

Lecture 3 SOAC

Numerical stability of numerical solution of the advection equation (the “CFL-condition”)

Exercise 2

Lax-Wendroff method

Spectral method

linear advection equation:

$$\frac{\partial C}{\partial t} + u_0 \frac{\partial C}{\partial x} = 0$$

velocity is constant

$C(x,t)$ =concentration

Numerical stability: von Neumann* stability analysis

First order numerical approximation to the linear advection equation:

$$\frac{C(x, t + \Delta t) - C(x, t)}{\Delta t} + u_0 \frac{C(x + \Delta x, t) - C(x, t)}{\Delta x} = 0$$

$u_0 > 0$ $\longrightarrow$ “first order downwind” approximation

Short notation:
$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + u_0 \frac{C_{j+1}^n - C_j^n}{\Delta x} = 0$$

time-index $\swarrow$
space-index $\swarrow$



- * John von Neumann (1903-1957) : initiator of numerical weather prediction
- See: <https://www.tandfonline.com/doi/abs/10.3402/tellusa.v2i4.8607>

Numerical stability: von Neumann stability analysis*

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* Insert a solution of the form: $C_j^n = a_k^n \exp(ikj\Delta x)$

(Fourier transformation)

$\uparrow$
wave number

Numerical stability: von Neumann stability analysis*

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* Insert a solution of the form: $C_j^n = a_k^n \exp(ikj\Delta x)$

$$\frac{a_k^{n+1} \exp(ikj\Delta x) - a_k^n \exp(ikj\Delta x)}{\Delta t} + u_0 \frac{a_k^n \exp(ik(j+1)\Delta x) - a_k^n \exp(ikj\Delta x)}{\Delta x} = 0$$

$$\frac{a_k^{n+1} - a_k^n}{\Delta t} + u_0 \frac{a_k^n \exp(ik\Delta x) - a_k^n}{\Delta x} = 0$$

Numerical stability

Repeat:

$$\frac{a_k^{n+1} - a_k^n}{\Delta t} + u_0 \frac{a_k^n \exp(ik\Delta x) - a_k^n}{\Delta x} = 0$$

Now assume: $a_k^{n+1} \equiv \rho a_k^n$



“amplification factor”

Numerical stability

Repeat:

$$\frac{a_k^{n+1} - a_k^n}{\Delta t} + u_0 \frac{a_k^n \exp(ik\Delta x) - a_k^n}{\Delta x} = 0$$

Now assume: $a_k^{n+1} \equiv \rho a_k^n$

$$\frac{a_k^n (\rho - 1)}{\Delta t} + u_0 \frac{a_k^n \exp(ik\Delta x) - a_k^n}{\Delta x} = 0$$

Amplification factor: $\rho = 1 - \frac{u_0 \Delta t}{\Delta x} (\exp(ik\Delta x) - 1)$

$$\sigma \equiv \frac{u_0 \Delta t}{\Delta x}$$

Numerical stability

Repeat:

$$\frac{a_k^{n+1} - a_k^n}{\Delta t} + u_0 \frac{a_k^n \exp(ik\Delta x) - a_k^n}{\Delta x} = 0$$

Now assume: $a_k^{n+1} \equiv \rho a_k^n$

$$\frac{a_k^n (\rho - 1)}{\Delta t} + u_0 \frac{a_k^n \exp(ik\Delta x) - a_k^n}{\Delta x} = 0$$

Amplification factor: $\rho = 1 - \frac{u_0 \Delta t}{\Delta x} (\exp(ik\Delta x) - 1)$ $\sigma \equiv \frac{u_0 \Delta t}{\Delta x}$

$$\rho = 1 - \sigma (\cos(k\Delta x) + i \sin(k\Delta x) - 1) = (1 + \sigma - \sigma \cos(k\Delta x)) - i \sigma \sin(k\Delta x)$$

Absolute value: $|\rho|^2 = (1 + \sigma - \sigma \cos(k\Delta x))^2 + (\sigma \sin(k\Delta x))^2$

Numerical stability *downwind scheme*

Repeat:

Absolute value: $|\rho|^2 = (1 + \sigma - \sigma \cos(k\Delta x))^2 + (\sigma \sin(k\Delta x))^2$

a little algebra

$$|\rho|^2 = 1 + 2\sigma(\sigma + 1)(1 - \cos(k\Delta x)) \quad \text{for every } k$$

Numerical stability *downwind scheme*

Repeat:

$$|\rho|^2 = (1 + \sigma - \sigma \cos(k\Delta x))^2 + (\sigma \sin(k\Delta x))^2$$

a little algebra

$$|\rho|^2 = 1 + 2\sigma(\sigma + 1)(1 - \cos(k\Delta x)) \quad \text{for every } k$$

Maximum ρ for $\cos(k\Delta x) = -1$

Amplification factor: $|\rho|^2 = 1 + 4\sigma(\sigma + 1) > 1$ where $\sigma \equiv \frac{u_0 \Delta t}{\Delta x} > 0$



downwind scheme is numerically *unstable* !

Numerical stability *upwind scheme*

If “**first-order upwind**”:

$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + u_0 \frac{C_j^n - C_{j-1}^n}{\Delta x} = 0 \quad \boxed{u_0 > 0}$$

Repeat derivation of the previous slides:

Result: $|\rho|^2 = 1 + 2\sigma(\sigma - 1)(1 - \cos(k\Delta x))$

If $\cos(k\Delta x) = -1$ $|\rho|^2 = 1 + 4\sigma(\sigma - 1)$

Numerical stability *upwind scheme*

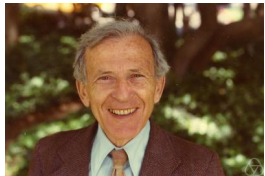
If “**first-order upwind**”:

$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + u_0 \frac{C_j^n - C_{j-1}^n}{\Delta x} = 0 \quad \boxed{u_0 > 0}$$

Repeat derivation of the previous slides:

Result: $\boxed{|\rho|^2 = 1 + 2\sigma(\sigma - 1)(1 - \cos(k\Delta x))}$

If $\cos(k\Delta x) = -1$ $\boxed{|\rho|^2 = 1 + 4\sigma(\sigma - 1)}$



Hans Friedrichs (1904-1988)



Kurt Lewy (1901-1982)



Richard Courant (1888-1972)

$$\boxed{|\rho|^2 < 1 \text{ if } \sigma \leq 1} \rightarrow \text{Numerical stability if } \boxed{\sigma \equiv \frac{u_0 \Delta t}{\Delta x} \leq 1}$$

Courant-Friedrichs-Lewy (1928) (“CFL-”) condition

Upwind scheme is numerically *stable* if “CFL”-condition is fulfilled

Numerical stability *implicit scheme*

If “first-order upwind”:

$$\frac{C_j^{n+1} - C_j^n}{\Delta t} + u_0 \frac{C_j^{n+1} - C_{j-1}^{n+1}}{\Delta x} = 0$$


$$u_0 > 0$$

Repeat derivation:

$$\text{Result: } |\rho|^2 = \{1 + 2\sigma(\sigma + 1)(1 - \cos(k\Delta x))\}^{-1} \leq 1$$

Implicit scheme is always numerically stable

With implicit scheme you can take large time steps, but accuracy is still determined by the time step. Unfortunately, this scheme is practically impossible to implement. Alternative is to make scheme “semi-implicit”, like Matsuno’s scheme. The stability of Matsuno’s scheme is revealed by the strong damping (see exercise 1).

Exercise 2

Lax-Wendroff scheme

Taylor series approximation to C_j^{n+1}

$$C_j^{n+1} = C_j^n + \Delta t \left(\frac{\partial C}{\partial t} \right)_j^n + \frac{\Delta t^2}{2} \left(\frac{\partial^2 C}{\partial t^2} \right)_j^n + \dots$$

Since (advection equation):

$$\frac{\partial C}{\partial t} + u_0 \frac{\partial C}{\partial x} = 0 \xrightarrow{\text{lecture 3}} \frac{\partial^2 C}{\partial t^2} = u_0^2 \frac{\partial^2 C}{\partial x^2}$$

We have:

$$C_j^{n+1} = C_j^n - u_0 \Delta t \left(\frac{\partial C}{\partial x} \right)_j^n + \frac{u_0^2 \Delta t^2}{2} \left(\frac{\partial^2 C}{\partial x^2} \right)_j^n + \dots$$

Exercise 2

Lax-Wendroff scheme

Previous slide:

$$C_j^{n+1} = C_j^n - u_0 \Delta t \left(\frac{\partial C}{\partial x} \right)_j^n + \frac{u_0^2 \Delta t^2}{2} \left(\frac{\partial^2 C}{\partial x^2} \right)_j^n + \dots$$



Last term: explicit numerical diffusion

Finite difference approximation to $\left(\frac{\partial^2 C}{\partial x^2} \right)_j^n$

$$\left(\frac{\partial^2 C}{\partial x^2} \right)_j^n \approx \frac{\frac{(C_{j+1}^n - C_j^n)}{\Delta x} - \frac{(C_j^n - C_{j-1}^n)}{\Delta x}}{\Delta x} \approx \frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\Delta x^2}$$

Exercise 2

Lax-Wendroff scheme

$$C_j^{n+1} = C_j^n - u_0 \Delta t \left(\frac{\partial C}{\partial x} \right)_j^n + \frac{u_0^2 \Delta t^2}{2} \left(\frac{\partial^2 C}{\partial x^2} \right)_j^n + \dots$$

$$\left(\frac{\partial^2 C}{\partial x^2} \right)_j^n \approx \frac{\frac{(C_{j+1}^n - C_j^n)}{\Delta x} - \frac{(C_j^n - C_{j-1}^n)}{\Delta x}}{\Delta x} \approx \frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\Delta x^2}$$

Finite difference approximation to the linear advection equation:

(Lax-Wendroff scheme)

$$\frac{C_j^{n+1} - C_j^n}{\Delta t} \approx -u_0 \left(\frac{C_{j+1}^n - C_{j-1}^n}{2\Delta x} \right) + \frac{u_0^2 \Delta t}{2} \left(\frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\Delta x^2} \right)$$

forward difference

central difference

central difference

Explicit numerical diffusion!

Numerically stabilizes the solution

Exercise 2

Spectral method

Set of uniformly spaced grid points in one spatial dimension:

$$\underline{x_j = j\Delta x}, \quad j=1,2,\dots,N, \quad \text{where } \underline{N\Delta x = L}.$$

The Fourier series of C , whose values are given only at N grid points, requires $(N+1)$ Fourier coefficients. The Fourier expansion of C is

$$C(x_j) = \operatorname{Re} \left\{ \sum_{k=0}^{N/2} C_k \exp(2\pi i k x_j / L) \right\} = \operatorname{Re} \left\{ \sum_{k=0}^{N/2} C_k \exp(2\pi i k j / N) \right\}$$

The Fourier coefficients are *complex*: $C_k = \alpha_k + i\beta_k$

Exercise 2

Spectral method

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The Fourier coefficients are *complex*: $C_k = \alpha_k + i\beta_k$

The **Fourier coefficients**, α_k and β_k for wave numbers, $k=0$ to $k=N/2$ can be found by evaluating

$$\alpha_k = \frac{2}{N} \sum_{j=0}^{N-1} C_j \cos(-2\pi k j / N), \quad \beta_k = \frac{2}{N} \sum_{j=0}^{N-1} C_j \sin(-2\pi k j / N) \quad \text{and} \quad \alpha_0 = \frac{1}{N} \sum_{j=0}^{N-1} C_j \quad (2)$$

$$(\beta_0 = 0)$$

Exercise 2

Spectral method

In the problem at hand (solving the linear advection equation) the **Fourier coefficients depend on time**:

$$C(x_j, t) = \operatorname{Re} \left\{ \sum_{k=0}^{N/2} C_k(t) \exp(2\pi i k j / N) \right\}$$

When we substitute this equation into the **linear advection equation**,

$$\frac{\partial C}{\partial t} = -u_0 \frac{\partial C}{\partial x}$$

we obtain **N independent first order ordinary differential equations for the time evolution of the Fourier coefficients** *:

$$\frac{dC_k}{dt} = -\frac{2\pi i u_0 k}{L} C_k \text{ for } k = 1, 2, 3, \dots, N/2 \quad (3)$$

Exercise 2

Spectral method

In the problem at hand (solving the linear advection equation) the Fourier coefficients depend on time:

$$C(x_j, t) = \operatorname{Re} \left\{ \sum_{k=0}^{N/2} C_k(t) \exp(2\pi i k j / N) \right\}$$

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we obtain N * independent first order ordinary differential equations for the time evolution of the Fourier coefficients :

$$\frac{dC_k}{dt} = -\frac{2\pi i u_0 k}{L} C_k \text{ for } k = 1, 2, 3, \dots, N/2 \quad (3)$$

* Since the Fourier coefficients are complex, (3) represents a system of N equations.

Exercise 2

Spectral method

$$\frac{\partial C}{\partial t} = -u_0 \frac{\partial C}{\partial x}$$

Previous slide:

$$\frac{dC_k}{dt} = -\frac{2\pi i u_0 k}{L} C_k \text{ for } k = 1, 2, 3, \dots, N/2 \quad (3)$$

The Fourier coefficients are complex: (3) represents a system of N equations.

At initial time we need to compute the complex Fourier coefficients from eq. (2).

Then we integrate the system (3) in time and, finally, transform back from “spectral space” to “physical space”.

Only a numerically approximation of the time derivative is needed.

For example the Euler forward time stepping scheme, but you will see that this will not work very well.

Heun's or Matsuno's time stepping scheme works much better (lecture 2).

Example:

Matsuno *predictor-corrector* time stepping scheme

Previous slide:

$$\frac{dC_k}{dt} = -\frac{2\pi i u_0 k}{L} C_k \text{ for } k = 1, 2, 3, \dots, N/2 \quad (3)$$

Step 1 (predictor step):

$$\frac{C_k^{*n+1} - C_k^n}{\Delta t} = -\frac{2\pi i u_0 k}{L} C_k^n$$

Step 2 (corrector step):

$$\frac{C_k^{n+1} - C_k^n}{\Delta t} = -\frac{2\pi i u_0 k}{L} C_k^{*n+1}$$

Note that, since the Fourier coefficients are complex, (3) represents a system of N equations.

Exercise 2: Molenkamp test (spectral in space and “Matsuno” is time)

Hint at the result for the spectral method

