

Dynamical Meteorology: lecture 6

Sections 1.17-1.18

Further introduction to project 3 (problem 1.14)

Box 1.9: Numerical approximation and model Python code

Box 1.10: model verification

Inertial oscillations and inertial frequency

Geostrophic balance and geostrophic flow

Introduction to stability of geostrophic balance

Tutorial after this lecture

Project 3

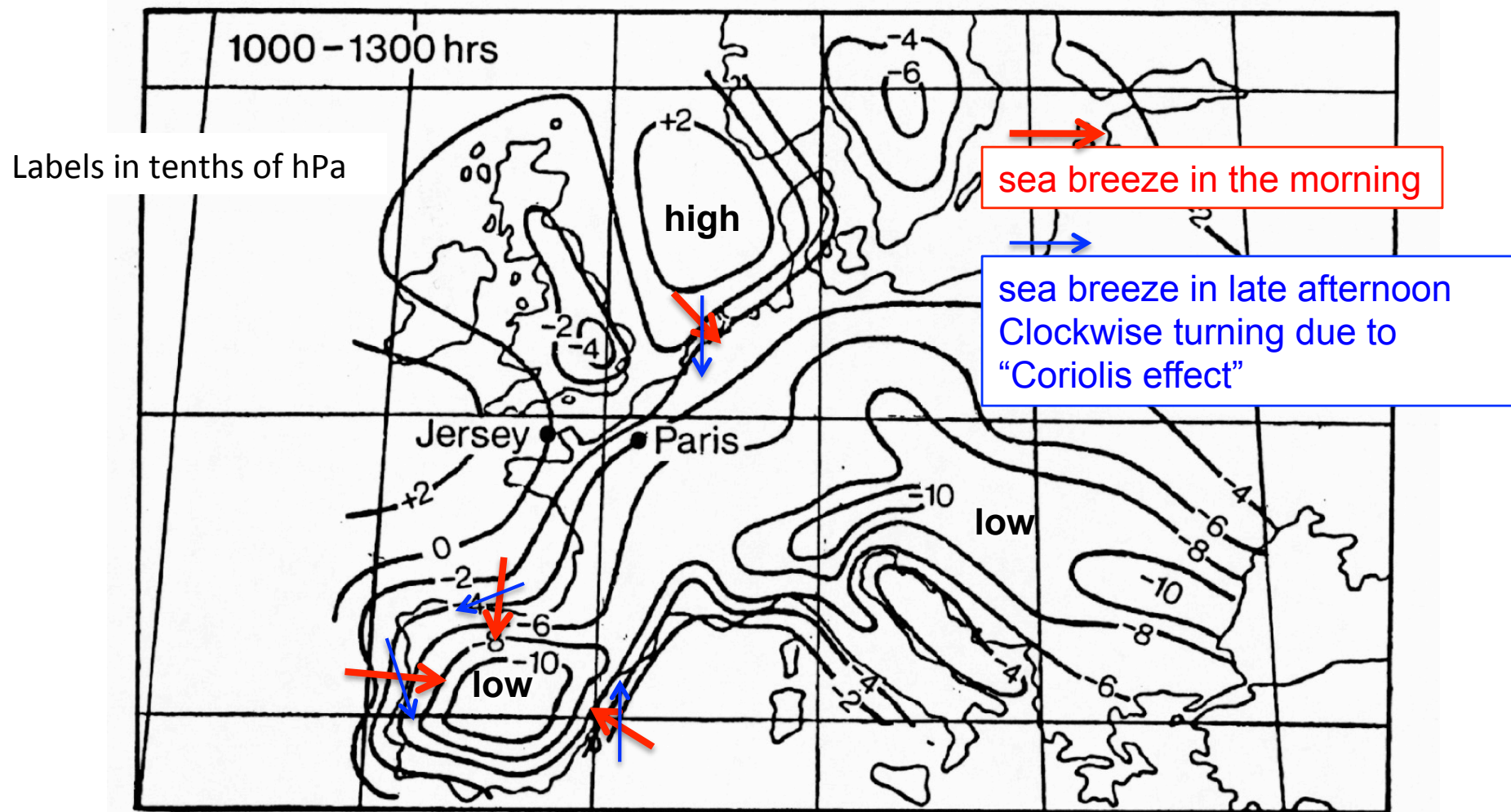
Exercises of Part C (the tephigram)

Deadline for project 3 is Fri. 9 December 2022

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(<http://www.uu.nl/~nvdelden/dynmeteorology.htm>)

Surface pressure change between 10 and 13 UTC in June and the sea breeze



See **figure 1.32**

Introduction to project 3 (problem 1.14)

Simulation of the sea breeze and its clockwise turning at the coast during two days in May

Compare the model outcome with the observations at IJmuiden

Add the effect of friction to make the simulation more realistic



A simplified but instructive seabreeze-model

Velocity components:

$$u \equiv \frac{dx}{dt}; \quad v \equiv \frac{dy}{dt}$$

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu$$

Pressure gradient force (per unit mass)

Coriolis force (per unit mass)

$$f = 2\Omega \sin \phi$$

ρ : density

p : pressure

Ω : Earth's angular velocity

ϕ : latitude

Prescribe the pressure gradient

pressure

Earth's angular velocity

$$\frac{\partial p}{\partial x} = A \cos(\Omega t + \varphi)$$

amplitude
(1 hPa per 100 km)

time

phase



Equations of motion

Back to the equations:

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu$$

Prescribe:

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} = A \cos(\Omega t + \varphi) \quad (\text{"parametrisation"})$$



$$\frac{du}{dt} = -\frac{A}{\rho} \cos(\Omega t + \varphi) + fv$$



$$\frac{dv}{dt} = -fu$$

Equations of motion

Previous slide:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = -fu$$

At the coast: $\frac{\partial u}{\partial x} = 0$ $\frac{\partial v}{\partial x} = 0$ (reasonable assumption)

Equations of motion

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At the coast: $\frac{\partial u}{\partial x} = 0$ $\frac{\partial v}{\partial x} = 0$ (reasonable assumption)

Equations are linearised: $\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)$

$$\frac{\partial v}{\partial t} = -fu$$

This is a “**Eulerian**” model, as opposed to the “**Lagrangian**” model of project 1.

Equations of motion

Previous slide:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

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At the coast: $\frac{\partial u}{\partial x} = 0$ $\frac{\partial v}{\partial x} = 0$ (reasonable assumption)

Equations are linearised:

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

$$\frac{\partial v}{\partial t} = -fu$$

This is a “**Eulerian**” model, as opposed to the “**Lagrangian**” model of project 1.

→ $\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)$ (*forced harmonic oscillator*)

The governing equation of the simplified “sea breeze model”

Previous slide:

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)$$

wind velocity perpendicular to the coast u
 amplitude of the daily cycle in pressure perpendicular to the coast A
 earth's angular velocity Ω
 phase φ
 Coriolis parameter: $f = 2\Omega \sin \phi$
 latitude ϕ

How well does the solution of this equation “simulate” sea breeze at IJmuiden?



Solution of the equation of the simplified “sea breeze model”

Previous slide:

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)$$

phase

$$\varphi = 0$$

Solution:

$$u(t) = C_1 \sin(ft) + C_2 \cos(ft) + \frac{A\Omega}{\rho(f^2 - \Omega^2)} \sin(\Omega t)$$

Boundary conditions: $u = 0$ and $v = 0$ at $t = 0$

(for example)



$$C_2 = 0$$

Solution of the equation of the simplified “sea breeze model”

Previous slide:

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)$$

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$$\varphi = 0$$

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(for example)

$$C_2 = 0$$

*

$$\frac{\partial u}{\partial t} = -\frac{A}{\rho} \text{ at } t = 0$$

*

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

Solution of the equation of the simplified “sea breeze model”

Previous slide:

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)$$

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$$\varphi = 0$$

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$$u(t) = C_1 \sin(ft) + C_2 \cos(ft) + \frac{A\Omega}{\rho(f^2 - \Omega^2)} \sin(\Omega t)$$

Boundary conditions: $u = 0$ and $v = 0$ at $t = 0$

(for example)

$$C_2 = 0$$

*

$$\frac{\partial u}{\partial t} = -\frac{A}{\rho} \text{ at } t = 0$$

$$C_1 = -\frac{Af}{\rho(f^2 - \Omega^2)}$$

*

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

Solution: two periods of oscillations

Therefore:

$$u(t) = + \frac{A}{\rho(f^2 - \Omega^2)} (\Omega \sin(\Omega t) - f \sin(ft))$$

Two time-scales: $2\pi/\Omega$ and $2\pi/f$

forcing period

natural (inertial) period

Analytical solution

$$u(t) = + \frac{A}{\rho(f^2 - \Omega^2)} (\Omega \sin(\Omega t) - f \sin(ft)) \quad (\text{eq. 108a, or 1a Box 1.9})$$

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t) \quad \rightarrow \quad v = \frac{1}{f} \frac{\partial u}{\partial t} + \frac{A}{f\rho} \cos(\Omega t)$$

$$v(t) = \frac{Af}{\rho(f^2 - \Omega^2)} (\cos(\Omega t) - \cos(ft)) \quad (\text{eq. 108a, or 1b Box 1.9})$$

This solution is used to test the numerical approximation to the time derivative in the original differential equations. In project 3 you are asked to add terms representing the effect of “friction”.

Solution of the model: clockwise turning of the wind vector

A wind-direction equal to 270° implies a sea breeze perpendicular to the coast.

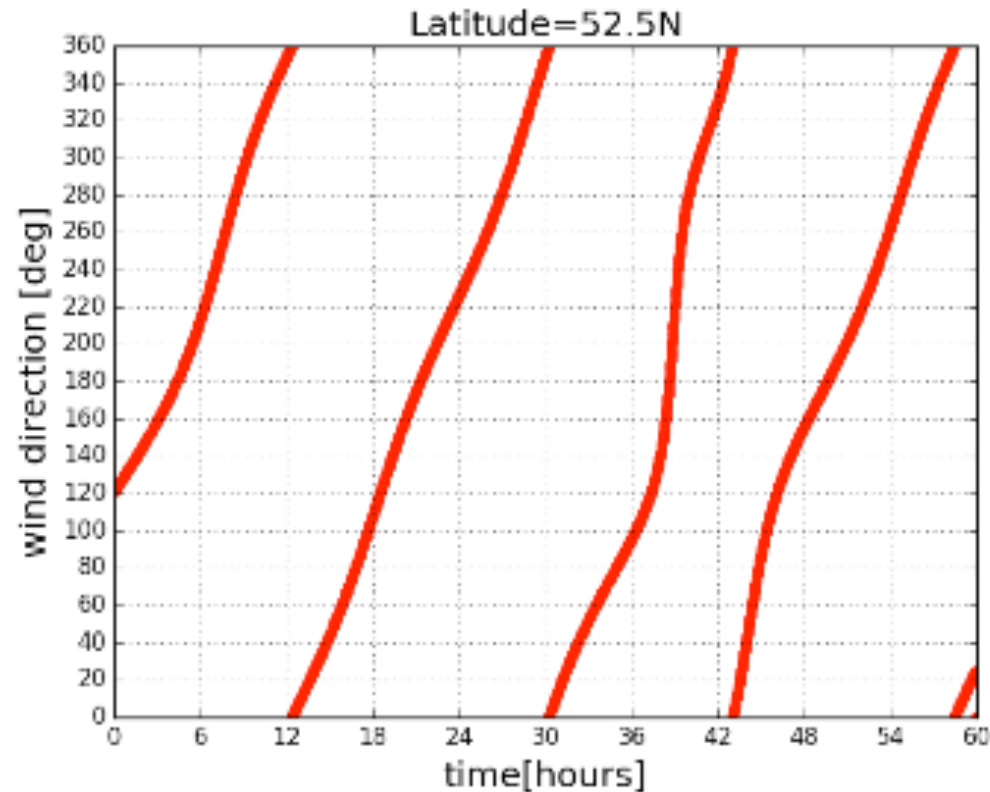


FIGURE 1.33. Wind direction, according to the **analytical solution** (eq. 1.108a,b), for latitude = 52.5°N , $A=0.001 \text{ Pa m}^{-1}$, $\phi=0$ and $\rho=1.16 \text{ kg m}^{-3}$, assuming that $u=v=0$ at $t=0$. The wind direction does not follow the same daily cycle each day and changes more quickly in theory (**this figure**) than in practice (see next slide, **figure 1.34**). **Rate of turning of the wind vector = 20° per hour in first 24 hours, but between $t=30$ hours and $t=43$ hours, this rate is nearly 30° per hour.**

Observations at IJmuiden: clockwise turning of the wind vector

Hourly wind direction (red circles) and temperature (black squares) at IJmuiden at the coast on May 7 and 8, 1976. Horizontal dashed line indicates the direction perpendicular to the coast; Horizontal solid line indicates the direction parallel to the coast.

In how far are the observations of the wind direction reproduced by the model (previous slide)?

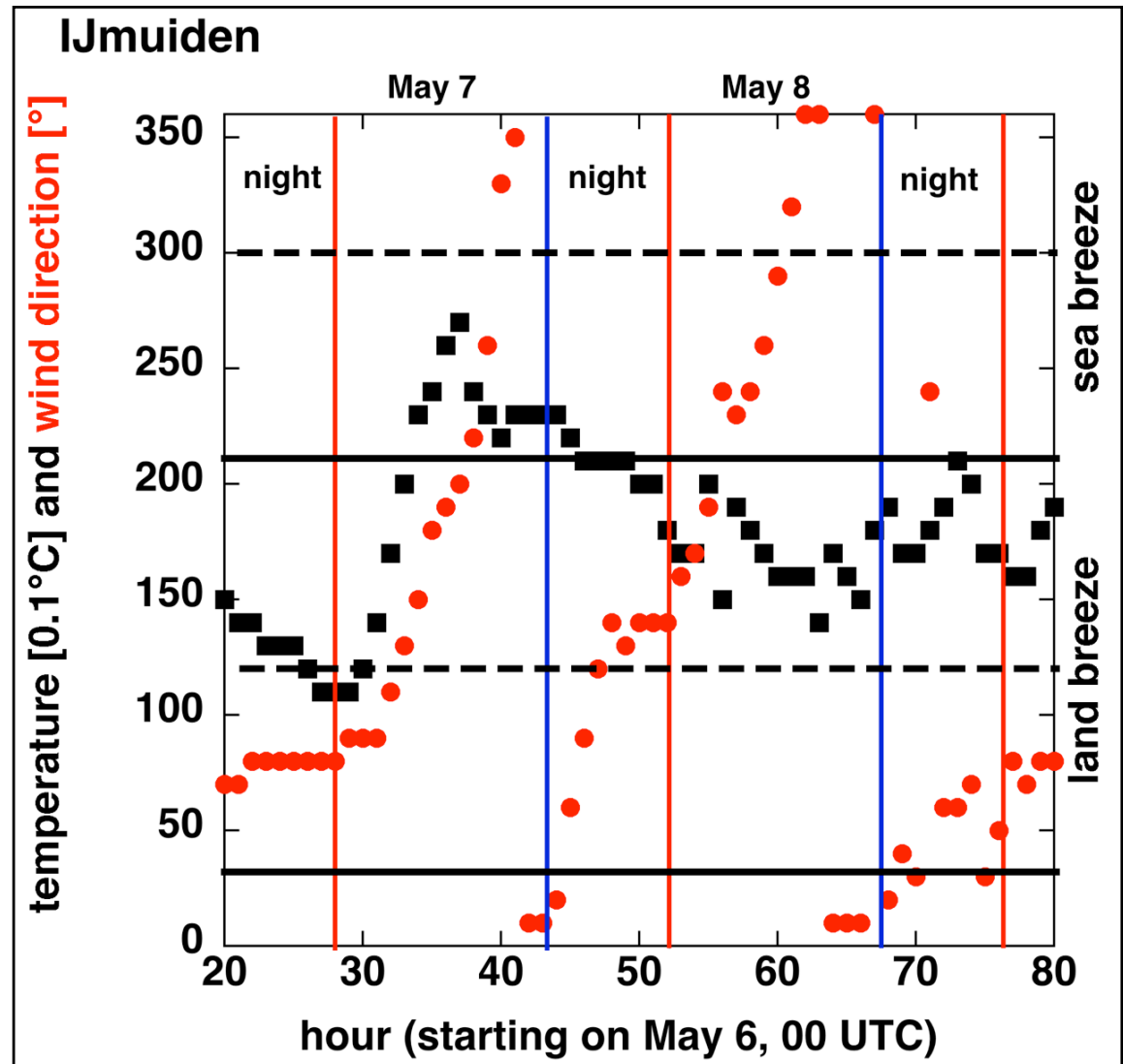


figure 1.34

PROJECT 3 - PROBLEM 1.14 in the lecture notes (p.126)
(in groups of max. 3 students) (15%)

Simulation of the sea/land breeze at the coast

In this small research project you will be concerned with a “Eulerian” numerical simulation model. You will be surprised about how close you can make a simple model correspond to observations!

The following eleven theoretical concepts or topics come up for discussion.

1. Coriolis force / acceleration
2. Inertial oscillation / inertial frequency or period
3. Geostrophic balance
4. Geostrophic wind and ageostrophic wind
5. Numerical approximations of the time derivative (Euler-forward, leap-frog, etc.)
6. Parametrisation
7. Rayleigh damping
8. Model initialisation
9. Model verification (Taylor diagram)
10. Correlation, standard deviation, root mean square (RMS) difference, bias, centred pattern RMS
11. Eulerian

Box 1.9

Seabreeze model: equations

Model equations: $\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi) \\ \frac{\partial v}{\partial t} = -fu \end{array} \right.$

Box 1.9

Seabreeze model: numerical approximation

Model equations: $\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi) \\ \frac{\partial v}{\partial t} = -fu \end{array} \right.$

Numerical approximation (divide time axis into discrete intervals):

time-step $\longrightarrow \Delta t$

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} = fv(t) - \frac{A}{\rho} \cos(\Omega t + \varphi)$$
$$\frac{v(t + \Delta t) - v(t)}{\Delta t} = -fu(t)$$

("forward finite difference approximation")

Box 1.9

Numerical sea-breeze model

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} = fv(t) - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

Euler forward scheme

$$\frac{v(t + \Delta t) - v(t)}{\Delta t} = -fu(t)$$

How accurate is this numerical approximation ?

see [box 1.9](#)

First part of project 3

Read **section 1.17** and [Boxes 1.9&1.10](#) of

Download the Python script: [SeaBreezeModel_SOAC2020.py](#) and run it on your computer and test the accuracy of the solution by comparing it with the analytical solution (the accuracy depends on the magnitude of the time step, among other)

Python script of the linear sea breeze model

```
# This script belongs to PROJECT 3 of the course, DYNAMICAL METEOROLOGY – 2021–2022
# BEGINNING OF THE PYTHON SCRIPT                               SeaBreezeModel[Project3DYME2022].py
# https://docs.python.org/2.7/
# IMPORT MODULES
import numpy as N # http://www.numpy.org
import matplotlib.pyplot as P # http://matplotlib.org
import math as M # https://docs.python.org/2/library/math.html

# Length of the simulation and time-step
time_max = 48.0 # length in hours of the simulation
dt = 360.0 # time step in s

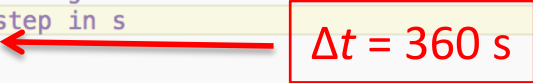
# PARAMETER VALUES
A = 0.001 # Pa m-1
phase = 0.0 # phase of surface pressure gradient in time
lat = 52 # latitude in degree
phi = 0.0 # phase of the pressure gradient
labda = 0.0 # Rayleigh damping coefficient

# CONSTANTS
omega = 0.000072792 # angular velocity Earth [s-1]
pi = M.pi
ro = 1.25 # density kg m-3
fcor = 2 * omega * M.sin(lat * pi/180) # Coriolis parameter
C1 = - (A/(fcor*ro)) * ( (M.pow(omega,2) / (M.pow(fcor,2) - M.pow(omega,2))) + 1)
C3 = A * omega / (ro*(M.pow(fcor,2) - M.pow(omega,2)))

# NUMBER OF TIME STEPS AND TIME-AXIS FOR PLOTTING
nt_len = int(time_max * 3600 / dt) # maximum number of time steps in the integration
time = N.zeros((nt_len))
time_axis = N.zeros((nt_len))

# DEFINE time, u, v and the analytical solution, u_ana, as arrays and fill them with zero's
u = N.zeros((nt_len)) # x-component velocity, numerical solution
v = N.zeros((nt_len)) # y-component velocity, numerical solution
u_ana = N.zeros((nt_len)) # analytical solution x-component velocity

# INITIAL CONDITION (t=0) : atmosphere in rest
nt = 0
time[nt] = 0
time_axis[nt] = 0
u[nt] = 0
v[nt] = 0
u_ana[nt] = 0
```



```

# INITIAL CONDITION (t=0) : atmosphere in rest
nt = 0
time[nt] = 0
time_axis[nt] = 0
u[nt] = 0
v[nt] = 0
u_ana[nt] = 0

# TIME LOOP EULER FORWARD SCHEME
for nt in range(len(time)-1):
    du = dt * (-(labda * u[nt]) + (fcor*v[nt]) - ((A/ro)* M.cos((omega*time[nt])+phase)))
    dv = dt * (-(labda * v[nt]) - (fcor * u[nt]))
    time[nt+1] = time[nt]+dt
    u[nt+1] = u[nt] + du
    v[nt+1] = v[nt] + dv
    u_ana[nt+1] = (C1 * M.sin(fcor * time[nt+1])) + ( C3* M.sin((omega * time[nt+1]) + phase) )

for nt in range(len(time)):
    time_axis[nt] = time[nt] / 3600. # time axis in hours

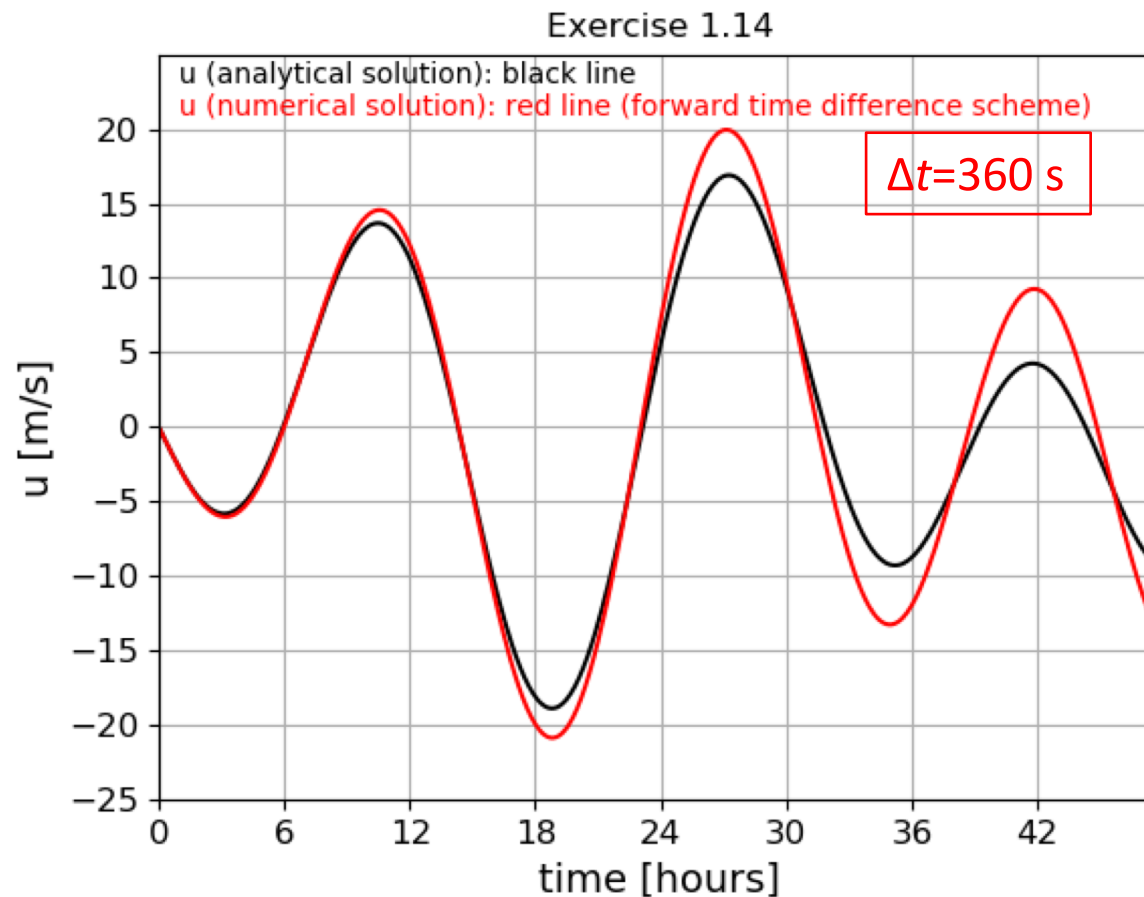
# MAKE PLOT of evolution in time of u, v and u_ana
P.plot(time_axis, u_ana, color='black')
P.plot(time_axis, u, color='red')
P.plot(time_axis, v, color='blue')
P.axis([0,time_axis[nt_len-1],-25.0,25.0]) # define axes
P.xticks(N.arange(0,time_axis[nt_len-1]+0.1,6), fontsize=12)
P.yticks(N.arange(-25.0,25.0,5), fontsize=12)
P.xlabel('time [hours]', fontsize=14) # label along x-axes
P.ylabel('velocity [m/s]', fontsize=14) # label along x-axes
P.title('SeabreezeSimulation') # Title at top of plot
P.text(1, 23, 'u (analytical solution) (no damping): black line', fontsize=10, color='black')
P.text(1, 21, 'u (numerical solution): red line (forward time difference scheme)', fontsize=10, color='red')
P.text(1, 19, 'v (numerical solution): blue line (forward time difference scheme)', fontsize=10, color='blue')
P.grid(True)
P.savefig("SeabreezeSimulation.png") # save plot as png-file
P.show() # show plot on screen

# END OF THE PYTHON SCRIPT

```

Numerical and analytical solution

Result of running SeaBreezeModel[Project3DYME2022].py:



You will get a **better numerical solution** if you reduce Δt .

Apply the model to IJmuiden on 7 & 8 May 1976

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} = fv(t) - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

Euler forward scheme

$$\frac{v(t + \Delta t) - v(t)}{\Delta t} = -fu(t)$$



Apply the model to Ijmuiden on 7 & 8 May 1976

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} = fv(t) - \frac{A}{\rho} \cos(\Omega t + \varphi)$$

Euler forward scheme

$$\frac{v(t + \Delta t) - v(t)}{\Delta t} = -fu(t)$$

From hourly observations **find the best value of A and φ .**

Observations are provided in the file, Hourly observations for 7-8 May 1976:

ObservationsAnalysisIjmuiden(AnswerProblem1_13AD).docx (see Blackboard).

Initialise the model with observed wind velocities at 0 UTC on 7 May 1976* and compare the outcome with the hourly observations

* $u = -9$ m/s; $v = -7.6$ m/s at Ijmuiden on 7 May 1976, 0 UTC

Add friction to make the model more realistic

$$\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi) - \lambda u$$

$$\frac{\partial v}{\partial t} = -fu - \lambda v$$

“Rayleigh damping”

Find the best value of the Rayleigh damping coefficient, λ , by correlating the solution with the observations of 7 and 8 May 1976.

see **Box 1.10**
(about model verification)

Extra problem: Write down the “stationary solution” of this system of equations and give an interpretation of this solution.

Geostrophic flow

The seabreeze circulation is part of a process of adjustment to geostrophic balance.

Equation of motion:
(slide 132)

$$\frac{d\vec{v}}{dt} = -\theta\vec{\nabla}\Pi - g\hat{k} - 2\vec{\Omega} \times \vec{v}$$

x-component:

$$\frac{du}{dt} = -\theta \frac{\partial \Pi}{\partial x} - fv$$

y-component:

$$\frac{dv}{dt} = -\theta \frac{\partial \Pi}{\partial y} + fu$$

(curvature of the flow is neglected)

Steady state
(geostrophic flow):

$$\theta \frac{\partial \Pi}{\partial x} + fv_g = 0$$

$$-\theta \frac{\partial \Pi}{\partial y} + fu_g = 0$$

Geostrophic zonal flow

Assume: no pressure gradient in x-direction

Assume a “geostrophic” **zonal flow**, u_g :

$$fu_g \equiv -\theta \frac{\partial \Pi}{\partial y}$$

Equations of motion:

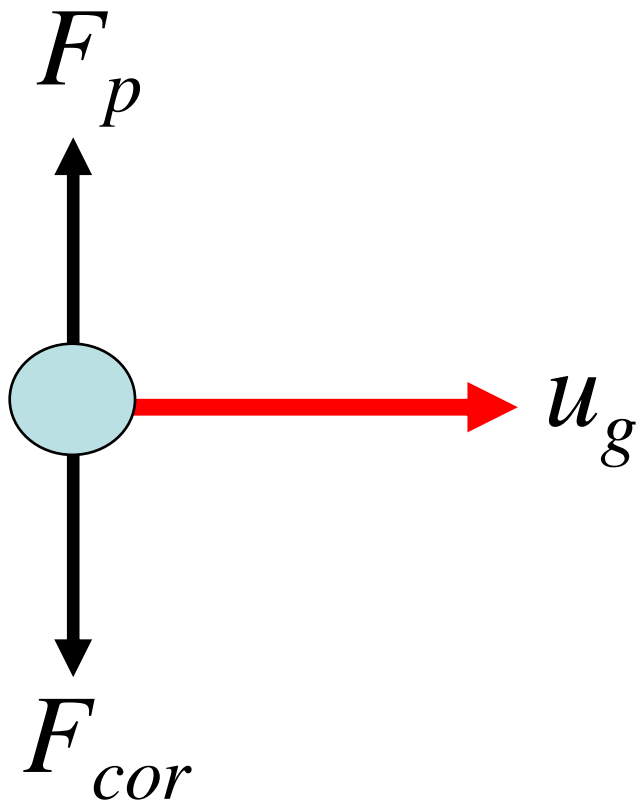
x-component: $\frac{du}{dt} = fv = f \frac{dy}{dt} \rightarrow du = fdy$

Conservation of linear momentum per unit mass: $M=u-fy$

y-component: $\frac{dv}{dt} = -\theta \frac{\partial \Pi}{\partial y} - fu = f(u_g - u)$

Stability of geostrophic zonal flow

Consider an air parcel in geostrophic balance, moving eastward with a velocity, u_g . What happens when we give it a small push in the positive y -direction? Its new latitude will be $y_0 + \partial y$.



At this new latitude it will have a zonal velocity, $u_g + f \partial y$, which is larger than the velocity, u_g , at the original latitude. If the pressure gradient force on the air parcel is the same at the new latitude, the air parcel will be forced back to its original equilibrium position, meaning that the equilibrium is stable. We will see in the next lecture that this is not always the case.

Next:

07/12/2022 (Wednesday, 13:15)

Dynamical Meteorology: lecture 7

Sections 1.18-1.21

Stability of geostrophic balance,

Inertial oscillations & inertial instability

The thermal wind

Meridional exchange of warm and cold air masses

“Optimum” slope of trajectories in the meridional plane & conversion of potential energy into kinetic energy

Conceptual model of a middle latitude cyclone (section 1.21)

Tutorial after this lecture (Friday 2 December 2022):

Project 3

Exercises of Part C (previous tutorial)

Deadline for project 3 is Fri. 9 December 2022

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<http://www.uu.nl/~nvdelden/dynmeteorology.htm>