Thermal convection of a stratified fluid under the influence of gravity

Lecture 6, course on "Turbulence in Fluids", Utrecht University

http://www.staff.science.uu.nl/~delde102/TinF.htms

2 March 2022

Part 1: Equations and analysis of the stability of the state of rest

Introduction
Basic equations
Linear equations for the time-evolution of small perturbations
The problem convection between parallel horizontal boundaries
Analysis of the stability of the state of rest and the onset of convection
Exercises part 1

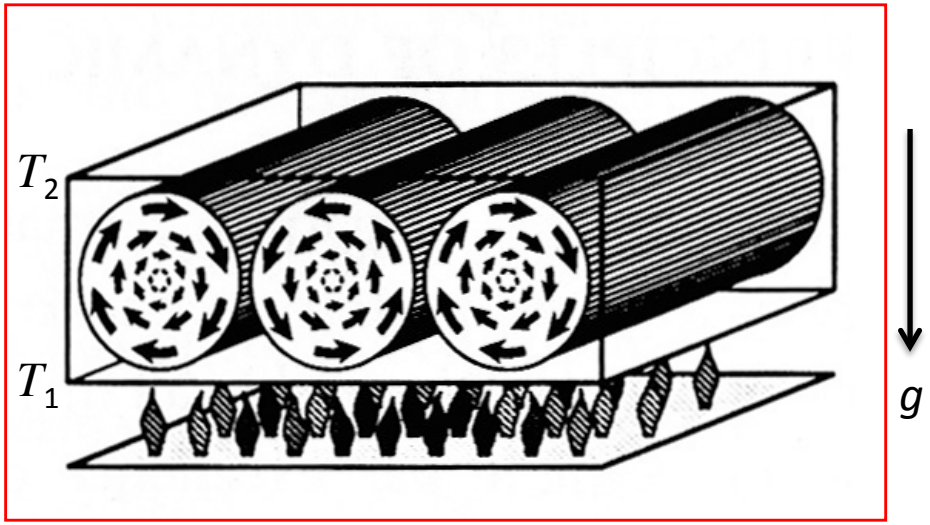
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Lecture notes: LectureNotes_Convection[TurbulenceFluids2022].pdf

Thermal convection of a stratified fluid under the influence of gravity

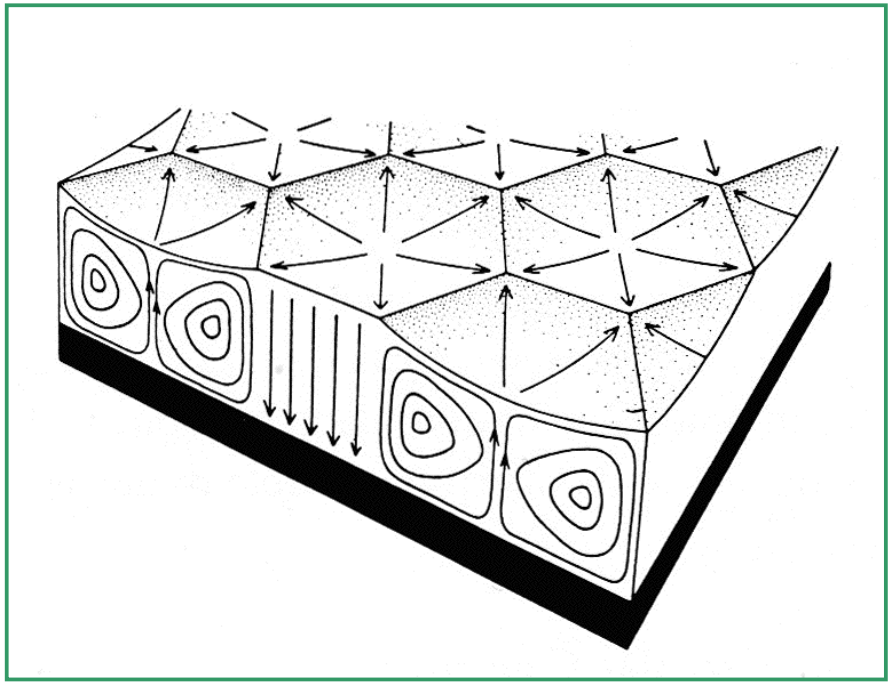
In this lecture we analyse the onset of convection currents

2-D convection cells



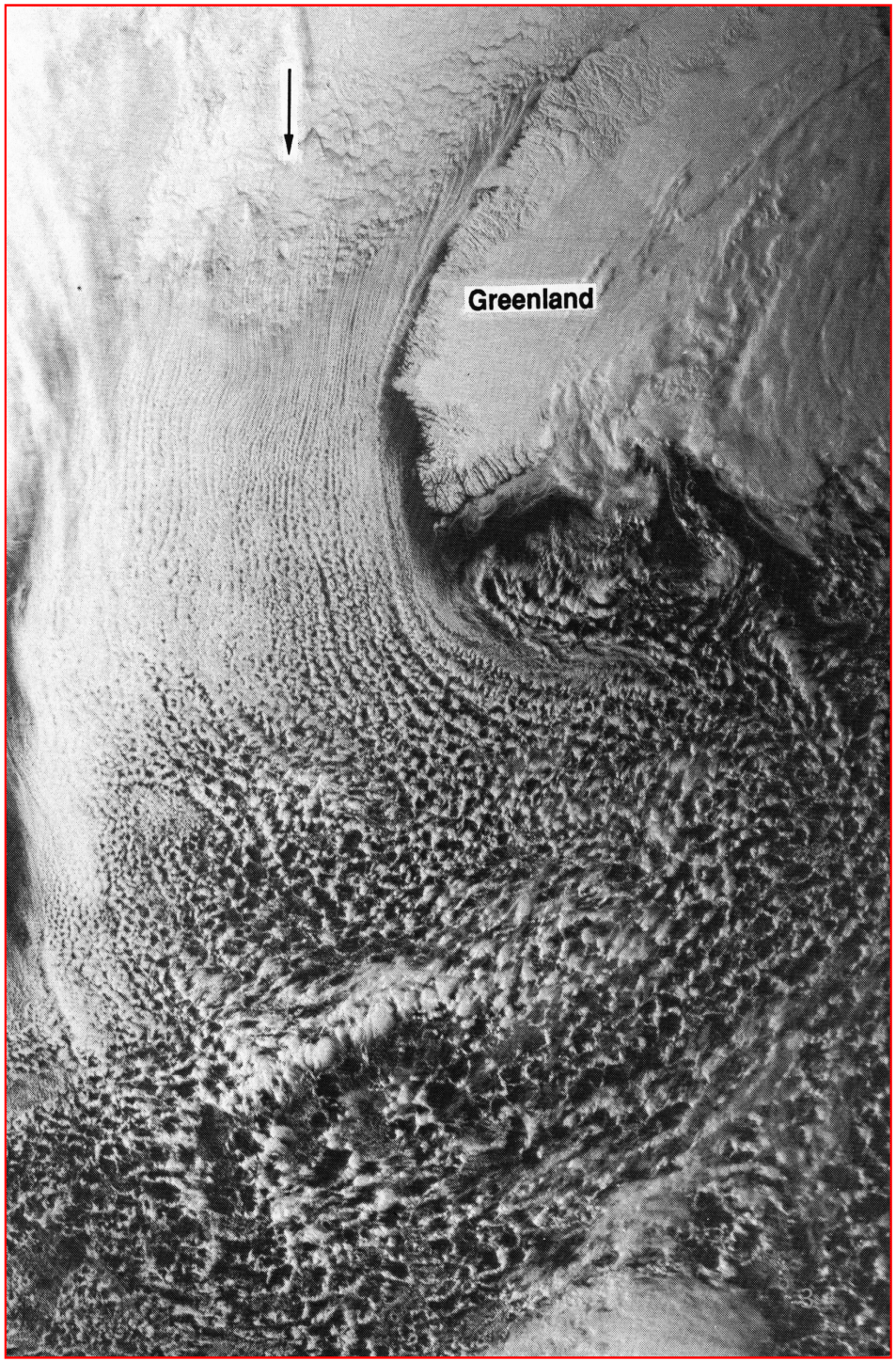
Convection if $T_1 > T_2$

3-D convection cells

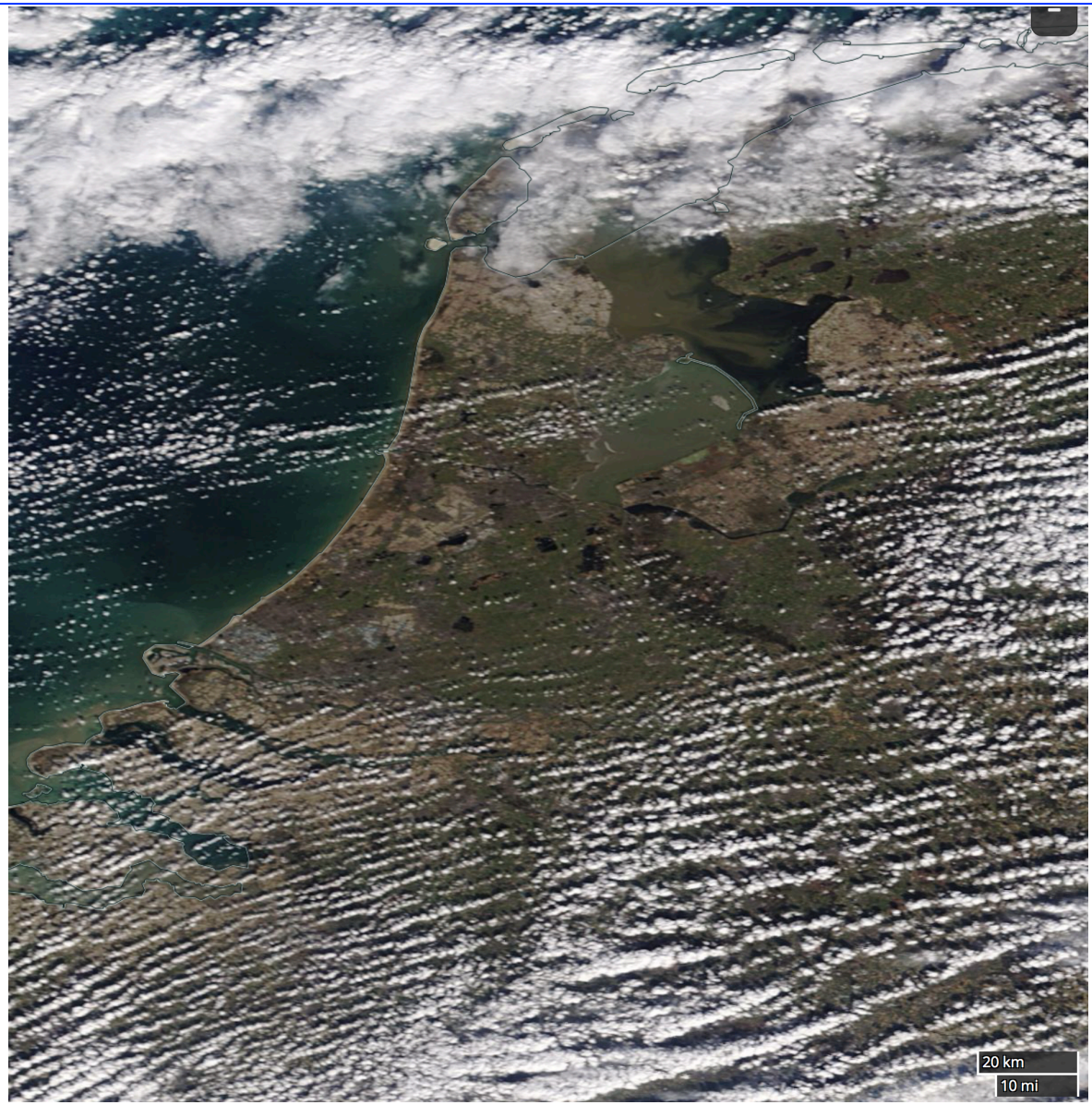


Thermal convection in the atmosphere

Two- and three-dimensional cloud patterns due to convection in the atmosphere over the North-West Atlantic Ocean and the Labrador Sea. The arrow indicates the direction of the average wind-vector. Cloud streets (two-dimensional convection) are observed in the Labrador Sea, while “open convection cells” (three-dimensional convection) are observed more towards the south over the Atlantic Ocean.



Monday 26-02-2018:

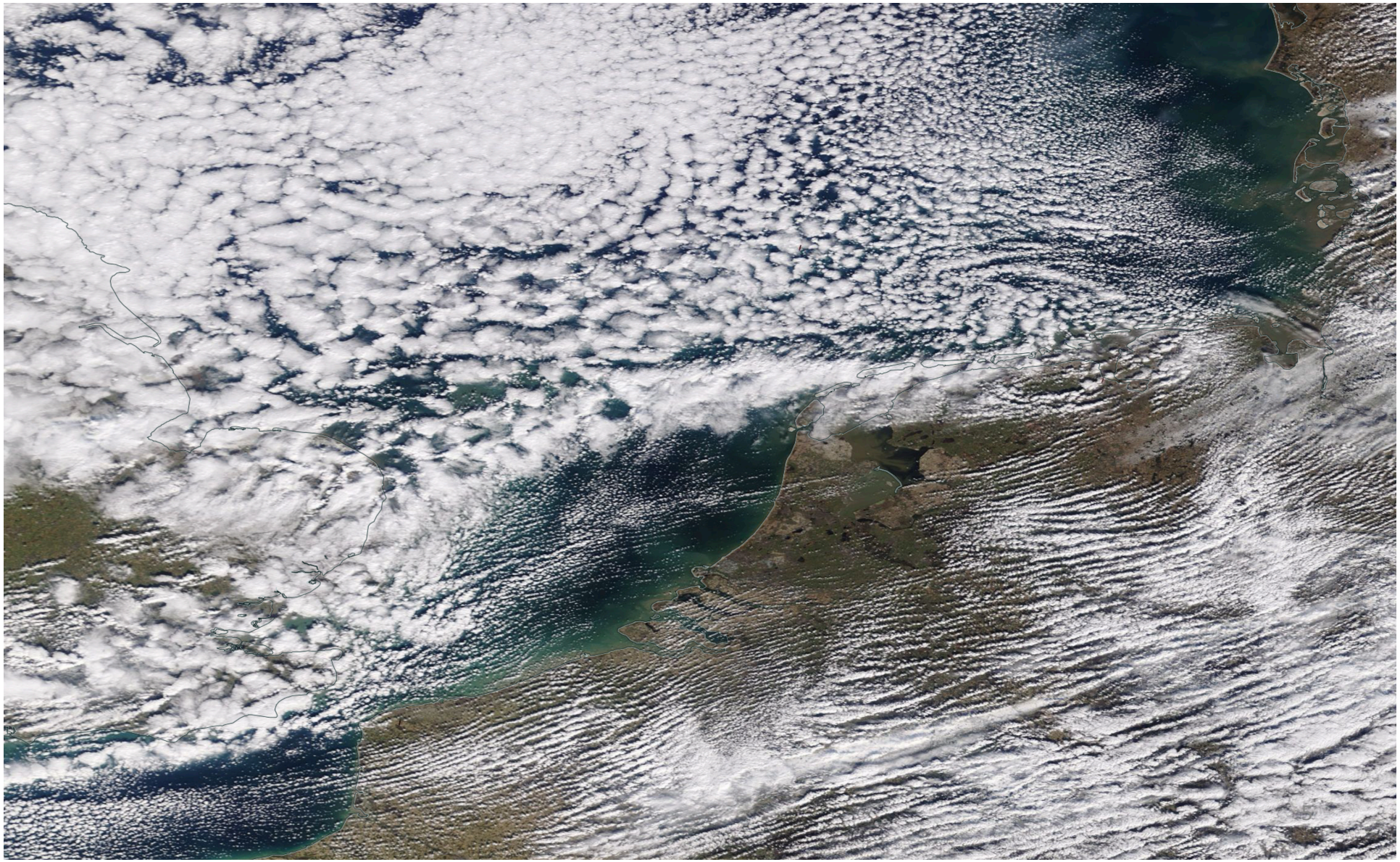


Cloud streets

Source: https://worldview.earthdata.nasa.gov

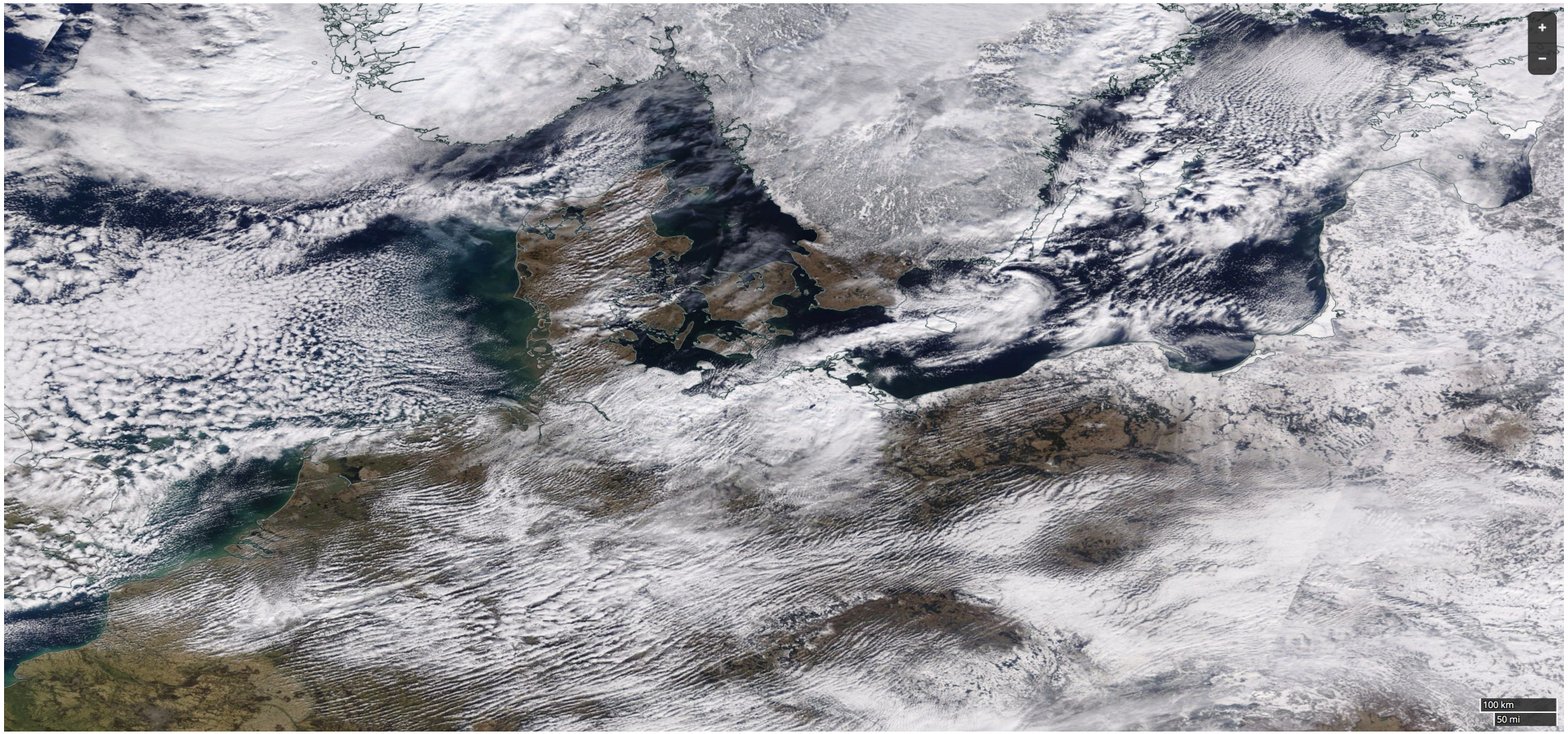
Zoom out

Monday 26-02-2018:



Zoom out

Monday 26-02-2018:



Source: https://worldview.earthdata.nasa.gov

Basic equations of fluid dynamics

(see first lecture of "Turbulence in Fluids" by Anna)

Equation of motion:

x-component:
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

y-component:
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v$$

z-component:
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w-g$$

Mass conservation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

(incompressible: density is constant)

Laplace operator: $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

A closed set of equations if we assume that **density is constant** !
In the next three weeks, we are going use these equations to study turbulence in a fluid of variable density. How is this possible?....

Linearisation around state of rest
(Boussinesq-approximation)

$$p = p_0(z) + p'(x,y,z,t)$$

(apply to all variables)

basic state + perturbation

state of rest

$$\boxed{\frac{\partial p_0}{\partial z} = -\rho_0 g}$$

(**hydrostatic balance**)

Linearisation around state of rest

(Boussinesq-approximation)

$$p = p_0(z) + p'(x,y,z,t)$$ (apply to all variables)

basic state + perturbation

state of rest

$$\frac{\partial p_0}{\partial z} = -\rho_0 g$$ (hydrostatic balance)

Vertical component of equation of motion:

Slide 7: → $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w - g$$

Multiply by ρ , substitute hydrostatic equation and assume that $\rho' << \rho_0$

Assume that perturbations in the velocity are so small that non-linear terms can be neglected

$$\rho_0 \frac{\partial w'}{\partial t} \approx -\frac{\partial p'}{\partial z} + \rho_0 \nu\nabla^2 w' - g\rho'$$

Buoyancy in terms of temperature (using the equation of state)

Previous slide: $$\rho_0 \frac{\partial w'}{\partial t} \approx -\frac{\partial p'}{\partial z} + \rho_0 \nu \nabla^2 w' - g\rho'$$

Equation of state: $\rho = \rho_0 (1 - \alpha(T - T_0))$

$$\rho' = -\alpha \rho_0 T'$$

Gay-Lussac: $V = V_0 (1 + \alpha(T - T_0))$

α = **thermal expansion coefficient**

$$V = \frac{m}{\rho} : \rho \approx \rho_0 (1 - \alpha(T - T_0))$$

α is a very small number!

Buoyancy: $g\rho' = -g\alpha\rho_0 T'$

$$\frac{\partial w'}{\partial t} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \nu \nabla^2 w' + g\alpha T'$$

→ next slide

Eliminate pressure with the continuity equation

Previous slide:

$$\frac{\partial w'}{\partial t} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \nu\nabla^2 w' + g\alpha T'$$

Now: eliminate pressure using the continuity equation as follows

Eliminate pressure with the continuity equation

First: $\nabla^2 \times$ $\frac{\partial w'}{\partial t} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \nu\nabla^2 w' + g\alpha T'$

$$\frac{\partial \nabla^2 w'}{\partial t} \approx -\frac{1}{\rho_{ref}}\frac{\partial \nabla^2 p'}{\partial z} + \nu\nabla^2\nabla^2 w' + g\alpha\nabla^2 T' \quad (1)$$

!!

Assume $\rho_0 = \rho_{ref}$ **=constant**. Makes first term on r.h.s. linear.

Eliminate pressure with the continuity equation

First: $\nabla^2 \times$ $\frac{\partial w'}{\partial t} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \nu\nabla^2 w' + g\alpha T'$

$$\frac{\partial \nabla^2 w'}{\partial t} \approx -\frac{1}{\rho_{ref}}\frac{\partial \nabla^2 p'}{\partial z} + \nu\nabla^2\nabla^2 w' + g\alpha\nabla^2 T' \quad (1)$$

!!

Second:

$\frac{\partial}{\partial x}$ $\frac{\partial u'}{\partial t} \approx -\frac{1}{\rho_{ref}}\frac{\partial p'}{\partial x} + \nu\nabla^2 u'$

$\frac{\partial}{\partial y}$ $\frac{\partial v'}{\partial t} \approx -\frac{1}{\rho_{ref}}\frac{\partial p'}{\partial y} + \nu\nabla^2 v'$

$\frac{\partial}{\partial z}$ $\frac{\partial w'}{\partial t} \approx -\frac{1}{\rho_{ref}}\frac{\partial p'}{\partial z} + \nu\nabla^2 w' + g\alpha T'$

!!

use continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (slide 7)

Adding three equations: $\nabla^2 p' = g\alpha\rho_{ref}\frac{\partial T'}{\partial z}$ (2)

Eliminate pressure with the continuity equation

(1) $$\frac{\partial \nabla^2 w'}{\partial t} \approx -\frac{1}{\rho_{ref}} \frac{\partial \nabla^2 p'}{\partial z} + \nu \nabla^2 \nabla^2 w' + g\alpha \nabla^2 T'$$

(2) $$\nabla^2 p' = g\alpha \rho_{ref} \frac{\partial T'}{\partial z}$$

Eliminate p'

$$\nabla_H{}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial \nabla^2 w'}{\partial t} \approx \nu \nabla^2 \nabla^2 w' + g\alpha \nabla_H{}^2 T'$$

Linear differential equation with **<u>two unknowns</u>**: w', T'

We need an additional equation for the temperature, T

Fourier's Law of thermal conduction

Heat flux, q, is proportional to temperature-gradient: $q = \kappa \nabla T$

Thermal conductivity

Apply Fourier's law to a fluid parcel: $\frac{dT}{dt} = \kappa \nabla^2 T$

Divergence of the heat flux

Material time-derivative of the temperature

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Local time-derivative of the temperature

Temperature equation

Previous slide:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa\nabla^2 T$$

Linearise about the state of rest for which, $\frac{\partial T_0(z)}{\partial z} = \frac{dT_0(z)}{dz} \equiv -\Gamma$

"**Lapse rate**":

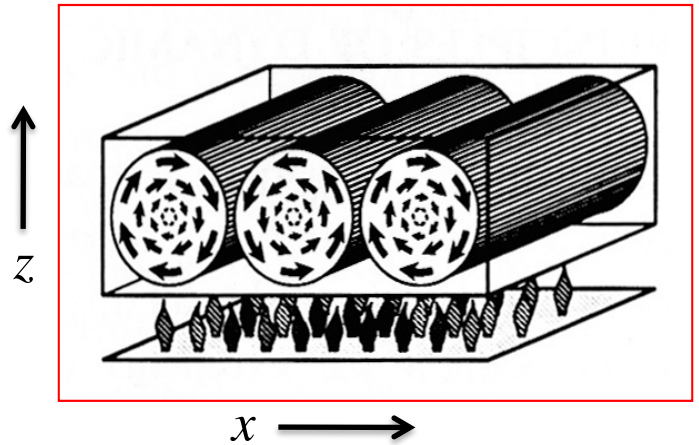
$$\frac{\partial T'}{\partial t} = \Gamma w' + \kappa\nabla^2 T'$$

Together with (previous slide)

$$\frac{\partial \nabla^2 w'}{\partial t} \approx \nu\nabla^2\nabla^2 w' + g\alpha\nabla_H{}^2 T'$$

Closed set of equations for time evolution of small perturbations to the state of rest

Restrict flow to two dimensions, x and z



Previous slide:

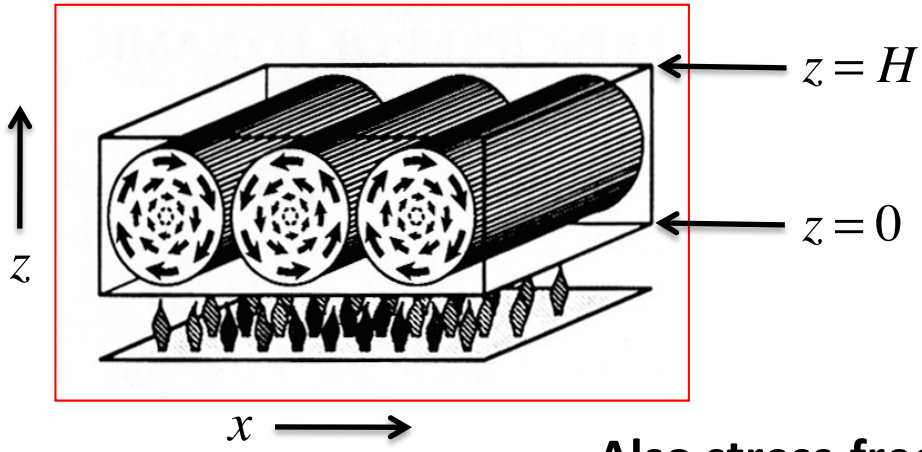
$$\frac{\partial T'}{\partial t} = \Gamma w' + \kappa \nabla^2 T'$$

$$\frac{\partial T'}{\partial t} = \Gamma w' + \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) T'$$

$$\frac{\partial \nabla^2 w'}{\partial t} \approx \nu \nabla^2 \nabla^2 w' + g\alpha \nabla_H{}^2 T'$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) w' = g\alpha \frac{\partial^2 T'}{\partial x^2} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)^2 w'$$

Boundary conditions

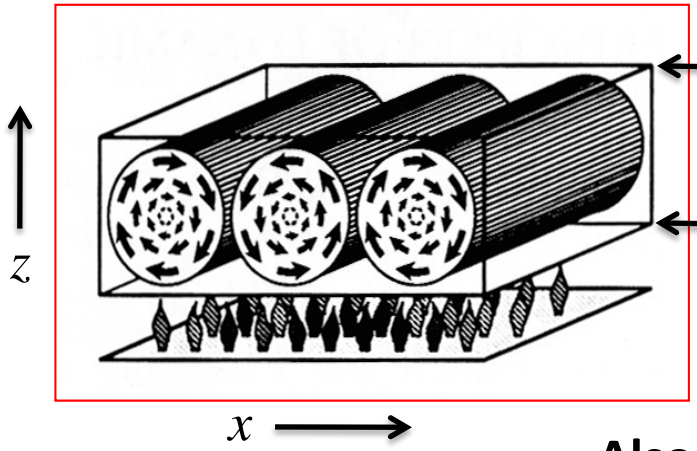


Closed upper and lower boundaries:

$$w' = 0 \text{ at } z = 0 \text{ and } z = H$$

Also stress-free: $\dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = H$

Boundary conditions



Closed upper and lower boundaries:

$$w' = 0 \text{ at } z = 0 \text{ and } z = H$$

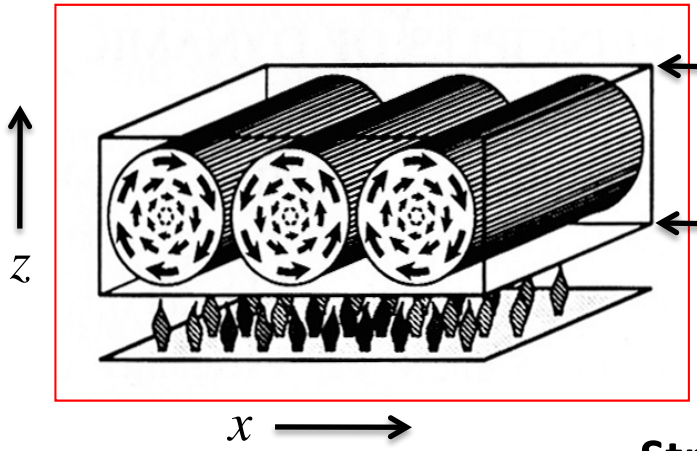
Also stress-free: $\dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = H$

Continuity equation implies $\dfrac{\partial^2 w'}{\partial z^2} = 0 \text{ at } z = 0 \text{ and } z = H$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(slide 7)

Boundary conditions



$$w' = 0 \text{ at } z = 0 \text{ and } z = H$$

Stress-free: $$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = H$$

Continuity equation implies $$\frac{\partial^2 w'}{\partial z^2} = 0 \text{ at } z = 0 \text{ and } z = H$$

Perfectly conducting: $T' = 0$ at $z = 0$ and $z = H$

With $$\frac{\partial T'}{\partial t} = \Gamma w' + \kappa \nabla^2 T'$$ this implies $$\frac{\partial^2 T'}{\partial z^2} = 0 \text{ at } z = 0 \text{ and } z = H$$

Boundary conditions

$$w' = 0 \text{ at } z = 0 \text{ and } z = H$$

$$T' = 0 \text{ at } z = 0 \text{ and } z = H$$

$$\frac{\partial^2 w'}{\partial z^2} = 0 \text{ at } z = 0 \text{ and } z = H$$

$$\frac{\partial^2 T'}{\partial z^2} = 0 \text{ at } z = 0 \text{ and } z = H$$

Assume: periodic boundary conditions in horizontal direction

Boundary conditions are satisfied if solution is of the form:

$$w' = W \exp\{ilx + \sigma t\} \sin\left(\frac{\pi n z}{H}\right); \quad T' = \Theta \exp\{ilx + \sigma t\} \sin\left(\frac{\pi n z}{H}\right)$$



Substitute solution into equations

→ Solution: $w' = W \exp\{ilx + \sigma t\} \sin\left(\frac{\pi n z}{H}\right)$; $T' = \Theta \exp\{ilx + \sigma t\} \sin\left(\frac{\pi n z}{H}\right)$

amplitude (complex) — wave number — growth rate — wave number

Equations: {

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) w' = g\alpha \frac{\partial^2 T'}{\partial x^2} + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2 w'$$

$$\frac{\partial T'}{\partial t} = \Gamma w' + \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) T'$$

(slide 17)

Substitute solution → {

$$-\sigma k^2 W = -g\alpha l^2 \Theta + \nu k^4 W$$

$$\sigma \Theta = \Gamma W - \kappa k^2 \Theta$$

"total wave-number", k :

$$k^2 \equiv l^2 + \left(\frac{\pi n}{H}\right)^2$$

Matrix notation

Previous slide:

$$-\sigma k^2 W = -g\alpha l^2 \Theta + \nu k^4 W$$

$$\sigma\Theta = \Gamma W - \kappa k^2 \Theta$$

System of linear equations can be written as….

$$MX = 0$$

with:

$$X \equiv \begin{pmatrix} W \\ \Theta \end{pmatrix}$$

$$M \equiv \begin{pmatrix} -\sigma k^2 - \nu k^4 & g\alpha l^2 \\ \Gamma & -\sigma - \kappa k^2 \end{pmatrix}$$

Non-trivial solution if determinant of M = 0

det(M)=0

Previous slide: $M = \begin{pmatrix} -\sigma k^2 - \nu k^4 & g\alpha l^2 \\ \Gamma & -\sigma - \kappa k^2 \end{pmatrix}$

Yields a **quadratic equation** in σ: $\sigma^2 + (\nu+\kappa)k^2\sigma + \nu\kappa k^4 - \frac{g\alpha l^2 \Gamma}{k^2} = 0$

Solve for σ: $\sigma = -\frac{(\nu+\kappa)k^2}{2} \pm \frac{1}{2}\left((\nu+\kappa)^2 k^4 - 4\left(\nu\kappa k^4 - \frac{g\alpha l^2 \Gamma}{k^2} \right) \right)^{1/2}$

Amplitude of the wave-like **perturbation will grow if $\sigma > 0$** (instability)

$$\sigma > 0 \text{ if } \quad -\left(\nu\kappa k^4 - \frac{g\alpha l^2 \Gamma}{k^2} \right) > 0$$

→ Therefore, **instability if** $\Gamma > \frac{\nu\kappa k^6}{g\alpha l^2} = \frac{\nu\kappa}{g\alpha l^2}\left(l^2 + \left(\frac{\pi n}{H} \right)^2 \right)^3$

Critical lapse rate

Previous slide:

$$\Gamma > \frac{\nu\kappa k^6}{g\alpha l^2} = \frac{\nu\kappa}{g\alpha l^2}\left(l^2 + \left(\frac{\pi n}{H}\right)^2\right)^3 \longrightarrow \Gamma_{crit} = \frac{\nu\kappa}{g\alpha l^2}\left(l^2 + \left(\frac{\pi n}{H}\right)^2\right)^3$$

Γ_{crit} has a **minimum** value if $n = 1$

and if $\dfrac{\partial \Gamma_{crit}}{\partial\left(l^2\right)} = 0 \longrightarrow l^2 = \dfrac{1}{2}\left(\dfrac{\pi}{H}\right)^2,$

with $l = \dfrac{2\pi}{L} \longrightarrow \boxed{L = 2\sqrt{2}H}$ ("preferred wavelength")

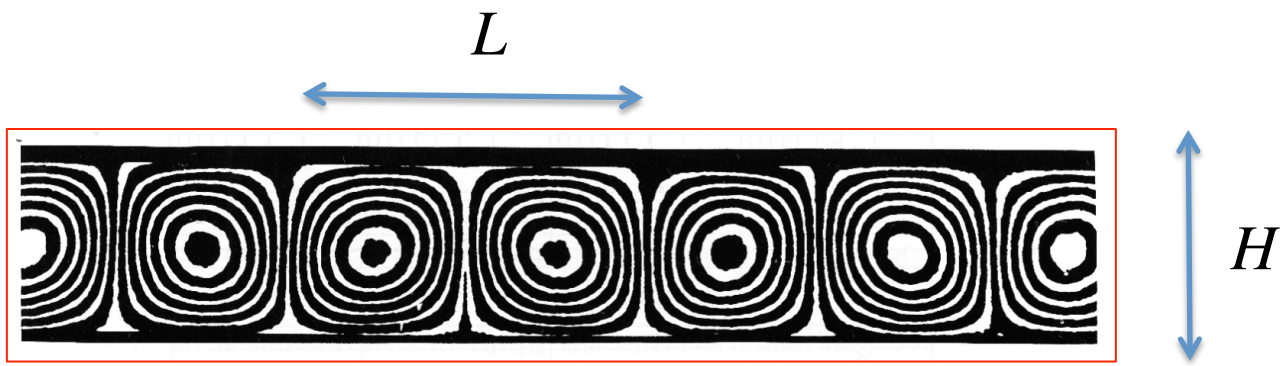
Exercise 2 (slide 29):
Express this criterion in terms of the non-dimensional ***Rayleigh number****:*

$$\mathrm{Ra} \equiv \frac{g\alpha\Gamma H^4}{\kappa\nu}$$

Aspect ratio, *L*/*H*, of convection cells

Previous slide:

$$L/H = 2\sqrt{2} \cong 2.8$$



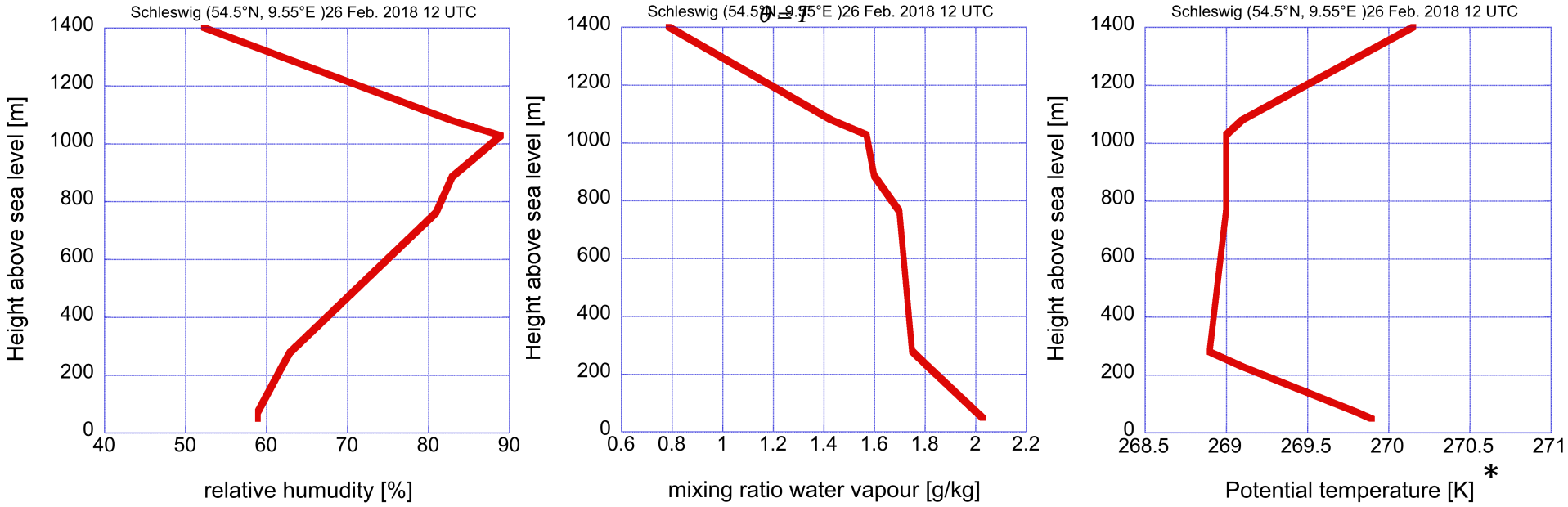
Side-view (differential interferograms) of two-dimensional convection cells in silicone oil in a rectangular box of relative dimensions 10:4:1, which is heated from below. Uniform heating produces convection “rolls” parallel to the shorter side.

Tutorial : exercises 1

(1)

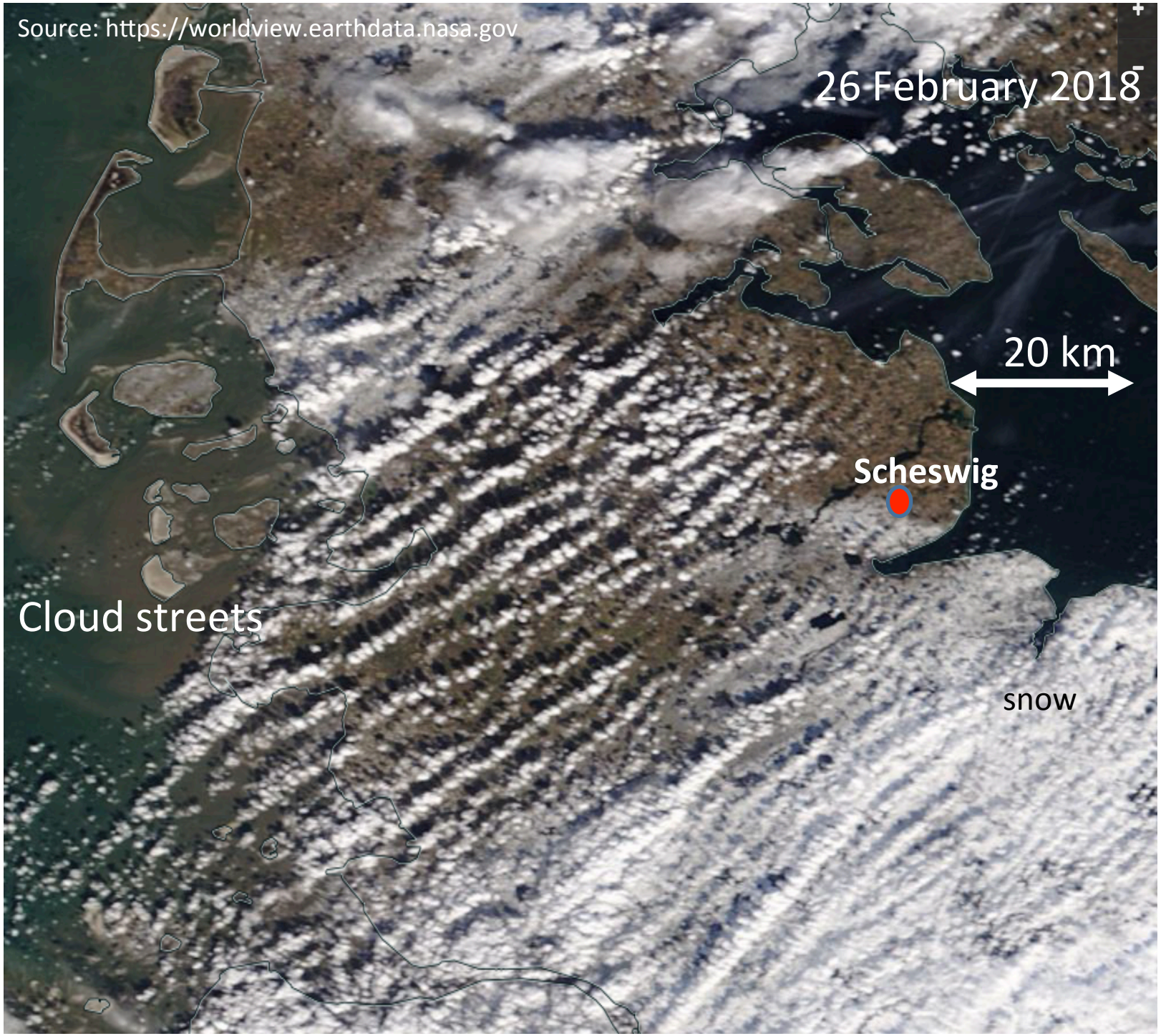
The distance between “cloud streets” over Northern Germany, seen on the satellite image on the next slide, is about 5-10 km. It can be deduced from the upper air observations in this area (at Schleswig), shown below, that cloud tops are located at levels between 1000 and 1250 m. How can we see this? What is the aspect ratio of the associated convection cells? This does not agree with the prediction made by Lord Rayleigh’s theory. Give at least three reasons for the disagreement?

Observations made at Schleswig of relative humidity, mixing ratio of water vapour and potential temperature



*potential temperature is materially conserved under adiabatic circumstances

Source of the data: http://weather.uwyo.edu


Source: https://worldview.earthdata.nasa.gov
26 February 2018
20 km
Scheswig
Cloud streets
snow

Tutorial: exercises 2 to 4

(after lecture 7)

(2)

The instability criterion (41) is commonly expressed in terms of the so-called **Rayleigh number**, Ra, which is a non-dimensional number, defined as

$$\mathrm{Ra} \equiv \frac{g\alpha\Gamma H^4}{\kappa\nu} \qquad (47)$$

What is the **critical value of the Rayleigh number** at the preferred horizontal wavelength? Plot the marginal or neutral value (at which the growth-rate is zero) of the Rayleigh number as a function of the aspect ratio, *L/H*, where the Rayleigh number is the ordinate. Identify the region of instability in this graph. Will this curve shift upwards or downwards if no-slip conditions are imposed on the upper and lower boundaries, instead of stress-free conditions (section 4)?

(3)

At which minimum Rayleigh number is a “mode” with *n*=2 unstable.

(4)

See page 11-12 of the lecture notes