source: [http://www.ecmwf.int/research/era/ERA-40_Atlas/](http://www.ecmwf.int/research/era/ERA-40_Atlas/).

Eq. (1.198) indicates that a non-zero meridional eddy heat flux is only possible if \( \delta \neq 0 \), that is if the waves are “tilted” with height. A positive value \( \delta \) (westward tilt with increasing height), as is seen in figure 1.98, is associated with a poleward heat flux, i.e. with a northward heat flux in the northern hemisphere and with a southward heat flux in the southern hemisphere (because \( f < 0 \) in the southern hemisphere).

**Problem 1.30. Consequence of variation of wave amplitude with height**

Derive an expression for \( T_0 \) for the case that \( A \) is not constant with height.
Transient waves are travelling, growing or decaying waves that are associated with baroclinic instability (e.g. figure 1.83). Baroclinic instability occurs only in troposphere. Nevertheless, the effects of transient waves are also observed very stronger above the tropopause, as can be observed in figures 1.100 and 1.101. These figures show the northward eddy heat transport on pressure surfaces, \( \vec{v} \cdot \vec{T} \) in DJF (figures 1.100) and in JJA (figures 1.101). Remember that the temperature of an air parcel that is moving horizontally on a pressure surface is conserved under adiabatic conditions. These figures indicate that waves propagate into the middle atmosphere only in the winter hemisphere. In the summer hemisphere eddy activity in the stratosphere, especially that associated with stationary eddies, is practically absent!

In chapter 11 a theory will be presented that indicates that stationary Rossby waves cannot propagate in the vertical direction if the zonal mean zonal wind, \( [u] \), is westward. It can be observed in figure 1.96, that the wind is indeed westward in the stratosphere of the summer hemisphere.

The stratospheric wind reversal from eastward in winter to westward in late spring and summer occurs during a so-called “final stratospheric warming” (section 1.29), after which vertical planetary wave propagation into the stratosphere practically ceases. The return of eastward winds in fall is associated with the slow cooling of the stratosphere in the Polar night, which sets in over the North Pole on September 21 and over the South Pole on March 21.

\[ \text{Energy gain} \quad \text{Energy loss} \]

\( \text{Implied energy transport} \)

\( \text{EQUATOR} \quad \text{POLE} \)

\( \text{Loss of angular momentum} \)

\( \text{Gain of angular momentum} \)

\[ W \]

\[ E \]

**Figure 1.104.** Latitudinal transport of (left) heat and (right) angular momentum. There is a net gain of energy in the tropics and a net loss of energy at higher latitudes. In order to balance the energy budget at each latitude, a poleward heat flux is required. The atmosphere gains angular momentum from the rotating Earth in low latitudes, where the surface winds are easterly (“the trade winds”), and loses it to the Earth in middle latitudes, where the surface winds are westerly. A poleward atmospheric flux of angular momentum in the atmosphere is thus implied. In the tropics momentum is transported upwards and polewards in the upper troposphere by the Hadley circulation. Eddies and waves take care of the poleward transport in the subtropics and mid-latitudes. Source: J.Marshall and R.A. Plumb, 2010: *Atmosphere, Ocean and Climate Dynamics: An Introductory Text*. Academis Press, 344 pp.
Figure 1.105. Schematic model of the general circulation in a meridional section and schematic fronts and streamlines at the Earth’s surface (from A. Defant, and F. Defant, 1958: Physikalische Dynamik der Atmosphäre, Akademische Verlagsgesellschaft mbH). The horizontal and vertical velocity components of the Ferrel circulation are $u$ and $\omega$, respectively. “PF” stands for Polar Front; “J,” stands for sub-tropical jet; “Jp,” stands for polar jet. The Ferrel circulation is a result of the zonal mean adjustment of the atmosphere to thermal wind balance in the presence of eddy fluxes of momentum and heat that disturb thermal wind balance.

In figure 1.102 and figure 1.103, which show the eddy transport of zonal momentum, we once again see that there is very little or no eddy activity in stratosphere of the summer hemisphere. In the southern hemisphere the transient eddy activity, which has its origin in the “Roaring Forties” (figure 1.68, lower panel), is much stronger than the stationary eddy activity. This is due to the practical absence of large-scale mountain ridges in the southern hemisphere in mid-latitudes. We also observe that eddy-momentum transport in the troposphere is in general poleward from the tropics to the mid-latitudes, approximately between the latitudes of 10°S and 60°S and 10°N and 60°N. Convergence of the eddy flux of westerly momentum occurs poleward of 35°N or 35°S. This leads to westerly winds in the midlatitudes.

The eddy transport of heat and momentum is responsible for the “wave-drag” that was discussed in connection with the Brewer Dobson circulation in section 1.29 (figures 1.65 and 1.66). The theory of this process is explained in chapter 11.

The general idea is that heat (energy) and (angular) momentum are transported poleward in the deep tropics by the Hadley circulation (figure 1.104). At higher latitudes this transport is taken over by eddies and waves with horizontal scales of 1000 km or larger. We observe in figure 1.102 and 1.103 that the meridional momentum transport is poleward from the tropics to about ±60° latitude. The Ferrel circulation (figures 1.45 and 1.105) plays an insignificant role in this respect, but it does seem to play a role in the downward transport of (angular) momentum to Earth in midlatitudes, thereby making the zonal mean zonal winds westerly also at the Earth’s surface in this latitude belt. The atmosphere gains (angular) momentum from the rotating Earth in the latitude belt of the Trade winds and loses it to the Earth in mid-latitudes in the westerly wind belt.
Figure 1.106. Pressure level climatologies (latitude-pressure projections) of the zonal mean diabatic heating, \( \frac{d\theta}{dt} \), for December-February (JJA) (upper panel) and for June-August (lower panel) for the years 1979-2002. Source: http://www.ecmwf.int/research/era/ERA-40_Atlas/. Cooling in the extra-tropics is due to emission of radiation by greenhouse gases. Heating near the lower boundary is due to sensible heat transfer from the Earth’s surface, which, in turn, is heated by the Sun. Heating in the tropical troposphere is mainly due to latent heat release. Heating in tropical and summer stratosphere is due to absorption of Solar radiation by ozone and oxygen.

Finally, a few words about the role of dynamics in the energy budget are appropriate here. There is a net energy gain by radiation in the tropics, while there is a net energy loss by radiation pole-ward of ±30°. It is very remarkable that the latter is also the case in the summer hemisphere for a considerable part of the atmosphere (below 20 hPa) (figure 1.106), despite the fact that the higher latitudes receive more Solar radiation in the summer months than the tropics (figure 1.107)! It appears that the temperature at high latitudes, both in the winter hemisphere and in the summer hemisphere, is maintained above that which is expected from radiative equilibrium due to adiabatic “heating” associated with poleward
eddy sensible heat transport and downwelling of potentially warm air. This has far reaching consequences for the dynamical structure of the atmosphere, as we will see in chapter 12.

**Figure 1.107.** Seasonal variation of the daily average insolation at the top of the atmosphere as a function of latitude. The units are W m$^{-2}$. Note that the polar cap receives more Solar radiation in summer than the tropics. The shaded area in the diagram indicates the polar night.

### Abstract of Chapter 1

Chapter 1 is intended to introduce the student to the concepts, assumptions, laws and resulting fundamental equations used to describe the dynamics of a rotating, stratified fluid, like the atmosphere. It also intends to give an overview of the circulations systems on different scales in the atmosphere.

A firm grasp of the meaning of the concepts of pressure, temperature, potential temperature, vorticity, hydrostatic balance, geostrophic balance and thermal wind balance is required to continue with the study of the subject matter in what follows.

The mathematical relation between potential temperature and Exner function, and temperature and pressure is the following.

\[
\theta = T \left( \frac{p_{\text{ref}}}{p} \right)^K = \frac{c_p T}{\Pi}.
\]

(1.53)&(1.54)

The basic closed set of three equations is the following.
\[
\frac{d\theta}{dt} = \frac{J}{\Pi}; \\
\frac{d\vec{v}}{dt} = -\theta\nabla \Pi - g\hat{k} - 2\Omega \times \vec{v} + \vec{F}_r; \\
\frac{c_v}{RT\Pi} \frac{d\Pi}{dt} = -\nabla \cdot \vec{v} + \frac{J}{\theta\Pi}.
\]

The unknown variables are \(\theta\) (potential temperature), \(\Pi\) (Exner function) and \(\vec{v}\) (velocity vector).

The equation of state,

\[
p\alpha = RT,
\]

\((\alpha = 1/\rho)\) has been used in deriving this set of three equations.

The distribution of moisture in the atmosphere is strongly constrained by the Clausius-Clapeyron relation:

\[
\frac{d\ln e_s}{dT} = \frac{L_v}{R_v T^2}.
\]

The vertical component of the vorticity vector, defined as

\[
\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},
\]

represents the vortical motion in the horizontal plain. A more useful measure of this vortical motion is the potential vorticity, defined in the hydrostatic case as

\[
Z_\theta \equiv \frac{\zeta + f}{\sigma},
\]

with the isentropic density defined as

\[
\sigma = -\frac{1}{g} \frac{\partial p}{\partial \theta}.
\]

Potential vorticity is materially conserved in adiabatic conditions.

The analysis of the stability of hydrostatic balance and of geostrophic balance, using the parcel method, yields the following two important frequencies:

\[
N = \sqrt{\frac{g}{\theta^*}} \frac{d\theta_0}{dz}; F = \sqrt{f \left( f - \frac{\partial u}{\partial y} \right)}.
\]

\(N\) is the Brunt-Väisälä frequency and \(F\) is the inertial frequency.

Furthermore, chapter 1 describes the reasons for the inhomogeneous distributions of ozone and water vapour. It describes the basic thermal structure of the atmosphere: the
**troposphere, stratosphere** and (tropical and extratropical) **tropopause**, including the division of the atmosphere into the **Overworld, Middleworld** and **Underworld** on the basis of the potential temperature- and potential vorticity distributions. It introduces the concepts of **isentrope, latent heat, equivalent potential temperature, dewpoint temperature, Lifting condensation level, CAPE, conditional instability, moist adiabatic lapse rate, thermal wind balance, potential vorticity, frontogenesis** (the **Q-vector**), **ageostrophic wind, spin down**, the **pressure coordinate system, wave propagation and dispersion** and **conceptual models** of weather patterns, such as the **front** and the **jetstreak**.

Chapter 1 also introduces and describes atmospheric phenomena such as, **the jet, the tropical cyclone, the tropopause, sea breeze, tides, the low level jet, Trade winds, Hadley circulation, ITCZ, Ferrel cell, Monsoon, Somali jet, ENSO, Walker circulation, subtropical anticyclone, annular belts of action, Annular Mode, polar night stratospheric vortex, Sudden Stratospheric Warming, cut-off low, surf zone, Brewer-Dobson circulation, QBO, thunderstorm, tornado, tropical cyclone, thunderstorm high, trough, ridge, planetary (Rossby) wave, baroclinic wave, cold-, warm- and back-bent front, warm sector, warm conveyor belt, dry intrusion, frontal occlusion, blocking high, wave breaking, wave drag and stationary and transient eddies**.

Finally this chapter introduces some elementary data analysis methods, such as **Fourier analysis, filtering** and **principal component analysis**. In addition to this, it advocates the use of the **programming language, Python**, as a tool to construct simple numerical models and to plot the results of computations.

---

**Further reading**

**Books**

Ambaum, M.H.P., 2010: **Thermal Physics of the Atmosphere**. Wiley-Blackwell, 239 pp. (A very good introduction to the thermodynamics of the atmosphere, including the role of water and its phase transitions and an explanation of the tephigram. Also introduces the theory of radiative emission and transfer)

Batchelor, G.K., 1970: **An Introduction to Fluid Dynamics**. Cambridge University Press, 615 pp. (Chapters 1-3) (The most mature and carefully composed existing book on fluid dynamics)

Cushman-Roissin, B., and J.M. Beckers, 2010: **Introduction to Geophysical Fluid Dynamics**. Academis Press. (This book is used in the third year Bachelors course “Geophysical Fluid Dynamics” at Utrecht University)


List of problems (chapter 1)

1.1 Buoyancy
1.2 Law of adiabatic expansion
Box 1.2 Energy conservation
1.3 Mixing ratio
1.4 Precipitable water and surface humidity
1.5 Ozone concentration profile according to Chapman’s theory
1.6 Continuity equation in terms of the Exner function
1.7 Reducing pressure to sea-level
1.8 Parcel model of buoyancy oscillation
1.9 Downdraughts
1.10 What can we do with a tephigram?
Box 1.5 Relative humidity and dew point
Box 1.6 (1) Cloud formation and vertical motion
Box 1.6 (2) Cloud formation and vertical motion
1.11 Parcel model of an inertial oscillation
Box 1.7 Conversion of potential energy into kinetic energy
1.12 Vorticity in, wind in, and formation of a tropical cyclone
1.13 Isentropic mixing across the isentropic tropopause
1.14 Mass of the atmosphere
Box 1.10 Spectral analysis
1.15 Probability density function
1.16 Cross-correlation- or covariance-matrix
1.17 Time filtering
1.18 Equatorial temperature anomalies associated with the QBO
1.19 Principal component analysis (PCA)
1.20 Influence of filtering out the trend in PCA
1.21 Analysis of observations
1.22 Geostrophic wind
1.23 Non-linear balance and gradient wind
1.24 Relative vorticity and horizontal divergence
1.25 Identifying areas of cold and warm air advection
1.26 Testing the thermal wind equation (1.148)
1.27 Identifying the location of fronts
1.28 The basic state zonal flow
1.29 Is the energy budget closed?
1.30 Consequence of variation of wave amplitude with height

This is the December 2014 edition of chapter 1 of the lecture notes on Atmospheric Dynamics., written by Aarnout van Delden (IMAU, Utrecht University, Netherlands, http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm