

Lecture 4 SOAC

The non-linear advection equation:
pseudo-spectral method

Pole problem

The shallow water equations: simulation of wave propagation
Errors in phase- and group velocity of simulated gravity waves

Error due to aliasing

Exercise 3: Numerical simulation of gravity wave propagation.

The *non-linear* advection equation: use the pseudo-spectral method

$$\frac{\partial C}{\partial t} = -u \frac{\partial C}{\partial x}$$



product of two Fourier series !!

In the pseudo-spectral method $\partial C/\partial x$ is calculated in spectral space and the result is transformed to physical (grid point) space. This provides an exact estimate of the spatial derivative at each grid point. The nonlinear term on the r.h.s. of the advection equation is calculated by multiplying $\partial C/\partial x$ with u at each grid point.

All spectral models are pseudo-spectral. Most operational numerical weather prediction models are pseudo-spectral, also the ECMWF-model.

(pseudo-) spectral method

Advantages of spectral methods:

1. Better approximation to spatial derivatives, including the pressure gradient
2. No Pole problem (see next slide)
3. Uniform resolution on the sphere

Disadvantages of the spectral methods:

1. Many calculations associated with processes that are local or conditional (like condensation of water vapour) must be done in physical space (on gridpoints).
2. Forward and backward transforms take time. A Fast Legendre Transform does not (yet) exist. The computational burden rises rapidly as a power of the resolution (number of wave numbers).
3. Difficult to handle non-periodic boundary conditions.

Pole problem

D. Williamson: Journal of the Meteorological Society of Japan, Vol. 85B, pp. 241--269, 2007

LATITUDE-LONGITUDE GRID

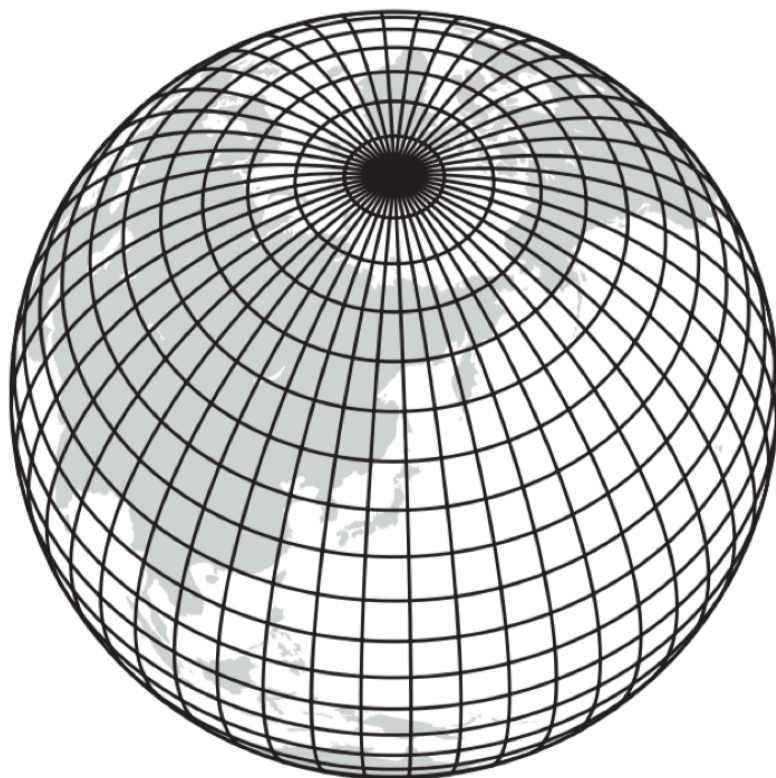


Fig. 1. A latitude-longitude grid consisting of equally spaced lines of constant latitude and longitude.

KURIHARA OR REDUCED GRID

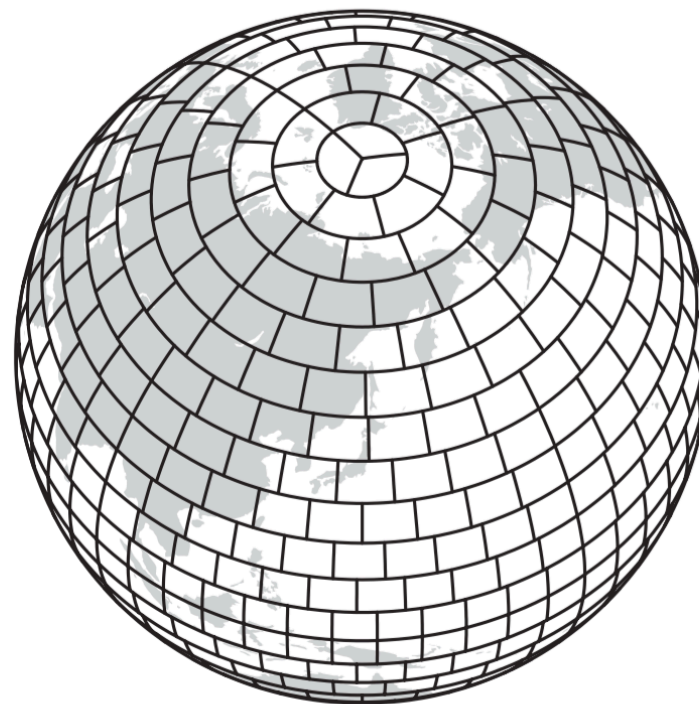


Fig. 4. A Kurihara or reduced grid in which the longitudinal grid interval measured in degrees is increased approaching the poles to keep the physical length as uniform as possible.

Pole problem

ECMWF-plans:

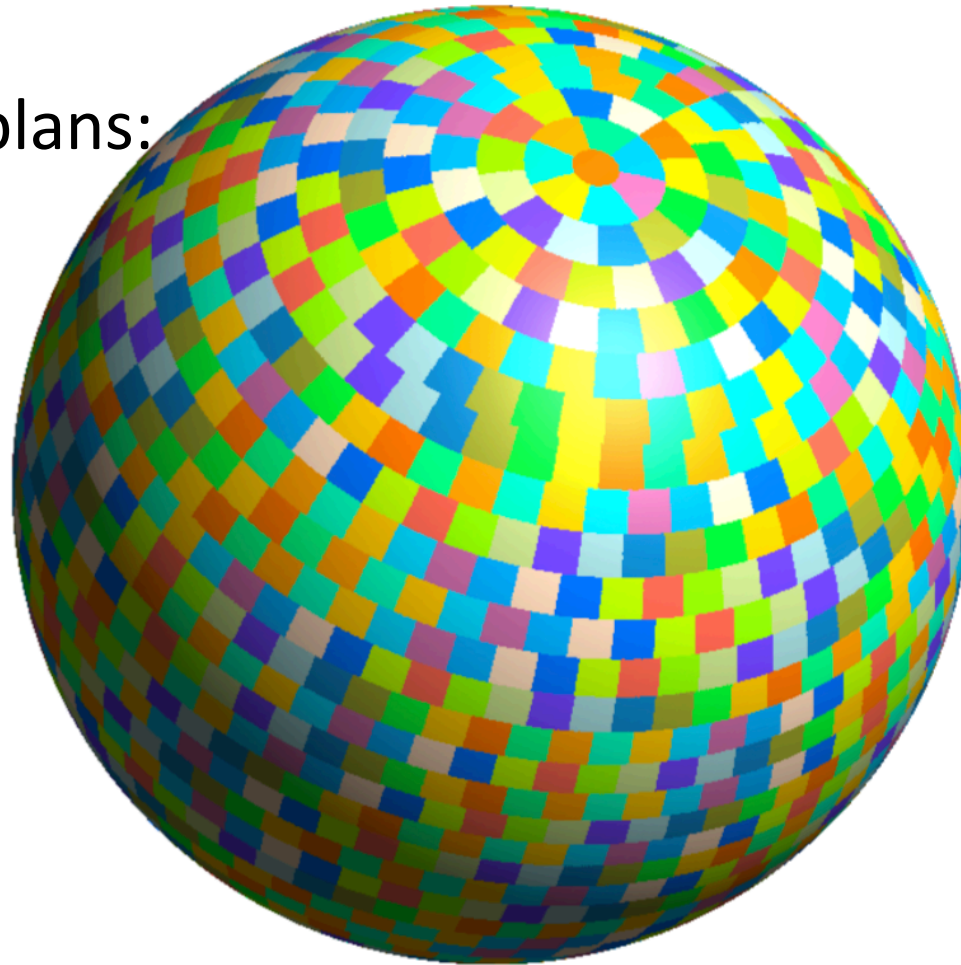


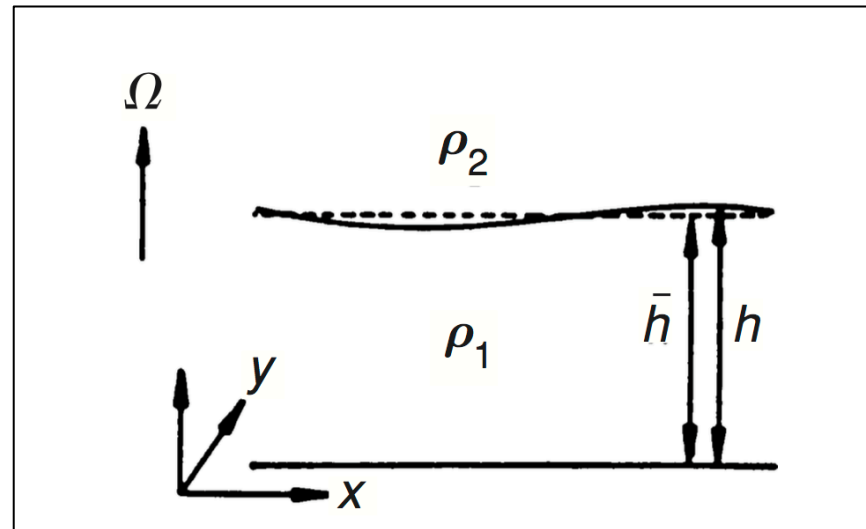
Figure 9: Equal-regions domain decomposition with 1600 partitions as typically applied in the new TCo1279 octahedral reduced Gaussian grid planned for the next horizontal resolution upgrade.

The shallow water equations: simulation of wave propagation

The canonical equations of geophysical fluid mechanics :

$$\frac{\partial u}{\partial t} = -g' \frac{\partial h}{\partial x} + fv; \quad \frac{\partial v}{\partial t} = -g' \frac{\partial h}{\partial y} - fu; \quad \frac{\partial h}{\partial t} = -H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

(see lecture 1)



$$\left(g' = \frac{\rho_1 - \rho_2}{\rho_1} g \right)$$

A simplified case (no background rotation):

One-dimensional linear shallow water equations with $f=0$:

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x}; \quad \frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x};$$

Substitute a solution of the form:

$$u(x,t) = u_0 \exp(i(kx - \omega t))$$

Dispersion relation: $\omega^2 = gHk^2$

Phase velocity: $c = \frac{\omega}{k} = \pm \sqrt{gH}$

Group velocity: $c_g = \frac{d\omega}{dk} = \pm \sqrt{gH}$

Group velocity=phase velocity

no dispersion!

One dimensional shallow water equation: numerical approximation

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x}; \quad \frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x};$$

Finite difference approximation x-derivative (central in space):

$$\frac{\partial h_j}{\partial t} = -H \frac{u_{j+1} - u_{j-1}}{2\Delta x}; \quad \frac{\partial u_j}{\partial t} = -g \frac{h_{j+1} - h_{j-1}}{2\Delta x} \quad (1)$$

Assume wave solution of the form: $u_j = u_0 \exp(i(kj\Delta x - \omega t))$

$$h_j = h_0 \exp(i(kj\Delta x - \omega t))$$

This yields

$$\begin{pmatrix} -(H \sin k\Delta x) / \Delta x & \omega \\ \omega & -(g \sin k\Delta x) / \Delta x \end{pmatrix} \begin{pmatrix} u_0 \\ h_0 \end{pmatrix} = 0$$

(derivation on see next 2 slides)

Equation:
$$\frac{\partial h_j}{\partial t} = -H \frac{u_{j+1} - u_{j-1}}{2\Delta x};$$

solution:
$$u_j = u_0 \exp(i(kj\Delta x - \omega t))$$

$$h_j = h_0 \exp(i(kj\Delta x - \omega t))$$

Substitute solution into equation:

$$-i\omega h_0 \exp(i(kj\Delta x - \omega t)) = -\frac{Hu_0}{2\Delta x} \exp(i(kj\Delta x - \omega t)) \{ \exp(i(k\Delta x)) - \exp(i(-k\Delta x)) \}$$

$$-i\omega h_0 = -\frac{Hu_0}{2\Delta x} \{ \exp(i(k\Delta x)) - \exp(i(-k\Delta x)) \}$$

$$-i\omega h_0 = -\frac{Hu_0}{2\Delta x} \{ \cos(k\Delta x) + i\sin(k\Delta x) - \cos(-k\Delta x) - i\sin(-k\Delta x) \}$$

$$-i\omega h_0 = -\frac{Hu_0}{2\Delta x} \{ 2i\sin(k\Delta x) \}$$

$$-\omega h_0 = -\frac{Hu_0}{\Delta x} \sin(k\Delta x)$$

$$-\left\{ \frac{H}{\Delta x} \sin(k\Delta x) \right\} u_0 + \omega h_0 = 0$$

Equation: $\frac{\partial h_j}{\partial t} = -H \frac{u_{j+1} - u_{j-1}}{2\Delta x};$

Other equation: $\frac{\partial u_j}{\partial t} = -g \frac{h_{j+1} - h_{j-1}}{2\Delta x}$

solution: $u_j = u_0 \exp(i(kj\Delta x - \omega t))$
 $h_j = h_0 \exp(i(kj\Delta x - \omega t))$

analogous derivation

Substitute solution into equation:

$$-i\omega h_0 \exp(i(kj\Delta x - \omega t)) = -\frac{Hu_0}{2\Delta x} \exp(i(kj\Delta x - \omega t)) \{ \exp(i(k\Delta x)) - \exp(i(-k\Delta x)) \}$$

$$-i\omega h_0 = -\frac{Hu_0}{2\Delta x} \{ \exp(i(k\Delta x)) - \exp(i(-k\Delta x)) \}$$

$$-i\omega h_0 = -\frac{Hu_0}{2\Delta x} \{ \cos(k\Delta x) + i \sin(k\Delta x) - \cos(-k\Delta x) - i \sin(-k\Delta x) \}$$

$$-i\omega h_0 = -\frac{Hu_0}{2\Delta x} \{ 2i \sin(k\Delta x) \}$$

$$-\omega h_0 = -\frac{Hu_0}{\Delta x} \sin(k\Delta x)$$

$$-\left\{ \frac{H}{\Delta x} \sin(k\Delta x) \right\} u_0 + \omega h_0 = 0$$

$$-\left\{ \frac{g}{\Delta x} \sin(k\Delta x) \right\} h_0 + \omega u_0 = 0$$

$$\begin{pmatrix} -(H \sin k\Delta x) / \Delta x & \omega \\ \omega & -(g \sin k\Delta x) / \Delta x \end{pmatrix} \begin{pmatrix} u_0 \\ h_0 \end{pmatrix} = 0$$

One dimensional shallow water equation: numerical approximation

Repeat the final result on the previous slide:

$$\begin{pmatrix} -(H \sin k\Delta x) / \Delta x & \omega \\ \omega & -(g \sin k\Delta x) / \Delta x \end{pmatrix} \begin{pmatrix} u_0 \\ h_0 \end{pmatrix} = 0$$

Non-trivial solutions are found if the **determinant of the matrix equals zero**

This yields the “**numerical**” **dispersion relation**:

$$\omega^2 = gH \left(\frac{\sin k\Delta x}{\Delta x} \right)^2$$

Numerical phase velocity:

$$c^* = \frac{\omega}{k} = c \frac{\sin k\Delta x}{k\Delta x}$$

Remember: $c = c_g = \pm \sqrt{gH}$

Numerical group velocity:

$$c_g^* = \frac{d\omega}{dk} = c_g \cos(k\Delta x)$$

(**derivation** on the next slide)

Derivation of expression for numerical group velocity on the previous slide:

Numerical frequency:

$$\omega^2 = gH \left(\frac{\sin k\Delta x}{\Delta x} \right)^2$$

$$2\omega \frac{d\omega}{dk} = \frac{2gH}{\Delta x} \sin k\Delta x \cos k\Delta x$$

$$\omega = \pm \sqrt{gH} \left(\frac{\sin k\Delta x}{\Delta x} \right)$$

$$\pm 2\sqrt{gH} \left(\frac{\sin k\Delta x}{\Delta x} \right) \frac{d\omega}{dk} = \frac{2gH}{\Delta x} \sin k\Delta x \cos k\Delta x$$

$$\frac{d\omega}{dk} = \pm \sqrt{gH} \cos k\Delta x$$



Numerical group velocity:

$$c_g^* = \frac{d\omega}{dk}$$

$$c_g^* = \pm \sqrt{gH} \cos k\Delta x$$

Analytical group velocity:

$$c_g = \pm \sqrt{gH}$$

Remember: $c = c_g = \pm \sqrt{gH}$

Overview of errors in numerical simulation of gravity waves

Numerical phase velocity:

$$c^* = \frac{\omega}{k} = c \frac{\sin k\Delta x}{k\Delta x}$$

Numerical group velocity:

$$c_g^* = \frac{d\omega}{dk} = c_g \cos(k\Delta x)$$

$$c = c_g = \pm\sqrt{gH} \longrightarrow c^* \neq c_g^*$$

Numerical phase velocity depends on the wave number: numerical dispersion
Numerical phase velocity is not equal to numerical group velocity

Overview of errors in numerical simulation of gravity waves

Numerical phase velocity:

$$c^* = \frac{\omega}{k} = c \frac{\sin k\Delta x}{k\Delta x}$$

Numerical group velocity:

$$c_g^* = \frac{d\omega}{dk} = c_g \cos(k\Delta x)$$

$$c = c_g = \pm\sqrt{gH} \longrightarrow c^* \neq c_g^*$$

Numerical phase velocity depends on the wave number: numerical dispersion
Numerical phase velocity is not equal to numerical group velocity

Long wave (k small): $c^* \approx c$ $c_g^* \approx c_g$

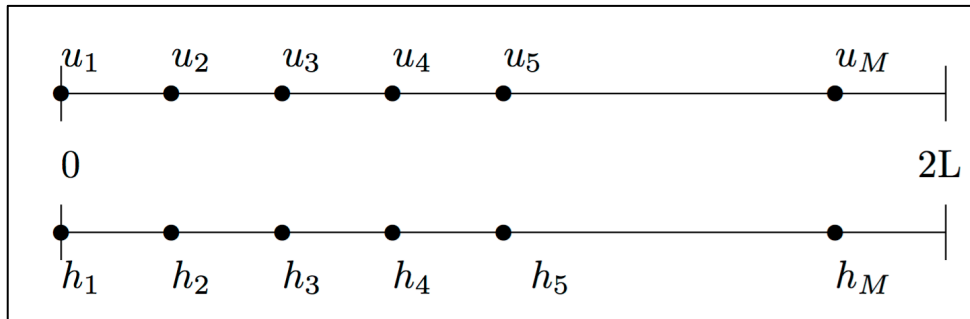
Representation of long waves in the numerical solution is **correct**

Short wave (k large): $c^* \approx 0$ $c_g^* = ?$

Representation of short waves in the numerical solution is **incorrect**

Improvements by using a “staggered” grid

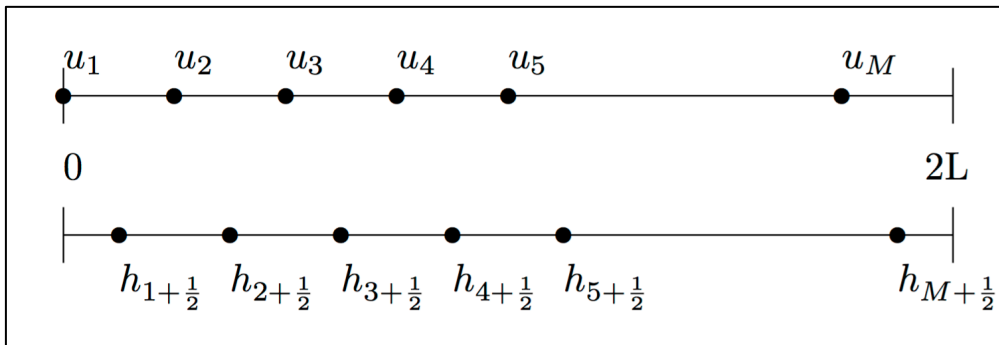
Unstaggered grid:



$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = -g \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x}$$

$$\frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} = -H \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

Staggered grid:



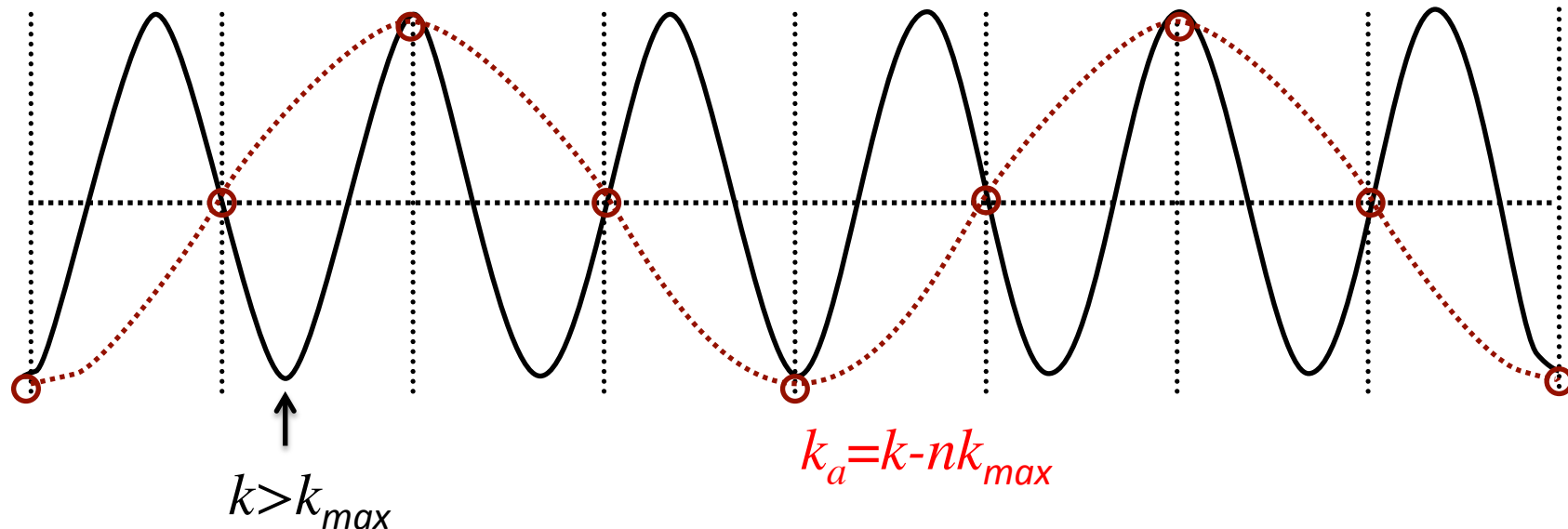
$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = -g \frac{h_{j+1/2}^n - h_{j-1/2}^n}{\Delta x}$$

$$\frac{h_{j+1/2}^{n+1} - h_{j+1/2}^{n-1}}{2\Delta t} = -H \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

- Advantages:**
1. Effective grid distance is halved, which reduces the error due to numerical dispersion.
 2. Eliminates possible decoupling of the solution for “even and odd grid points”.

“Aliasing” errors

This topic is discussed in the course, “MAIO”



The shortest resolved wave (one wavelength) must “cover” two gridpoints, so $k_{max} = 2\pi/2\Delta x$

When two resolved waves k_1 and k_2 (both smaller k_{max}) interact, a higher harmonic wave is excited with wave number, $k_1 + k_2 > k_{max}$. This wave is interpreted by the grid as a longer wave, as is demonstrated in the figure above. This leads to so-called aliasing errors (see **MAIO**), and may give unstable growth of waves that the grid can just resolve.

Exercise 3:

Numerical Simulation of wave propagation

Solve the linear one-layer shallow water equations numerically:

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x}; \quad \frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x}$$

on a **periodic spatial domain** with length, $L=10^4$ km, and with $H=1000$ m and $g=10 \text{ m s}^{-2}$, employing an unstaggered spatial grid. The fluid is at rest initially and a perturbation in h is inserted in the middle of the domain at $x=0$ according to

$$h = H + \eta_0 \exp\left(-\left(\frac{x}{a_0}\right)^2\right) \quad (-5000 \leq x \leq 5000 \text{ km}, \quad \eta_0 = 50 \text{ m and } a = 200 \text{ km})$$

Perform integrations for different values of the grid distance, Δx .

The philosophy of these experiments is similar to that of the Molenkamp test. You should observe *two* perturbations in the free surface that propagate in both directions with constant phase velocity without loss of form (no dispersion). If the boundary conditions are periodic, the waves will meet after a certain time and the state at this time, in terms of the height of the free surface, h , should be exactly as at $t=0$.

Exercise 3: continued

- (a) What is the phase speed of the gravity waves on the free surface?
- (b) At what time will the waves be at the same position as at initial time?
- (c) Perform numerical integrations, lasting this time (see part b), for different spatial grid resolutions by varying the total number of grid points between 100 and 1000. Do this for a fixed time step, $\Delta t = 50$ s. Plot the result at initial time ($t=0$) and at the end of the integration for each integration and interpret the result in the light of “the overview of errors in the numerical simulation of gravity waves” on slide 91.
- (d) Can you improve on the result by reducing Δt ?
- (e) Repeat exercise (c) with a staggered grid in space with the same resolution. In which way does this improve the numerical solution?

SOAC 2021

Deadline hand-in exercises of Part 1

Exercise 1: Thursday 16 September 2021

Exercises 2 and 3: Thursday 7 October 2021

Final remarks

This concludes the SOAC-lectures on numerical methods.

The course continues with a research project supervised by an IMAU-researcher and coordinated by Michiel Baatsen.

Thank you for your attention !