## 9

# Baroclinic Waves, Cyclogenesis and Frontogenesis 


#### Abstract

of a landmark article on the theory of cyclogenesis

By obtaining complete solutions, satisfying all relevant simultaneous differential equations and boundary conditions, representing small disturbances of simple states of steady baroclinic large-scale atmospheric motion it is shown that these simple states of motion are almost invariably unstable. An arbitrary disturbance (corresponding to some inhomogeneity of an actual system) may be regarded as analysed into "components" of a certain simple type, some of which grow exponentially with time. In all the cases examined there exists one particular component that grows faster than any other. It is shown how, by a process analogous to "natural selection", this component becomes dominant in that almost any disturbance tends eventually to a definite size, structure and growthrate (and to a characteristic life-history after the disturbance has ceased to be "small"), which depends only on the broad characteristics of the initial (unperturbed) system. The characteristic disturbances (forms of breakdown) of certain types of initial system (approximating to those in practice) are identified as the ideal forms of the observed cyclone waves and long waves of middle and high latitudes.


E.T. Eady (1949), "Long waves and cyclone waves" (Tellus, 1, 33-52)
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http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm
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### 9.1 Introduction

The formation or genesis of cyclones (cyclogenesis) tends to occur in zones that exhibit a strong lowlevel horizontal temperature gradient. These so-called baroclinic zones are encountered principally in winter in middle latitudes. This chapter begins by describing a typical example of cyclogenesis in a baroclinic zone in middle latitudes and then continues by presenting a theory that explains the mechanism of cyclogenesis in a baroclinic zone. This theory emerged in the years after the Second World War, together with the so-called "quasi-geostrophic approximation". This approximation, which provides a closed set of only two equations, governing the behaviour of large-scale quasibalanced circulation systems (section 1.30), is introduced in section 9.3. Based on the quasi-geostrophic approximation, a diagnostic equation for vertical velocity (the so-called "omega equation") is derived (section 9.5), which again (see the "Sawyer-Eliassen equation" in chapter 8) highlights the intimate relation between frontogenesis and vertical motion. In a statically stable atmosphere vertical motion occurs as a response to the destruction of thermal wind balance by frontogenesis or frontolysis. This response is required in order to maintain thermal wind balance. Under certain circumstances this response may be "unstable", i.e. baroclinically unstable. In section 9.6 a two-level model of the atmosphere is introduced, and used to study this baroclinically unstable response applied to Rossby waves in a baroclinic zone. We find that the amplitude of Rossby waves grows exponentially if the horizontal temperature gradient exceeds a threshold value, which depends on the wavelength of the waves.

Potential vorticity, which should also be a useful concept in this context, returns into the discussion at the end of this chapter. This reflects the two different viewpoints on middle latitude atmospheric dynamics that presently exist side by side. The first viewpoint employs pressure as a vertical coordinate, while the second viewpoint employs potential temperature as a vertical coordinate and is thus known as the "PV- $\theta$ viewpoint" ( $\mathrm{PV}=$ potential vorticity). Both viewpoints assume that the atmosphere is in a state of "quasi-balance".

### 9.2 An example of cyclogenesis

An example of cyclogenesis just south of Ireland on March 21995 is discussed in this section. This example was also used in sections 1.30 and 1.36. Figure 9.1 shows the GFS ${ }^{74}$-analysis of the 500 hPa geopotential (indicated in colours), sea-level pressure and the $1000-500 \mathrm{hPa}$ "thickness" on March 2, 1995, at 00 UTC, 06 UTC, 12 UTC and 18 UTC. The "thickness" (indicated by the solid black contours in figure 9.1), which is defined as the distance between two constant pressure surfaces, is a measure of the average temperature of the layer in between these constant pressure surfaces (Box 9.1). A narrow zone, characterised by a strong isobaric temperature gradient (i.e. closely packed thickness contours), is present over the Atlantic Ocean. This zone is referred to as the polar front. Embedded in the polar front is a weak trough in the surface pressure, which is indicated by the red arrow. This disturbance, which is identified as a "baroclinic Rossby wave" (section 9.7), propagates in eastward direction along the polar front without change of amplitude until it reaches Western Europe at 12 UTC on March 2, 1995. At that point in time the disturbance starts to amplify suddenly, i.e. the surface pressure in the disturbance decreases rapidly and a cyclone, with closed isobars, develops over Southern England and the North Sea (figure 9.2). This process is called "cyclogenesis".

[^0]

Figure 9.1. continued on next page.


FIGURE 9.1. GFS reanalysis (source: http://www.wetter3.de/Archiv/) of the 500 hPa geopotential (see colour coding on the right with units in dm ), the distance between 1000 hPa - and 500 hPa -levels (black contours; labels in dm ) and sea-level pressure (white contours; labels in hPa) on 2-3 March 1995. Red arrow points to the trough of a travelling baroclinic Rossby-wave.


FIGURE 9.2. GFS reanalysis (source: http://www.wetter3.de/Archiv/) of the sea-level pressure and the wind at 10 m above the surface on 2 March 1995. The contour interval is 2 hPa . The intensifying trough (indicated by a red arrow) propagates in easterly direction.

The sudden decrease of the surface pressure in the disturbance appears to be connected to the presence of a jet exit at upper levels. Cyclogenesis begins on March 2 at 12 UTC, when the disturbance is collocated with the left exit region of a jet (figure 9.3) (section 1.32). Frontogenesis and reduction of the static stability in the left exit of the jetstreak helps to promote baroclinic instability, a process, which was studied in sections 1.20 and 1.21 , and put forward as the process leading to the formation of cyclones in middle latitudes. This chapter elaborates further upon this topic.

Detailed surface analyses of this event are shown in figure 9.4. Corresponding satellite images are shown in figure 1.97. The cloud band is observed over the Atlantic Ocean at 1344 UTC on March 2, 1995. To the north of this cloud band, convection cells are observed indicating the presence of a relatively cold air mass. The cloud band ends over the southern part of the British Isles in what could be referred to as a "cloud head", following the terminology of Browning and Roberts (1994) (figure 9.6). This is the left exit region of the jet. Forced upward motion in this area is inducing convergence and therefore cyclogenesis at the earth's surface. The subsequent evolution of the cyclone is illustrated nicely by the satellite images displayed in figures 1.97b and 1.97c. The life cycle of the depression corresponds quite closely to the conceptual model proposed by Shapiro and Keyser (1990), which is shown in figure 1.100 and figure 9.5. In the early stages of the life cycle of the cyclone, the classical pattern of the cold front moving towards the southeast overtaking the warm front is observed. The cold front, which is oriented almost at right angles to the warm front, seems to propagate away from the cyclone centre, leaving behind a bent back warm front that eventually "wraps around" the cyclone centre. This process is referred to as frontal fracture.


Figure 9.3. Sea-level pressure (black contours, labeled in hPa ), height of the 300 hPa isobaric level (blue contours, labeled in metres) and wind speed at 300 hPa (light red: $50-60 \mathrm{~m} / \mathrm{s}$, medium red: 60-70 m/s; dark red: $70-80 \mathrm{~m} / \mathrm{s}$ ), on 2 March 1995, 12 UTC. Based on the ERA-Interim reanalysis.

The cold and warm currents (arrows), sketched in the lower panel of figure 9.5, correspond to portions of the cold conveyor belt (CCB) and warm conveyor belt (WCB) where these are close to the Earth's surface. The WCB, beyond the tip of the dashed arrow in the lower panel of figure 9.5, goes on to ascend above the warm front, as shown in figure 9.6a and c. Also shown in figure 9.6b and $\mathbf{c}$ is the so-called dry intrusion and associated dry slot. The cloud-free dry slot can also be recognised in the satellite image in figure $\mathbf{1 . 9 7 b}$. The dry intrusion is a mass of dry air that is associated with the decending branch of the left entrance region of the jet streak (see figure 1.88). Frequently, this air is of stratospheric origin. The satellite image in figure 1.97 c shows the typical spiral cloud pattern associated with the mature depression with warm and dry air caught in the centre. Clouds associated with the bent back warm front on 3 March 1995 activated over the relatively warm North Sea and caused significant snowfall in The Netherlands during the night of 3-4 March 1995.

### 9.3 Quasi-geostrophic approximation

In the 1940's a theory was put forward by Jule Charney and, independently, by Eric Eady (see the quote at the beginning of this chapter) to explain the genesis of midlatitude cyclones. Both these theoreticians proposed that the genesis of typical mid-latitude cyclones is due to an instability of the jet stream. The two-dimensional version of this instability, called baroclinic instability, was discussed in sections 1.20 and 1.21. To set the stage for the presentation of the theory of three-dimensional baroclinic instability, we must make the "quasi-geostrophic" approximation.

## PROBLEM 9.1. Analyis of the surface fronts.

Using the conceptual model in figure 9.5 or figure 1.100, draw the position the fronts (warm and cold) in figure 9.4.


## 2 March 1995, 18 UTC

Figure 9.4. Continued on next page.


## 3 March 1995, 00 UTC

Figure 9.4. Continued on next page.

## PROBLEM 9.2. Upper air analysis of the structure of a baroclinic depression.

Analyse the temperature and the geopotential in the plotted $500 \mathrm{hPa}-$ and 850 hPa chart displayed in figure 9.7. In the case of the 500 hPa chart draw isotherms every $2.5^{\circ} \mathrm{C}$ starting at $-22.5^{\circ} \mathrm{C}$ and draw isopleths of equal geopotential height every 5 decametres starting at 515 decametre. In the case of the 850 hPa chart draw isotherms every $2.5^{\circ} \mathrm{C}$ starting at $-7.5^{\circ} \mathrm{C}$ and draw isopleths of equal geopotential height 125 decametre. c. Perform the analysis of the geopotential and the temperature for the 850 hPa level. Draw isotherms every $2.5^{\circ} \mathrm{C}$ starting at $-7.5^{\circ} \mathrm{C}$ and draw isopleths of equal geopotential height every 5 decametres starting at 125 decametre. Interpret the analysis.


## 3 March 1995, 06 UTC

Figure 9.4. Continued on next page.


## 3 March 1995, 12 UTC

Figure 9.4. Surface weather maps of 2-3 March 1995, (a) 18 UTC (2 March), (b) 00 UTC ( 3 March), (c) 06 UTC (3 March), (d) 12 UTC ( 3 March) and (e) 18 UTC ( 3 March). Isobars are shown by grey solid lines (interval: 2 hPa ). The letters L denotes sea-level pressure minimum. A square marks the position of the weather station as well as the end point of a wind-vector. The barbs attached to the wind-vector indicate the 10 minute mean wind speed, ff, measured at a height of 10 m : no barb corresponds to $\mathrm{ff}<2.5$ knots ( 1 knot= $0.5 \mathrm{~m} / \mathrm{s}$ ); each whole barb corresponds to an extra 10 knots; each half a barb corresponds to an extra 5 knots. If the windspeed is equal or greater than $25 \mathrm{~m} / \mathrm{s}$, this is indicated by a triangle attached to the wind vector. Also indicated are the temperature ( ${ }^{\circ} \mathrm{C}$ ) (upper left), the dew point $\left({ }^{\circ} \mathrm{C}\right)$ (lower left), the sea level pressure ( hPa ) (upper right), the change in the sea level pressure over the past 3 hours and the cloudiness (octa's) (within the square). ${ }^{75}$

[^1]

## 3 March 1995, 18 UTC

Figure 9.4. Caption on previous page.

[^2]

FIGURE 9.5. The life cycle of an extra-tropical cyclone according to M.A. Shapiro and D. Keyser (1990) ${ }^{76}$ : (I) incipient frontal cyclone; (II) frontal fracture; (III) bent-back warm front and frontal T-bone; (IV) warm-core frontal seclusion. Upper: sea level isobars (solid lines); fronts (bold lines) and cloud signature (shaded). Lower: isotherms (solid lines); cold and warm currents (solid arrows: cold conveyor belt and dashed arrows: warm conveyor belt). See also figure 1.100.

The dynamical equations in isobaric coordinates, derived in section 1.23, are summarised in Box 9.1. For reference they are repeated below. The horizontal momentum equation, the hydrostatic equation, the continuity equation, and the thermodynamic energy equation are, dropping the subscript $p$,
$\frac{d \vec{v}}{d t}=-f \hat{k} \times \vec{v}-\vec{\nabla} \Phi$,
$\frac{\partial \Phi}{\partial p}=-\frac{R T}{p}$,
$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$,
$\frac{\partial T}{\partial t}+u \frac{\partial T}{\partial x}+v \frac{\partial T}{\partial y}-S_{p} \omega=\frac{J}{c_{p}}$,
where
$S_{p} \equiv \frac{R T}{c_{p} p}-\frac{\partial T}{\partial p}=-\frac{T}{\theta} \frac{\partial \theta}{\partial p}$,
The horizontal derivatives are performed at constant pressure.

[^3](a)

(c)


Figure 9.6. Air flow in a developing extratropical cyclone. (a) Sea-level pressure and frontal analysis. Part of the bent back front is shown as a cold front. The unshaded cold frontal symbols indicate where the front is detectable only aloft. (b) Cloud analysis. Precipitation features are depicted as follows: solid hatching - cloud head precipitation, cross-hatched where convective (moderate to heavy); broken hatching - warm conveyor belt precipitation (mainly light intensity); broken cross-hatching - midlevel convective precipitation associated with the upper cold front (patchy moderate rain); solid shading - narrow cold frontal rainbands (heavy rain associated with intense shallow line convection). (c) Conveyor belt analysis. System-relative motion for the key moist flows: (i) main warm conveyor belt (WCB) labelled W1; (ii) Lower part of WCB which peels off and ascends as flow W2 in the top part of the cloud head; (iii) Cold conveyor belt (CCB) whose ascending diffluent flow is responsible for a large proportion of the clou head and associated precipitation. The westward component of the flow within W1 and the CCB is associated with the ageostrophic transverse circulation at the exit of an upper level jet streak. (d) Dry intrusion analysis. System relative motion for the dry air that descends from near the tropopause upwind and reascends while approaching the cyclone centre where it overruns a shallow moist zone associated with W2 (see smz in (c)) (based on Browning, K.A., and N.M. Roberts, 1994: Structure of a frontal cyclone. Quart.J.Roy.Meteor.Soc., 120, 1535-1557).


FIGURE 9.7. Plotted weather maps of 3 March 1995, 12 UTC for two isobaric levels: $\mathbf{5 0 0} \mathbf{h P a}$ (panel on this page) and 850 hPa (panel on next page). The position of a radiosonde station is indicated by a square. Also indicated are the temperature $\left({ }^{\circ} \mathrm{C}\right)$ (upper left), the dew point depression $\left({ }^{\circ} \mathrm{C}\right)$ (lower left) and geopotential height (dm) (upper right). Barbs, attached to the wind vector, which points towards the station position, indicate the windspeed, ff. No barb corresponds to $f f<2.5$ knots ( 1 knot is $0.5 \mathrm{~m} / \mathrm{s}$ ); each whole (half) barb corresponds to an extra 10 (5) knots. If the windspeed is equal or greater than $25 \mathrm{~m} / \mathrm{s}$, this is indicated by a triangle attached to the wind vector, while the value of the wind speed (in $\mathrm{m} / \mathrm{s}$ ) is indicated to the right of the plot in bold numbers.


Figure 9.7. Caption on previous page.

## Box 9.1 Primitive equations in pressure coordinates

With the ideal gas law (eq. 1.10b), $p=R \rho T$, the hydrostatic equation ( $\partial p / \partial z=-\rho g$ ) can be expressed as follows.

$$
\begin{equation*}
\frac{\partial \Phi}{\partial p}=-\frac{R T}{p}, \tag{1}
\end{equation*}
$$

where the "geopotential" is
$\Phi \equiv g z$.
Integration of (1) in the vertical yields the hypsometric equation:
$z_{T} \equiv z_{2}-z_{1}=\frac{R}{g} \int_{p_{2}}^{p_{1}} T d(\ln p)$,
The quantity, $z_{\mathrm{T}}$, is the thickness of an atmospheric layer between the pressure surfaces $p_{2}$ and $p_{1}$. Defining the layer mean temperature as
$\langle\boldsymbol{T}\rangle \equiv \int_{p_{2}}^{p_{1}} T d(\ln p)\left[\int_{p_{2}}^{p_{1}} d(\ln p)\right]^{-1}$,
we obtain
$z_{T}=\frac{R}{g}\langle T\rangle \ln \frac{p_{1}}{p_{2}}$.

Thus, the thickness of an atmospheric layer, bounded by isobaric surfaces, is proportional to the mean temperature of this layer. We can also eq. 5a as
$z_{T} \equiv H \ln \frac{p_{1}}{p_{2}}$.

Here, $H$ is the layer scale height, defined as
$H \equiv \frac{R\langle T\rangle}{g}$.
The thermodynamic energy in pressure coordinates is (eq. 1.195):
$\frac{\partial T}{\partial t}+u\left(\frac{\partial T}{\partial x}\right)_{p}+v\left(\frac{\partial T}{\partial y}\right)_{p}+\omega \frac{\partial T}{\partial p}-\frac{\alpha \omega}{c_{p}}=\frac{J}{c_{p}}$,
where the differentiation with respect to $x$ and $y$ is carried out at constant pressure, and where $c_{\mathrm{p}}$ is the specific heat at constant pressure $\left(c_{\mathrm{p}}-c_{\mathrm{v}}=R\right)$. Eq. 7 is rewritten as
$\frac{\partial T}{\partial t}+u\left(\frac{\partial T}{\partial x}\right)_{p}+v\left(\frac{\partial T}{\partial y}\right)_{p}-S_{p} \omega=\frac{J}{c_{p}}$,
with
$S_{p} \equiv \frac{R T}{c_{p} p}-\frac{\partial T}{\partial p}=-\frac{T}{\theta} \frac{\partial \theta}{\partial p}$,
which is the static stability parameter for the isobaric system.
With pressure as a vertical coordinate, the horizontal components of the momentum are (eqs. 1.191) are
$\frac{d u}{d t}-\frac{u v \tan \phi}{a}=-\left(\frac{\partial \Phi}{\partial x}\right)_{p}+f v+F_{x}$,
$\frac{d v}{d t}+\frac{u^{2} \tan \phi}{a}=-\left(\frac{\partial \Phi}{\partial y}\right)_{p}-f u+F_{y}$.
The terms, $F_{x}$ and $F_{y}$ represent friction.
The material derivative with respect to time is expanded as (eq. 1.176)
$\frac{d}{d t}=\frac{\partial}{\partial t}+u\left(\frac{\partial}{\partial x}\right)_{p}+v\left(\frac{\partial}{\partial y}\right)_{p}+\omega \frac{\partial}{\partial p}$.
If applied to large scale motion systems in mid-latitudes, eqs. 10a,b are frequently simplified to (Box 1.2)
$\frac{d u}{d t}=-\frac{\partial \Phi}{\partial x}+f v$
$\frac{d v}{d t}=-\frac{\partial \Phi}{\partial y}-f u$

The continuity equation in the pressure coordinate system is (eq. $5, \mathbf{B o x} 1.9$ )
$-\frac{v \tan \phi}{a}+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_{p}+\frac{\partial \omega}{\partial p}=0$,
which, when applied to middle latitudes, is usually simplified to
$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_{p}+\frac{\partial \omega}{\partial p}=0$.
The continuity equation (14), the hydrostatic relation (1), the thermodynamic equation (8) and the momentum equation ( $12 \mathrm{a}, \mathrm{b}$ ) form a closed set of equations, called the "primitive equations".

Equations 9.1-4 can be further simplified by assuming that the horizontal flow is nearly geostrophic and that the magnitude of the vertical velocity is much smaller than the magnitude of the horizontal velocity. We may write (as in eq. 1.234)

$$
\vec{v}=\vec{v}_{g}+\vec{v}_{a} .
$$

Here, the geostrophic wind is determined by
$\vec{v}_{g} \equiv f_{0}^{-1} \hat{k} \times \vec{\nabla} \Phi$,
with $f_{0}=f\left(y_{0}\right)$, where $y_{0}$ is a chosen reference value of $y$, corresponding to chosen reference latitude, $\phi_{0}$ (say: $45^{\circ}$ ).

The Coriolis parameter is written as
$f \equiv f_{0}+\frac{d f}{d y}\left(y-y_{0}\right)=f_{0}+\beta\left(y-y_{0}\right)$,

Approximation (9.7) is usually referred to as the middle latitude beta-plane approximation, which was first introduced by Rossby in 1939) (see also section 1.34). The beta-parameter,
$\beta=\frac{2 \Omega \cos \phi_{0}}{a}$.
Eqs. 9.6 and 9.7 represent the first two parts of the quasi-geostrophic approximation.
In the third part of the quasi-geostrophic approximation we define the fluid parcel acceleration as being subject to the "geostrophic constraint", i.e.
$\frac{d \vec{v}}{d t} \approx \frac{d_{g} \vec{v}_{g}}{d t}$,
where
$\frac{d_{g}}{d t} \equiv \frac{\partial}{\partial t}+\vec{v}_{g} \cdot \vec{\nabla}=\frac{\partial}{\partial t}+u_{g} \frac{\partial}{\partial x}+v_{g} \frac{\partial}{\partial y}$.

With assumptions 9.7 and 9.9 we write eq. 9.1 as

$$
\frac{d_{g} \vec{v}_{g}}{d t}=-f \hat{k} \times \vec{v}-\vec{\nabla} \Phi=-\left(f_{0}+\beta\left(y-y_{0}\right)\right) \hat{k} \times\left(\vec{v}_{g}+\vec{v}_{a}\right)+f_{0} \hat{k} \times \vec{v}_{g}=-\left(f_{0}+\beta\left(y-y_{0}\right)\right) \hat{k} \times \vec{v}_{a}-\beta\left(y-y_{0}\right) \hat{k} \times \vec{v}_{g}
$$

or
$\frac{d_{g} \vec{v}_{g}}{d t} \approx-f_{0} \hat{k} \times \vec{v}_{a}-\beta\left(y-y_{0}\right) \hat{k} \times \vec{v}_{g}$.

Here, we have assumed that $f_{0} \gg \beta\left(y-y_{0}\right)$, which represents the fourth part of the quasi-geostrophic approximation, which implies that the flow is retricted to a relatively narrow mid-latitude channel with a meridional width of about 1000 km , since $f_{0} \approx 10^{-4} \mathrm{~s}^{-1}$ and $\beta \approx 10^{-11} \mathrm{~s}^{-1} \mathrm{~m}^{-1}$. This retriction is, in fact also imposed by the approximation leading to eq. 9.6.

The assumption that the ageostrophic wind can always be neglected compared to the geostrophic wind is in fact very dubious, especially in the case of a quickly intensifying and/or fast travelling cyclone, which is usually associated with strong isallobaric effects (section 1.31 ). Nevertheless, this assumption is made here in order to make progress in constructing a tractable analytical theory of the initial growth-phase of midlatitude cyclones. The test of this theory is not whether it is able to describe these systems quantitatively very accurately, but whether it is able to reproduce and explain the qualitative features of the mechanism of cyclogenesis.

Because the horizontal divergence of the geostrophic wind in the quasi-geostrophic approximation, as defined in eq. 9.6, is equal to zero, the continuity equation (9.3) can be written as
$\frac{\partial u_{a}}{\partial x}+\frac{\partial v_{a}}{\partial y}+\frac{\partial \omega}{\partial p}=0$,
which shows that vertical motion is determined by the ageostrophic part of the wind.
In the thermodynamic equation (9.4) horizontal advection is approximated by its geostrophic value, as in eq. 9.9). Vertical temperature advection is retained, however, because adiabatic heating or cooling, owing to vertical motion, is usually of the same order of magnitude as horizontal temperature
advection, despite the smallness of the vertical velocity. The term can be somewhat simplified, though, by writing the temperature as a sum of a basic state, which is only dependent on pressure, and a perturbation, as in the Boussinesq approximation (chapter 3), i.e.
$T(x, y, z, t)=T_{0}(p)+T^{\prime}(x, y, z, t)$, with $T^{\prime} \ll T_{0}$.

Therefore eq. 9.4 becomes
$\frac{\partial T}{\partial t}+u_{g} \frac{\partial T}{\partial x}+v_{g} \frac{\partial T}{\partial y}-\frac{\sigma p}{R} \omega=\frac{J}{c_{p}}$,
where , in this chapter, using Holton's (2004) notation, the static stability parameter is
$\sigma \equiv-\frac{R T_{0}}{p} \frac{d \ln \theta_{0}}{d p}$.
In this equation $\theta_{0}$ is the potential temperature corresponding to the basic state temperature $T_{0}$. The quasi-geostrophic approximation assumes that static stability, $\sigma$, is constant in time. Note that the symbol, $\sigma$, is used to denote isentropic density in chapters 1 and 7 (section 1.23). Eqs. 9.2 (hydrostatic balance), 9.7 (beta-plane approximation), $9.11,9.12$ and 9.14 constitute the quasi-geostrophic equations.

### 9.4 Quasi-geostrophic vorticity- and thermodynamic equations

The quasi-geostrophic equations are further simplified by deriving a quasi-geostrophic vorticity equation as follows. First, we define the geostrophic wind as follows:
$u_{g} \approx-\frac{1}{f_{0}} \frac{\partial \Phi}{\partial y} ; v_{g} \approx \frac{1}{f_{0}} \frac{\partial \Phi}{\partial x}$.

With this the vertical component of the relative vorticity, $\zeta=\partial v / \partial x-\partial u / \partial y$, is approximated "quasigeostrophically" as
$\zeta_{g} \equiv \frac{\partial v_{g}}{\partial x}-\frac{\partial u_{g}}{\partial y}=\frac{1}{f_{0}}\left(\frac{\partial^{2} \Phi}{\partial x^{2}}+\frac{\partial^{2} \Phi}{\partial y^{2}}\right) \equiv \frac{1}{f_{0}} \nabla^{2} \Phi$.

The quasi-geostrophic vorticity equation can be obtained from the quasi-geostrophic momentum equation. The two components of the quasi-geostrophic momentum equation (9.11) are
$\frac{d_{g} u_{g}}{d t}=f_{0} v_{a}+\beta y v_{g}$,
$\frac{d_{g} v_{g}}{d t}=-f_{0} u_{a}-\beta y u_{g}$,

From this it follows that
$\frac{d_{g} \zeta_{g}}{d t}=-f_{0}\left(\frac{\partial u_{a}}{\partial x}+\frac{\partial v_{a}}{\partial y}\right)-\beta v_{g}$,

This equation can also be written as, using (9.12) and (9.17)
$\frac{d_{g} \nabla^{2} \Phi}{d t}=f_{0}^{2} \frac{\partial \omega}{\partial p}-\beta \frac{\partial \Phi}{\partial x}$.
The quasi-geostrophic thermodynamic equation (9.14) can be written, using the hydrostatic relation (9.2), as
$\frac{d_{g} \frac{\partial \Phi}{\partial p}}{d t}=-\sigma \omega-\frac{R J}{c_{p} p}$.

If the atmosphere is adiabatic (i.e. $J=0$ ), eqs. 9.21 and 9.22 form a closed set of equations in only two dependent variables, $\Phi$ and $\omega$. The fact that we need to know the distribution of only two variables at a certain point in time in order to predict the future thermodynamic evolution of the atmosphere is the strength of the quasi-geostrophic approximation. However, one of these two variables, i.e, the vertical velocity, $\omega$, is practically impossible to measure directly. Several methods have been devised to estimate the vertical velocity from other variables that can be measured more easily. The following section discusses one of these methods, which is due to Hoskins, Draghici and Davies (1978) and Hoskins and Pedder (1980) ${ }^{77}$.

## PROBLEM 9.3. Divergence, vertical motion and phase speed of a planetary wave

Given the following expression for the geopotential field:
$\Phi=\Phi_{0}(p)+U f_{0}\left\{-y\left[\cos \left(\frac{\pi p}{p_{\text {ref }}}\right)+1\right]+\frac{1}{l} \sin (l(x-c t))\right\}$
where $\Phi_{0}$ is a function of $p$ alone, $c$ is a phase speed, $U$ has the dimensions of velocity, $l$ is a zonal wave number, $p_{\text {ref }}=1000 \mathrm{hPa}$ and the Coriolis parameter $f=f_{0}+\beta y$, where $f_{0}$ and $\beta$ are constants.
(a) Obtain the horizontal divergence field, which is consistent with this $\Phi$-field. Use the quasigeostrophic vorticity equation in pressure coordinates:
$\frac{d_{g} \zeta_{g}}{d t}=-f_{0}\left(\frac{\partial u_{a}}{\partial x}+\frac{\partial v_{a}}{\partial y}\right)-\beta v_{g}$,
where $u_{a}$ and $v_{a}$ are the ageostrophic horizontal wind components and, furthermore,
$\frac{d_{g}}{d t} \equiv \frac{\partial}{\partial t}+u_{g} \frac{\partial}{\partial x}+v_{g} \frac{\partial}{\partial y}, \zeta_{g}=\frac{\partial v_{g}}{\partial x}-\frac{\partial u_{g}}{\partial y}, u_{g}=-\frac{1}{f_{0}} \frac{\partial \Phi}{\partial y}$ and $v_{g}=\frac{1}{f_{0}} \frac{\partial \Phi}{\partial x}$.

[^4](b) Derive an expression for $\omega(x, y, p, t)$ by integrating the continuity equation,
$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$,
upwards from the earth's surface. Assume that the vertical velocity at the earth's surface (i.e. at $p=p_{\text {ref }}$ ), $\omega\left(x, y, p_{r e f}\right)=0$.
(c) Under which condition is the expression for $\omega(x, y, p, t)$, derived in $\mathbf{1}(\mathbf{b})$, consistent with the boundary condition $\omega=0$ at $p=0$ ? Give an interpretation of this expression.

### 9.5 Quasi-geostrophic adjustment to thermal wind balance: the "omega-equation"

This section demonstrates that, in the absence of orography and under adiabatic conditions ( $J=0$ ), large scale vertical motion in a statically stable atmosphere (i.e. $\sigma>0$; see eq. 9.15 ) occurs only as a response to frontogenesis (or frontolysis).

In order to derive a diagnostic equation for $\omega$, we shall first consider the tendency of geostrophic motion to destroy thermal wind balance. Thermal wind balance on the $f$-plane ( $f$ is assumed constant) can be written as (see eq. 1.267)

$$
\begin{equation*}
\frac{\partial v_{g}}{\partial p}=-\frac{R}{p f_{0}} \frac{\partial T}{\partial x} ; \frac{\partial u_{g}}{\partial p}=\frac{R}{p f_{0}} \frac{\partial T}{\partial y} . \tag{9.23}
\end{equation*}
$$

We use the $x$-component of the equation of motion (9.18) and the temperature equation (9.14). For didactical purposes we begin by neglecting ageostrophic motion ( $v_{a}, \omega$ ). If, for simplicity, $\beta$ and $J$ are set equal to zero, it can easily be shown, using (9.23), that
$\frac{d_{g}}{d t}\left(\frac{\partial u_{g}}{\partial p}\right)=-\frac{\partial u_{g}}{\partial p} \frac{\partial u_{g}}{\partial x}-\frac{\partial v_{g}}{\partial p} \frac{\partial u_{g}}{\partial y}=\frac{R}{p f_{0}}\left(-\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial y}+\frac{\partial u_{g}}{\partial y} \frac{\partial T}{\partial x}\right)$
$\frac{d_{g}}{d t}\left(\frac{\partial T}{\partial y}\right)=-\frac{\partial u_{g}}{\partial y} \frac{\partial T}{\partial x}-\frac{\partial v_{g}}{\partial y} \frac{\partial T}{\partial y}$.

Subtracting eqs 9.24b from 9.24a yields
$\frac{d_{g}}{d t}\left(\frac{f_{0} p}{R} \frac{\partial u_{g}}{\partial p}-\frac{\partial T}{\partial y}\right)=2\left(\frac{\partial u_{g}}{\partial y} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial y} \frac{\partial T}{\partial y}\right) \equiv-2 Q_{g 2} \equiv-2 \frac{d_{g}}{d t}\left(\frac{\partial T}{\partial y}\right)$,
where we use the fact that
$\frac{\partial u_{g}}{\partial x}+\frac{\partial v_{g}}{\partial y}=0$,

Eq. 9.25 demonstrates that thermal wind balance is destroyed by frontogenetic or frontolytic processes. These frontogenetic or frontolytic processes are associated with horizontal shear and/or confluence of the geostrophic wind.

Geostrophic motion tends to destroy thermal wind balance. We will now see that the role of ageostrophic motion is to restore or "conserve" thermal wind balance.

Allowing for ageostrophic motion, i.e. if $v_{\mathrm{a}}$ and $\omega$ are not neglected in (9.14) and (9.18), we find (instead of eq. 9.25):

$$
\begin{equation*}
\frac{d_{g}}{d t}\left(\frac{f_{0} p}{R} \frac{\partial u_{g}}{\partial p}-\frac{\partial T}{\partial y}\right)=-2 Q_{g 2}+\frac{f_{0}^{2} p}{R} \frac{\partial v_{a}}{\partial p}-\frac{p \sigma}{R} \frac{\partial \omega}{\partial y} \tag{9.27}
\end{equation*}
$$

We now assume that thermal wind balance, which is destroyed by frontogenetic processes embodied in the first term on the r.h.s. of (9.27), is maintained by ageostrophic motion, i.e.
$\frac{f_{0}^{2} p}{R} \frac{\partial v_{a}}{\partial p}-\frac{p \sigma}{R} \frac{\partial \omega}{\partial y}=2 Q_{g 2}$.

A similar equation can be deduced from the $y$-component of the momentum equation (9.19), yielding

$$
\begin{equation*}
\frac{f_{0}^{2} p}{R} \frac{\partial u_{\mathrm{a}}}{\partial p}-\frac{p \sigma}{R} \frac{\partial \omega}{\partial x}=2 Q_{\mathrm{g}_{1}}=2 \frac{\mathrm{~d}_{\mathrm{g}}}{\mathrm{~d} t}\left(\frac{\partial T}{\partial x}\right) \tag{9.29}
\end{equation*}
$$

with
$Q_{g 1}=-\left(\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}\right)$.

If we now take $\partial / \partial y(9.28)+\partial / \partial x(9.29)$ and use (9.12), we find the omega equation:
$\sigma \nabla^{2} \omega+f_{0}^{2} \frac{\partial^{2} \omega}{\partial p^{2}}=-\frac{2 R}{p} \vec{\nabla} \cdot \vec{Q}_{g}$
where
$\vec{Q}_{g} \equiv\left(Q_{g 1}, Q_{g 2}\right)=-\left(\frac{\partial \vec{v}_{g}}{\partial x} \cdot \vec{\nabla} T, \frac{\partial \vec{v}_{g}}{\partial y} \cdot \vec{\nabla} T\right)$
is referred to as the geostrophic $\mathbf{Q}$-vector (section 1.37). Expanding the r.h.s. of eq. 9.32, we find
$Q_{g 1}=-\left(\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}\right) ; Q_{g 2}=-\left(\frac{\partial u_{g}}{\partial y} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial y} \frac{\partial T}{\partial y}\right)$.

[^5]The omega equation (9.31) is a diagnostic equation for the field of vertical motion in terms of the instantaneous fields of geopotential and temperature. The omega equation, unlike the continuity eq. 9.12, provides a method of estimating the vertical motion that does not depend on measurements of wind, which are much less accurate than measurements of temperature and geopotential. A drawback of the omega equation is that it contains products of second order derivatives. Accurate estimation of such terms from noisy observational data can be problematic.

The omega equation (9.31) is one of several versions of this equation, which have appeared in the literature in the twentieth century. Holton's book (see "further literature") also discusses an older still popular version. The omega equation, as it appears in (9.31) is, in the view of this author, the most useful, because it links vertical motion in a quasi-balanced atmosphere to the intensification or weakening of the isobaric temperature gradient and/or the rotation of the isobaric temperature gradient, hence clearly demonstrating that large scale vertical motion is a manifestation of adjustment to thermal wind balance.

The omega equation (9.31) is an elliptic partial differential equation, similar to the PV-inversion equation 9 in Box 1.12 and the Eliassen-Sawyer equation eq. 8.32. The solution of the omega equation indicates that frontogenesis at a front will induce a field of vertical motion over an appreciable area in the vicinity of the front, in the same manner as a potential vorticity anomaly induces a wind field at an appreciable distance from the PV-anomaly. In other words, the idea of "action at a distance" is applicable to the solution of the omega equation too. The explicit solution of the omega equation 9.31 is discussed in section 9.9.

Let us perform a first rough analysis of what we may expect, by assuming that $\omega$ has a sinusoidal dependence in the horizontal as well as in the vertical as
$\omega=W_{0} \sin \left(\frac{\pi p}{p_{r e f}}\right) \sin (k x) \sin (l y)$,
(so that $\omega=0$ at $p=0$ and $p=p_{\text {ref }}$ ) where $k$ and $l$ are the wavenumbers and $W_{0}$ is a constant, we can write

$$
\begin{equation*}
\left(\nabla^{2}+\frac{f_{0}^{2}}{\sigma} \frac{\partial^{2}}{\partial p^{2}}\right) \omega=-\left[k^{2}+l^{2}+\frac{1}{\sigma}\left(\frac{f_{0} \pi}{p_{0}}\right)^{2}\right] \omega \tag{9.35}
\end{equation*}
$$

This equation shows that the 1.h.s. of (9.31) is proportional to $-\omega$, for the solution (9.34). By recalling that $\omega=-\rho g w$ ( $\omega<0$ implies upward motion), we see that the l.h.s. of the omega equation (9.31) is proportional to the vertical velocity. Therefore, forcing of vertical motion is represented simply by the pattern of the $Q_{\mathrm{g}}$-vector. More specifically, regions where the $Q_{\mathrm{g}}$-vector is convergent (divergent) correspond to ascent (descent).

Let us, for example, consider an ideal jetstreak aligned along the $x$-axis, with isotherms also aligned along the $x$-axis (see figure 9.8). In the entrance region we have confluent motion (i.e. $\partial \nu_{g} / \partial y<0$ ). If the temperature decreases with increasing $y$, the $Q_{\mathrm{g}}$-vector points toward the south. This implies convergence of the $Q_{\mathrm{g}}$-vector in the right entrance region, looking in the flow direction. Therefore, we expect upward motion in the right entrance region of the jetstreak. In the exit region the motion is diffluent (i.e. $\partial v_{\mathrm{g}} / \partial y>0$ ). Therefore, the $Q_{\mathrm{g}}$-vector points towards the north, implying that there is upward motion in the left exit region.

In general it is less easy to accurately estimate direction, magnitude and divergence of the $Q_{\mathrm{g}}$-vector, even though it is a straightforward matter to compute the $Q_{\mathrm{g}}$-vector from gridded data of geopotential and temperature by approximating the derivatives by finite differences. The use of $Q$-vectors will be illustrated further in section 9.8 and chapter 10.


FIGURE 9.8. Orientation of Q-vectors (bold arrows) for confluent (jet entrance) flow. Dashed lines are isotherms (After Sanders, F., B.J. Hoskins, 1990: An easy method for the estimation of Q-vectors from weather maps. Wea.Forecasting, 5, 346-353).

### 9.6 The two level model

This section discusses the role of the instability of thermal wind balance in accounting for the genesis and initial intensification of mid-latitude cyclones. Following Phillips (1954) ${ }^{79}$, we adopt the most simplified model of the atmosphere that can incorporate three-dimensional baroclinic processes. The atmosphere in this model is represented by two layers, bounded by surfaces, numbered 0,2 and 4 , as shown in figure 9.9.

It is convenient to define a geostrophic streamfunction, $\psi=\phi / f_{0}$. The geostrophic wind (9.6) and the geostrophic vorticity (9.17) can be expressed respectively as
$\vec{v}_{\psi}=\hat{k} \times \vec{\nabla} \psi, \zeta_{g}=\nabla^{2} \psi$.

Remember that $\nabla^{2} \equiv \nabla_{h}^{2}$.
In terms of $\psi$, the quasi-geostrophic vorticity equation becomes
$\frac{\partial \nabla^{2} \psi}{\partial t}+\vec{v}_{\psi} \cdot \vec{\nabla}\left(\nabla^{2} \psi\right)+\beta \frac{\partial \psi}{\partial x}=f_{0} \frac{\partial \omega}{\partial p}$

Using the hydrostatic relation (9.2) and neglecting diabatic heating ( $J=0$ ), the quasi-geostrophic thermodynamic equation (9.14) can be written as
$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial p}\right)+\vec{v}_{\psi} \cdot \vec{\nabla}\left(\frac{\partial \psi}{\partial p}\right)=-\frac{\sigma}{f_{0}} \omega$

We now "apply" (9.37) at levels 1 and 3 . To do this, we must estimate the divergence term at these levels using the following finite difference approximations to the vertical derivatives.

[^6]

FIGURE 9.9. Arrangement of variables in the vertical direction for the two-level model (from Holton, 2004).
$\left(\frac{\partial \omega}{\partial p}\right)_{1} \approx \frac{\omega_{2}-\omega_{0}}{\delta p},\left(\frac{\partial \omega}{\partial p}\right)_{3} \approx \frac{\omega_{4}-\omega_{2}}{\delta p}$.
Here $\delta p$ is the pressure interval between the levels 0 and 2 , and 2 and 4 . Subscripts indicate the vertical level for each dependent variable.

The resulting vorticity equations are
$\frac{\partial \nabla^{2} \psi_{1}}{\partial t}+\vec{v}_{1} \cdot \vec{\nabla}\left(\nabla^{2} \psi_{1}\right)+\beta \frac{\partial \psi_{1}}{\partial x}=\frac{f_{0}}{\delta p} \omega_{2}$,
$\frac{\partial \nabla^{2} \psi_{3}}{\partial t}+\vec{v}_{3} \cdot \vec{\nabla}\left(\nabla^{2} \psi_{3}\right)+\beta \frac{\partial \psi_{3}}{\partial x}=-\frac{f_{0}}{\delta p} \omega_{2}$.

Here, we have used the fact that $\omega_{0}=0$ and assumed that $\omega_{4}=0$.
We next write the thermodynamic energy equation (9.38) at level 2 . We must evaluate $\partial \psi / \partial p$, using the approximate formula
$\left(\frac{\partial \psi}{\partial p}\right)_{2} \approx \frac{\psi_{3}-\psi_{1}}{\delta p}$.
This yields
$\frac{\partial}{\partial t}\left(\psi_{1}-\psi_{3}\right)=-\vec{v}_{2} \cdot \vec{\nabla}\left(\psi_{1}-\psi_{3}\right)+\frac{\sigma \delta p}{f_{0}} \omega_{2}$.
The first term on the r.h.s. in (9.43) represents advection of the $250-750 \mathrm{hPa}$ thickness by the wind at 500 hPa . Unfortunately, the 500 hPa streamfunction, $\psi_{2}$, is not a predicted field in this model. Therefore, $\psi_{2}$ must be obtained by linearly interpolating between 250 hPa and 750 hPa as follows.
$\psi_{2} \approx \frac{\left(\psi_{1}-\psi_{3}\right)}{2}$.

If this formula is used, (9.40), (9.41) and (9.43) become a closed set of prediction equations in the variables $\psi_{1}, \psi_{3}$ and $\omega_{2}$.

### 9.7 Linear analysis: Rossby waves and baroclinic instability

In order to simplify the analysis as much as possible we assume that the streamfunctions, $\psi_{1}$ and $\psi_{3}$, can be expressed as follows
$\psi_{1}=-U_{1} y+\psi_{1}{ }^{\prime}(x, y, t)$,
$\psi_{3}=-U_{3} y+\psi_{3}{ }^{\prime}(x, y, t)$,
$\omega_{2} \approx \omega_{2}{ }^{\prime}(x, y, t)$.

The "background" geostrophic zonal velocities at levels 1 and 3 are constants with the values $U_{1}$ and $U_{3}$, respectively.

Substituting (9.45) into (9.40-41) and (9.43) and linearising yields the perturbation equations,
$\left(\frac{\partial}{\partial t}+U_{1} \frac{\partial}{\partial x}\right) \nabla_{h}^{2} \psi_{1}^{\prime}+\beta \frac{\partial \psi_{1}^{\prime}}{\partial x}=\frac{f_{0}}{\delta p} \omega_{2}^{\prime}$,
$\left(\frac{\partial}{\partial t}+U_{3} \frac{\partial}{\partial x}\right) \nabla_{h}^{2} \psi_{3}^{\prime}+\beta \frac{\partial \psi_{3}^{\prime}}{\partial x}=\frac{-f_{0}}{\delta p} \omega_{2}^{\prime}$,
$\left(\frac{\partial}{\partial t}+U_{m} \frac{\partial}{\partial x}\right)\left(\psi_{1}^{\prime}-\psi_{3}^{\prime}\right)-U_{T} \frac{\partial}{\partial x}\left(\psi_{1}^{\prime}+\psi_{3}^{\prime}\right)=\frac{\sigma \delta p}{f_{0}} \omega_{2}^{\prime}$,
with
$\nabla_{h}^{2} \equiv \frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}$.

To express $v_{2}$ in terms of $\psi_{1}$ and $\psi_{3}$, we linearly interpolate as follows.
$v_{2}=v_{2}^{\prime}=\frac{1}{2}\left(v_{1}^{\prime}+v_{3}^{\prime}\right)=\frac{1}{2} \frac{\partial}{\partial x}\left(\psi_{1}^{\prime}+\psi_{3}^{\prime}\right)$,

We define
$U_{M} \equiv \frac{\left(U_{1}+U_{3}\right)}{2}$ and $U_{T} \equiv \frac{\left(U_{1}-U_{3}\right)}{2}$.

Thus, $U_{\mathrm{M}}$ and $U_{\mathrm{T}}$ are respectively, the vertically averaged zonal wind and the mean thermal wind for the interval between levels 1 and 3 .

The dynamical properties of the system (9.46-48) are more clearly expressed if $\omega_{2}^{\prime}$ is eliminated. First we write (9.46) and (9.47) as

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\left(U_{M}+U_{T}\right) \frac{\partial}{\partial x}\right) \nabla_{h}^{2} \psi_{1}^{\prime}+\beta \frac{\partial \psi_{1}^{\prime}}{\partial x}=\frac{f_{0}}{\delta p} \omega_{2}^{\prime} \tag{9.50}
\end{equation*}
$$

$\left(\frac{\partial}{\partial t}+\left(U_{M}-U_{T}\right) \frac{\partial}{\partial x}\right) \nabla_{h}^{2} \psi_{3}{ }^{\prime}+\beta \frac{\partial \psi_{3}{ }^{\prime}}{\partial x}=-\frac{f_{0}}{\delta p} \omega_{2}{ }^{\prime} \quad$,

We now define the barotropic and baroclinic perturbations as
$\psi_{M} \equiv \frac{\left(\psi_{1}+\psi_{3}\right)}{2}$ and $\psi_{T} \equiv \frac{\left(\psi_{1}-\psi_{3}\right)}{2}$.

Adding (9.50) and (9.51) and using the definitions in (9.52) yields
$\left(\frac{\partial}{\partial t}+U_{M} \frac{\partial}{\partial x}\right) \nabla_{h}^{2} \psi_{M}+\beta \frac{\partial \psi_{M}}{\partial x}+U_{T} \frac{\partial}{\partial x} \nabla_{h}^{2} \psi_{T}=0$.

Subtracting (9.51) from (9.50) and combining with (9.48) to eliminate $\omega_{2}^{\prime}$ yields
$\left(\frac{\partial}{\partial t}+U_{M} \frac{\partial}{\partial x}\right)\left(\nabla_{h}^{2} \psi_{T}-2 \lambda^{2} \psi_{T}\right)+\beta \frac{\partial \psi_{T}}{\partial x}+U_{T} \frac{\partial}{\partial x}\left(\nabla_{h}^{2} \psi_{M}+2 \lambda^{2} \psi_{M}\right)=0$,
where
$\lambda^{2} \equiv \frac{f_{0}^{2}}{\sigma(\delta \rho)^{2}}$.

The parameter, $\lambda$, is the inverse of the Rossby radius of deformation, within the context of this model. Equations (9.53) and (9.54) govern, respectively, the time-evolution of the barotropic (vertically averaged) vorticity and the baroclinic (thermal) vorticity of the perturbation.

As in section 3.3, we assume that wavelike solutions exist of the form
$\psi_{M}=A \exp [i(l x+m y-\omega t)] ; \psi_{T}=B \exp [i(l x+m y-\omega t)]$.

Substituting these solutions into (9.53) and (9.54), yields a pair of simultaneous linear algebraic equations for the coefficients $A$ and $B$ :
$\left[\left(c_{x}-U_{M}\right) k^{2}+\beta\right] A-U_{T} k^{2} B=0$,
$U_{T}\left(k^{2}-2 \lambda^{2}\right) A-\left[\left(c_{x}-U_{M}\right)\left(k^{2}+2 \lambda^{2}\right)+\beta\right] B=0$,
where
$k^{2} \equiv l^{2}+m^{2}$ and $c_{x} \equiv \omega / l$.

Non-trivial solutions will exist only if the determinant of the coefficients of $A$ and $B$ is zero. Thus the phase speed $c$ must satisfy the condition
$k^{2}\left(k^{2}+2 \lambda^{2}\right)\left(c_{x}-U_{M}\right)^{2}+2 \beta\left(k^{2}+\lambda^{2}\right)\left(c_{x}-U_{M}\right)+\left[\beta^{2}-U_{T}^{2} k^{2}\left(k^{2}-2 \lambda^{2}\right)\right]=0$,

The dispersion relation (9.59) yields for the phase speed
$c_{x}=U_{M}-\frac{\beta\left(k^{2}+\lambda^{2}\right)}{k^{2}\left(k^{2}+2 \lambda^{2}\right)} \pm \sqrt{\delta}$
where
$\delta \equiv \frac{\beta^{2} \lambda^{4}}{k^{4}\left(k^{2}+2 \lambda^{2}\right)^{2}}-\frac{U_{T}^{2}\left(2 \lambda^{2}-k^{2}\right)}{\left(k^{2}+2 \lambda^{2}\right)}$,

Although (9.60) appears to be rather complicated, it is immediately apparent thatthe phase speed will have an imaginary part if $\delta<0$, in which case perturbations will amplify exponentially.

Let us consider the special case, $\beta=0$. In this case

$$
\begin{equation*}
c_{x}=U_{M} \pm U_{T}\left(\frac{k^{2}-2 \lambda^{2}}{k^{2}+2 \lambda^{2}}\right)^{1 / 2} \tag{9.62}
\end{equation*}
$$

For waves with zonal wave numbers satisfying $k^{2}<2 \lambda^{2}$, (9.62) has an imaginary part. Therefore, all waves with wavelengths greater than the critical wavelength $L_{\mathrm{c}}=\sqrt{2} \pi / \lambda$ will amplify. The growth rate of this amplification is equal to ( $\mathrm{i} \omega$ ). Based on the definition of $\lambda$ (9.55), we can write
$L_{c}=\frac{\pi(2 \sigma)^{1 / 2} \delta p}{f_{0}}$.

For typical tropospheric conditions, $(2 \sigma)^{1 / 2} \approx 2 \times 10^{-3} \mathrm{~N}^{-1} \mathrm{~m}^{3} \mathrm{~s}^{-1}$. Therefore, with $\delta p=50000 \mathrm{~Pa}$ and $f_{0}=10^{-4}$ $\mathrm{s}^{-1}$ we find that $\mathrm{L}_{\mathrm{c}}=3000 \mathrm{~km}$, which is of the same order of magnitude as the typical wavelength in the longitudinal direction of observed synoptic disturbances. Eq. 9.63 also reveals that the critical wavelength for baroclinic instability increases with the static stability, $\sigma$.

Let us now consider the special case, $\boldsymbol{U}_{T}=\mathbf{0}$. In this so-called barotropic case (9.60) reduces to either
$c_{x}=U_{M}-\frac{\beta}{k^{2}}$
or
$c_{x}=U_{M}-\frac{\beta}{\left(k^{2}+2 \lambda^{2}\right)}$

In the absence of baroclinicity, the two-level model has two free (normal mode) small amplitude solutions, which represent oscillations or waves, which exist due to the $\beta$-effect. These waves are termed Rossby waves (section 1.34). According to (9.62) the phase of Rossby waves propagates in westerly direction relative to the basic barotropic current, $U_{M}$. Rossby waves can be identified with the troughs and ridges that are characteristic of upper air charts of the geopotential height at, for instance 500 hPa (figure 1.93).

If we set $\beta=0$ and $U_{T}=0$ in eq. 9.61, we find that the quasigeostrophic equations do not support waves. This implies that acoustic waves, gravity waves and inertial waves are filtered out as a solution by
the quasi-geostrophic approximation, which includes the hydrostatic approximation. The principal reason for developing the quasi-geostrophic theory in the 1940's was to find a system of equations, which could be integrated numerically without too much computational expense, while still retaining as much of the meteorologically interesting phenomena as possible, i.e. the Rossby waves. Sound waves, which, because of their high phase velocity and high frequency, represent a significant computational burden, if they are included in a numerical solution, are less interesting, because they do not play a key role in the formation of precipitation systems. Buoyancy waves also form a significant computational burden, because they require a high spatial resolution, while only having a significant influence on largescale flow through the wave drag effect (section 3.5).

In the general case, where all terms in (9.60) are retained, the stability criterion is most easily understood by computing the neutral curve, which connects all values of $U_{T}$ and $k$ for which $\delta=0$, so that the flow is neurtally or marginally stable. The condition $\delta=0$ implies that
$\frac{\beta^{2} \lambda^{4}}{k^{4}\left(k^{2}+2 \lambda^{2}\right)}=U_{T}^{2}\left(2 \lambda^{2}-k^{2}\right)$,

## PROBLEM 9.4. Standing Rossby waves

Demonstrate that standing barotropic Rossby waves as a response to flow over orography are not possible in easterly flow.

## PROBLEM 9.5. The properties of baroclinic Rossby waves in the real atmosphere

Investigate the properties of Rossby waves during a specific period of time, using reanalysis data from NCEP (see http://www.esrl.noaa.gov/psd/data/gridded/data.ncep.reanalysis2.html) or ECMWF (http://apps.ecmwf.int/datasets/). You may choose any period of one or several months that you think is interesting, such as the period containing the case discussed in section 9.2 (2-3 March 1995) or the period in November 2002, which was marked by very well defined wave packets (chapter 11) (see http://www.wetter3.de/Archiv/).
(a) Retrieve the four time daily 500 hPa meridional wind, $v$, and the four time daily 500 hPa geopotential height for the latitude band $45-55^{\circ} \mathrm{N}$ ( or $40-60^{\circ} \mathrm{N}$ ), for a particular set of consecutive days and construct a Hovmöller diagram as in figure 1.93. The reanalysis is available on a "lat-lon" grid, with grid-distances of $1^{\circ}$ in both latitudinal and longitudinal direction. The latitude band $45^{\circ}-55^{\circ} \mathrm{N}$ corresponds to a grid of 360 points in the longitudinal direction by 11 points in the latitudinal direction. In order to construct a Hovmöller diagram you must average the data in latitudinal direction for each time, so that you obtain an array with dimensions correponding to the number of analysis times (e.g. 28 if you take 7 days) and the number of points in the longitudinal direction (360, if you take a full circle around the globe). Compute the zonal mean for each time, and then compute the anomaly with respect to the zonal mean.
(b) Estimate the average wavelength and eastward phase velocity of the waves in the geopotential from this Hovmöller diagram over the Atlantic/European sector. Can you identify a group of ridges and troughs? If so, estimate the group velocity.
(c) Estimate the zonal average zonal velocity in this sector. Compare this with an estimate of the zonal average zonal geostrophic velocity for the same region. Check the validity of the dispersion relations, (9.64a) and (9.64b). To test the validity of (9.64b) you need an estimate of $\lambda$ (or $\sigma$ ).
(d) Identify episodes of baroclinic instability, by identifying wave-amplitude growth. Are these episodes connected to enhanced baroclinicity?


Figure 9.10. Neutral stability curve for the two-level baroclinic model. Source of this figure: Holton (2004) (see the list of references at the end of this chapter).

This complicated relationship between $U_{\mathrm{T}}$ and $k$ can best be displayed in a graph by solving for $k^{4} / 2 \lambda^{4}$, yielding

$$
\begin{equation*}
\frac{k^{4}}{2 \lambda^{4}}=1 \pm\left[1-\frac{\beta^{2}}{4 \lambda^{4} U_{T}^{2}}\right]^{1 / 2} \tag{9.66}
\end{equation*}
$$

Equation (9.66) is displayed graphically in figure 9.10. The nondimensional quantity $k^{2} / 2 \lambda^{2}$ is plotted along the horizontal axis. The non-dimensional parameter $2 \lambda^{2} U_{\mathrm{T}} / \beta$ is plotted along the vertical axis. The latter parameter is proportional to the thermal wind or meridional basic state temperature gradient. Baroclinic waves are always stable if the quantity, $k^{2} /\left(2 \lambda^{2}\right)=2 \pi^{2} \sigma(\delta p)^{2} /\left(L_{x}^{2} f_{0}^{2}\right)>1$, or if
$A_{R} \equiv \frac{L_{x}}{2 \delta p}<\frac{\pi \sqrt{\sigma}}{\sqrt{2} f_{0}}$.

Here, $A_{R}$ represents the "aspect ratio" (horizontal scale [ m ] relative to vertical scale [Pa]) of the wave. On the right hand side of this inequality we recognise a quantity that was identified as the Rossby ratio in section 7.7. Propagating baroclinic Rossby waves are associated with a secondary vertical circulation (section 9.9). If this circulation has an aspect ratio, which is smaller than the Rossby ratio, the amplitude of the wave will not grow due to baroclinic instability.

The neutral curve in figure 9.10 separates the unstable region in the $U_{\mathrm{T}}-k$ plane from the stable region. The inclusion of the $\beta$-effect serves to stabilise the flow, because unstable roots exist only for $\left|U_{\mathrm{T}}\right|>\beta /\left(2 \lambda^{2}\right)$. The $\beta$-effect stabilises the long-wave end of the wave spectrum. The minimum value of $U_{\mathrm{T}}$, required for unstable growth, depends strongly on $k$. The flow is always stable for waves shorter
than the critical wavelength, $L_{c}=\pi \sqrt{2} / \lambda$ (eq. 9.63). By differentiating (9.65) with respect to $k$ and setting $\mathrm{d} U_{\mathrm{T}} / \mathrm{d} k=0$, we find that the minimum value of $U_{T}$, for which unstable waves may exist, occurs when $k^{2}=\sqrt{2} \lambda^{2}$. This wave number corresponds to the wave, which becomes unstable for the lowest value of the thermal wind. Observed growing waves should have a wave number that lies close to this wave number of "maximum instability", because, if $U_{T}$ is gradually raised from zero, the flow becomes unstable first for perturbations of wave number $k=2^{1 / 4} \lambda$. These perturbations amplify and in this process remove energy from the mean thermal wind, thereby decreasing $U_{T}$ and stabilizing the flow. Under normal conditions of static stability, the theoretical wavelength of maximum instability is approximately 4000 km , which is of similar order of magnitude as the wavelength of amplifying midlatitude baroclinic waves in reality.

The thermal wind, required for marginal stability at the wavelength of maximum instability, is about $U_{\mathrm{T}} \approx 4 \mathrm{~m} / \mathrm{s}$. This implies a shear of $8 \mathrm{~m} / \mathrm{s}$ between 250 and 750 hPa . Comparing this with the example, which is discussed in section 9.2, we find ${ }^{80}$ that the mean zonal wind speed on March 3, 1995 at 00 UTC within the trough or core of the jetstreak is $16 \mathrm{~m} / \mathrm{s}$ at 750 hPa , and $24 \mathrm{~m} / \mathrm{s}$ at 250 hPa , implying a shear of $8 \mathrm{~m} / \mathrm{s}$ between these levels, which perhaps by fortune is exactly the value for marginal instability quoted above. This seems to verify the hypothesis that the growth of the baroclinic Rossby wave, shown in figures 9.1 and 9.2, originates from "small perturbations" of a baroclinically unstable basic current.

It is, however, doubtful whether this is really the case. Numerical weather prediction models generally predict cyclogenesis with a surprising degree of accuracy many days in advance. If the time and location of cyclogenesis were really dependent on the presence of a random infinitesimal perturbation, the performance of numerical weather prediction models would be very much worse than it actually is. It is, therefore, very likely that cyclogenesis requires finite amplitude (relatively intense) perturbations. These perturbations are in fact frequently observed as potential vorticity anomaliesnear the tropopause.

We must not, therefore, interpret the theory of baroclinic instability too literally, but more as giving valuable information about the criteria (in terms of barocinicity, wavelength of an initial disturbance and static stability) that need to be fullfilled to initiate cyclogenesis as well as about the connections between the preferred length-scales of planetary Rossby waves and the constraints imposed on the atmosphere by stratification and rotation.

PROBLEM 9.6. Maximum growthrate of a baroclinic disturbance (taken from Holton, 2004).
Show, using eq. 9.62, that the maximum growth rate for baroclinic instability when $\beta=0$ occurs for
$k^{2}=2 \lambda^{2}(\sqrt{2}-1)$.

How long does it take the most rapid growing wave to amplify by a factor of $e$ if $\lambda=2 \times 10^{-6} \mathrm{~m}^{-1}$ and $\mathrm{U}_{T}=20 \mathrm{~m} \mathrm{~s}^{-1}$.

PROBLEM 9.7. Phase tilt of a baroclinic disturbance (taken from Holton, 2004).
For the case $\beta=0$ determine the phase difference between the 250 hPa and the 750 hPa geopotential fields for the most unstable baroclinic wave (see problem 9.6).

[^7]
### 9.8 Vertical motion in small-amplitude baroclinic waves

The basic physics of linear baroclinic instability can be distilled from the analysis presented in the previous section by using the omega equation (9.31) and Q -vectors (eq. 9.33).

The Q -vector, linearised around a basic state baroclinic homogeneous zonal current, as defined in (9.45), becomes (using $\partial u_{\mathrm{g}} / \partial x+\partial v_{\mathrm{g}} / \partial y=0$ ),

$$
\begin{equation*}
\left(Q_{g 1}, Q_{g 2}\right)=\left(-\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y},+\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial y}\right) \tag{9.68}
\end{equation*}
$$

In the trough of a Rossby wave we have $\partial v_{\mathrm{g}} / \partial x>0$. In the northern hemisphere $\partial T / \partial y<0$. Therefore, the Q-vector points in eastward direction in a trough. In the ridge we have $\partial v_{g} / \partial x<0$, which implies that the $\underline{\mathbf{Q}}$-vector points in westward direction in a ridge. Neglecting for simplicity the accelerations and decelerations of the zonal current, we see that the Q -vector converges to the east of the trough and diverges to the west of the trough, which implies upward motion to the east of the trough and downward motion to the west of the trough.

The perturbation analysis of a baroclinic zonal current in the two-level model yields the same conclusion if we assume, for simplicity, that the meridional wave number, $m=0$. This assumption implies (eqs. 9.43 and 9.54) that the geostrophic zonal velocities at levels 1 and 3 are constants with the values $U_{1}$ and $U_{3}$, respectively. This, in turn implies that (eq. 9.30),
$Q_{g 1}=-\left(\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}\right) \approx-\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}$.

The most important frontogenetic effect, which is left in the linear analysis of the previous section, is the frontogenetic effect of the rotation of isotherms, due to the shear term, $\partial v_{g} / \partial x \partial T / \partial y$. This frontogenetic effect turns the isotherms from the longitudinal direction into the meridional direction. Because the atmosphere is constrained (in the theory) to remain in thermal wind balance, this requires a simultaneous turning of the thermal wind into the meridional direction (see section 1.35). According to eq. 9.31 , this is taken care of by an ageostrophic "circulation" in the $x-p$ plane. In other words, the ageostrophic circulation must conform to eq. 9.29 , which is repeated below:
$\frac{f_{0}^{2} p}{R} \frac{\partial u_{a}}{\partial p}-\frac{p \sigma}{R} \frac{\partial \omega}{\partial x}=2 Q_{g 1} \approx-2 \frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}$.

In the trough, $\partial v_{\mathrm{g}} / \partial x>0$ and $\partial T / \partial y<0$, thus $Q_{\mathrm{g} 1}>0$. In the ridge, $\partial v_{\mathrm{g}} / \partial x<0$ and $\partial T / \partial y<0$, thus $Q_{\mathrm{g} 1}<0$. Since $m=0, Q_{\mathrm{g} 2}=0$. Therefore, there is convergence of the geostrophic Q-vector east of the trough and west of the ridge. According to the omega equation (9.31) this implies upward motion east of the trough and west of the ridge. Within the framework of the two-level quasi-geostrophic model this is attended with an increase of the geostrophic relative vorticity $\left(=\partial v_{\mathrm{g}} / \partial x\right.$ because $m=0$ ) at low levels (below 500 hPa ), due to mass convergence. Due to this, the trough at low levels will tend to shift towards the east.

Let us apply the omega equation to level 2 (figure 9.9). The $\mathrm{Q}_{\mathrm{g}}$-vector for the two-level model can be derived simply from (9.31). We first estimate the second term on the left hand side by finite differencing in $p$. Using (9.39), we obtain

$$
\begin{equation*}
\frac{\partial^{2} \omega}{\partial p^{2}} \approx \frac{\left[\frac{\partial \omega}{\partial p}\right]_{3}-\left[\frac{\partial \omega}{\partial p}\right]_{1}}{\delta p} \approx-\frac{2 \omega_{2}}{(\delta p)^{2}} \tag{9.71}
\end{equation*}
$$



Figure 9.11. Structure of a baroclinic wave at midlevels (level 2 in the two-layer model, i.e. 500 hPa ). The relative vorticity and the meridional velocity are shown as a function of longitude. The meridional velocity is $90^{\circ}$ out of phase with the vorticity. See the text for further explanation.

With eq. 1 in Box (9.1) we have
$T=-\frac{p}{R} \frac{\partial \Phi}{\partial p}=-\frac{f_{0} p}{R} \frac{\partial \psi}{\partial p}$

Applying this to level 2 of the two-level model we obtain
$T_{2}=\frac{f_{0}}{R}\left(\psi_{1}-\psi_{3}\right)$.
The omega equation (9.31), applied to model level 2, becomes
$\sigma\left(\nabla^{2}-2 \lambda^{2}\right) \omega_{2}=-\frac{2 R}{p} \vec{\nabla} \vec{Q}_{g}$,
Substituting (9.73) in (9.68), using (9.45) and (9.49), we obtain
$\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}-2 \lambda^{2}\right) \omega_{2}=-\frac{4 f_{0}}{\sigma \delta p} U_{T} \frac{\partial^{2} v_{2}{ }^{\prime}}{\partial x^{2}}$.
Observing that
$\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}-2 \lambda^{2}\right) \omega_{2}^{\prime} \alpha-\omega_{2}{ }^{\prime}$


FIGURE 9.12. Illustration of mechanism of baroclinic instability (see text in this section and in section 1.21).
(see the arguments leading to eq. 9.35), we may interpret (9.75) physically by noting that
$-\omega_{2}{ }^{\prime} \propto w_{2}^{\prime} \propto-f_{0} U_{T} \frac{\partial^{2} v_{2}^{\prime}}{\partial x^{2}}$.

It is sometimes said that vertical motion is associated with "advection of disturbance (relative) vorticity by the basic state thermal wind". If the thermal wind is eastward (i.e. $U_{T}>0$ ), the motion is upward on the east side of the trough (where $\partial \zeta / \partial x<0$ and downward on the west side of the trough (where $\partial \zeta / \partial x>0$ ).

Figure 9.11 shows the schematic structure of a baroclinic wave at 500 hPa (level 2 in the two layer model) in terms of relative vorticity, $\zeta$, and the meridional velocity component, $v$. Vertical motion is forced by frontogenesis, due to the horizontal rotation of the temperature gradient (a vector). This rotation is counterclockwise (cyclonic) in the trough and clockwise (anticyclonic) in the ridge. The associated Q -vectors point eastward in the trough and westward in the ridge, leading to Q-vector divergence to the west of the trough and Q -vector convergence to the east of the trough. Convergence of the $\mathrm{Q}_{g}$-vector is associated with upward motion. This explains the observed persistent ascent of poleward moving warm air to the east of the trough. If this warm air travels upward and poleward, as shown by the solid arrow in figure 9.12, it replaces colder air. This is required for further growth of the wave, i.e. for conversion of potential energy into kinetic energy (see also section 1.21). However, if warm air travels poleward and upward, as shown by the dashed arrow in figure 9.12, it replaces warmer air. This implies a cooling of the warm sector of the wave, which will weaken both the warm front and the cold front and arrest further growth of the amplitude of the baroclinic wave.

If we assume that eq. 9.77 has wave-like solutions in vertical velocity and meridional velocity, i.e. $v_{2}^{\prime} \approx A \exp \{i(l x+m y-\omega t)\}$, (as in eq. 9.56), we find that
$w_{2}{ }^{\prime} \propto f_{0} U_{T} l^{2} v_{2}{ }^{\prime}$.

This equation reveals that (1) poleward motion is upward and equatorward motion is downward. And (2) that trajectories in the meridional plane will steepen with increasing zonal wavenumber, or decreasing zonal wavelength. In other words, short waves are stabilised. This explains the high wave number cutoff in figure 9.10.

### 9.9 Solution of the omega equation for idealized situations

Let us investigate two possible, but hypothetical, distributions of the geopotential. The first case is
$\Phi(x, y, p, t)=\Phi_{0}(p)-f_{0} U_{0}\left(y-y_{0}\right) \cos \left(\frac{\pi p}{2 p_{0}}\right)+\frac{f_{0} V_{0}}{k} \sin (k x-c t)$

Here, $k=2 \pi / L_{x}, c$ is the zonal component of the phase velocity, $V_{0}$ and $U_{0}$ are constant velocities, $p_{0}=1000 \mathrm{hPa}, y_{0}$ is a value of $y$ corresponding the reference latitude where $f=f_{0}$ and $\Phi_{0}$ is a function of pressure (compare this with the expression for $\Phi$ in problem 9.3).

The second case is
$\Phi(x, y, p, t)=\Phi_{0}(p)+\frac{1}{2} f_{0} U_{0}\left(y-y_{0}\right) \tanh \left(\frac{p-p_{\text {front }}}{p_{\text {scale }}}\right)+\frac{f_{0} V_{0}}{k} \sin (k x-c t)$.

Here, the parameters $p_{\text {front }}$ and $p_{\text {scale }}$ have the units of pressure ( Pa ). The physical meaning of these parameters will become apparent below.

The geostrophic velocity components, which are associated with these distributions of geopotential, are
$u_{g}=-\frac{1}{f_{0}} \frac{\partial \Phi}{\partial y}=U_{0} \cos \left(\frac{\pi p}{2 p_{0}}\right)$ and $v_{g}=\frac{1}{f_{0}} \frac{\partial \Phi}{\partial x}=V_{0} \cos (k x-c t)$
for the first case and


FIGURE 9.13: Zonal geostrophic wind, $u_{\mathrm{g}}$, in the two cases which are investigated in this section. Case 1 corresponds to the geopotential given in eq. 9.76. Case 2 corresponds to the geopotential given in eq. 9.77. In both cases the vertical shear of the geostropic wind approaches zero at the top of the atmosphere. In case 1 the vertical shear of the geostropic wind has its highest value at the surface. In case 2 the vertical shear of the geostropic wind is concentrated around a prescribed pressure level, $p=p_{\text {front }}$. In this figure $p_{\text {front }}=500 \mathrm{hPa}$ and $p_{\text {scale }}=100 \mathrm{hPa}$.
$u_{g}=-\frac{1}{2} U_{0} \tanh \left(\frac{p-p_{\text {front }}}{p_{\text {scale }}}\right)$ and $v_{g}=V_{0} \cos (k x-c t)$
for the second case. Note that the amplitude of the meridional velocity is independent of $k$ (by construction)

In the first hypothetical situation, the eastward geostrophic wind increases with height from zero at the surface $\left(\right.$ at $\left.p=p_{0}\right)$ to $U_{0}$ at the "top of the atmosphere", at $p=0$ (figure 9.13). In the second idealized hypothetical situation there is a shallow shear zone in the geostrophic wind, centred at the pressure level, $p=p_{\text {front }}$.

In both cases there is a barotropic wavelike perturbation in the meridional component of the wind with a zonal wavelength equal to $L_{x}$, which is superposed on the zonal geostrophic flow. This perturbation is associated with a perturbation in the geostrophic relative vorticity, which is identical in both cases, i.e.,
$\zeta_{g}=-\frac{1}{f_{0}}\left(\frac{\partial^{2} \phi}{\partial x^{2}}+\frac{\partial^{2} \phi}{\partial y^{2}}\right)=-V_{0} k \sin (k x-c t)$.

The thermal wind equation (9.23),
$\frac{\partial u_{g}}{\partial p}=\frac{R}{p f_{0}} \frac{\partial T}{\partial y}$ and $\frac{\partial v_{g}}{\partial p}=-\frac{R}{p f_{0}} \frac{\partial T}{\partial x}$,
yields the following equation for the meridional temperature gradient:
$\frac{\partial T}{\partial y}=-\frac{\pi U_{0} f_{0}}{2 R} \frac{p}{p_{0}} \sin \left(\frac{\pi p}{2 p_{0}}\right)$ and $\frac{\partial T}{\partial x}=0$
in the first case, and
$\frac{\partial T}{\partial y}=-\frac{U_{0} f_{0}}{2 R} \frac{p}{p_{\text {scale }}}\left\{1-\tanh ^{2}\left(\frac{p-p_{\text {front }}}{p_{\text {scale }}}\right)\right\}$ and $\frac{\partial T}{\partial x}=0$
in the second case. The above relations imply that
$Q_{g 2}=-\left(\frac{\partial u_{g}}{\partial y} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial y} \frac{\partial T}{\partial y}\right)=0$,
in both cases. Furthermore, in the first case,
$Q_{g 1}=-\left(\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}\right)=-\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}=-\frac{\pi U_{0} f_{0} V_{0} k}{2 R} \frac{p}{p_{0}} \sin \left(\frac{\pi p}{2 p_{0}}\right) \sin (k x-c t)$,
and in the second case,
$Q_{g 1}=-\left(\frac{\partial u_{g}}{\partial x} \frac{\partial T}{\partial x}+\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}\right)=-\frac{\partial v_{g}}{\partial x} \frac{\partial T}{\partial y}=-\frac{U_{0} f_{0} V_{0} k}{2 R} \frac{p}{p_{\text {scale }}}\left\{1-\tanh ^{2}\left(\frac{p-p_{\text {front }}}{p_{\text {scale }}}\right)\right\} \sin (k x-c t)$

This demonstrates that the perturbation in the geostrophic relative vorticity induces a frontogenetic effect, which acts as a disturbance to thermal wind balance. This frontogenetic effect is associated with the rotation of the isotherms from the zonal direction into the meridional direction.

Now, let us solve the omega equation for these two hypothetical situations. For simplicity and because the $y$-component of the $Q$-vector is zero, we neglect the meridional ( $y$-) derivative, so that the omega equation in the first situation becomes
$\sigma \frac{\partial^{2} \omega}{\partial x^{2}}+f_{0}^{2} \frac{\partial^{2} \omega}{\partial p^{2}}=\frac{-2 R}{p} \frac{\partial Q_{g 1}}{\partial x}=\frac{\pi U_{0} f_{0} V_{0} k^{2}}{p_{0}} \sin \left(\frac{\pi p}{2 p_{0}}\right) \cos (k x-c t)$.

Dividing this equation by $f_{0}^{2}$, we get
$\frac{\partial^{2} \omega}{\partial p^{2}}+\frac{\sigma}{f_{0}^{2}} \frac{\partial^{2} \omega}{\partial x^{2}}=\frac{\pi U_{0} V_{0} k^{2}}{p_{0} f_{0}} \sin \left(\frac{\pi p}{2 p_{0}}\right) \cos (k x-c t)$.

Following the two-level model linear stability analysis of section 9.7, let us impose the following simple boundary conditions
$\omega=0$ at $p=p_{0}$ and $\omega=0$ at $p=0$.

We should keep in mind that the lower boundary condition is not necessarily correct. Furthermore, we assume an infinite (or periodic) domain in the $x$-direction. If $\sigma$ is constant, we may, in view of the factor $\cos (k x)$ in the "source term" on the r.h.s. of eq. 9.88 , substitute the following solution:
$\omega=W(p) \cos (k x-c t)$.

The vertical motion is $90^{\circ}$ out of phase with the relative vorticity. We now obtain an equation for the pressure dependence of the amplitude, $W$, of $\omega$ :
$\frac{d^{2} W}{d p^{2}}-\frac{\sigma}{f_{0}^{2}} k^{2} W=\frac{\pi U_{0} V_{0} k^{2}}{p_{0} f_{0}} \sin \left(\frac{\pi p}{2 p_{0}}\right)$.

It is easy to see that the equation for $W$ in the second case becomes
$\frac{d^{2} W}{d p^{2}}-\frac{\sigma}{f_{0}^{2}} k^{2} W=\frac{U_{0} V_{0} k^{2}}{p_{\text {scale }} f_{0}}\left\{1-\tanh ^{2}\left(\frac{p-p_{\text {front }}}{p_{\text {scale }}}\right)\right\}$.

Since the factor, $\sigma k^{2} / f_{0}^{2}$, is positive, eqs. 9.91 and 9.92 are elliptic partial differential equations. For analytical purposes the second case is the most interesting. If we assume that $p_{\text {scale }}$ is very small, the frontogenetic forcing will be restricted to a shallow layer around the pressure level, $p=p_{\text {front }}$. Outside this layer we have,
$\frac{d^{2} W}{d p^{2}}-\frac{\sigma}{f_{0}^{2}} k^{2} W=0$,
instead of (9.92). Eq. 9.93 is a Helmholtz equation, like equation 3.23. The solution of this equation is

$$
\begin{equation*}
W=C_{1} \exp \left(\frac{k \sqrt{\sigma}\left(p-p_{\text {front }}\right)}{f_{0}}\right)+C_{2} \exp \left(-\frac{k \sqrt{\sigma}\left(p-p_{\text {front }}\right)}{f_{0}}\right) . \tag{9.94}
\end{equation*}
$$

Here $C_{1}$ and $C_{2}$ are constants determined by the boundary conditions. In view of the boundary condition (9.89), $W$ should go to zero at $p=0$ and at $p=p_{0}$. Therefore
$W=C_{1} \exp \left(\frac{k \sqrt{\sigma}\left(p-p_{\text {front }}\right)}{f_{0}}\right)$ for $p<p_{\text {front }}$,
and
$W=C_{2} \exp \left(-\frac{k \sqrt{\sigma}\left(p-p_{\text {front }}\right)}{f_{0}}\right)$ for $p>p_{\text {front }}$.

In both cases ( 9.95 and 9.96) we note an exponential decrease of $W$ with $p$ away from the front. The characteristic pressure scale, i.e. the e-folding distance, associated with this exponential decay is,
$\Delta p=\frac{f_{0}}{k \sqrt{\sigma}}=\frac{f_{0} L_{x}}{2 \pi \sqrt{\sigma}}$.

This vertical scale, which is also introduced in chapter 7, is referred to as the "Rossby height". This vertical scale represents the characteristic depth over which adjustment to thermal wind balance takes place. Apparently, this depends on the horizontal scale of the wave, which is inducing frontogenesis by rotating the isentropes. A dynamically more fundamental scale is, therefore, the aspect ratio,
$\frac{L_{x}}{\Delta p}=\frac{2 \pi \sqrt{\sigma}}{f_{0}}$,
which is referred to in chapter 7 as "Rossby's ratio" (comment: also compare eq. 9.63 with eq. 9.98 ). With a typical value of $\sigma$ in the troposphere of $2 \times 10^{-6} \mathrm{~m}^{2} \mathrm{~Pa}^{-2} \mathrm{~s}^{-2}$ and $f_{0}=10^{-4} \mathrm{~s}^{-1}$ we obtain a value of 100 m $\mathrm{Pa}^{-1}$ for "Rossby's ratio" If $L_{\mathrm{x}}=1000 \mathrm{~km}$, the characteristic pressure scale of the vertical influence of frontogenesis at the specified level $p=p_{\text {front }}$ is about 100 hPa . This means that the dynamical influence of frontogenesis is "felt" at levels 100 hPa above and below $p=p_{\text {front }}$. In other words, frontogenesis "acts at a distance" is the same way as a potential vorticity anomaly.

With $\Delta p=-\rho g \Delta z$ (hydrostatic balance) and the definition of the static stability parameter, $\sigma(9.15)$, we find that the aspect ratio of large scale circulation systems is

$$
\begin{equation*}
\frac{L_{x}}{\Delta z}=\frac{2 \pi N}{f_{0}}, \tag{9.99}
\end{equation*}
$$

where the Brunt-Väisälä frequency is

$$
\begin{equation*}
N=\sqrt{\frac{g}{\theta_{0}} \frac{\partial \theta_{0}}{\partial z}} . \tag{9.100}
\end{equation*}
$$

Eq. 9.99 is the quasi-geostrophic version of eq. 7.76. The typical aspect ratio, i.e. the horizontal scale divided by the vertical scale, of large-scale circulation systems, therefore, is in the order of $N / f_{0} \approx 100$. The ratio $f / N$ is sometimes called "Prandtl's ratio of scales".


FIGURE 9.14: Amplitude, $W(p)$, of the vertical velocity, $\omega$, in case (2) (figure 9.13) for $L_{\mathrm{x}}=1000 \mathrm{~km}$ and for $L_{\mathrm{x}}=5000 \mathrm{~km}$. Other parameter values are given in the text. The solution is found numerically by the relaxation method, described in the text below, where the $p$-axis is divided into 41 points, so that $\mathrm{D} p=25 \mathrm{hPa}$.

For $p \approx p_{\text {front }}$ the equation governing the amplitude of the vertical velocity becomes
$\frac{d^{2} W}{d p^{2}}-\frac{\sigma}{f_{0}^{2}} k^{2} W=\frac{c k^{2} U_{0} f_{0}}{p_{\text {scale }}}$.
Because the r.h.s. of the above equation is positive (if $f_{0}>0$ ), the solutions in the two domains can only be matched if $W<0$. This impies that $\omega$ is out of phase with $v_{\mathrm{g}}$. In other words, the vertical velocity, $w$, is in phase with $v_{\mathrm{g}}$, which means that poleward motion is upward, while equatorward motion is downward.

The full amplitude equation (9.92) can be solved numerically for arbitrary value of $p_{\text {scale }}$. By defining equidistant gridpoints on the $p$-axis with index, $j$, running from $j=1$ at $p=0$ to $j=j \max$ at $p=p_{0}$ and approximating the second order derivative for each gridpoint in this equation as
$\frac{\partial^{2} W}{\partial p^{2}} \approx \frac{1}{\Delta p}\left[\frac{W[j-1]-W[j]}{\Delta p}-\frac{W[j]-W[j+1]}{\Delta p}\right]=\frac{W[j+1]+W[j-1]-2 W[j]}{\Delta p^{2}}$,
we obtain the following numerical approximation of (9.92):

$$
\begin{equation*}
\frac{W[j+1]+W[j-1]-2 W[j]}{\Delta p^{2}}-\frac{\sigma}{f_{0}^{2}} k^{2} W[j]=\frac{U_{0} V_{0} k^{2}}{f_{0} p_{\text {scale }}}\left\{1-\tanh ^{2}\left(\frac{p[j]-p_{\text {front }}}{p_{\text {scale }}}\right)\right\}, \tag{9.103}
\end{equation*}
$$

which can be rewritten as

$$
\begin{equation*}
\frac{W[j+1]+W[j-1]}{\Delta p^{2}}-\left(\frac{\sigma}{f_{0}^{2}} k^{2}+\frac{2}{\Delta p^{2}}\right) W[j]-\frac{U_{0} V_{0} k^{2}}{f_{0} p_{\text {scale }}}\left\{1-\tanh ^{2}\left(\frac{p[j]-p_{\text {front }}}{p_{\text {scale }}}\right)\right\}=0 . \tag{9.104}
\end{equation*}
$$

By imposing the boundary conditions, $W(0)=W\left(p_{0}\right)=0$, eq. 9.104 can be solved interatively. The iteration
is started by assuming trial values $W[j]=0$. Evaluation of the l.h.s. of the numerically approximated equation (9.104) will yield a residual, $\Delta R$. This residual is used to make a new estimate of $W[j]$, i.e.
$W_{\text {new }}[j]=W_{\text {old }}[j]+\Delta R\left(\frac{\sigma}{f_{0}^{2}} k^{2}+\frac{2}{\Delta p^{2}}\right)^{-1}$,
such that the new residual is equal to zero at the gridpoint under consideration. By repeatedly applying this procedure to each gridpoint we will converge to the correct solution (see problem 5.1 and boxes 7.1 and 7.3). The solution for $L_{\mathrm{x}}=1000 \mathrm{~km}$ and for $L_{\mathrm{x}}=5000 \mathrm{~km}$ is shown in figure 9.14. The other parameter values are $U_{0}=80 \mathrm{~m} \mathrm{~s}^{-1}, V_{0}=5 \mathrm{~m} \mathrm{~s}^{-1}, f_{0}=10^{-4} \mathrm{~s}^{-1}, \sigma=2 \times 10^{-6} \mathrm{~m}^{2} \mathrm{~Pa}^{-2} \mathrm{~s}^{-2}, p_{\text {scale }}=50 \mathrm{hPa}, p_{\text {front }}=500 \mathrm{hPa}$ and $p_{0}=1000 \mathrm{hPa}$. It appears that the vertical velocity is relatively weak for the longer wave (large $L_{\mathrm{x}}$ ). This scale-dependence of the amplitude of vertical motion in a baroclinic wave explains the "short-wave cutoff" of baroclinic instability, which is revealed by eq. 9.62 and in figure 9.10. The meridional slope of parcel trajectories in relatively short waves is larger than meridional slope of the background isentropes (figure 9.12), so that kinetic energy is converted into potential energy (stability) instead of the reverse (instability) (section 1.21).

Another point to note is that the frontogenetic forcing is determined (among other) by the meridional velocity $\left(V_{0}\right)$ (r.h.s. of eqs. 9.91 and 9.92). This is because the meridional component of the motion turns the zonally oriented isotherms into the meridional direction.


FIGURE 9.15: Q-vector (upper panel) and vertical velocity, $\omega$ (lower panel), in case 2, i.e. when the front is located in a shallow layer centred at 500 hPa (red is positive; blue is negative). The frontogenetic forcing of the vertical motion is also concentrated in this shallow layer. The arrows in the upper panel indicate the direction of the Q -vector, while they indicate the direction of the vertical motion in the lower panel. The contours in the upper panel represent the value of $\mathrm{Q}_{\mathrm{g} 1}$. The contour interval is $2 \times 10^{-9} \mathrm{~K} \mathrm{~m}^{-1} \mathrm{~s}^{-1}$. The contour interval in the lower panel is 2 hPa per hour. The parameter values are $U_{0}=80 \mathrm{~m} \mathrm{~s}^{-1}, L_{\mathrm{x}}=5000 \mathrm{~km}, V_{0}=5 \mathrm{~m} \mathrm{~s}^{-1}, f_{0}=10^{-4} \mathrm{~s}^{-1}, \sigma=2 \times 10^{-6} \mathrm{~m}^{2} \mathrm{~Pa}^{-2} \mathrm{~s}^{-2}, p_{\text {scale }}=50 \mathrm{hPa}, p_{\text {front }}=500 \mathrm{hPa}$ and $p_{0}=1000 \mathrm{hPa}$.


Figure 9.16: Q -vector (upper panel) and vertical velocity, $\omega$ (lower panel), in case 1, i.e. when the front is most intense at the surface of the Earth (red is positive; blue is negative). The arrows in the upper panel indicate the direction of the Q -vector, while they indicate the direction of the vertical motion in the lower panel. The contours in the upper panel represent the value of $\mathrm{Q}_{\mathrm{g} 1}$. The contour interval is $0.5 \times 10^{-9} \mathrm{~K} \mathrm{~m}^{-1} \mathrm{~s}^{-1}$. The contour interval in the lower panel is 1 hPa per hour. The parameter values are $U_{0}=80 \mathrm{~m} \mathrm{~s}^{-1}, L_{\mathrm{x}}=5000 \mathrm{~km}, V_{0}=5 \mathrm{~m} \mathrm{~s}^{-1}, f_{0}=10^{-4} \mathrm{~s}^{-1}, \sigma=2 \times 10^{-6} \mathrm{~m}^{2} \mathrm{~Pa}^{-2} \mathrm{~s}^{-2}$, and $p_{0}=1000 \mathrm{hPa}$.

The full solution (9.90) of the omega equation for $L_{x}=5000 \mathrm{~km}$ (for case 2) is shown in the lower panel of figure 9.15. The upper panel of this figure shows the value of $\mathrm{Q}_{\mathrm{g} 1}$. We find upward motion in regions where the $Q$-vector converges and downward motion in regions where the $Q$-vector diverges. This response to frontogenetic forcing is, however, not restricted to the thin sheet of air between 400 hPa and 600 hPa where the "frontogenetic forcing" is acting, but is spread out over the full depth of the atmosphere. This "action at distance" can also be observed in case (1) (figure 9.16). Here, the frontogenetic forcing is largest at the Earth's surface and decays to zero in the upper half of the atmosphere. Nevertheless, relatively strong vertical motions are also observed in the upper half of the atmosphere.

### 9.10 "PV- $\theta$ viewpoint" of forcing of vertical motion

Our qualitative knowledge of the characteristics of the solution of the PV-inversion equation for a steady PV-anomaly can provide illuminating insight into the relation between large-scale vertical motion and a propagating PV-anomaly. This is sometimes referred to as the "PV- $\theta$ viewpoint". Consider a positive potential vorticity anomaly embedded in a westerly mean flow with linear shear in height. The associated cyclonic circulation and the attraction of the isentropes towards the centre of the anomaly are sketched in figure 9.17. Let us assume that the PV-anomaly is located at a particular discrete height, it is represented mathematically by positive $\delta$-function. If the coordinate system is such that the mean flow is zero at this level, then the circulation is a stationary solution of the equations of
motion. Assuming adiabatic conditions, this implies that the isentropes do not move and that the air must flow among them. The air above the anomaly must flow down the isentrope to the west and up the isentrope to the east, as shown in figure 9.17a. Similarly, the air below the anomaly must flow up the isentropic surface to the east and down the isentropic surface to the west. Also, associated with the shear and thermal wind balance, isentropic surfaces must slope upwards towards the pole. Hence, as indicate in figure 9.17a, the poleward moving air to the east of the anomaly must ascend and the equatorward moving air to the west of the anomaly must descend. Therefore, the total "isentropic upglide" vertical motion is positive (ascent) to the east and negative (descent) to the west.


FIGURE 9.17. Schematic west-east vertical sections illustrating the effect of a positive potential vorticity $\delta$-function superimposed on a westerly flow with a linear shear in height, z. Dashed lines represent isentropes. Horizontal and curved arrows sketch the horizontal circulation. Vertical pointing arrows indicate the vertical motion associated with isentropic upglide (continuous) and isentropic displacement (dashed). As viewed in a frame of reference in which (a) the anomaly appears stationary and (b) the zonal flow on the lower isentrope is zero, so that the anomaly appears to be moving from the left to the right (Hoskins, B.J., M. Pedder and D.W. Jones, 2003: The omega equation and potential vorticity. Q.J.R.Meteorol.Soc., 129, 3277-3303).

Suppose that the coordinate system is chosen such that the shear flow is zero in the neighbourhood of the lower isentrope (figure 9.17b). The components of the vertical motion associated with isentropic upglide in the meridional direction are unchanged. However, the westerly component now gives upglide vertical velocity at the lower isentrope, and larger values of isentropic upglide at the upper isentrope. Of course, the vertical motion is not dependent on the coordinate system. Therefore, there must be an additional part of the full vertical motion associated with the translation of the potential vorticity anomaly in this coordinate system. We refer to this as the "isentropic displacement" vertical motion. As the PV-anomaly moves to the east the isentropes on the eastern side above the anomaly must move down and those below must move up (figure 9.17b). Similarly, on the western side the isentropes must return to their undisturbed level through ascent above and descent below. In this coordinate system it is the sum of the isentropic upglide and the isentropic displacement vertical motions that gives the same vertical motion as the isentropic upglide vertical motion in a frame of reference within which the translation of the anomaly appears stationary.

We have thus split the vertical motion into two components: the isentropic upglide associated with translation of air relative to PV-anomaly and the isentropic displacement associated with translation of a potential vorticity anomaly in a chosen reference frame and with its development (changing intensity). If the frame of reference is chosen such that the potential vorticity anomaly is stationary, the isentropic displacement vertical velocity is zero unless the potential vorticity anomaly is "developing" (changing intensity).


Figure 9.18. Topography of isentropic surfaces associated with eastward moving upper PV-anomalies. The dark lines mark the intersection of the tropopause with the 10 km level, separating air of PV>2 PVU to the north from air of $\mathrm{PV}<2 \mathrm{PVU}$ to the south. Shown also are system relative isentropic up-and downgliding (arrows directed along $\theta$ surfaces) and vertical motion due to induced bulging of the isentropic surface (vertical arrows) when viewed from Earth-relative perspective. The dashed line represents a fixed latitude. Figure due to E.B. Carroll (published in Meteorol.Apl., 10 (2003), p. 285.)

Figure 9.18 gives a three-dimensional view of what is meant by these concepts, where the potential vorticity anomaly pattern consists of a series of positive (associated with troughs in the northern hemisphere) and negative (associated with ridges in the northern hemisphere) anomalies embedded in a meridional potential vorticity gradient. The induced wind field is wavelike. There is upgliding meridional motion on the east side of the trough and downgliding meridional motion on the west side of the trough. The isentropic-displacement-component of the vertical motion is associated with the westeast (zonal) propagation of the series of troughs and ridges and depends on the phase speed of this wavelike pattern relative to the actual zonal velocity component of the air parcels.

Consider a frame of reference moving at some constant horizontal velocity $\left(c_{\mathrm{x}}, c_{\mathrm{y}}\right)$. The potential temperature equation in this frame of reference is as follows.
$\frac{d \theta}{d t}=\frac{\partial \theta}{\partial t}+\left(u-c_{x}\right) \frac{\partial \theta}{\partial x}+\left(v-c_{y}\right) \frac{\partial \theta}{\partial y}+w \frac{\partial \theta}{\partial z}=\frac{J}{\Pi}$

If this equation is divided by $\partial \theta / \partial z$ and taking into account that the slope of the isentrope in, for instance, the $x$-direction is
$\frac{\partial z_{\theta}}{\partial x}=-\frac{\partial \theta / \partial x}{\partial \theta / \partial z}$
( $z_{\theta}$ is the height of the isentropic surface), the following equation for $w$ is obtained.

$$
\begin{equation*}
w=\left(\frac{\partial z_{\theta}}{\partial t}\right)+\left(u-c_{x}\right)\left(\frac{\partial z_{\theta}}{\partial x}\right)+\left(v-c_{y}\right)\left(\frac{\partial z_{\theta}}{\partial y}\right)+\frac{J}{\Pi}\left(\frac{\partial \theta}{\partial z}\right)^{-1} . \tag{9.108}
\end{equation*}
$$

The first term on the right hand side of eq. 9.108 is the isentropic displacement vertical motion. This term is sometimes described as "the vaccuum cleaner effect": in adiabatic conditions, air rises in advance of an approaching upper level positive PV-anomaly, while air sinks in the lee of a retreating positive PV -anomaly. Isentropic upglide/downglide vertical motion (the second and third term in eq. 9.108) may, however, counter the vaccuum cleaner effect. The fourth term in eq. 9.108 represents the contribution of heating or cooling to vertical motion.

## ABSTRACT OF CHAPTER 9

Chapter 9 is concerned with the theory of mid-latitude baroclinic flow. The observed life cycle of a midlatitude baroclinic cyclone is discussed. The quasi-geostrophic approximation is introduced. The quasi-geostrophic vorticity- and thermodynamic equations are derived (eqs. 9.21 and 9.22). These two equations represent a closed set with the geopotential height and the vertical velocity as unknown variables.

Since the vertical velocity is not, in general, measured on the synoptic scale, a diagnostic equation is derived (the omega-equation) (eq. 9.31), which relates the vertical velocity to the measurable quantities, geopotential height and temperature. The solution of this elliptic differential equation gives insight into the physical relation between forced vertical motion in a statically stable atmosphere and frontogenesis (the geostrophic Q -vector). The general rule is that Q -vector-convergence is associated with upward motion. In a trough, which is usually embedded in a baroclinic zone with cold air poleward of warm air, the Q -vector points eastward. In ridge the Q -vector points westward. Thus, according to the solution of the omega equation the upward motion should be observed to the east of the trough and to the west of a ridge, which is indeed very frequently the case in reality. The solution of the omega-equation also shows that frontogenetic forcing of vertical motion is spread out over a vertical distance, which is proportional to the wavelength of the wave and the ratio of the Coriolis parameter to the Brunt-Väisälä frequency, $f / N$, which is called Prandtl's ratio of scales. The typical aspect ratio of large-scale balanced circulation systems, therefore, is in the order of $N / f \approx 100$. This aspect ratio is referred to as "Rossby's ratio".

The quasi-geostrophic approximation is used to formulate a two-layer model of the atmosphere in middle latitudes. The linear stability of a middle latitude zonal flow, in thermal wind balance with a meridional temperature gradient, is analysed using the method of normal modes. The effect of the meridional variation of the Coriolis parameter (the beta-effect) is included. This analysis reveals the existence, in the two-level model, of so-called "barotropic and baroclinic Rossby waves". The amplitude of baroclinic Rossby waves grows exponentially in time if (1) the vertical shear of the zonal wind exceeds a threshold value, (2) the zonal wavelength is larger than a critical value and, (3) the wave tilts westward with increasing height (problem 9.7). The instability of baroclinic Rossby waves is invoked as a theory of middle latitude cyclogenesis.

The pattern of vertical motion in a baroclinic Rossby wave, with upward motion east of the trough and downward motion west of the trough, is understood from two perspectives: (1) the "quasigeostrophic viewpoint" (omega equation and Q-vectors) and (2) the so-called "PV- $\theta$ viewpoint". We will use both viewpoints in chapters 10,11 and 12 to understand the life-cycle of a baroclinic Rossby wave (chapter 10), the interaction between zonal mean flow and waves (chapter 11) and the interaction between adiabatic transport of heat and momentum (or mass and potential vorticity substance) and diabatic processes in the general circulation of the atmosphere (chapter 12).

## Further reading

## Textbooks

Holton, J.R., 2004: An Introduction to Dynamic Meteorology. Academic Press, 529 pp . (chapter 6: quasi-geostrophic equations; chapter 8: two-level model and baroclinic instability)

Lackmann, G., 2011: Midlatitude Synoptic Meteorology. American Meteorological Society. 345 pp. (see chapter 3 on isentropic analysis and chapter 4 on potential vorticity).

## Articles

Carroll, E.B., 2003: Thermal advection, vorticity advection and potential vorticity advection in extratropical synoptic-scale development. Meteorol.Appl., 10, 281-292. (Potential vorticity viewpoint of forcing of vertical motion)

Charney, J.G., 1949: On the physical basis for numerical prediction of large-scale motions in the atmosphere. J.Meteorol., 6, 372-385. (The original article on the quasi-geostrophic approximation)

Clough, S.A., C.S.A. Davitt and A.J. Thorpe, 1996: Attribution concepts applied to the omega equation. Quart.J.Roy.Meteor.Soc., 122, 1943-1962. (An interesting paper that discusses the solution of the omega equation and the idea that frontogenesis induces circulations that "act at a distance", as was also shown in section 8.3 of these lecture notes)
van Delden, A., and R. Neggers, 2003: A case study of tropopause cyclogenesis. Meteorol.Appl., 10, 197-209. (Application of theoretical concepts introduced in this chapter to a case of cyclogenesis at the tropopause)

Hoskins, B., 1997: A potential vorticity view of synoptic development. Meteorol.Appl., 4, 325-334. (Explains the potential vorticity viewpoint of the development of large scale circulation systems with a minimum of mathematics: recommended literature)

Davies, H.C., 1997: Emergence of the mainstream cyclogenesis theories. Meteorol.Zeitschrift, N.F. 6, 261-274. (This article gives an interesting account of the history of cyclogenesis theories)

Hoskins, B.J., I. Draghici and H.C. Davies, 1978: A new look at the omega-equation. Quart.J.Roy.Meteor.Soc., 104, 31-38. (The original paper that derived the most useful version of the omega equation, which shows the connection between frontogenesis and vertical motion)

Pedder, M.A., 1997: The omega equation: Q-G interpretations of simple circulation features. Meteorol Appl., 4, 335-344. (Compares different quasi-geostrophic versions of the omega equation: recommended literature)

Semple, A.T., 2003: A review and unification of conceptual models of cyclogenesis. Meteorol.Appl., 10, 39-59.

## List of problems (chapter 9)

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"Le soleil couchant" by Claude Monet, North Carolina Museum of Art. High layered clouds hint at an approaching warm front at sunset in Etretat (Normandy, France).

This is the December 2016 edition of chapter 9 of the lecture notes on Atmospheric Dynamics, written by Aarnout van Delden (IMAU, Utrecht University, Netherlands, http://www.staff.science.uu.nl/~delde102/AtmosphericDynamics.htm .


[^0]:    74 GFS stands for "Global Forecasting System", which is the numerical forcasting system of the National Center for Environmental Prediction in the United States (http://www.nco.ncep.noaa.gov/pmb/products/).

[^1]:    75 The analysis of the surface pressure has been performed by computer using "optimal" linear interpolation. The interpolation is done on a grid with a grid distance of $0.5^{\circ}$ in both horizontal directions. For each gridpoint the nearest nine stations, giving "reasonable" measurements, are found. All possible permutations of three stations are formed. This yields 84 overlapping station-triangles. If a triangle contains an angle smaller than $25^{\circ}$ it is discarded, yielding

[^2]:    $N \leq 84$ station triangles. This yields $N$ possible values of the pressure at the gridpoint. From a histogram of these values one "optimal" value is selected (see elsewhere for the details). The resulting analysis is smoothed with 9 point smoother (see Haltiner, G.J., and R.T. Williams, 1980: Numerical Prediction and Dynamic Meteorology. Second edition. John Wiley \& Sons. 477 pp., p. 397)

[^3]:    76 A.T. Semple, 2003: A review and unification of conceptual models of cyclogenesis. Meteorol.Appl., 10, 39-59.

[^4]:    77 Hoskins, B.J., I. Draghici and H.C. Davies, 1978: A new look at the omega-equation. Quart.J.Roy.Meteor.Soc., 104, 31-38.
    Hoskins and M.A. Pedder, 1980: The diagnosis of middle latitude synoptic development. Quart.J.Roy.Meteor.Soc., 106, 707-719.

[^5]:    ${ }^{78}$ For those readers, which have studied chapter 8 : section 8.3 discusses a situation in which $\partial T / \partial x=0, \partial u g / \partial x=A$ and $\partial v \mathrm{~g} / \partial y=-A$ and $\partial v \mathrm{~g} / \partial x=0$ (see eq. 8.23 ). The $Q$-vector in that case is
    $\vec{Q}_{g} \equiv\left(Q_{g 1}, Q_{g 2}\right)=\left(0, A \frac{\partial T}{\partial y}\right)$.

    Thus, we see that the forcing term on the r.h.s. of equation (8.32), describing the vertical circulation perpendicular to a front aligned in the $x$-direction, is in fact the $y$-component of the geostrophic $Q$-vector. This is not surprising because ( 8.32 ) describes the vertical circulation, which is needed to preserve thermal wind balance in the presence of a geostrophic wind field, which acts frontogenetically and, therefore, continuously acts to destroy thermal wind balance.

[^6]:    79 Phillips, N.A., 1954: Energy transformations and meridional circulations associated with simple baroclinic waves in a two-level, quasi-geostrophic model. Tellus, 6, 272-286.

[^7]:    ${ }^{80}$ See the sounding made at Brest and Camborne (http://weather.uwyo.edu/upperair/sounding.html).

