

The advection equation

Lecture 5, course on “Turbulence in Fluids”, Utrecht University

<http://www.staff.science.uu.nl/~delde102/TinF.htm>

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Introduction

One-Dimensional Advection Equation: General Properties

Solving the One-Dimensional Advection Equation: the Spectral Method

A Low-Order Model of the One-Dimensional Advection Equation

Numerical solutions and spectral energy transfer

Lecture notes:

<http://www.staff.science.uu.nl/~delde102/LectureOnTheAdvectionEquation.pdf>

Contents of lectures 6-10

LectureNotes_Convection[TurbulenceFluids2021].pdf (blackboard, under course content, lecture 5)

Lectures on thermal convection and turbulence of a stratified fluid

Third year bachelor course, "Turbulence in Fluids" (NS376B)

Department of Physics, Utrecht University.

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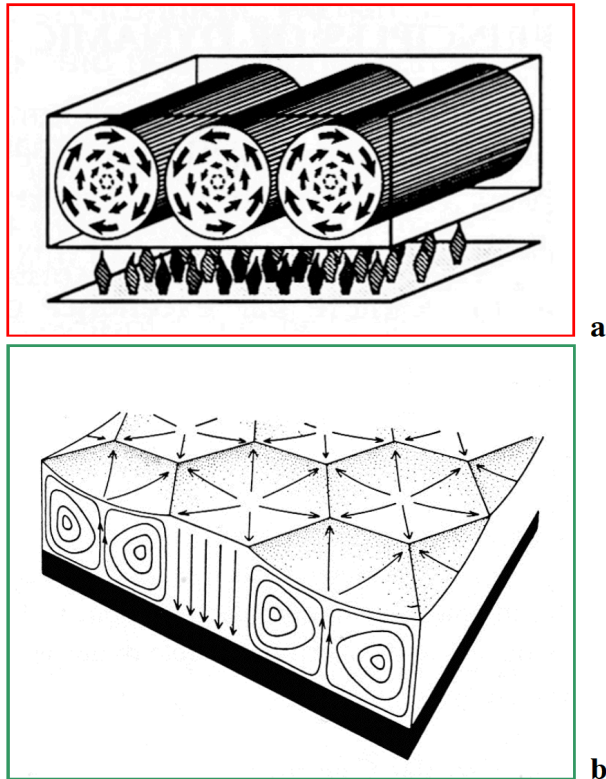


Figure 1. Schematic diagram illustrating (a) two-dimensional and (b) three-dimensional hexagonal convection patterns in a fluid, which is heated from below and cooled from above

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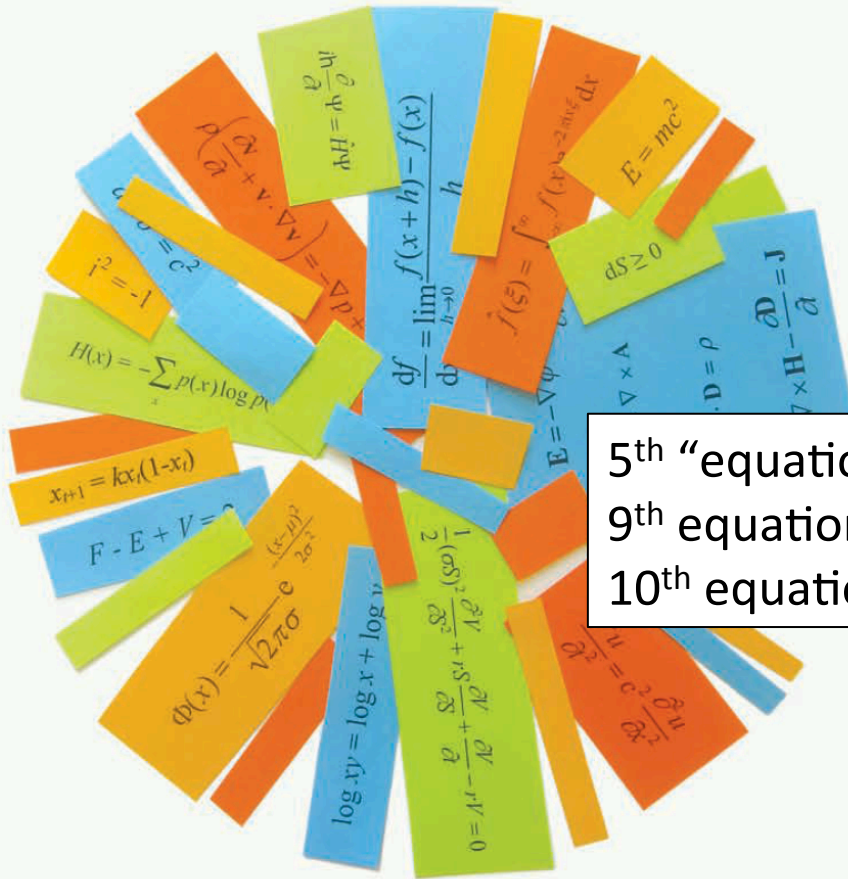
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In Pursuit of the Unknown

17 Equations That Changed the World



Ian Stewart



Ian Stewart

5th “equation: definition of i (complex numbers)
9th equation: Fourier transform
10th equation: Navier-Stokes equation

5

Portent of the ideal world

The Square Root of Minus One

$$\overset{\text{imaginary number}}{i} \overset{\text{squared}}{^2} = \overset{\text{minus one}}{-} \overset{\text{one}}{1}$$

What does it say?

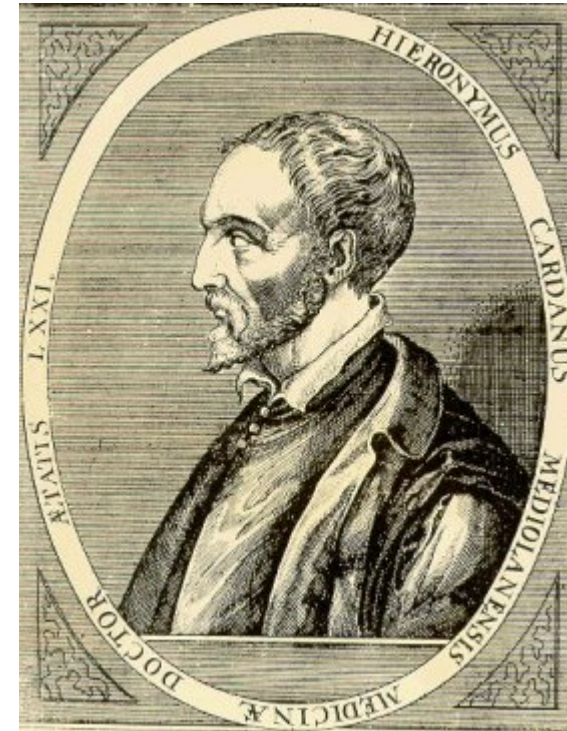
Even though it ought to be impossible, the square of the number i is minus one.

Why is that important?

It led to the creation of complex numbers, which in turn led to complex analysis, one of the most powerful areas of mathematics.

What did it lead to?

Improved methods to calculate trigonometric tables. Generalisations of almost all mathematics to the complex realm. More powerful methods to understand waves, heat, electricity, and magnetism. The mathematical basis of quantum mechanics.



Gerolamo Cardano (1501-1576)

9

Ripples and blips

Fourier Transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

Diagram illustrating the Fourier Transform equation with labels:

- $\hat{f}(\xi)$: transform of function
- ξ : frequency
- $\int$: integral
- $-\infty$: minus infinity
- ∞ : infinity
- $f(x)$: function
- x : space
- $e^{-2\pi i x \xi}$: square root of -1
- dx : frequency

What does it say?

Any pattern in space and time can be thought of as a superposition of sinusoidal patterns with different frequencies.

Why is that important?

The component frequencies can be used to analyse the patterns, create them to order, extract important features, and remove random noise.

What did it lead to?

Fourier's technique is very widely used, for example in image processing and quantum mechanics. It is used to find the structure of large biological molecules like DNA, to compress image data in digital photography, to clean up old or damaged audio recordings, and to analyse earthquakes. Modern variants are used to store fingerprint data efficiently and to improve medical scanners.



Jean-Baptiste Joseph Fourier (1768-1830)

10 The ascent of humanity

Navier–Stokes Equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$$

Diagram illustrating the Navier–Stokes Equation with labels for its components:

- ρ : density
- $\frac{\partial \mathbf{v}}{\partial t}$: time derivative
- $\mathbf{v}$: velocity
- ∇ : gradient
- $\mathbf{v} \cdot \nabla$: dot product
- p : pressure
- $\nabla \cdot \mathbf{T}$: divergence
- $\mathbf{T}$: stress
- $\mathbf{f}$: body forces

What does it say?

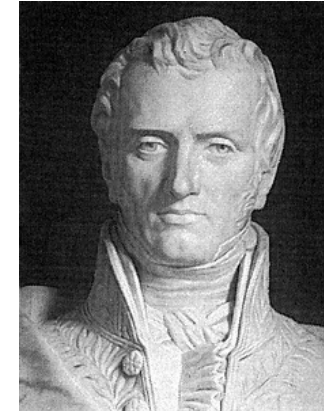
It's Newton's second law of motion in disguise. The left-hand side is the acceleration of a small region of fluid. The right-hand side is the forces that act on it: pressure, stress, and internal body forces.

Why is that important?

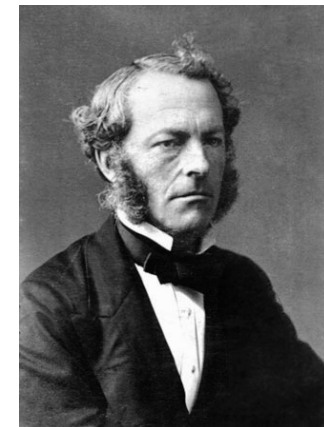
It provides a really accurate way to calculate how fluids move. This is a key feature of innumerable scientific and technological problems.

What did it lead to?

Modern passenger jets, fast and quiet submarines, Formula 1 racing cars that stay on the track at high speeds, and medical advances on blood flow in veins and arteries. Computer methods for solving the equations, known as computational fluid dynamics (CFD), are widely used by engineers to improve technology in such areas.



Navier(1785-1836)



Stokes (1819-1903)

Navier-Stokes equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$$

Diagram illustrating the Navier-Stokes equation with labels for its components:

- ρ : density
- $\frac{\partial \mathbf{v}}{\partial t}$: time derivative
- $\mathbf{v} \cdot \nabla \mathbf{v}$: dot product (velocity) and gradient
- $-\nabla p$: pressure
- $\nabla \cdot \mathbf{T}$: divergence (stress)
- $\mathbf{f}$: body forces

advection



Right hand side = 0 (**no forces**):

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = 0$$

Advection equation

The same as:

$$\frac{d\vec{v}}{dt} = 0$$

Advection term is non-linear: source of turbulence

Advection of **velocity** (momentum per unit mass):

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v} \cdot \nabla \vec{v}$$

Advection of a **passive scalar** (concentration, C):

$$\frac{\partial C}{\partial t} = -\vec{v} \cdot \nabla C$$

Non-linear term

C-contour advection

2002/09/03



One-dimensional advection

The bare minimum model of advection

Advection of **velocity** (momentum per unit mass):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad (1)$$

Boundary condition:

$$u(0, t) = u(L, t) = 0$$

Initial condition:

$$u(x, 0) = u_0 \sin\left(\frac{2\pi x}{L}\right)$$

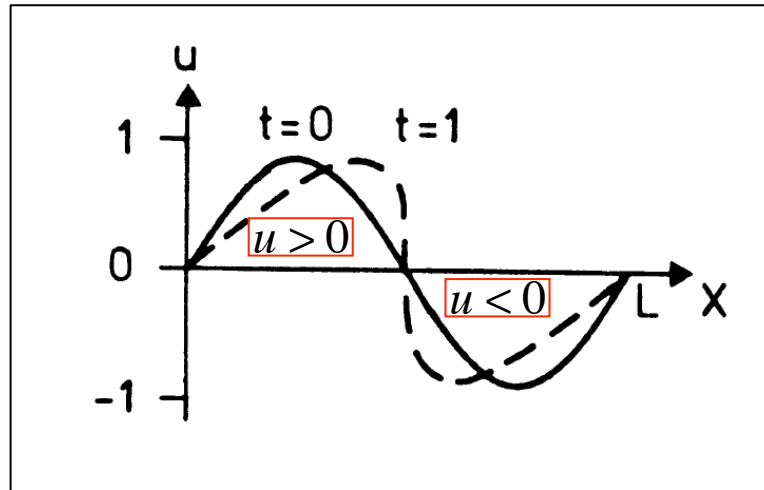


FIGURE 1a. Initial distribution of u ($t=0$, full line) and distribution of u just at the onset of “breaking” ($t=1$, dashed line) ($L=2\pi$ and $u_0=1$).

Method of characteristics

Along a characteristic : $u = dx/dt = \text{constant}$

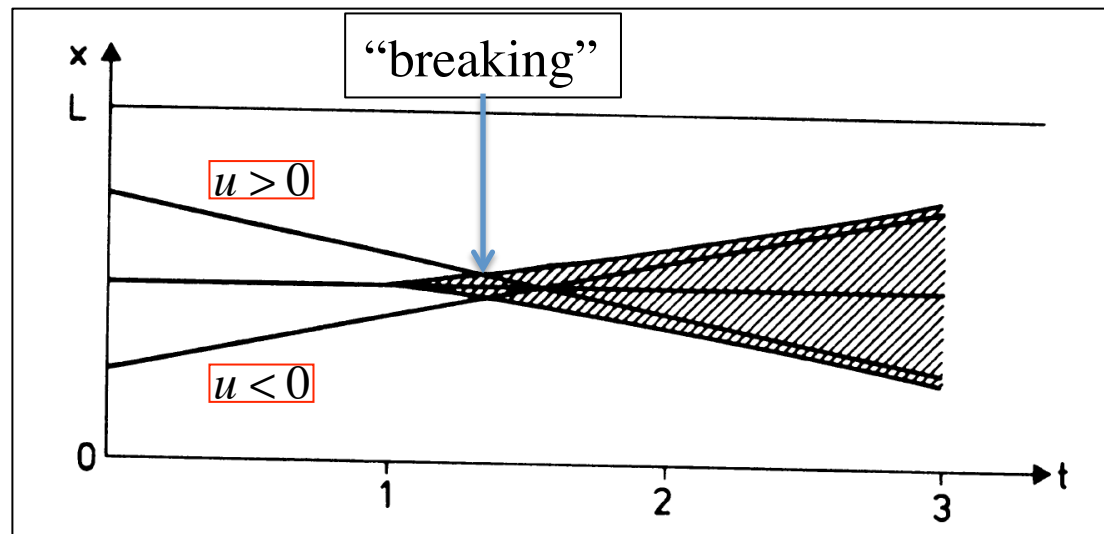


FIGURE 1b. Characteristics of eq. (1) in the t - x plane, assuming that $L=2\pi$ and $u_0=1$. Inside the hatched, cusp-shaped region the solution is multi-valued.

“Slope” equation

Slope: $S \equiv \frac{\partial u}{\partial x}$

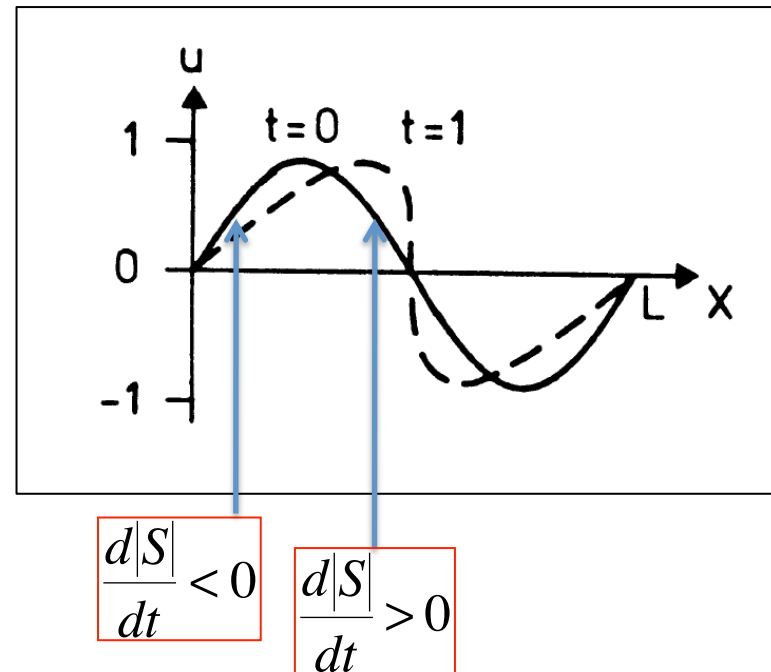
Slope-advection equation:

$$\frac{dS}{dt} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)$$

$$\frac{dS}{dt} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)$$

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x}$$

$$\frac{dS}{dt} = \frac{\partial}{\partial x} \left(-u \frac{\partial u}{\partial x} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \longrightarrow \frac{dS}{dt} = -S^2$$



Slope equation with *damping*

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\epsilon u$$

damping coefficient

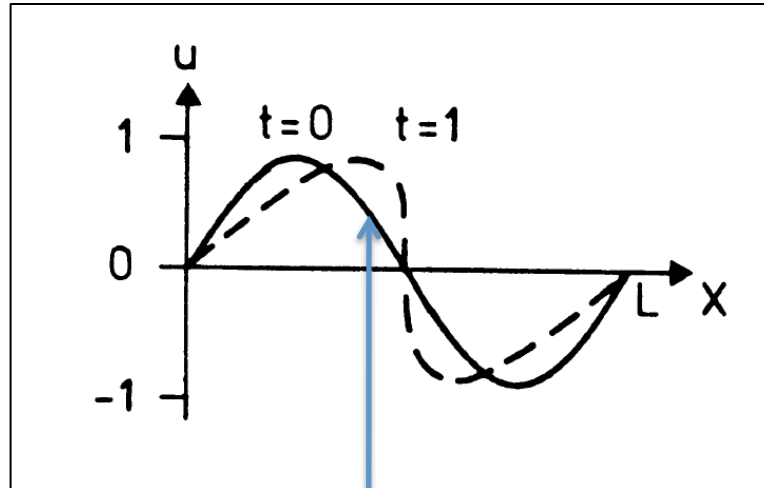
$$S \equiv \frac{\partial u}{\partial x}$$

Previous slide:

$$\frac{dS}{dt} = \frac{\partial}{\partial x} \left(-u \frac{\partial u}{\partial x} - \epsilon S \right) + u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)$$

$$\frac{dS}{dt} = -S^2 - \epsilon S$$

This damping-term always reduces the slope



$$\frac{d|S|}{dt} < 0 \text{ if } \epsilon > S$$

Spectral (Fourier) method

Fourier series:

$$u(x,t) \equiv \sum_{\alpha} U_{\alpha}(t) S_{\alpha}(x)$$

amplitude

Basis function

Basis functions:

$$S_{\alpha}(x) \equiv \exp(2\pi i l_{\alpha} x / L)$$

$$S_{\alpha}(x) \equiv \exp(2\pi i l_{\alpha} x / L) = \cos(2\pi l_{\alpha} x / L) + i \sin(2\pi l_{\alpha} x / L)$$

$\sum_{\alpha}$ represents a sum over all $\alpha \equiv l_{\alpha}$

integer between $-\infty$ and $+\infty$



Fourier (1768-1830)

Spectral method: imposing b.c.'s

Fourier series:

$$u(x,t) \equiv \sum_{\alpha} U_{\alpha}(t) S_{\alpha}(x)$$

amplitude

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Fourier (1768-1830)

Boundary conditions (b.c.'s): $u(0,t) = u(L,t) = 0$

Boundary conditions satisfied if $S_{\alpha}(x) = i \sin(2\pi l_{\alpha} x / L)$

Spectral method: imposing b.c.'s

Fourier series:

$$u(x,t) \equiv \sum_{\alpha} U_{\alpha}(t) S_{\alpha}(x)$$

Basis functions:

$$S_{\alpha}(x) \equiv \exp(2\pi i l_{\alpha} x / L)$$

$$S_{\alpha}(x) \equiv \exp(2\pi i l_{\alpha} x / L) = \cos(2\pi l_{\alpha} x / L) + i \sin(2\pi l_{\alpha} x / L)$$

Boundary conditions (b.c.'s): $u(0,t) = u(L,t) = 0$

Boundary conditions satisfied if $S_{\alpha}(x) = i \sin(2\pi l_{\alpha} x / L)$

Amplitude is complex number: $U_{\alpha}(t) = U_{\alpha}^R + i U_{\alpha}^I$

u is real physical variable if $U_{\alpha}(t) S_{\alpha}(x) = (U_{\alpha}^R + i U_{\alpha}^I) i \sin(2\pi l_{\alpha} x / L)$

with $U_{\alpha}^R = 0$



Fourier (1768-1830)

Spectral transform of advection equation

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x}$$

substitute

$$u(x,t) \equiv \sum_{\alpha} U_{\alpha}(t) S_{\alpha}(x)$$

$$S_{\alpha}(x) \equiv \exp(2\pi i l_{\alpha} x / L)$$

yields:

$$\sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} \equiv - \left\{ \sum_{\alpha} U_{\alpha} S_{\alpha} \right\} \left\{ \sum_{\beta} (2\pi i l_{\beta} / L) U_{\beta} S_{\beta} \right\}$$

same as

$$\sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} \equiv - \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} S_{\alpha} S_{\beta}$$



$\sum_{\alpha, \beta}$ is a double sum over all positive and negative wave numbers

Spectral transform of advection equation

Previous slide:

$$\longrightarrow \sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} \equiv - \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} S_{\alpha} S_{\beta}$$

$\sum_{\alpha, \beta}$ is a double sum over all positive and negative wave numbers

We can separate this equation into a set of nonlinearly coupled ordinary differential equations by making use of **orthogonality of the basis functions**:

Spectral transform advection equation

Previous slide:

$$\sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} \equiv - \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} S_{\alpha} S_{\beta}$$

$\sum_{\alpha, \beta}$ is a double sum over all positive and negative wave numbers

We can separate this equation into a set of nonlinearly coupled ordinary differential equations by making use of **orthogonality of the basis functions**:

Orthogonality of basis functions:

$$\frac{1}{L} \int_0^L S_{\alpha} S_{\beta}^* dx = \delta_{\alpha, \beta}$$

complex conjugate of S_{β}

Kronecker-delta:

$$\delta_{\alpha, \beta} \equiv 1 \text{ if } \alpha = \beta$$

$$\delta_{\alpha, \beta} \equiv 0 \text{ if } \alpha \neq \beta$$

Slide 15:

$$S_{\beta}(x) \equiv \exp(2\pi i l_{\beta} x / L)$$

$$S_{\beta}^*(x) \equiv \exp(-2\pi i l_{\beta} x / L)$$

Spectral transform advection equation

Previous slide:

$$\longrightarrow \sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} \equiv - \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} S_{\alpha} S_{\beta}$$

$\sum_{\alpha, \beta}$ is a double sum over all positive and negative wave numbers

We may separate this equation into a set of nonlinearly coupled ordinary differential equations by making use of a special property of the basis functions

Orthogonality of basis functions: $\frac{1}{L} \int_0^L S_{\alpha} S_{\beta}^* dx = \delta_{\alpha, \beta}$

Multiply the equation by S_{γ} and integrate over domain $0 \leq x \leq L$:

$$\frac{1}{L} \int_0^L \left\{ \sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} S_{\gamma}^* \right\} dx \equiv - \frac{1}{L} \int_0^L \left\{ \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} S_{\alpha} S_{\beta} S_{\gamma}^* \right\} dx \longrightarrow$$

With this mathematical operation we are projecting the equation on the wave number, γ .

Spectral transform advection equation

Previous slide: 

$$\frac{1}{L} \int_0^L \left\{ \sum_{\alpha} \frac{dU_{\alpha}}{dt} S_{\alpha} S_{\gamma}^* \right\} dx \equiv - \frac{1}{L} \int_0^L \left\{ \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} S_{\alpha} S_{\beta} S_{\gamma}^* \right\} dx$$

Orthogonality of basis functions: $\frac{1}{L} \int_0^L S_{\alpha} S_{\beta}^* dx = \delta_{\alpha, \beta}$

Also: $S_{\alpha} S_{\beta} = S_{\alpha + \beta}$

$$\frac{dU_{\gamma}}{dt} = - \sum_{\alpha, \beta} (2\pi i l_{\beta} / L) U_{\alpha} U_{\beta} \delta_{\gamma, \alpha + \beta}$$

next slide 

A set of first-order differential equations,
describing the evolution in time of the amplitudes, U_{γ} .

Spectral transform advection equation

Previous slide:

$$\frac{dU_\gamma}{dt} = - \sum_{\alpha, \beta} (2\pi i l_\beta / L) U_\alpha U_\beta \delta_{\gamma, \alpha+\beta}$$

Slide 14:

Due to **boundary conditions and “physical reality”**: $U_\alpha(t) = U_\alpha^R + iU_\alpha^I = iU_\alpha^I$

$$\frac{dU_\gamma^I}{dt} = \sum_{\alpha, \beta} (2\pi l_\beta / L) U_\alpha^I U_\beta^I \delta_{\gamma, \alpha+\beta}$$

or:
$$\frac{dU_\gamma^I}{dt} = \sum_{\alpha, \beta} I_{\gamma\alpha\beta} U_\alpha^I U_\beta^I$$

next slide

Interaction coefficient: $I_{\gamma\alpha\beta} \equiv (2\pi l_\beta / L) \delta_{\gamma, \alpha+\beta}$

Wave-interactions obey selection rule

Previous slide:

$$\frac{dU_{\gamma}^I}{dt} = \sum_{\alpha, \beta} I_{\gamma\alpha\beta} U_{\alpha}^I U_{\beta}^I$$

Interaction coefficient: $I_{\gamma\alpha\beta} \equiv (2\pi l_{\beta} / L) \delta_{\gamma, \alpha+\beta}$

selection rule: $\gamma = \alpha + \beta$

Selection rule determines which wave-interactions contribute to the time rate of change of U_{γ}^I

Truncation: a low order model of advection

$$\frac{dU_{\gamma}^I}{dt} = \sum_{\alpha, \beta} I_{\gamma\alpha\beta} U_{\alpha}^I U_{\beta}^I$$

$$I_{\gamma\alpha\beta} \equiv (2\pi l_{\beta} / L) \delta_{\gamma, \alpha+\beta}$$

Truncation at wave number 2 (for example) implies that this sum includes wave-number -2,-1,0,1,2.

Because of the boundary conditions: $U_{-\gamma}^I = -U_{\gamma}^I$

Therefore, equations needed only for U_1^I and U_2^I .

Triad interactions:

γ	α	β
1	2	-1
1	-1	2
2	1	1

} $\gamma = \alpha + \beta$ (selection rule)

Table 1. Wave number combinations giving non-zero contributions to the interaction coefficient in the lowest order *nonlinear* truncation of the advection equation, which takes only wave numbers 1 and 2 inside the truncation.

Lowest order nonlinear model: $l_{\max}=2$

Triad interactions:

γ	α	β
1	2	-1
1	-1	2
2	1	1

$$\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \gamma = \alpha + \beta \quad (\text{selection rule})$$

Table 1. Wave number combinations giving non-zero contributions to the interaction coefficient in the lowest order *nonlinear* truncation of the advection equation, which takes only wave numbers 1 and 2 inside the truncation.

$$\frac{dU_{\gamma}^I}{dt} = \sum_{\alpha, \beta} C l_{\beta} U_{\alpha}^I U_{\beta}^I \delta_{\gamma, \alpha + \beta}$$

To shorten notation:

$$X \equiv U_1^I; \quad Y \equiv U_2^I \quad \text{and} \quad C \equiv \frac{2\pi}{L}$$

wave number 1

$$\frac{dX}{dt} = -CXY$$

wave number 2

$$\frac{dY}{dt} = CX^2$$

Lowest order nonlinear model: $l_{\max}=2$

Triad interactions:

γ	α	β
1	2	-1
1	-1	2
2	1	1

$$\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \gamma = \alpha + \beta \quad (\text{selection rule})$$

Table 1. Wave number combinations giving non-zero contributions to the interaction coefficient in the lowest order nonlinear truncation of the advection equation, which takes only wave numbers 1 and 2 inside the truncation.

$$\frac{dU_{\gamma}^I}{dt} = \sum_{\alpha, \beta} C l_{\beta} U_{\alpha}^I U_{\beta}^I \delta_{\gamma, \alpha + \beta}$$

To shorten notation:

$$X \equiv U_1^I; \quad Y \equiv U_2^I \quad \text{and} \quad C \equiv \frac{2\pi}{L}$$

wave number 1

$$\frac{dX}{dt} = -CXY$$

wave number 2

$$\frac{dY}{dt} = CX^2$$

Why are the interactions between α and β contributing to $\gamma=0$ not included in the table above?

Energy conservation by the low-order model

$$l_{\max}=2$$

wave number 1 $\frac{dX}{dt} = -CXY$

wave number 2 $\frac{dY}{dt} = CX^2$

Kinetic energy is conserved: $\frac{d}{dt} \left[\frac{1}{2} (X^2 + Y^2) \right] \equiv \frac{dK}{dt} = 0$

Kinetic energy is determined by initial state: $K = \frac{1}{2} (X(0)^2 + Y(0)^2)$

Solution lies on a **circle in phase space**: $(X^2 + Y^2) = 2K$

Steady states

$$l_{\max}=2$$

wave number 1 $\frac{dX}{dt} = -CXY = 0$

wave number 2 $\frac{dY}{dt} = CX^2 = 0$

$$X_0 = 0 \text{ and } X_0 Y_0 = 0$$



Subscript "0" stands for steady state

Two steady states: $(X, Y) = (0, Y_0)$ and $(0, -Y_0)$

with $Y_0^2 = 2K$

In both steady state solutions all kinetic energy is in the smallest scale of motion

Linear stability of steady state

Perturb the steady state:

$$l_{\max}=2$$

$$X = (X_0 + \delta X)$$

$$Y = (Y_0 + \delta Y)$$

Substitute in the governing equation:

$$\frac{d}{dt}(X_0 + \delta X) = -C(X_0 + \delta X)(Y_0 + \delta Y)$$

$$\frac{d}{dt}(Y_0 + \delta Y) = C(X_0 + \delta X)^2$$

$$C \equiv \frac{2\pi}{L}$$

Linear stability of steady state

Perturb the steady state:

$$l_{\max}=2$$

$$X = (X_0 + \delta X)$$

$$Y = (Y_0 + \delta Y)$$

Substitute in the governing equation:

$$\frac{d}{dt}(X_0 + \delta X) = -C(X_0 + \delta X)(Y_0 + \delta Y)$$

$$\frac{d}{dt}(Y_0 + \delta Y) = C(X_0 + \delta X)^2$$

$$C \equiv \frac{2\pi}{L}$$

Yields equations governing the evolution of the perturbations:

$$\frac{d\delta X}{dt} = -C(X_0\delta Y + Y_0\delta X + \delta X\delta Y) \approx -C(X_0\delta Y + Y_0\delta X) = -CY_0\delta X$$



The steady state is stable if $Y_0 > 0$ and unstable if $Y_0 < 0$

$$\frac{d\delta Y}{dt} = C(2X_0\delta X + (\delta X)^2) \approx 2CX_0\delta X = 0 \longrightarrow$$

Another look at the linear stability of steady state

Previous slide:



$$\frac{d\delta Y}{dt} = C(2X_0\delta X + (\delta X)^2) \approx 2CX_0\delta X = 0$$

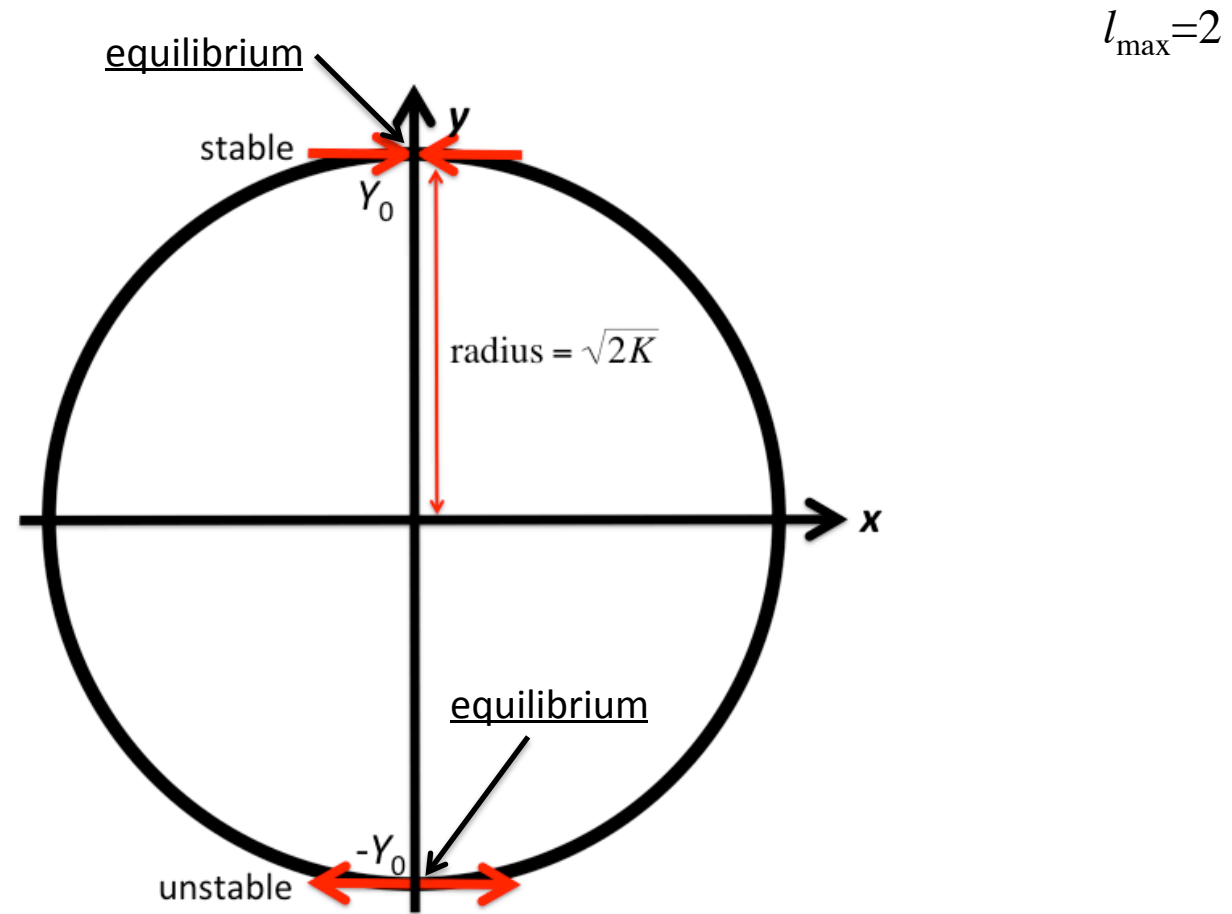
neutral stability!?



When the steady state, which lies on the Y -axis, is perturbed in the Y -direction, the radius (r) of the circle, representing the solution in phase space, is perturbed also. Due to conservation of kinetic energy, $K = \frac{1}{2}r^2$, the solution cannot return to the original steady state.

Figure 4

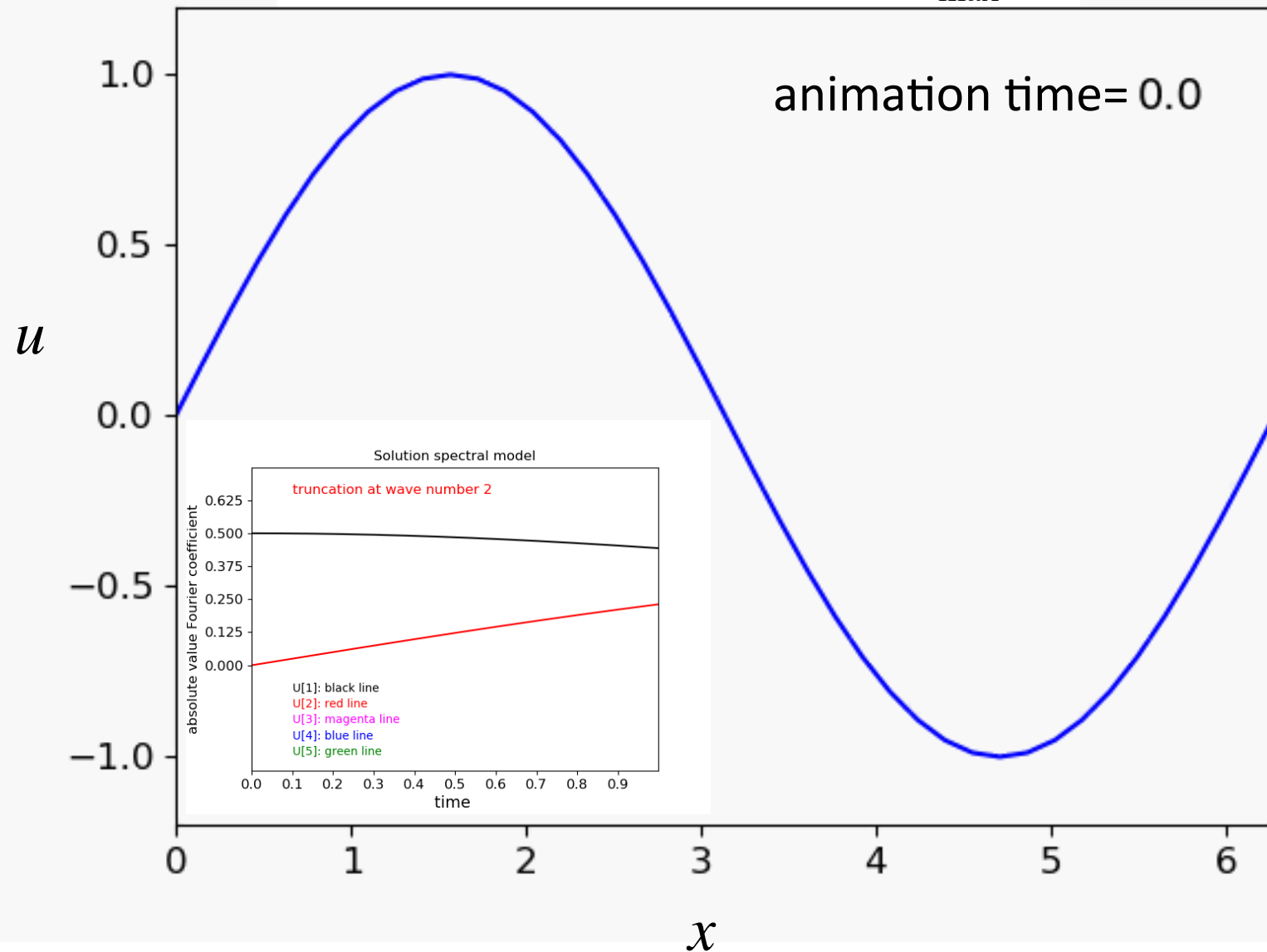
Solution in phase space



Solution lies on a circle in phase space: $(X^2 + Y^2) = 2K$

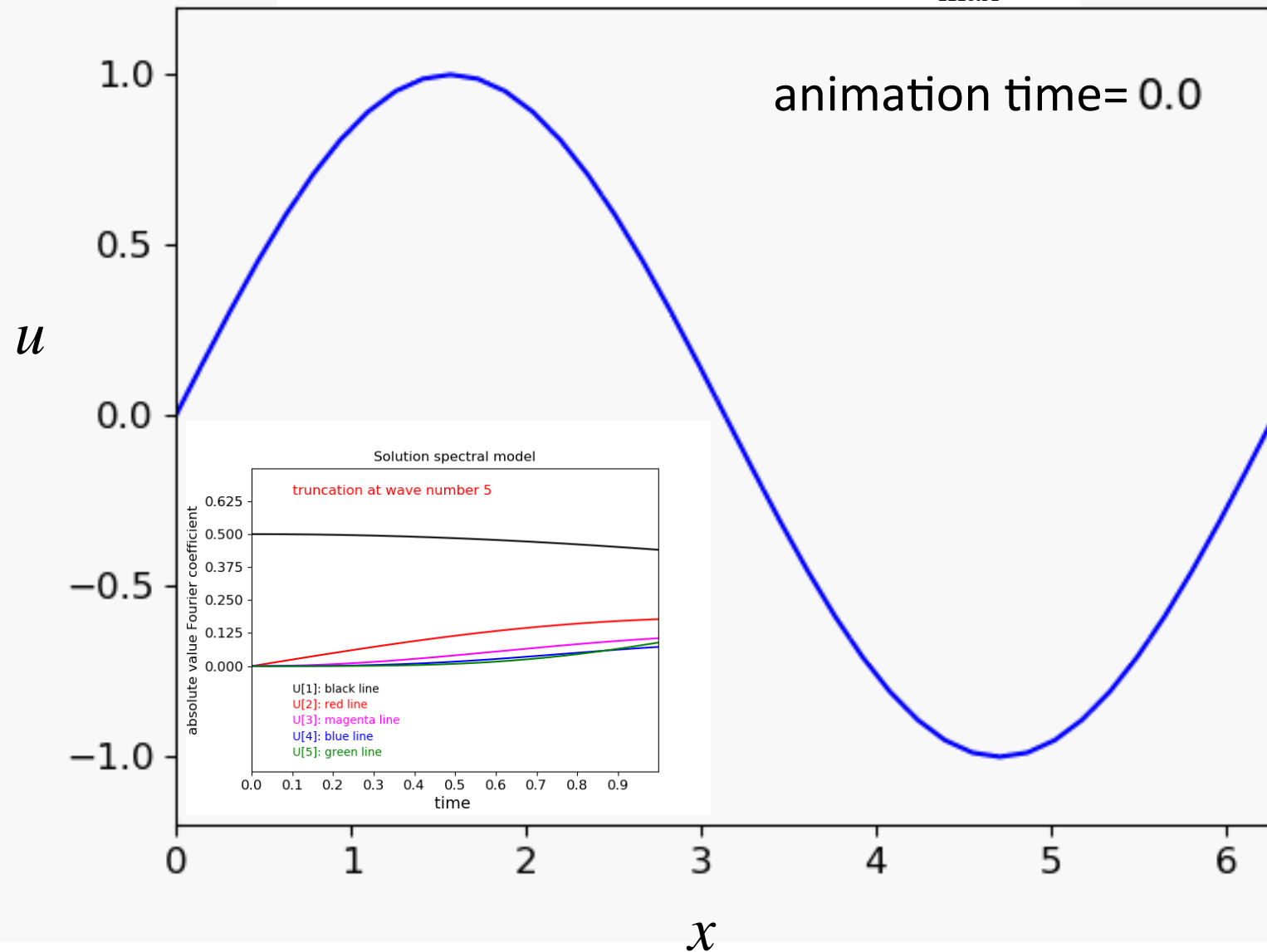
Animation of solution in physical space

Low order non-linear model: $l_{\max}=2$



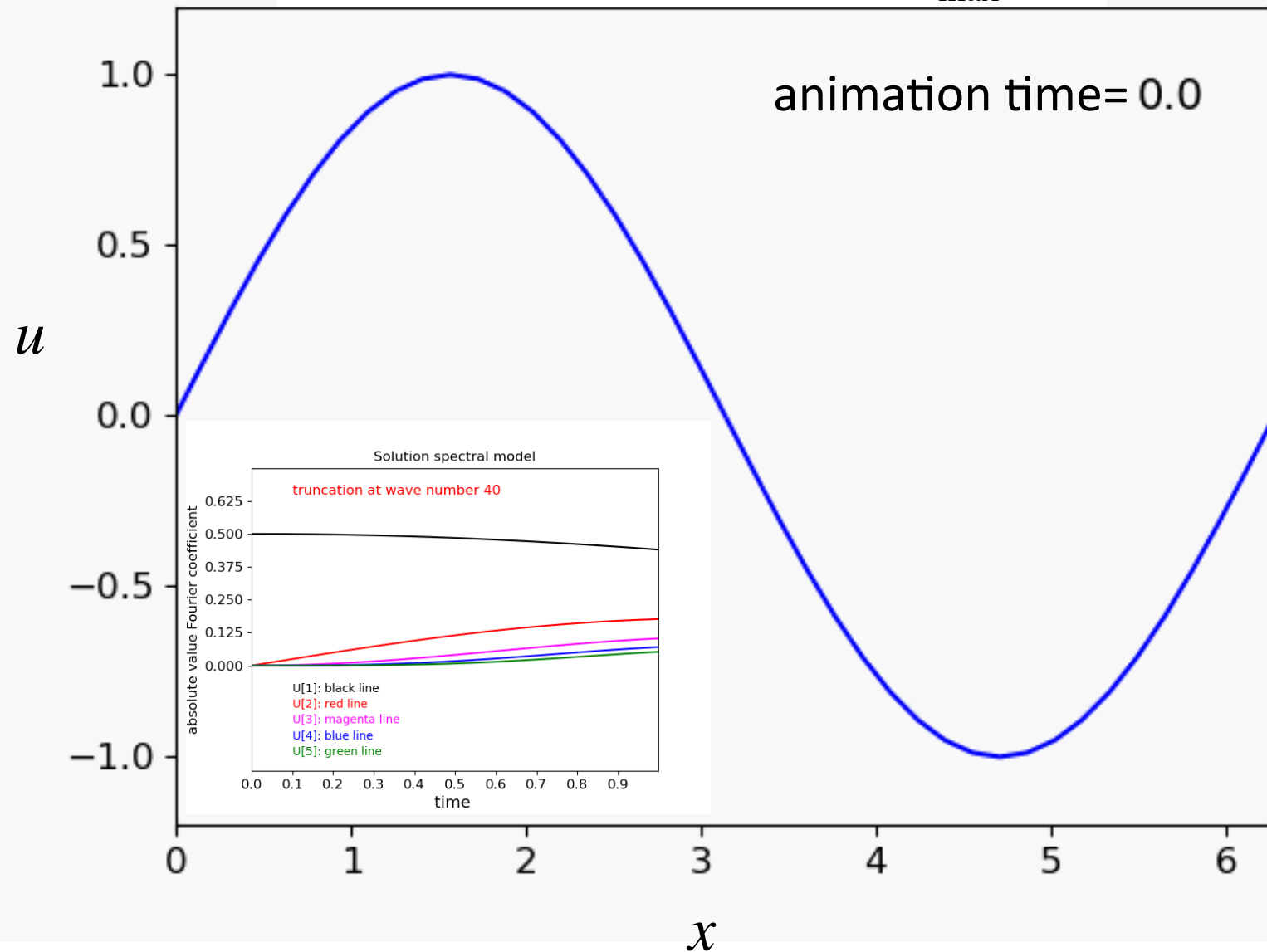
Animation of solution in physical space

Low order non-linear model: $l_{\max}=5$



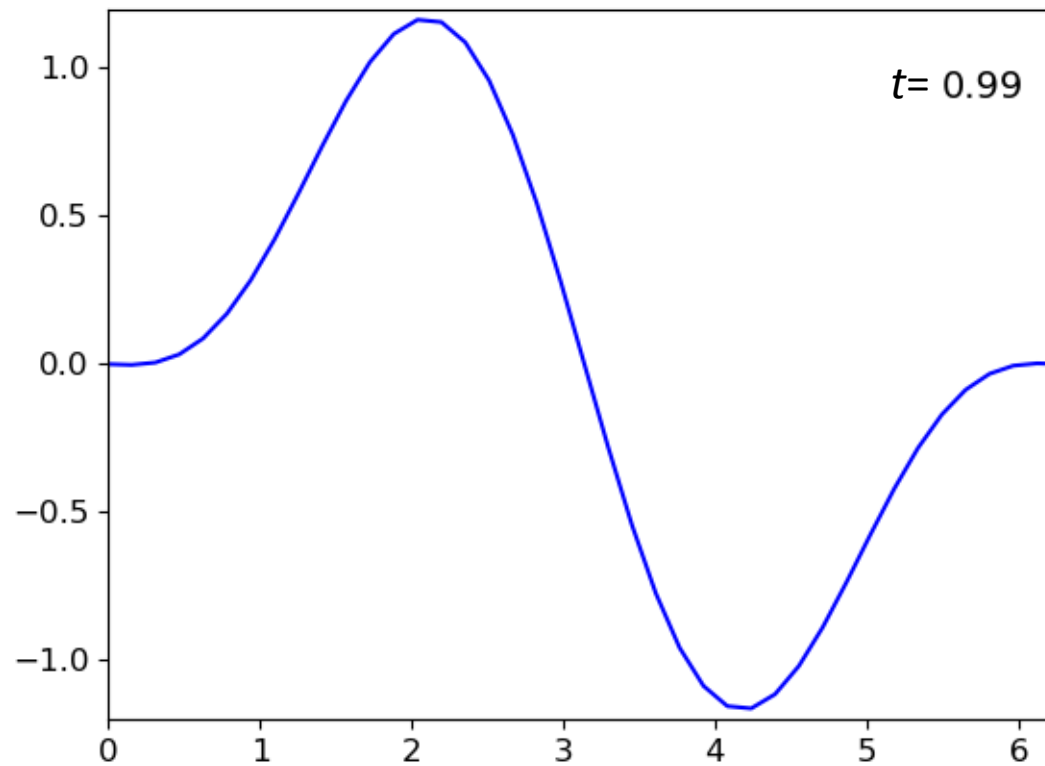
Animation of solution in physical space

Low order non-linear model: $l_{\max}=40$



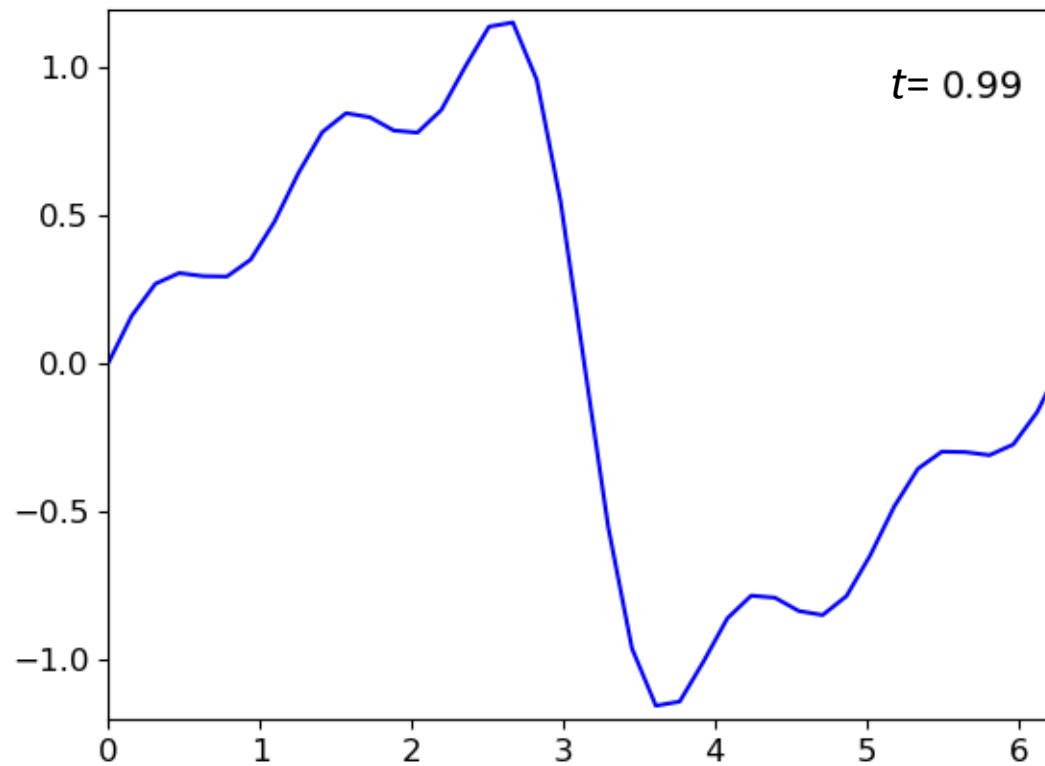
Final state at $t=0.99$

Low order non-linear model: $l_{\max}=2$



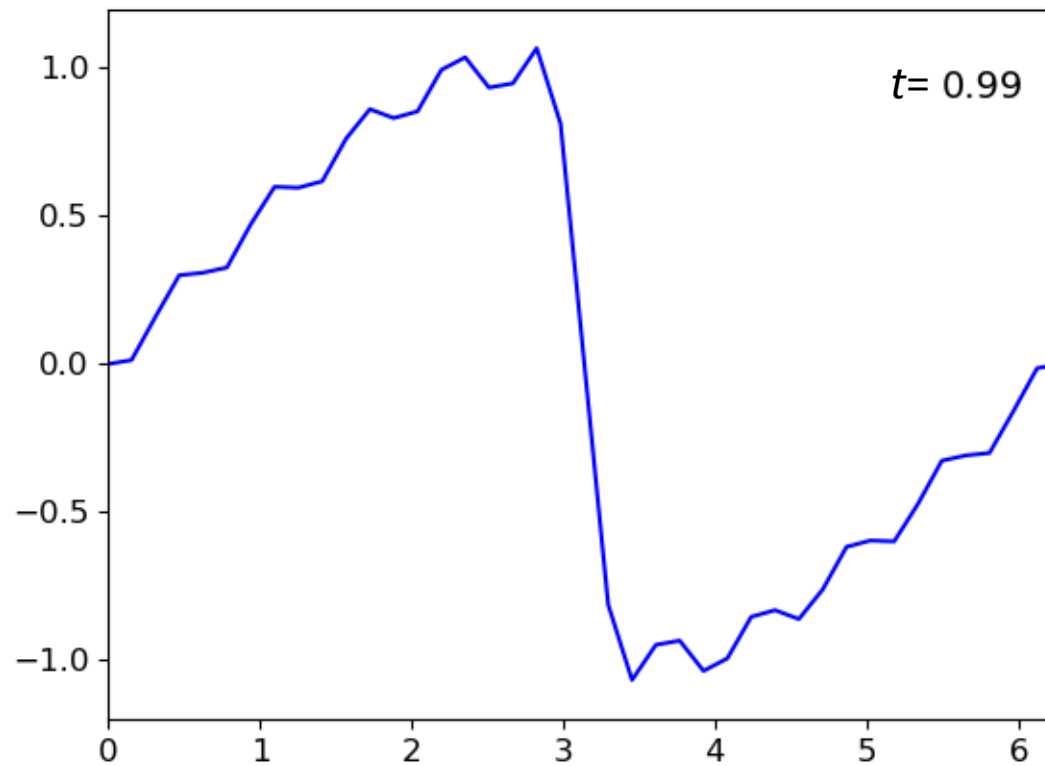
Final state at $t=0.99$

Low order non-linear model: $l_{\max}=5$



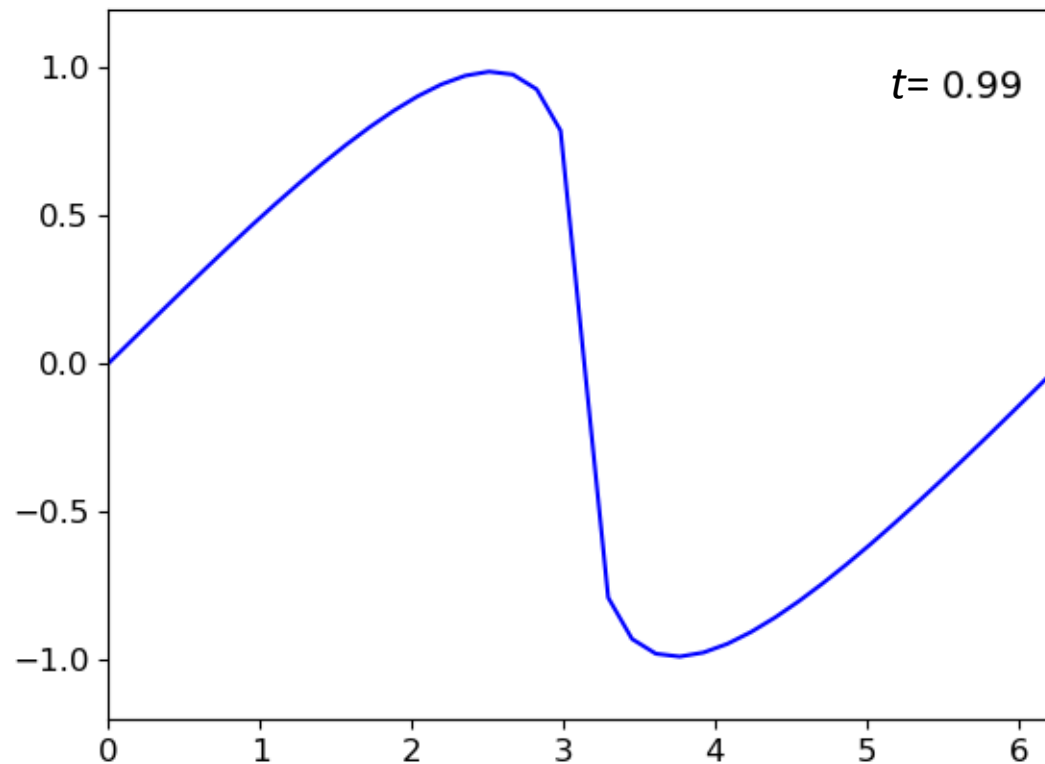
Final state at $t=0.99$

Low order non-linear model: $l_{\max}=10$

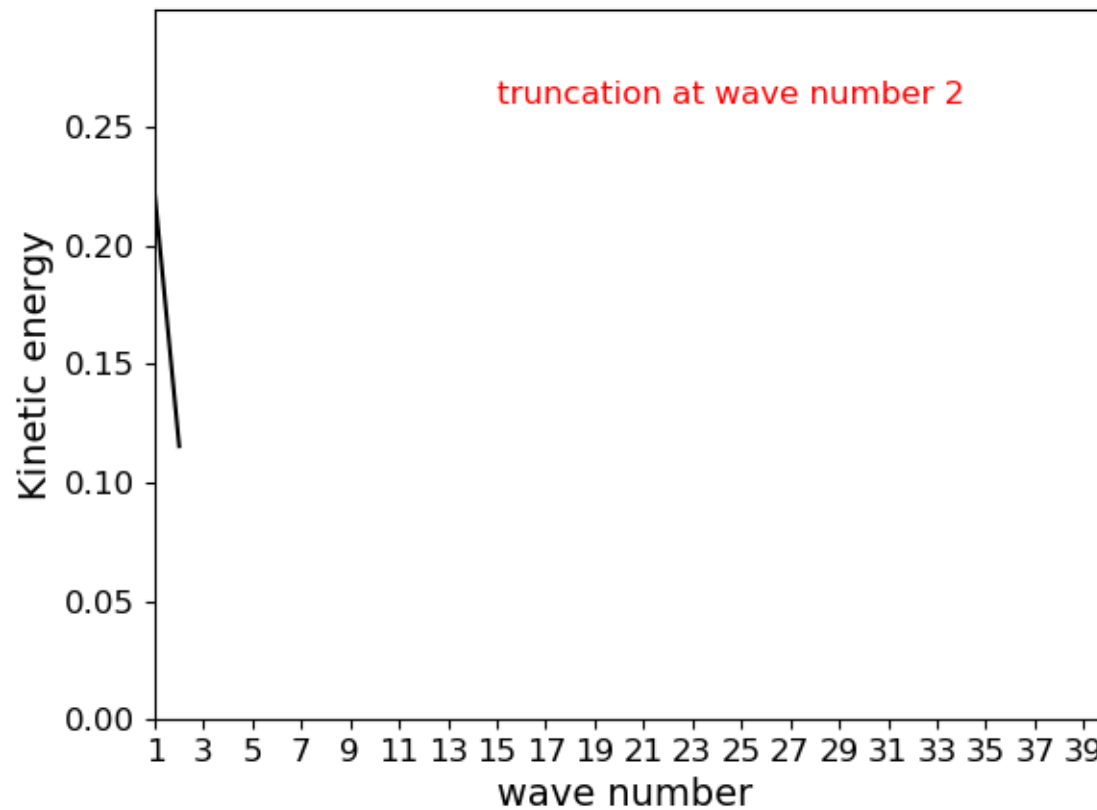


Final state at $t=0.99$

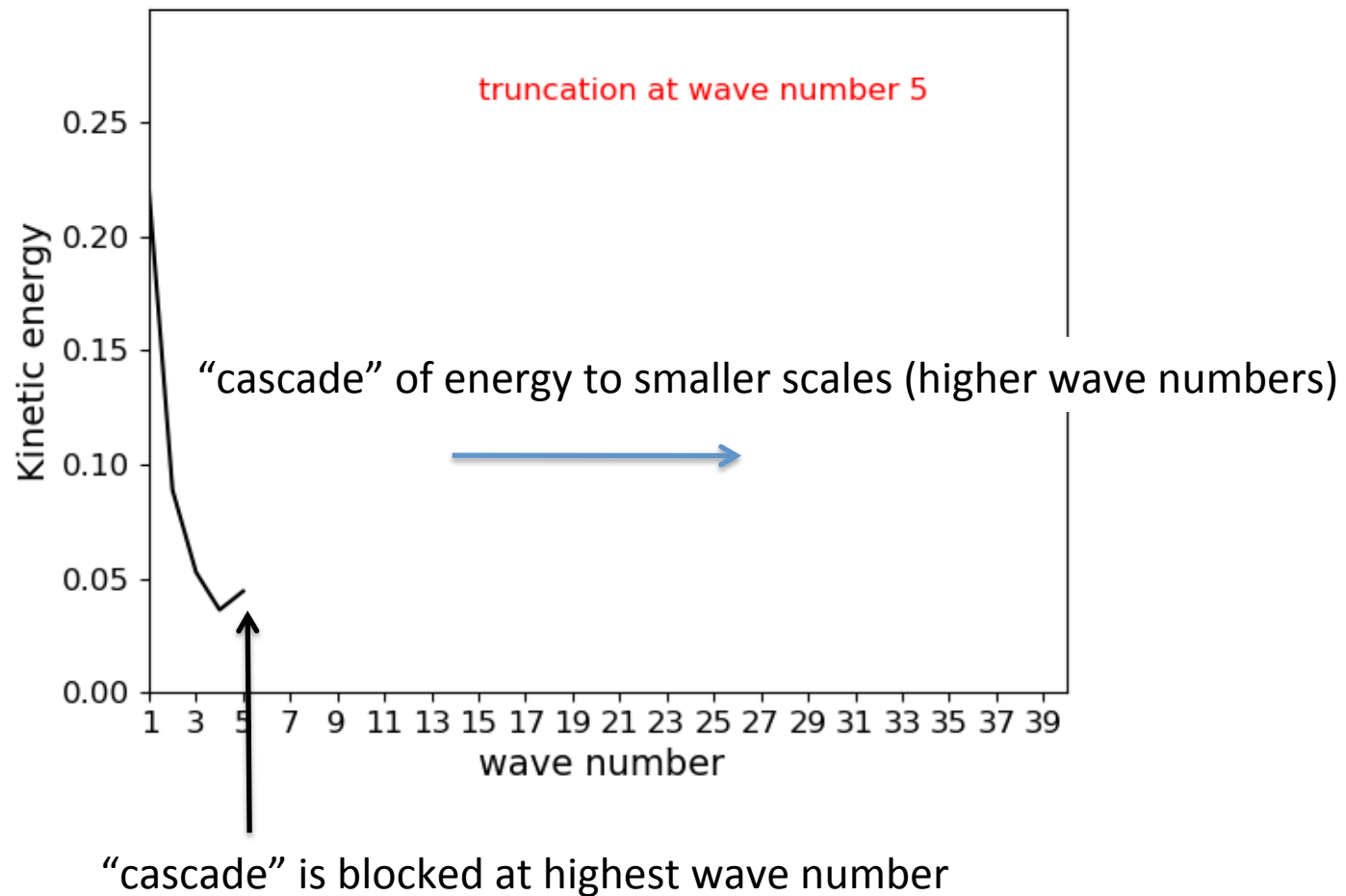
Low order non-linear model: $l_{\max}=40$



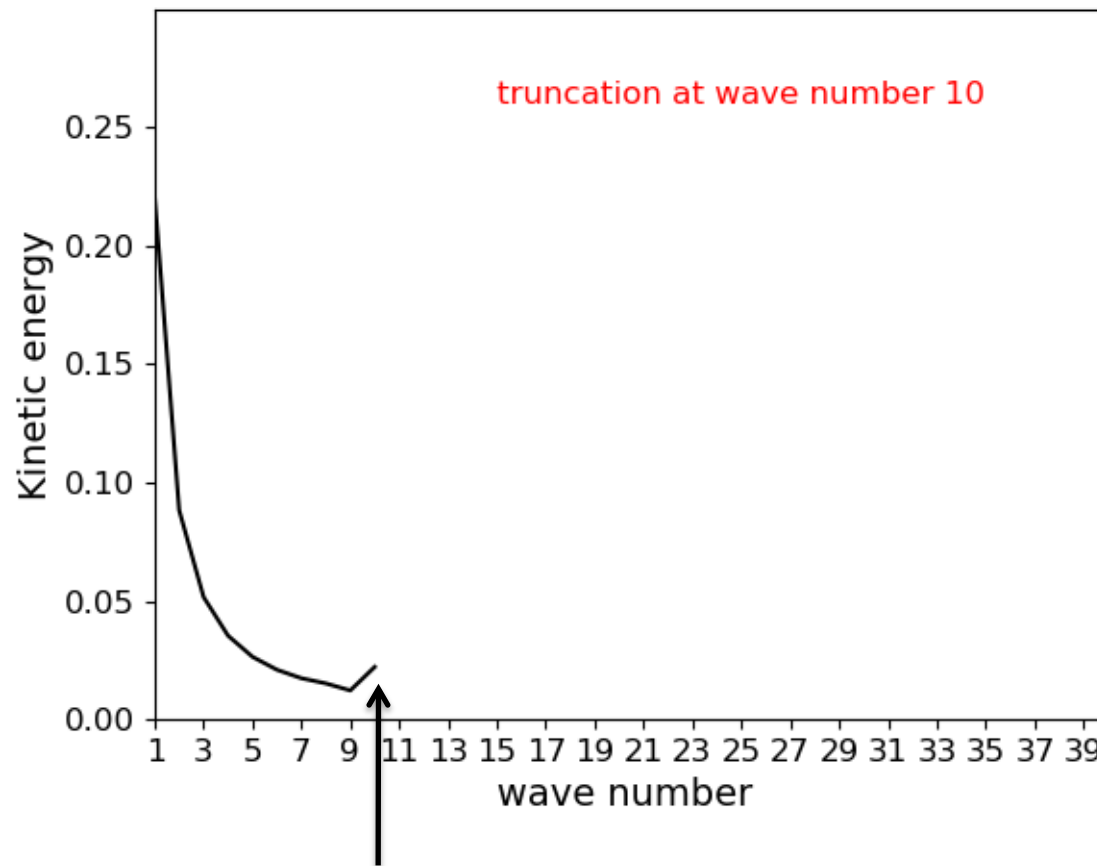
Kinetic energy spectrum, $l_{\max}=2$



Kinetic energy spectrum, $l_{\max}=5$

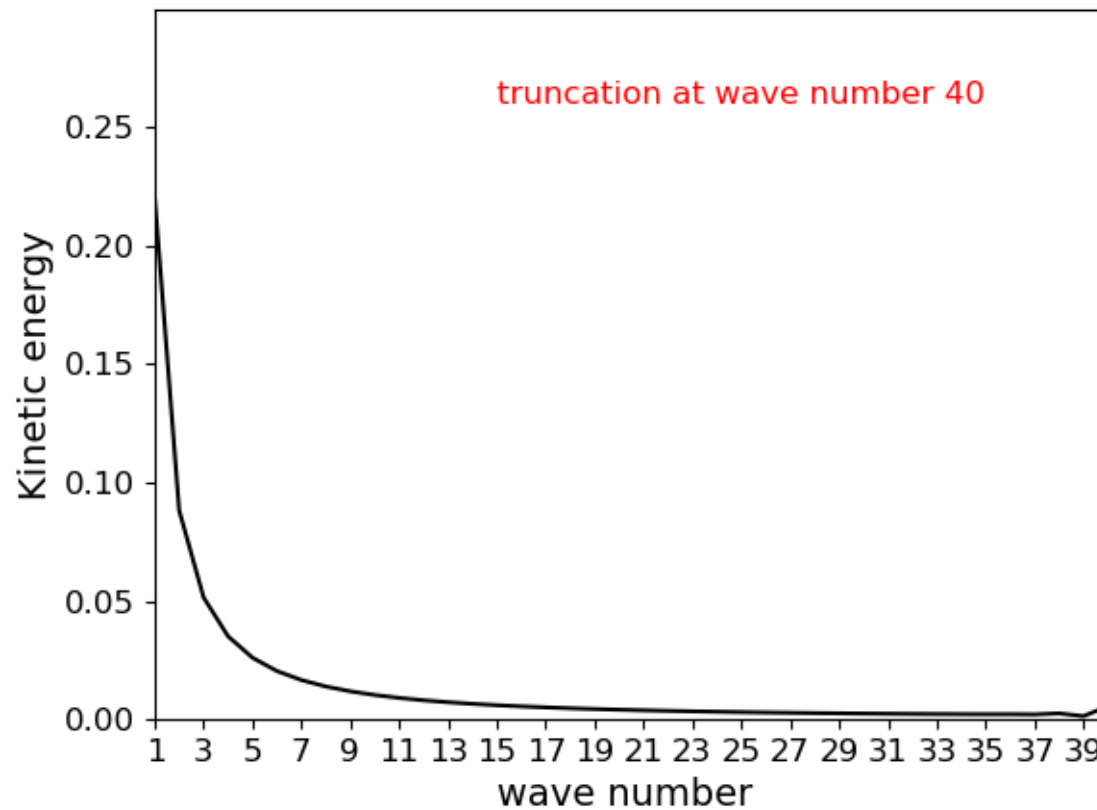


Kinetic energy spectrum, $l_{\max}=10$



“cascade” is blocked at highest wave number

Kinetic energy spectrum, $l_{\max}=40$



Concluding remarks

This lecture is intended to introduce

- 1) the **spectral method**,
- 2) to illustrate some typical properties of non-linear systems, in particular of fluid advection, such as **multiple equilibrium states** and **transfer of energy between scales of motion**.
- 3) **linear stability analysis**

Tutorial on Wednesday, 2 March 2022: exercises 1, 2 and 3