

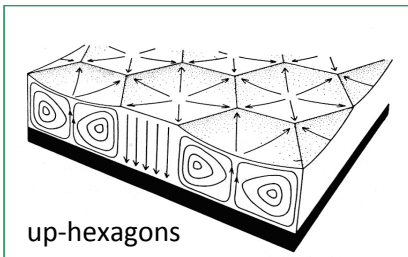
Thermal convection of a stratified fluid under the influence of gravity

Lectures 10, course on “Turbulence in Fluids”, Utrecht University

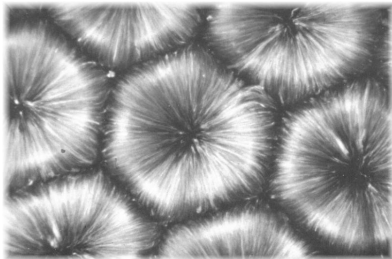
<http://www.staff.science.uu.nl/~delde102/TinF.html>

16 March 2022

Part 5: Pattern formation in thermal convection



Hexagonal cells in the laboratory



The horizontal pattern of thermal convection

Fourier representation of certain patterns in three spatial dimensions

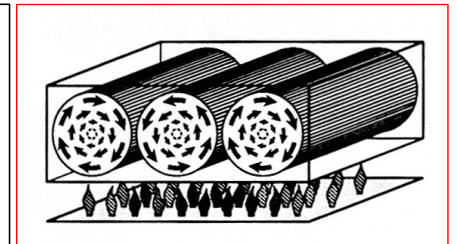
The equations for governing three-dimensional convection

Three-dimensional “poloidal flow” in spectral space

A low-order model of poloidal convection: plan-form selection

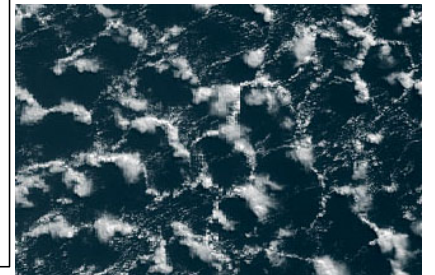
An experiment with the low-order model (ensemble prediction)

Exercises / about the exam...



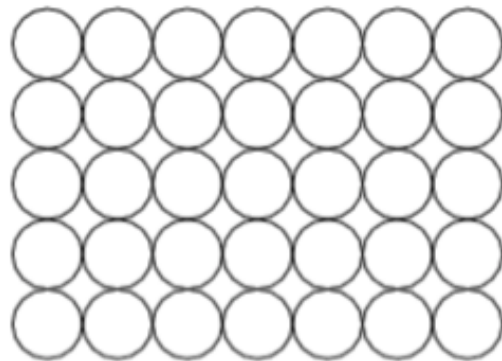
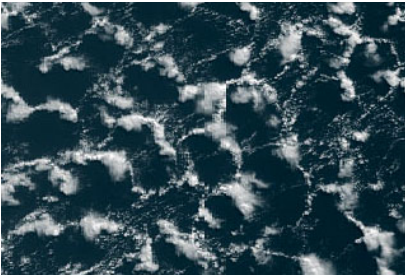
rolls

Open cells in the atmosphere

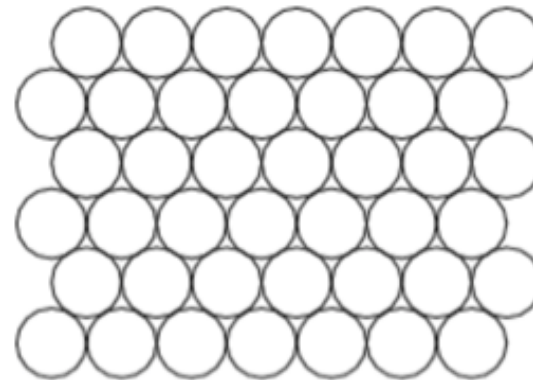


Closest packing of circles = updraughts

Open cells in the atmosphere

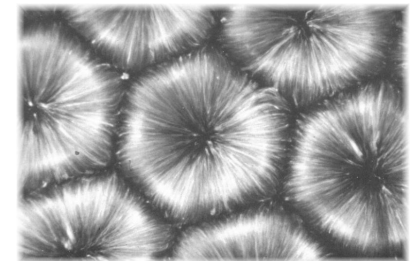


square packing



hexagonal packing

Hexagonal cells in the laboratory



An arrangement of circles inside a given boundary such that no two circles overlap and all circles are mutually tangent. The hexagonal packing is closest (most efficient) because the area covered by circles divided by the total area is a minimum ($=\pi/(2\sqrt{3})=0.91$). This ratio in the case of square packing ($\pi/4=0.79$).

The hexagonal packing is most “efficient”. But space must be made for the downward motion to satisfy continuity

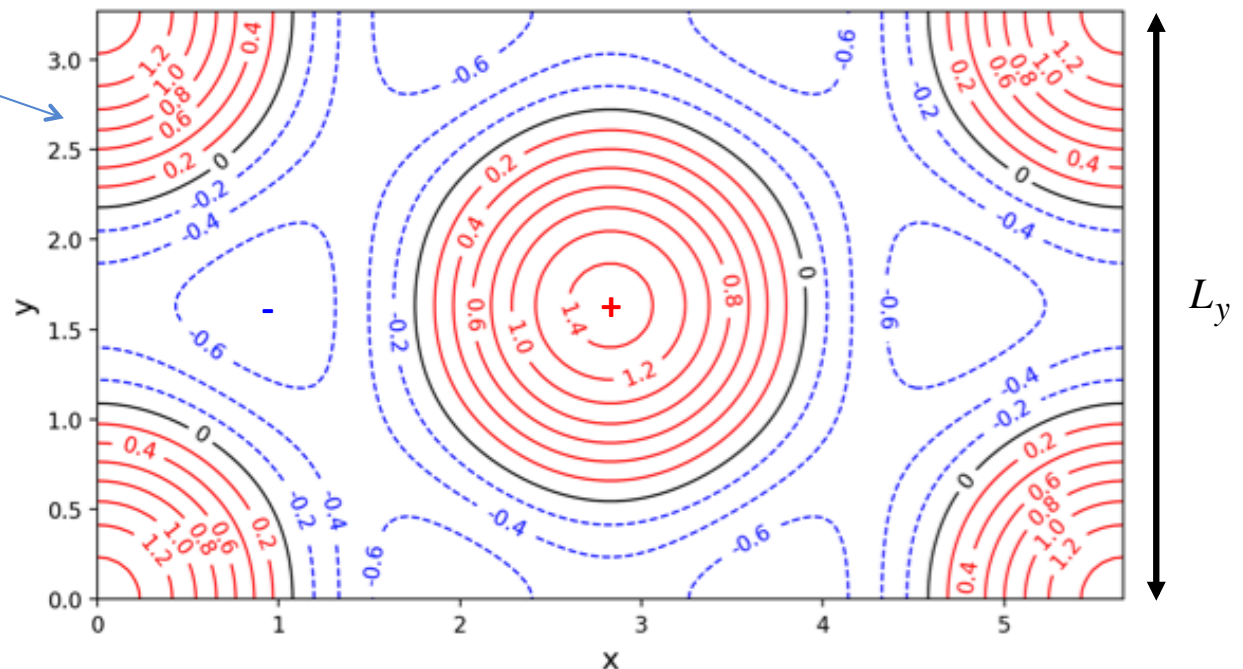
Constructing a hexagonal pattern taking continuity into account

Vertical velocity: $w(x,y) = \cos \pi a_x l x \cos \pi a_y m y + \frac{1}{2} \cos 2\pi a_x l x$ (eq. 116a)

$$a_y \equiv \frac{2H}{L_y} = \sqrt{3}a_x; \quad a_x \equiv \frac{2H}{L_x} = \frac{1}{2\sqrt{2}} \longrightarrow L_x = 4\sqrt{2}H = \sqrt{3}L_y$$

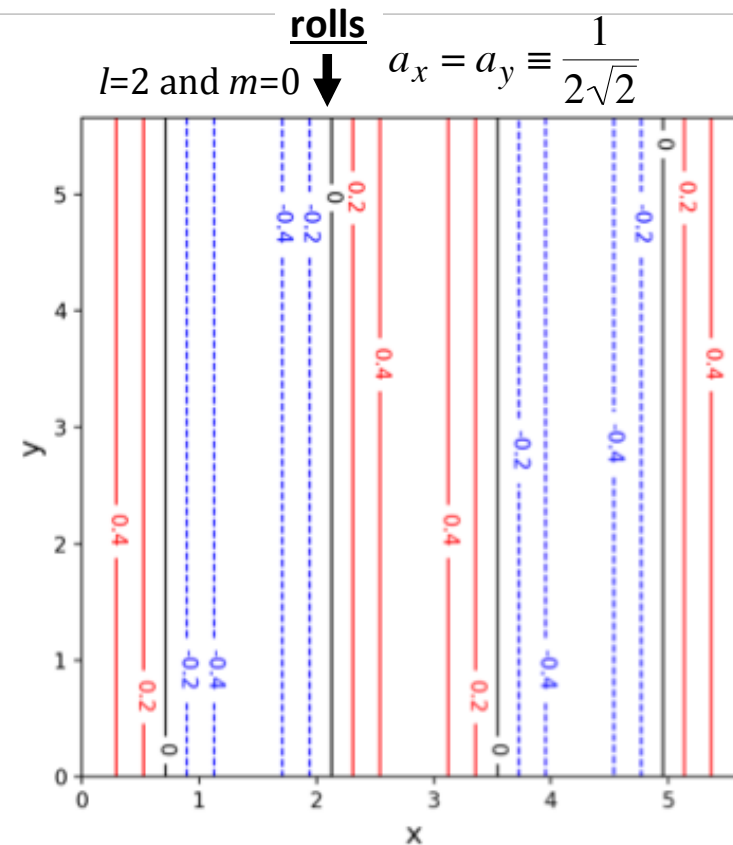
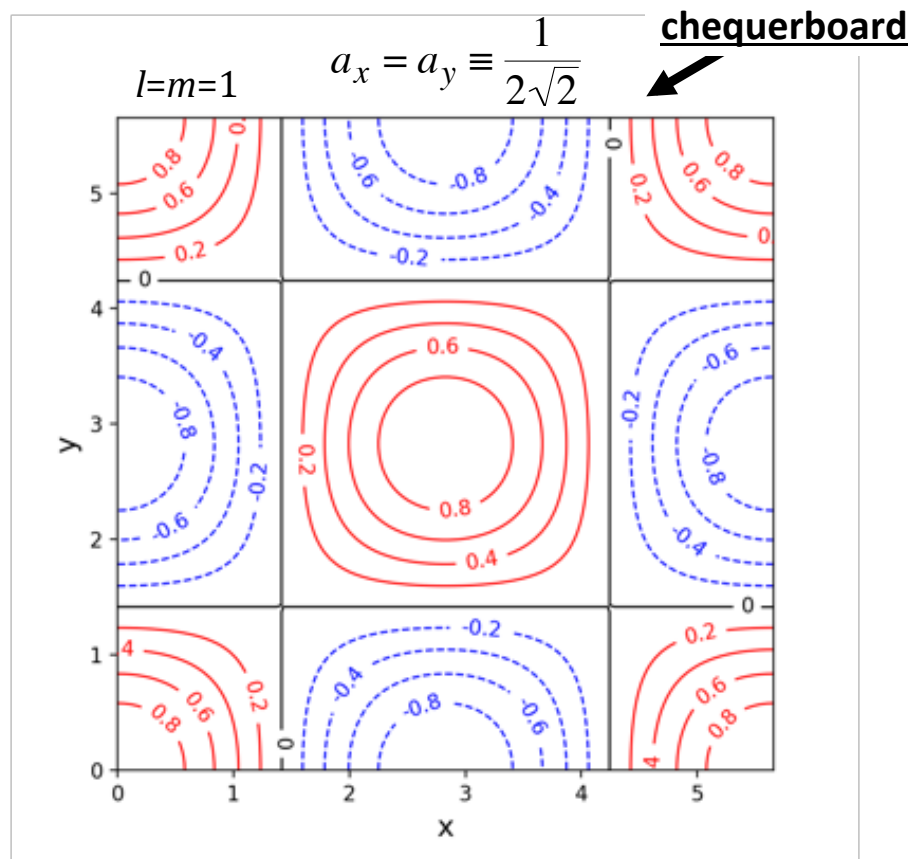
Up-hexagon, or “l-cell”

The hexagonal pattern of vertical velocity, w , according to eq. 116, with $l=m=1$, and a plus sign in front of the second term $a_y = \sqrt{3}a_x$. The upper panel shows an up-hexagon (eq. 116a). The area covered by circular updraught divided by the total area is equal to 0.4 (see figure 18, lecture notes).



Hexagonal pattern = superposition of two patterns

Vertical velocity: $w(x,y) = \cos \pi a_x l x \cos \pi a_y m y + \frac{1}{2} \cos 2\pi a_x l x$ (eq. 116a)



Partial differential equations governing 3D convection

Eliminate pressure from the governing equations following the procedure of lecture 6:

4 equations, 4 unknowns (u , v , w and ζ)

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Vertical comp. of vorticity

$$\frac{d\zeta}{dt} = \zeta \frac{\partial w}{\partial z} - \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} + \text{Pr} \nabla^2 \zeta$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vertical velocity

$$\begin{aligned} \frac{d\nabla^2 w}{dt} = & \left(2 \frac{\partial^2 u}{\partial z^2} - 2 \frac{\partial^2 u}{\partial x \partial z} - \nabla^2 u \right) \frac{\partial w}{\partial x} + \left(2 \frac{\partial^2 v}{\partial z^2} - 2 \frac{\partial^2 v}{\partial x \partial z} - \nabla^2 v \right) \frac{\partial w}{\partial y} - \nabla^2 w \frac{\partial w}{\partial z} \\ & + 2 \left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 w}{\partial x \partial y} \right) \frac{\partial v}{\partial x} + 2 \left(\frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial v}{\partial y} \\ & + 2 \left(\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial u}{\partial x} + 2 \left(\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 w}{\partial x \partial y} \right) \frac{\partial u}{\partial y} + \text{Pr} \nabla_h^2 \theta + \text{Pr} \nabla^4 w \end{aligned}$$

Fourier transformation of the equations

Diagnostic equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Fourier series:

$$\begin{Bmatrix} u \\ v \\ w \\ \zeta \\ \theta \end{Bmatrix} = \sum_{\alpha} \begin{Bmatrix} U_{\alpha}(t) \\ V_{\alpha}(t) \\ W_{\alpha}(t) \\ Z_{\alpha}(t) \\ \Theta_{\alpha}(t) \end{Bmatrix} S_{\alpha}(x, y, z)$$

These linear equations transform into

$$U_{\alpha} = -\frac{a_x n_{\alpha} l_{\alpha}}{q_{\alpha}^2} W_{\alpha} + \frac{a_y m_{\alpha}}{\pi q_{\alpha}^2} i Z_{\alpha}$$

$$V_{\alpha} = -\frac{a_y n_{\alpha} m_{\alpha}}{q_{\alpha}^2} W_{\alpha} - \frac{a_x l_{\alpha}}{\pi q_{\alpha}^2} i Z_{\alpha}$$

Basis function:

$$S_{\alpha}(x, y, z) = \exp\{i\pi(a_x l_{\alpha} x + a_y m_{\alpha} y + n_{\alpha} z)\}$$

Domain:

$$0 \leq x \leq \frac{L_x}{H}; \quad 0 \leq y \leq \frac{L_y}{H}$$

$$a_x = \frac{2H}{L_x}; \quad a_y = \frac{2H}{L_y}$$

Orthonormality:

$$\int S_{\alpha} S_{\beta}^* d\sigma = \delta_{\alpha, \beta}$$

Poloidal 3D flow: vorticity in horizontal plane=zero

Diagnostic relations in spectral space

$$U_{\alpha} = -\frac{a_x n_{\alpha} l_{\alpha}}{q_{\alpha}^2} W_{\alpha} + \frac{a_y m_{\alpha}}{\pi q_{\alpha}^2} i Z_{\alpha}$$

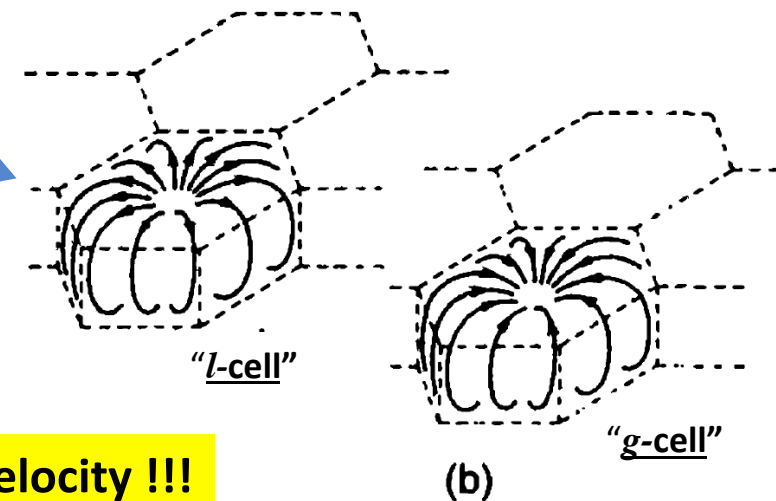
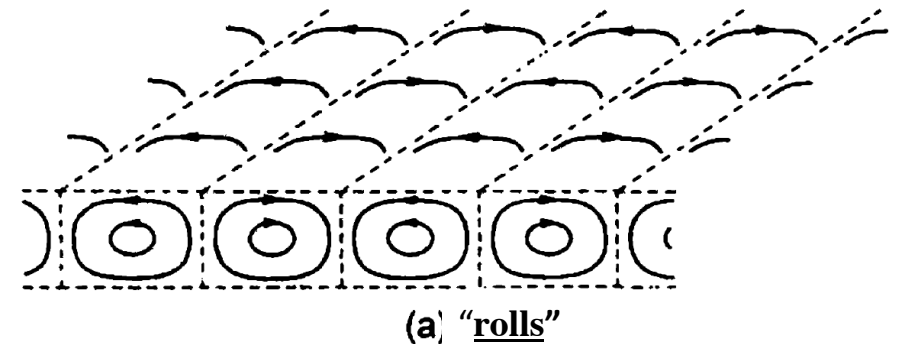
$$V_{\alpha} = -\frac{a_y n_{\alpha} m_{\alpha}}{q_{\alpha}^2} W_{\alpha} - \frac{a_x l_{\alpha}}{\pi q_{\alpha}^2} i Z_{\alpha}$$

Simplify by assuming that the flow is “**poloidal**”:

$$Z_{\alpha} = 0$$

$$U_{\alpha} = -\frac{a_x n_{\alpha} l_{\alpha}}{q_{\alpha}^2} W_{\alpha}$$

$$V_{\alpha} = -\frac{a_y n_{\alpha} m_{\alpha}}{q_{\alpha}^2} W_{\alpha}$$



Velocity is completely determined by vertical velocity !!!

Spectral form of the vertical velocity equation

$$\frac{dW_\gamma}{dt} = \sum_{\alpha,\beta} \left\{ \pi N_{\gamma\alpha\beta} i W_\alpha W_\beta \right\} + \text{Pr} \frac{\pi^2 q_\gamma^2}{k_\gamma^2} \Theta_\gamma - \text{Pr} k_\gamma^2 W_\gamma$$



Interaction coefficient: function of wave-numbers and aspect ratios
Contains the selection rule.

Total wave-numbers $q_\gamma^2 = a_x^2 l_\gamma^2 + a_y^2 m_\gamma^2$ $k_\gamma^2 = \pi^2 (q_\gamma^2 + n_\gamma^2)$

Spectral form of the temperature equation

$$\frac{d\Theta_\gamma}{dt} = \sum_{\alpha,\beta} \left\{ \pi L_{\gamma\alpha\beta} i W_\alpha \Theta_\beta \right\} + \text{Ra} W_\gamma - k_\gamma^2 \Theta_\gamma$$



Interaction coefficient: function of wave-numbers and aspect ratios
Contains the selection rule.

Closed system is spectral space

$$\frac{d\Theta_\gamma}{dt} = \sum_{\alpha,\beta} \left\{ \pi L_{\gamma\alpha\beta} i W_\alpha \Theta_\beta \right\} + \text{Ra} W_\gamma - k_\gamma^2 \Theta_\gamma$$

$$\frac{dW_\gamma}{dt} = \sum_{\alpha,\beta} \left\{ \pi N_{\gamma\alpha\beta} i W_\alpha W_\beta \right\} + \text{Pr} \frac{\pi^2 q_\gamma^2}{k_\gamma^2} \Theta_\gamma - \text{Pr} k_\gamma^2 W_\gamma$$

Impose boundary conditions: as in previous lectures -> symmetry conditions on Fourier coefficients

Truncate the Fourier series: include the fundamental wave vectors, (1,1,1) and (0,2,1), which make a hexagon

Interaction of these modes will create (1,1,2) and (0,2,2) and, of course, higher harmonics.

We start by deriving a low-order model, which includes only these 4 modes for w .

For temperature we add (0,0,1), (0,0,2), (0,0,3) and (0,0,4)

Table showing all interactions inside truncation

Class 1 interactions			Class 2 interactions			Class 3 interactions		
γ	α	β	γ	α	β	γ	α	β
(0, 0, 1)	(1, 1, -1)	(-1, -1, 2)	(1, 1, 1)	(1, 1, 2)	(0, 0, -1)	(0, 2, 1)	(1, 1, -1)	(-1, 1, 2)
(0, 0, 1)	(-1, -1, -1)	(1, 1, 2)	(1, 1, 1)	(1, 1, -1)	(0, 0, 2)	(0, 2, 1)	(-1, 1, -1)	(1, 1, 2)
(0, 0, 1)	(-1, 1, -1)	(1, -1, 2)	(1, 1, 1)	(1, 1, -2)	(0, 0, 3)	(1, 1, 1)	(1, -1, -1)	(0, 2, 2)
(0, 0, 1)	(1, -1, -1)	(-1, 1, 2)	(0, 2, 1)	(0, 2, 2)	(0, 0, -1)	(1, 1, 2)	(1, -1, 1)	(0, 2, 1)
(0, 0, 2)	(1, 1, 1)	(-1, -1, 1)	(0, 2, 1)	(0, 2, -1)	(0, 0, 2)	(0, 2, 2)	(-1, 1, 1)	(1, 1, 1)
(0, 0, 2)	(-1, 1, 1)	(1, -1, 1)	(0, 2, 1)	(0, 2, -2)	(0, 0, 3)	(1, 1, 1)	(1, -1, 2)	(0, 2, -1)
(0, 0, 3)	(1, 1, 1)	(-1, -1, 2)	(1, 1, 2)	(1, 1, 1)	(0, 0, 1)			
(0, 0, 3)	(-1, -1, 1)	(1, 1, 2)	(1, 1, 2)	(1, 1, -1)	(0, 0, 3)			
(0, 0, 3)	(-1, 1, 1)	(1, -1, 2)	(1, 1, 2)	(1, 1, -2)	(0, 0, 4)			
(0, 0, 3)	(1, -1, 1)	(-1, 1, 2)	(0, 2, 2)	(0, 2, 1)	(0, 0, 1)			
(0, 0, 4)	(1, 1, 2)	(-1, -1, 2)	(0, 2, 2)	(0, 2, -1)	(0, 0, 3)			
(0, 0, 4)	(-1, 1, 2)	(1, -1, 2)	(0, 2, 2)	(0, 2, -2)	(0, 0, 4)			
(0, 0, 1)	(0, 2, -1)	(0, -2, 2)						
(0, 0, 1)	(0, -2, -1)	(0, 2, 2)						
(0, 0, 2)	(0, 2, 1)	(0, -2, 1)						
(0, 0, 3)	(0, 2, 1)	(0, -2, 2)						
(0, 0, 3)	(0, -2, 1)	(0, 2, 2)						
(0, 0, 4)	(0, 2, 2)	(0, -2, 2)						

Low-order model of poloidal convection (12 equations)

temperature
vert. velocity

$$\begin{aligned}
 \frac{d\Theta_{111}}{dt} &= -\frac{3}{2}\pi W_{111}\Theta_{022} - \frac{3}{2}\pi W_{021}\Theta_{112} + \pi\Theta_{001}W_{112} - 2\pi\Theta_{002}W_{111} - 3\pi\Theta_{003}W_{112} + \text{Ra}W_{111} - \pi^2(4a^2 + 1)\Theta_{111} \\
 \frac{d\Theta_{021}}{dt} &= -3\pi W_{111}\Theta_{112} + \pi\Theta_{001}W_{022} - 2\pi\Theta_{002}W_{021} - 3\pi\Theta_{003}W_{022} + \text{Ra}W_{021} - \pi^2(4a^2 + 1)\Theta_{021} \\
 \frac{d\Theta_{112}}{dt} &= \frac{3}{2}\pi W_{021}\Theta_{111} + \frac{3}{2}\pi W_{111}\Theta_{021} + \pi\Theta_{001}W_{111} - 3\pi\Theta_{003}W_{111} - 4\pi\Theta_{004}W_{112} + \text{Ra}W_{112} - \pi^2(4a^2 + 4)\Theta_{112} \\
 \frac{d\Theta_{022}}{dt} &= 3\pi W_{111}\Theta_{111} + \pi\Theta_{001}W_{021} - 3\pi\Theta_{003}W_{021} - 4\pi\Theta_{004}W_{022} + \text{Ra}W_{022} - \pi^2(4a^2 + 4)\Theta_{022} \\
 \frac{d\Theta_{001}}{dt} &= -4\pi W_{111}\Theta_{112} - 4\pi W_{112}\Theta_{111} - 2\pi W_{021}\Theta_{022} - 2\pi W_{022}\Theta_{021} - \pi^2(\Theta_{001} - \bar{\Theta}_{001}) \\
 \frac{d\Theta_{002}}{dt} &= 8\pi W_{111}\Theta_{111} + 4\pi W_{021}\Theta_{021} - 4\pi^2\Theta_{002} \\
 \frac{d\Theta_{003}}{dt} &= 12\pi W_{111}\Theta_{112} + 12\pi W_{112}\Theta_{111} + 6\pi W_{021}\Theta_{122} + 6\pi W_{022}\Theta_{021} - 9\pi^2\Theta_{003} \\
 \frac{d\Theta_{004}}{dt} &= 16\pi W_{112}\Theta_{112} + 8\pi W_{022}\Theta_{022} - 16\pi^2\Theta_{004} \\
 \frac{dW_{111}}{dt} &= -\frac{3}{2}\pi W_{022}W_{111} - \frac{3}{2}\pi W_{021}W_{112} + \frac{4a^2 \text{Pr}}{4a^2 + 1}\Theta_{111} - \pi^2 \text{Pr}(4a^2 + 1)W_{111} \\
 \frac{dW_{021}}{dt} &= -3\pi W_{112}W_{111} + \frac{4a^2 \text{Pr}}{4a^2 + 1}\Theta_{021} - \pi^2 \text{Pr}(4a^2 + 1)W_{021} \\
 \frac{dW_{112}}{dt} &= 3\pi \frac{4a^2 + 1}{4a^2 + 4}W_{111}W_{021} + \frac{4a^2 \text{Pr}}{4a^2 + 4}\Theta_{112} - \pi^2 \text{Pr}(4a^2 + 4)W_{112} \\
 \frac{dW_{022}}{dt} &= 3\pi \frac{4a^2 + 1}{4a^2 + 4}W_{111}^2 + \frac{4a^2 \text{Pr}}{4a^2 + 4}\Theta_{022} - \pi^2 \text{Pr}(4a^2 + 4)W_{022}
 \end{aligned}
 \tag{163}$$

Hexagons if:

$$\Theta_{111} = \Theta_{021}; \Theta_{112} = \Theta_{022};$$

$$W_{111} = W_{021}; W_{112} = W_{022}.$$

Substitute into equations:

$$\frac{d\Theta_{111}}{dt} = \frac{d\Theta_{021}}{dt}; \frac{d\Theta_{112}}{dt} = \frac{d\Theta_{022}}{dt};$$

$$\frac{dW_{111}}{dt} = \frac{dW_{021}}{dt}; \frac{dW_{112}}{dt} = \frac{dW_{022}}{dt}.$$

$$a_x^2 = 3a_y^2 = 3a^2$$

```
# LOW-ORDER MODEL OF POLOIDAL CONVECTION
# (SECTION 36 of Lecture notes on Thermal Convection for "Turbulence in Fluids")
# Numerical integration of equation-set 163 (lecture notes)
# Skeleton version of the script (no graphics)
```

```
# IMPORT MODULES
import numpy as np
import matplotlib.pyplot as plt
import math as M
from numpy import random
```

Model code*

PoloidalConvectionModel[vrs1].py
(see Blackboard)

```
#-----
# FUNCTION THAT COMPUTES TIME-DERIVATIVES

def time_derivative(W111_,W112_,W021_,W022_,T111_,T112_,T021_,T022_,T001_,T002_,T003_,T004_,T_MEAN_001,Ra,Pr,a):
# SHORT NOTATION OF TOTAL WAVE NUMBERS (eqs. 142 and 143)
pi = M.pi
pi2 = M.pow(pi,2.0)
k41 = (4.0*M.pow(ay,2.0)) + 1.0
k44 = (4.0*M.pow(ay,2.0)) + 4.0
a2 = M.pow(ay,2.0)

fFT111 = -((3./2.)*pi*W111_*T022_) - ((3./2.)*pi*W021_*T112_) + (pi*T001_*W112_) - (2*pi*T002_*W111_) - (3*pi*T003_*W112_) + (Ra*W111_) - (pi2*k41 * T111_)
fFT112 = ((3./2.)*pi*W021_*T111_) + ((3./2.)*pi*W111_*T021_) + (pi*T001_*W111_) - (3*pi*T003_*W111_) - (4*pi*T004_*W112_) + (Ra*W112_) - (pi2*k44 * T112_)
fFT021 = -(3*pi*W111_*T112_) + (pi*T001_*W022_) - (2*pi*T002_*W021_) - (3*pi*T003_*W022_) + (Ra*W021_) - (pi2*k41 * T021_)
fFT022 = (3*pi*W111_*T111_) + (pi*T001_*W021_) - (3*pi*T003_*W021_) - (4*pi*T004_*W022_) + (Ra*W022_) - (pi2*k44 * T022_)
fFT001 = -(4*pi*W111_*T112_) - (4*pi*W112_*T111_) - (2*pi*W021_*T022_) - (2*pi*W022_*T021_) - (pi2*(T001_-T_MEAN_001))
fFT002 = (8*pi*W111_*T111_) + (4*pi*W021_*T021_) - (4*pi2*T002_)
fFT003 = (12*pi*W111_*T112_) + (12*pi*W112_*T111_) + (6*pi*W021_*T022_) + (6*pi*W022_*T021_) - (9*pi2*T003_)
fFT004 = (16*pi*W112_*T112_) + (8*pi*W022_*T022_) - (16*pi2*T004_)
fFW111 = -((3./2.)*pi*W022_*W111_) - ((3./2.)*pi*W021_*W112_) + ((4*a2*Pr/k41)*T111_) - (pi2*Pr*k41*W111_)
fFW112 = ((3*pi*k41/k44)*W111_*W021_) + ((4*a2*Pr/k44)*T112_) - (pi2*Pr*k44*W112_)
fFW021 = -(3*pi*W112_*W111_) + ((4*a2*Pr/k41)*T021_) - (pi2*Pr*k41*W021_)
fFW022 = (3*pi*(k41/k44)*W111_*W111_) + ((4*a2*Pr/k44)*T022_) - (pi2*Pr*k44*W022_)

return fFT111, fFT112, fFT021, fFT022, fFT001, fFT002, fFT003, fFT004, fFW111, fFW112, fFW021, fFW022
#RETURNS TIME DERIVATIVES

# END OF FUNCTION -----
```

*first part of the code

Down (up)-hexagon associated with mean heating (cooling)

Horizontal mean temperature is $\theta(t, z) = -2\{\Theta_{001} \sin \pi z + \Theta_{002} \sin 2\pi z + \Theta_{003} \sin 3\pi z + \Theta_{004} \sin 4\pi z\}$

```
# TIME-PARAMETERS
nrun_len = 25 # NUMBER OF RUNS (e.g. 250; see page 56 of the lecture notes)
dt=0.005 # time step
nt_len = 5000 # number of time steps
t_len = nt_len * dt # total time of integration

# NON-DIMENSIONAL NUMBERS
Ra = 5000.0 # Rayleigh number
Pr = 5.0 # 5.0 Prandtl number
T_MEAN_001 = 0.0 # IMPOSED NON-LINEARITY STATIC TEMPERATURE PROFILE
```

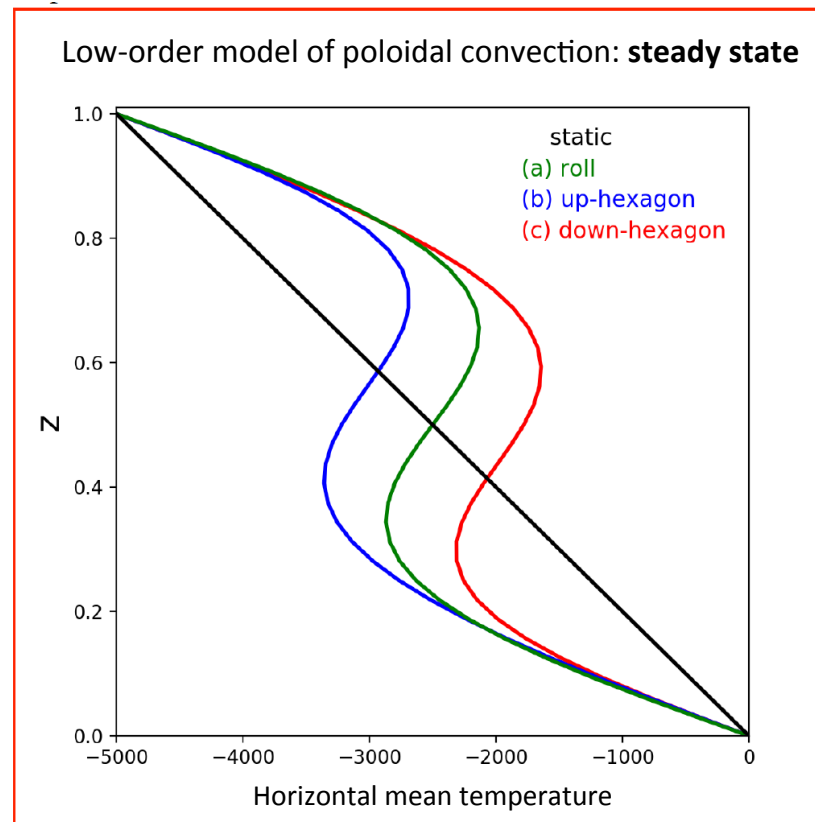
Up-hexagons: mean cooling*

Down-hexagons: mean warming*

Rolls: no mean temperature change

*In the atmosphere: radiative cooling

*In the atmosphere: latent heat release due to condensation

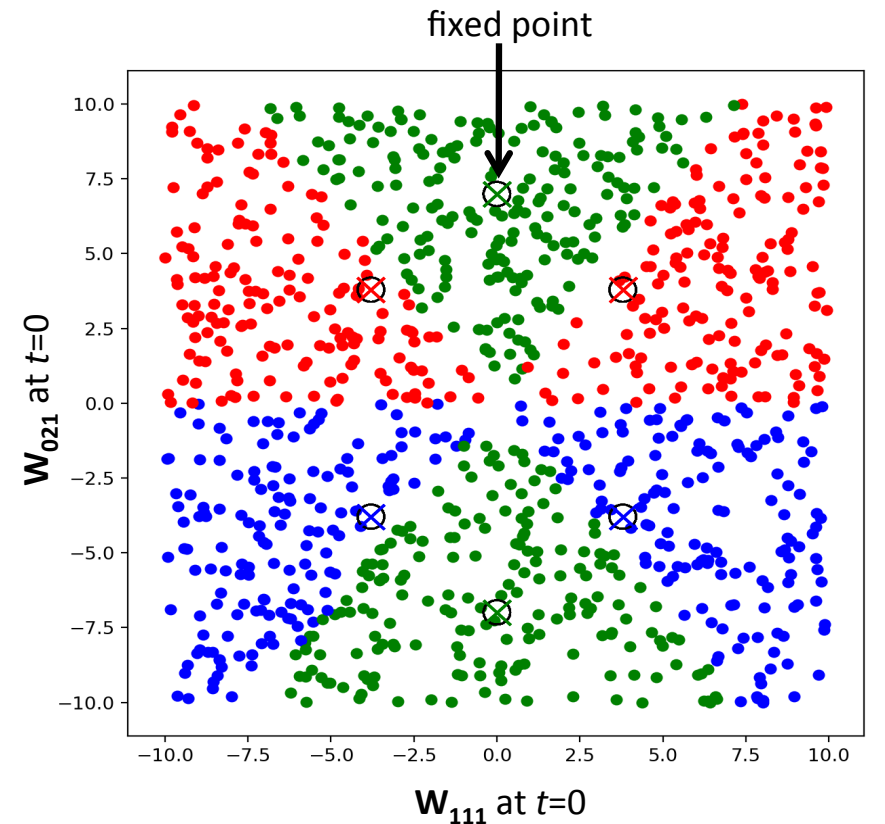


Exercise 22

“Experimental mathematics”

Basins of attraction

The scatter plot shows the result of 1000 integrations of the low-order model of poloidal convection (eqs. 163), each lasting 25 time units, for $Ra=5000$, $Pr=5$, $\alpha=1/(2\nu^2)$ and $\bar{\Theta}_{001} = 0$. The **black circles** with coloured crosses represent the **steady states** in terms of W_{111} and W_{021} at these parameter values, where red is a **down hexagon**, blue is an **up-hexagon** and green is a **roll**. The steady state of rest ($W_{111}=W_{021}=0$) is not indicated explicitly because it is linearly unstable. The **red dots** represent the **initial values of W_{111} and W_{021}** of runs, which have a **down-hexagon as final stable steady state**. The **blue dots** represent the initial values of W_{111} and W_{021} of runs, which have a **up-hexagon as final stable steady state**. The **green dots** represent the initial values of W_{111} and W_{021} of runs, which have a **roll convection pattern as final stable steady state**. The 1000 integrations lead to rolls in 386 cases, up-hexagons in 298 cases and down-hexagons in 315 cases. The fixed points associated with steady state hexagons appear to lie very close to the edge of their respective basins of attraction, bordering the basins of attraction of rolls. Sensitivity to initial conditions is greatest near the state of rest.



Up-hexagon: $W_{021} < 0$; down-hexagon: $W_{021} > 0$; $W_{111} = 0$: roll

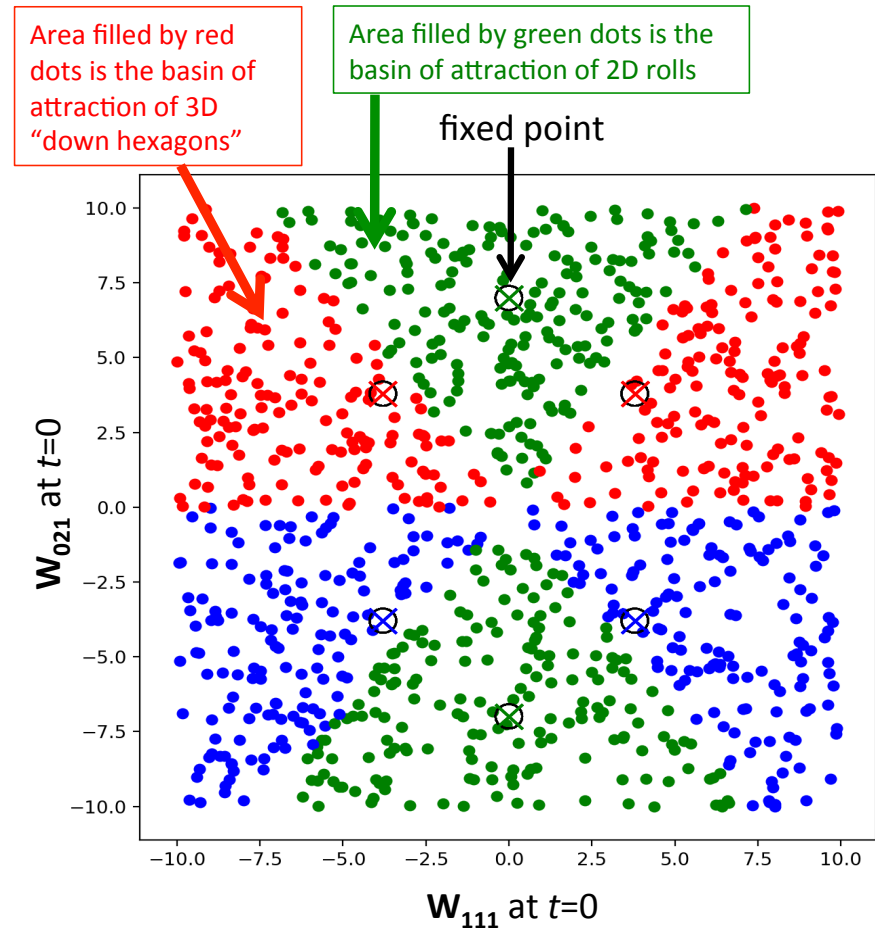
The “essence of turbulence”

According to David Ruelle (1991), **turbulence** is described by **strange attractors** (after transients die out due to dissipation).

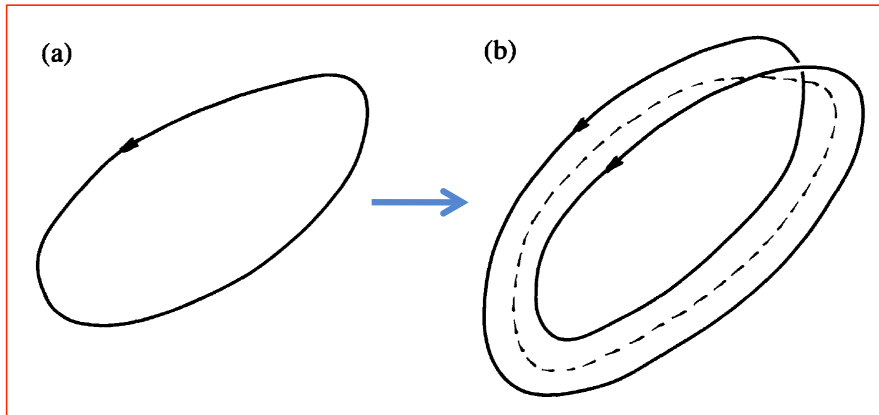
Deterministic “motion” through phase space **on a strange attractor** has **sensitive dependence on initial conditions**, which means that nearby trajectories in phase space separate exponentially fast, i.e. the system has a positive “Lyapunov exponent.”

Deterministic means that the system has no randomly varying term or input parameters

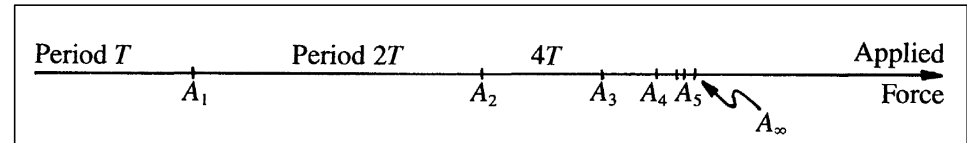
However, **sensitive dependence on initial conditions** does not necessarily depend on the existence of a strange attractor. For example, the diagram on the right (also previous slide) demonstrates that a nonlinear system, which may have several fixed points at identical parameter values, exhibits “basins of attraction”. If the initial condition lies very close to the border between two basins of attraction, one can say that the solution will be sensitively dependent on initial conditions. This idea has occupied weather forecasters...



Transition to turbulence/chaos in a nonlinear system



Period doubling: (a) projection of a periodic orbit in phase space; (b) after increasing the forcing, this orbit is replaced by another one, which is approximately twice as long.



The period doubling cascade. When the force applied to the system is varied, period doubling occurs at values marked $A_1, A_2, A_3, \dots$, accumulating at A_∞ .

Feigenbaum discovered that:

$$\lim_{n \rightarrow \infty} \frac{A_n A_{n+1}}{A_{n+1} A_{n+2}} = 4.66920 \dots$$

This result stands out in the field of “Chaos and Complexity”, and is yet another example of the very **rich behaviour** of nonlinear systems.

I hope this course is giving you an appreciation of this rich behaviour.

More inspiration ?

May, R.M., 1976: Simple mathematical models with very complicated dynamics. **Nature**, 261, 459-467.

Ruelle, D., 1991: **Chance and Chaos**. Princeton University Press. 195 pp.

Lorenz, E.N., 1993: **The Essence of Chaos**. University of Washington Press.

(see Blackboard)

Exercises

Tutorial 10-11

Part 4

Exercise 17: code provided on Blackboard



(of 10-component model of 2D-convection)

- TransitionToTurbulentConvection[Krishnamurti1973].pdf
- (see blackboard)

27. Exercises, part 4

(16)

The **total enstrophy** (E) (the area integrated square-vorticity) in two-dimensional flow is

$$E = \frac{1}{2} \int (\eta^2) d\sigma$$

(see **lecture 9**). Express E in terms the Fourier coefficients of the stream function (see **exercise 10**). In **section 26** it is shown that kinetic energy is conserved in the two triad interactions in the 10-component model. Show that enstrophy is also conserved in these two triad interactions. Assume that mode B in this model possesses the lowest critical Rayleigh number. Show that the middle scale in both triad interactions in the 10-component model loses/gains more kinetic energy to/from larger scale than to/from the smaller scale

(17)

For this exercise you are given the python-code of the 10-component model of two-dimensional convection, which integrates the twelve equations (105), and asked to install and run this code on your computer. Oscillations with a period in the range between 1 and 10 non-dimensional viscous time-units (H^2/ν) are observed in a layer of fluid with low Pr and Rayleigh numbers which are just super-critical to several times super-critical (Krishnamurti, 1973)*. The 10-component model also exhibits oscillations at low Pr ($=0.1$), with $a_x=1/2\sqrt{2}$ and for Rayleigh number 1.5 times supercritical.

- (a) Can you qualitatively foresee the existence of these oscillations from eqs. 112 and 113?
- (b) Investigate these oscillations in the 10-component model (105). Are the oscillations qualitatively in accord with your predictions based on eqs. 112 and 113?
- (c) Do you get oscillations with periods similar to periods observed by Krishnamurti?
- (d) Is this solution “chaotic”? If so, why?
- (e) Make a scatter plot (“map”) of the peak-values of A as a function of the previous peak-value of A . Compare this “map” with the Lorenz map (**figure 12**).
- (f) Plot the time between successive peaks as a function of time half-way these peaks. What do you conclude from this?

Exercises

Tutorial 10-11

Part 5

(18)

Show that the steady state solutions of the Lorenz (1963) model are also solutions of the 12-component model of poloidal thermal convection (eqs. 163). Derive expressions for these solutions.

(19)

(a) Make a vertical cross section of the vertical velocity and the associated temperature distribution at $x=L_x/2$, as a function of y and z , for steady hexagonal convection according to the 12-component model of poloidal convection (eqs.163). The corresponding values of the Fourier coefficients are: $W_{111}=-3.93$, $W_{021}=-3.93$, $W_{112}=0.22$, $W_{022}=0.22$, $\Theta_{111}=-179.3$, $\Theta_{021}=-179.3$, $\Theta_{112}=-1.2$, $\Theta_{022}=-1.2$, $\Theta_{001}=564.8$, $\Theta_{002}=672.8$, $\Theta_{003}=-21.6$, $\Theta_{004}=-0.1$ ($Ra=5000$, $a=1/(2\sqrt{2})$ and $\bar{\Theta}_{001}=500$). Units are non-dimensional (eq. 48). Is this an up- or a down-hexagon?

(b) Make a plot of the associated horizontal mean temperature profile, as in [figure 19](#).

(20)

The total kinetic energy is defined as

$$K \equiv \frac{1}{2} \int (u^2 + v^2 + w^2) d\sigma.$$

Here $d\sigma$ is a volume element. The integration extends over the region defined by eq. 123. The relation between the different velocity components in spectral space is given in eqs 140 and 141.

(a) Transform the above expression for the total kinetic energy to spectral space, assuming that the flow is poloidal.

(b) Show that total kinetic energy is conserved by the nonlinear terms in the low-order 12-component model of poloidal thermal convection (eqs. 163).

Exercises

Tutorial 10-11

Part 5

Exercise 22: code provided on Blackboard



(of 12-component model of poloidal convection)

(21)

Apply the adiabatic elimination technique to the latter 4 model equations in the equation set (163), following the procedure that is presented in **section 25**. Assume that W_{111} and W_{021} are self-excited and that W_{112} and W_{022} are damped. The damped modes are slaved to the self-excited modes. Note that the positive relaxation coefficients for W_{111} and W_{021} are equal, as are the negative relaxation coefficients for W_{112} and W_{022} . Identify the steady state convection patterns and analyse their stability, using the resulting two equations (**figure 19**).

(22)

For this exercise you are given the python-code of the low-order model of poloidal convection, which integrates the twelve equations (163), and asked to install and run this code on your computer. As described on page 56, identify the basins of attraction in the plane, W_{111} versus W_{021} , of the three possible steady state solutions of the 12-component model (eqs. 163): rolls, up-hexagons, and down-hexagon. Do this for $Pr=5$, $Ra=5000$ and $a=1/(2\sqrt{2})$ and $\bar{\Theta}_{001}=0$.

(23)

Investigate the stability of the finite amplitude convective steady state solution of the Lorenz model (exercise 18) in the low-order model of poloidal convection as a function of Ra and Pr , by performing model-integrations in which the initial steady state solution is perturbed slightly, assuming that $a=1/(2\sqrt{2})$ and $\bar{\Theta}_{001}=0$.

Exercises to prepare for the exam question

The exam on Monday 11 April 2022, 8:30-10:30 will consist of two parts: a numerical part, for which you have to bring your laptop, and a theoretical part.

The numerical part will be modeled on the following exercises

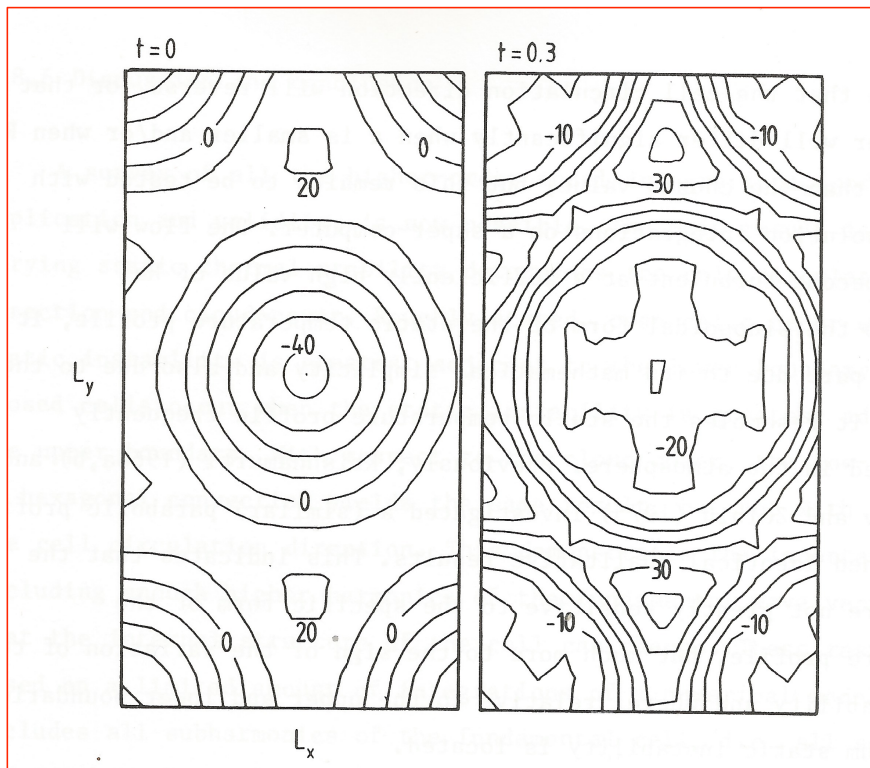
Exercises 17 and 22

and

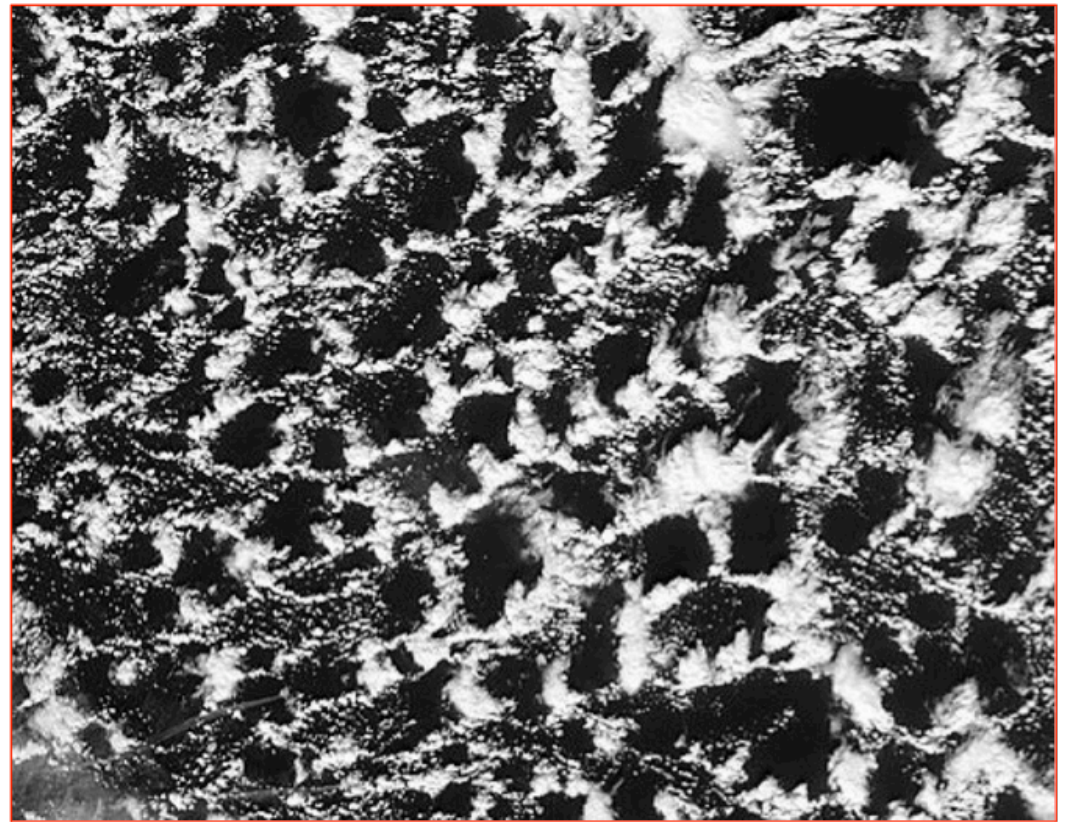
ExamQuestionTurbulenceInFluidsApril2021.pdf
(see Blackboard)

My interest comes from my PhD research project on the question of pattern selection in atmospheric convection

A down-hexagon becomes an “open convection cell”



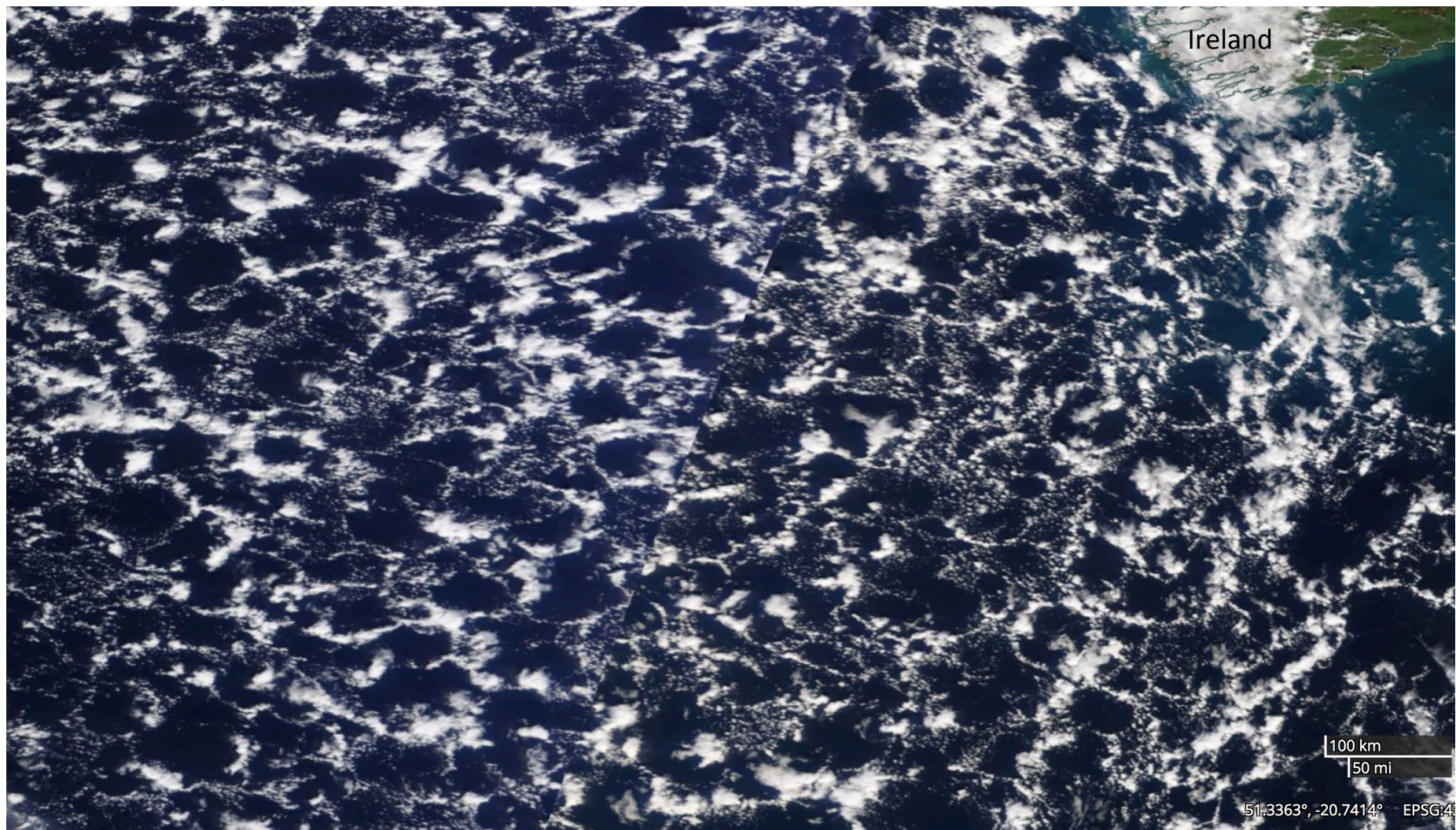
Vertical velocity halfway the convective layer in non-dimensional units (eq. 48) at the beginning (left panel) and at the end (right panel) of a run with a high-order model $Ra=5000$, $Pr=5$, $\alpha=1/(2\sqrt{2})$ and $\beta=2500$ ([figure 21](#)).



Open convection cells ("g-cells") over the Atlantic Ocean ([figure 22](#))

3 March 2022: open cells over the Atlantic Ocean

Note: very large aspect ratio and low cloud cover!



3 March 2022

(earthdata.nasa.gov)

Thank you!

200 km

200 mi

44.4379°, -3.0227° EPSG: