

Lecture 1 SOAC

Introduction to Exercise 1

Canonical equation of Fluid Dynamics

(Navier-Stokes equation)

Canonical equation of Geophysical Fluid Dynamics (GFD)

(shallow water equations)

Simplification of the shallow water equations

Inertial Oscillations

Analytical solution

Numerical approximation of the time-derivative

Comparison of analytical and numerical solution

Recommended book:

In Pursuit of the Unknown

17 Equations That
Changed the World



Ian Stewart

10 The ascent of humanity

Navier–Stokes Equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$$

density velocity pressure stress body forces

time derivative dot product gradient divergence

add *Coriolis force* to r.h.s.

Advection: exercise 2

What does it say?

It's Newton's second law of motion in disguise. The left-hand side is the acceleration of a small region of fluid. The right-hand side is the forces that act on it: pressure, stress, and internal body forces.

Why is that important?

It provides a really accurate way to calculate how fluids move. This is a key feature of innumerable scientific and technological problems.

Equation of motion for a fluid parcel

spatial coordinate
↓
 $u(x, t)$
↑
x-component velocity
time ↗

Inviscid equation of motion (Euler equation*)

acceleration due to pressure gradient due to Coriolis effect

↓ ↓ ↓

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv$$

density velocity perpendicular to u

Coriolis parameter:
 $f \equiv 2\Omega \sin \phi$

* [https://en.wikipedia.org/wiki/Euler_equations_\(fluid_dynamics\)](https://en.wikipedia.org/wiki/Euler_equations_(fluid_dynamics))

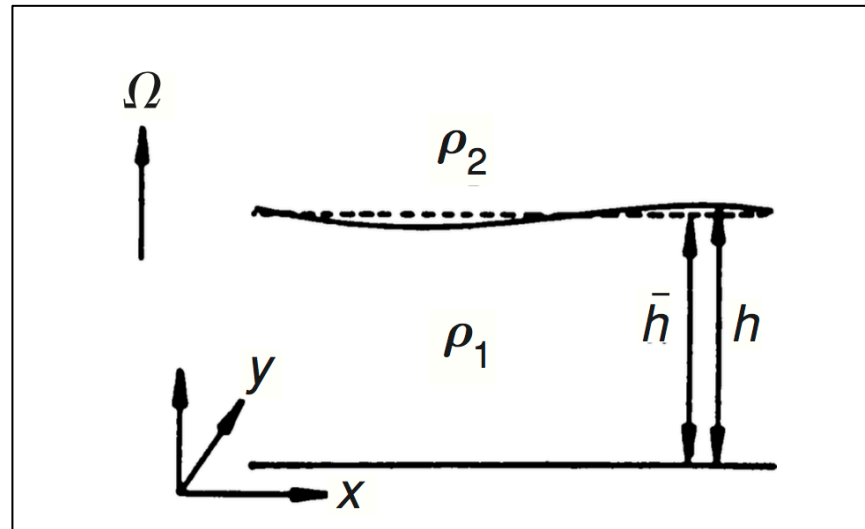
Canonical equation of GFD:

Shallow water equations

Rotating stratified layer of fluid

$$\begin{aligned}\frac{du}{dt} &= -g' \frac{\partial h}{\partial x} + fv; \\ \frac{dv}{dt} &= -g' \frac{\partial h}{\partial y} - fu; \\ \frac{dh}{dt} &= -h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)\end{aligned}$$

↑
Last equation: **mass conservation**



“Reduced gravity:” $\left(g' = \frac{\rho_1 - \rho_2}{\rho_1} g \right)$

$f = \text{constant}$

Canonical equation of GFD:

Shallow water equations: advection terms

Rotating stratified layer of fluid

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -g' \frac{\partial h}{\partial x} + fv; \\ \frac{dv}{dt} &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -g' \frac{\partial h}{\partial y} - fu; \\ \frac{dh}{dt} &= \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} = -h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)\end{aligned}$$

↑
advection term: non-linear !

Canonical equation of GFD:

Shallow water equations in flux form

For simplicity neglect variations in the y-direction

$$\begin{aligned}\frac{\partial hu}{\partial t} &= -\frac{\partial hu^2}{\partial x} - g'h \frac{\partial h}{\partial x} + fhv; \\ \frac{\partial hv}{\partial t} &= -\frac{\partial huv}{\partial x} - fhu; \\ \frac{\partial h}{\partial t} &= -\frac{\partial hu}{\partial x}\end{aligned}$$

In short:

$$\begin{aligned}\frac{\partial U}{\partial t} &= -\frac{\partial uU}{\partial x} - g'h \frac{\partial h}{\partial x} + fV; \\ \frac{\partial V}{\partial t} &= -\frac{\partial vU}{\partial x} - fU; \\ \frac{\partial h}{\partial t} &= -\frac{\partial U}{\partial x}\end{aligned}$$

$$U \equiv hu; \quad V \equiv hv$$

Canonical equation of GFD:

Shallow water equations: linearised

For simplicity assume that $h = H + h'$; $h' \ll H$

$\frac{\partial u^2}{\partial x}$ and $\frac{\partial uv}{\partial x}$ are small compared to linear terms

$$\begin{aligned}\frac{\partial hu}{\partial t} &= -\frac{\partial hu^2}{\partial x} - g'h \frac{\partial h}{\partial x} + fhv; \\ \frac{\partial hv}{\partial t} &= -\frac{\partial huv}{\partial x} - fhu; \\ \frac{\partial h}{\partial t} &= -\frac{\partial hu}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial t} &\approx -g' \frac{\partial h'}{\partial x} + fv; \\ \frac{\partial v}{\partial t} &\approx -fu; \\ \frac{\partial h}{\partial t} &\approx -H \frac{\partial u}{\partial x}\end{aligned}$$

We'll come back to these equations in **lecture 4**

Analytical solution in a simplified case

(neglecting variations in h)

Simplified equations: $\left\{ \begin{array}{l} \frac{\partial u(t)}{\partial t} = fv(t) \\ \frac{\partial v(t)}{\partial t} = -fu(t) \end{array} \right.$

Analytical solution:

$$\frac{\partial^2 u}{\partial t^2} = f \frac{\partial v}{\partial t} = -f^2 u \longrightarrow \frac{\partial^2 u}{\partial t^2} + f^2 u = 0$$

solution

$$u(t) = C_1 \sin(ft) + C_2 \cos(ft)$$

(inertial oscillations)

$$v(t) = C_1 \cos(ft) - C_2 \sin(ft)$$

If $u(0)=10$ m/s and $v(0)=0$ m/s, then $C_1=0$ and $C_2=10$ m/s.

$$u(t) = 10 \cos(ft) \quad v(t) = -10 \sin(ft)$$

Numerical approximation: time difference-scheme

Example equation:

$$\frac{\partial u(t)}{\partial t} = fv(t)$$

Taylor series approximation to time-derivative (left hand side):

$$u(t + \Delta t) = u(t) + \Delta t \frac{\partial u}{\partial t}(t) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(t) + O(\Delta t^3)$$

divide by Δt



$$\frac{u(t + \Delta t) - u(t)}{\Delta t} - \frac{\partial u}{\partial t}(t) = \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2}(t) + O(\Delta t^2)$$



“Euler-forward” finite difference
approximation



Error: “first order” (linear) in Δt

Numerical approximation

Equations: $\left\{ \begin{array}{l} \frac{\partial u(t)}{\partial t} = fv(t) \\ \frac{\partial v(t)}{\partial t} = -fu(t) \end{array} \right.$

Numerical approximation with Euler forward scheme:

time-step $\xrightarrow{\Delta t}$ $\frac{u(t + \Delta t) - u(t)}{\Delta t} = fv(t)$

$$\frac{v(t + \Delta t) - v(t)}{\Delta t} = -fu(t)$$

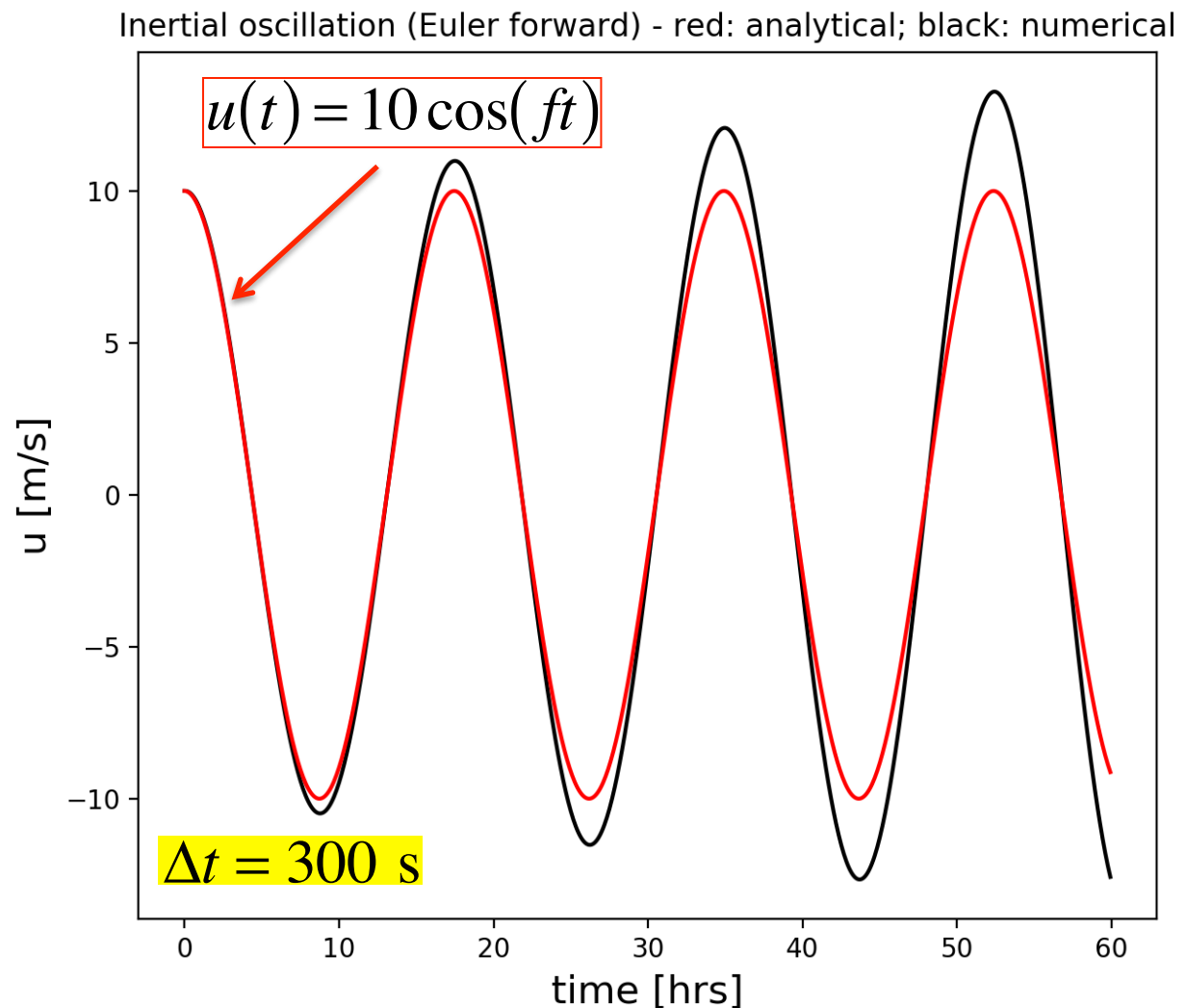
Error: “first order” in Δt

Comparison numerical approximation with analytical solution: numerical stability

Numerical solution with Euler- forward time-stepping scheme

Amplitude of the oscillation *increases* while it should be constant. This is a manifestation of **numerical instability** !

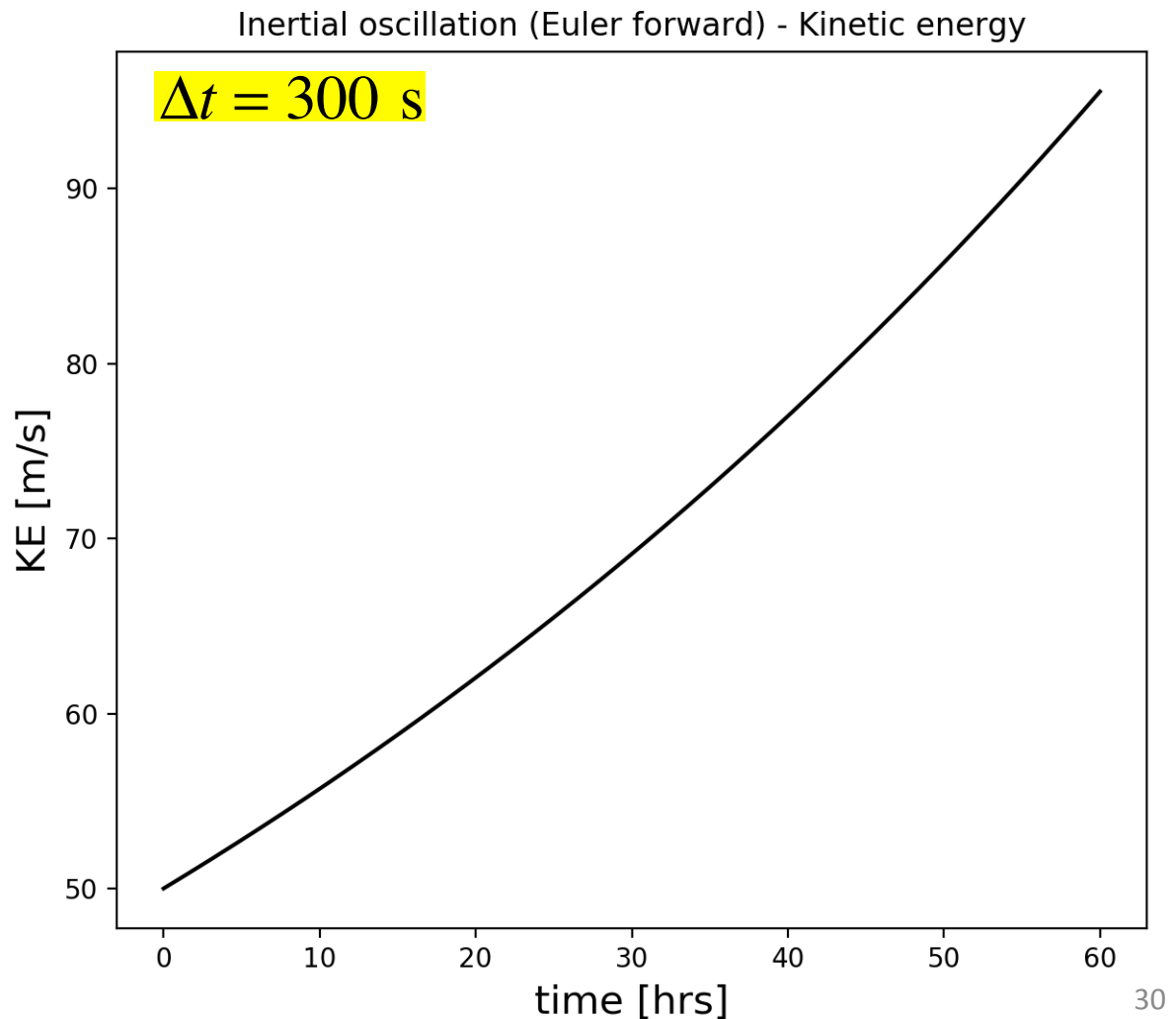
Note: numerical solution is perfectly correlated with analytical solution, but still in error!



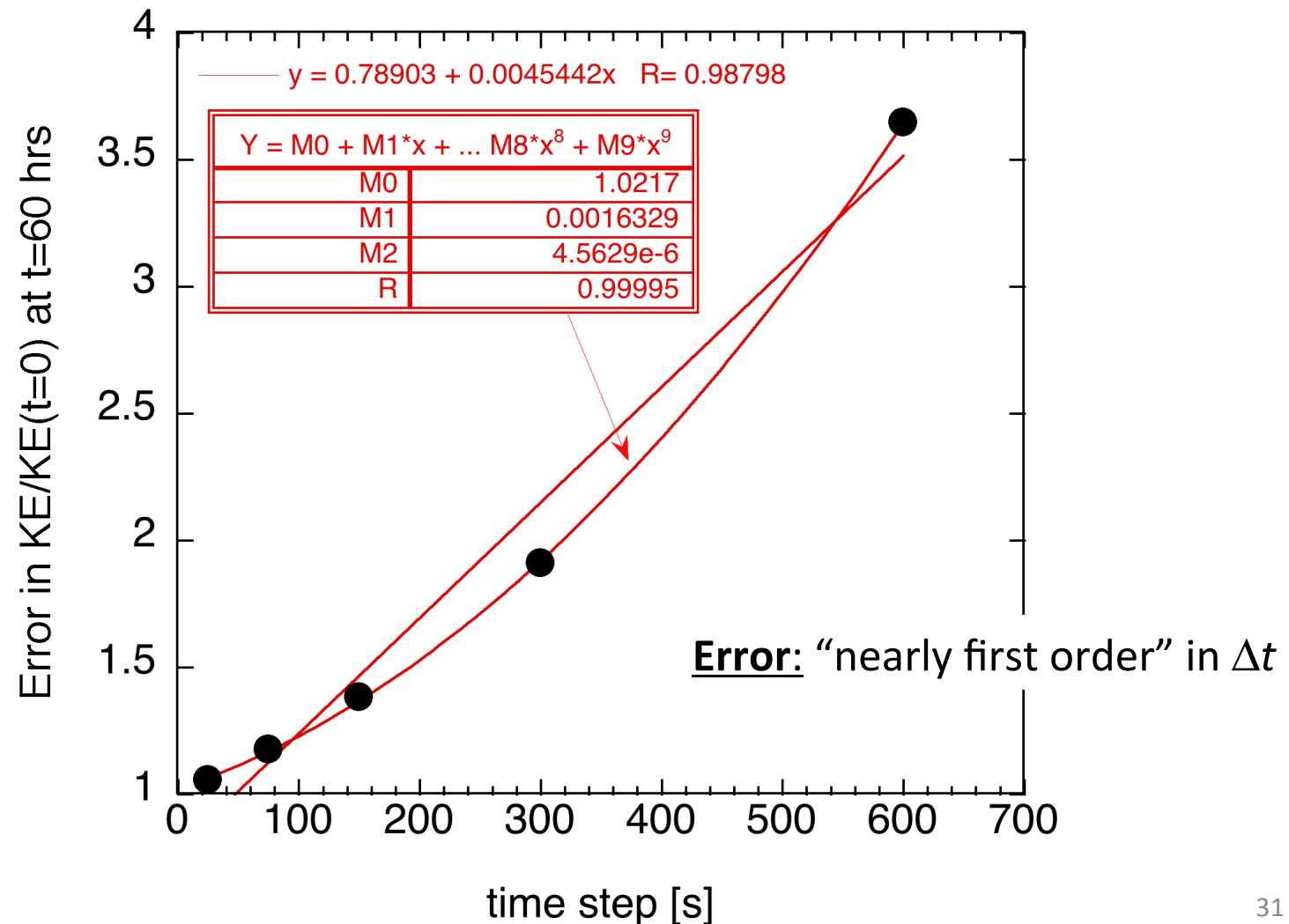
Conservation of kinetic energy, KE ? No!

$$KE(t=0) \equiv \frac{1}{2} \left(u(0)^2 + v(0)^2 \right) = 50 \text{ m}^2\text{s}^{-2}$$

$KE(t=60 \text{ hr})$ increases in time. This is due to the **truncation error** and



Truncation error Euler forward scheme



Python script numerical solution (forward Euler scheme)

```
1  import numpy as N
2  import matplotlib.pyplot as P
3  import math as M
4
5  nt_len = 720
6  dt = 300.0
7
8  fCor = 0.0001
9
10 u_ana = N.zeros((nt_len))
11 u = N.zeros((nt_len))
12 v = N.zeros((nt_len))
13 time = N.zeros((nt_len))
14
15 # time loop
16
17 u[0]= 10.0
18 u_ana[0]= 10.0
19 for nt in range(1,nt_len):
20     time[nt] = time[nt-1] + dt
21     u[nt] = u[nt-1] + (fCor * v[nt-1] * dt)
22     v[nt] = v[nt-1] - (fCor * u[nt-1] * dt)
23     u_ana[nt] = u[0] * M.cos(fCor * time[nt])
24
25 # make graph
26 P.figure(figsize=(7,6))
27
28 P.xlabel('time [hrs]',fontsize=15)
29 P.ylabel('u [m/s]',fontsize=15)
30 P.title('Inertial oscillation (Euler forward) - red: analytical; black: numerical')
31 P.plot(time/3600, u, color='black')
32 P.plot(time/3600, u_ana, color='red')
33 P.show() # show plot on screen
```

Next lecture

More about time-stepping schemes and their truncation error and numerical stability.

The linear advection equation: analytical solution

Numerical approximation of the linear advection equation:
Truncation error due to numerical diffusion and numerical dispersion.

Exercise 1

Introduction to exercise 2