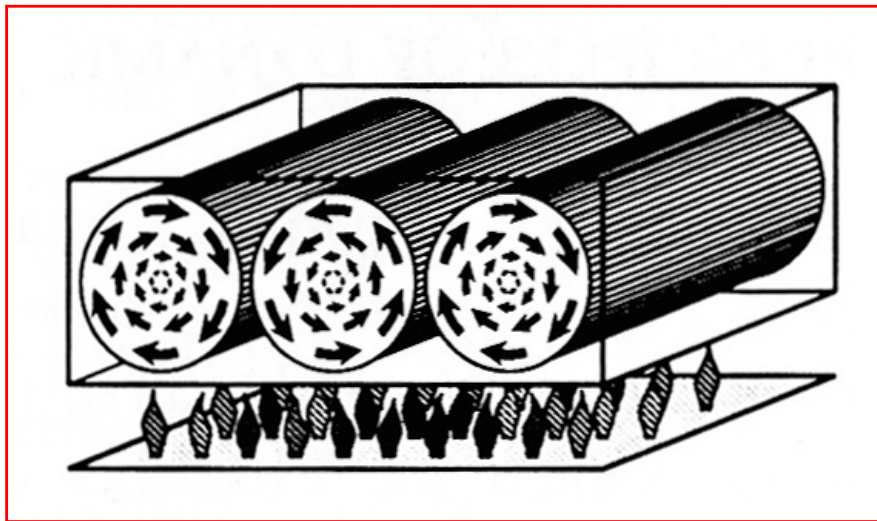


Thermal convection of a stratified fluid under the influence of gravity

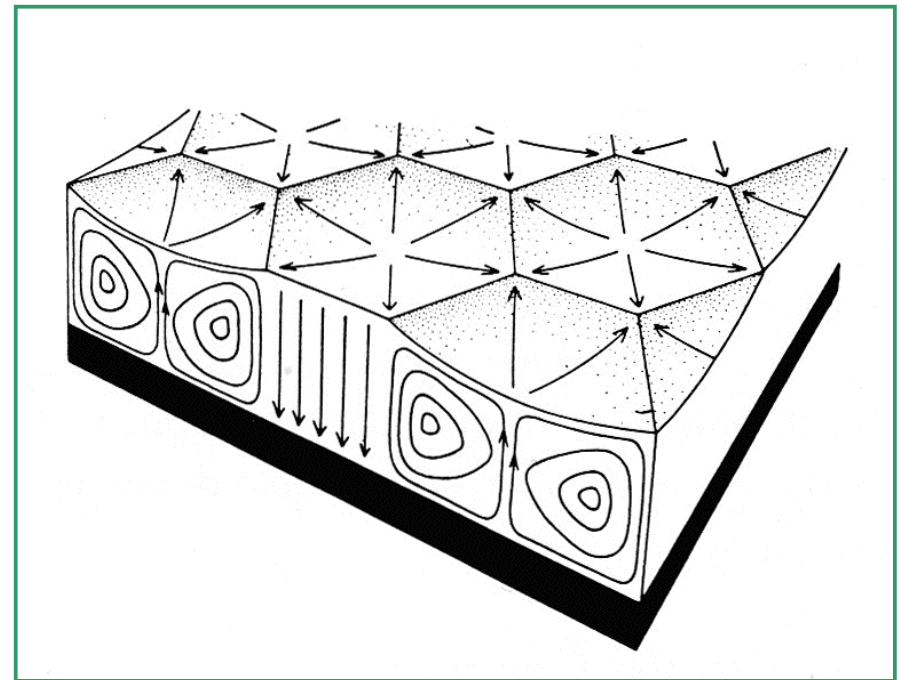
Lectures 7, course on “Turbulence in Fluids”, Utrecht University

<http://www.staff.science.uu.nl/~delde102/TinF.htm>

Part 2: finite amplitude convection



2-D convection cells



3-D convection cells

Lectures 6: convection if

$$\Gamma > \Gamma_{crit} = \frac{\nu \kappa}{g \alpha l^2} \left(l^2 + \left(\frac{\pi n}{H} \right)^2 \right)^3 \rightarrow \text{Preferred wavelength:}$$
$$L = 2\sqrt{2}H$$

Thermal convection of a stratified fluid under the influence of gravity

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7 March 2022

Part 2: finite amplitude convection

Finite amplitude convection

Non-dimensional units and parameters

Fourier transformation of the governing equations

Spectral energy cascade in thermal convection

Lorenz (1963) model of two dimensional convection

Exercises part 2

Two-dimensional convection ($\partial(\dots)/\partial y=0$):
introduce a streamfunction, ψ

Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ is satisfied if $u \equiv -\frac{\partial \psi}{\partial z}; w \equiv \frac{\partial \psi}{\partial x}$

Two-dimensional convection : equation governing streamfunction

Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ is satisfied if $u \equiv -\frac{\partial \psi}{\partial z}$; $w \equiv \frac{\partial \psi}{\partial x}$

Apply Boussinesq approximation to horizontal components of the equation of motion (see lecture 6):

$$\begin{aligned} \frac{\partial}{\partial z} \quad & \boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_{ref}} \frac{\partial p'}{\partial x} + \nu \nabla^2 u} \\ \rightarrow \quad & - \\ \frac{\partial}{\partial x} \quad & \boxed{\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_{ref}} \frac{\partial p'}{\partial z} + \nu \nabla^2 w - g \alpha T'} \end{aligned}$$

$\downarrow$ **eliminate:** p', u, w :

$$\boxed{\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial z} = \alpha g \frac{\partial T'}{\partial x} + \nu \nabla^2 \nabla^2 \psi}$$

Introduce streamfunction into the **temperature equation:**

Temperature:
$$\frac{\partial T'}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial T'}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial T'}{\partial z} = \Gamma \frac{\partial \psi}{\partial x} + \kappa \nabla^2 T'$$

Together with equation for **streamfunction** (previous slide):

$$\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial z} = \alpha g \frac{\partial T'}{\partial x} + \nu \nabla^2 \nabla^2 \psi$$

Closed set of equations

Non-dimensional units

Previous slide:
$$\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial z} = \alpha g \frac{\partial T'}{\partial x} + \nu \nabla^2 \nabla^2 \psi$$

Non-dimensional units: length = H ; time = $\frac{H^2}{\kappa}$; temperature = $\frac{\kappa \nu}{g \alpha H^3}$

$$\psi [\text{m}^2 \text{s}^{-1}] = H^2 \frac{\kappa}{H^2} \psi^*$$

Non-dimensional streamfunction

Equations in non-dimensional units

Previous slide:
$$\frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial z} = \alpha g \frac{\partial T'}{\partial x} + \nu \nabla^2 \nabla^2 \psi \longrightarrow$$

Non-dimensional units: length = H ; time = $\frac{H^2}{\kappa}$; temperature = $\frac{\kappa \nu}{g \alpha H^3}$

$$\psi [\text{m}^2 \text{s}^{-1}] = H^2 \frac{\kappa}{H^2} \psi^*$$

$\longrightarrow$
$$\frac{\partial \nabla^2 \psi^*}{\partial t} - \frac{\partial \psi^*}{\partial z} \frac{\partial \nabla^2 \psi^*}{\partial x} + \frac{\partial \psi^*}{\partial x} \frac{\partial \nabla^2 \psi^*}{\partial z} = \text{Pr} \frac{\partial \theta}{\partial x} + \text{Pr} \nabla^2 \nabla^2 \psi^*$$

The symbols, ψ^* and θ , are non-dimensional streamfunction and temperature

Prandtl number: $\text{Pr} \equiv \frac{\nu}{\kappa}$

Equations in non-dimensional units

Temperature:
$$\frac{\partial T'}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial T'}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial T'}{\partial z} = \Gamma \frac{\partial \psi}{\partial x} + \kappa \nabla^2 T'$$

Non-dimensional units:

$$\text{length} = H; \text{ time} = \frac{H^2}{\kappa}; \text{ temperature} = \frac{\kappa \nu}{g \alpha H^3}$$

→
$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi^*}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial \psi^*}{\partial x} \frac{\partial \theta}{\partial z} = \text{Ra} \frac{\partial \psi^*}{\partial x} + \nabla^2 \theta$$

The symbol, θ , is used to indicate the non-dimensional temperature perturbation

Rayleigh number:
$$\text{Ra} \equiv \frac{g \alpha \Gamma H^4}{\kappa \nu}$$

Fourier transformation of the equations

Physical domain in non-dimensional units: $0 \leq x \leq L/H$ $0 \leq z \leq 1$

Define “basis functions” as: $S_{\alpha}(x, z) \equiv \exp\{i\pi(a_x l_{\alpha} x + n_{\alpha} z)\}$

$a_x \equiv 2H/L$ l_{α} and n_{α} are integer wave numbers

Fourier transformation of the equations

Physical domain in non-dimensional units: $0 \leq x \leq L/H$ $0 \leq z \leq 1$

Define “basis functions” as: $S_{\alpha}(x,z) \equiv \exp\{i\pi(a_x l_{\alpha} x + n_{\alpha} z)\}$

$a_x \equiv 2H/L$ l_{α} and n_{α} are integer wave numbers

Any variable can be expressed as a “Fourier series” as follows

$$\theta(x,z,t) = \sum_{\alpha} \Theta_{\alpha}(t) S_{\alpha}(x,z)$$

Sum over all integral lattice points in the volume of $\alpha = (l_{\alpha}, n_{\alpha})$

wave-vector

Orthonormality of basis functions

The stream function and the temperature are represented by Fourier series as follows

$$\begin{pmatrix} \psi \\ \theta \end{pmatrix} = \sum_{\alpha} \begin{bmatrix} \Psi_{\alpha} \\ \Theta_{\alpha} \end{bmatrix} S_{\alpha}(x, z)$$

Orthonormality of the basis-functions:

$$\int S_{\alpha} S_{\beta}^{*} d\sigma = \delta_{\alpha, \beta}$$

where the asterisk designates a complex conjugate, while δ is the kroncker delta.

The **integration** extends over the **area**,

$$0 \leq x \leq \frac{2}{a_x}, \quad 0 \leq z \leq 2$$



!!! mathematical trick

Physical domain:

$$0 \leq x \leq L/H$$

$$0 \leq z \leq 1$$

$$a_x \equiv 2H/L$$

and $d\sigma$ is an area element divided by the total area, $4/a_x$

Transform equations to “spectral space”

Take the temperature equation:

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi^*}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial \psi^*}{\partial x} \frac{\partial \theta}{\partial z} = \text{Ra} \frac{\partial \psi^*}{\partial x} + \nabla^2 \theta$$

Note: from now on we drop the asterisk:

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial z} = \text{Ra} \frac{\partial \psi}{\partial x} + \nabla^2 \theta$$

Substitute:

$$\begin{pmatrix} \psi \\ \theta \end{pmatrix} = \sum_{\alpha} \begin{bmatrix} \Psi_{\alpha} \\ \Theta_{\alpha} \end{bmatrix} S_{\alpha}(x, z)$$

$$S_{\alpha}(x, z) \equiv \exp\{i\pi(a_x l_{\alpha} x + n_{\alpha} z)\}$$

Non-linear terms become

$$\sum_{\alpha, \beta} \{-i\pi n_{\alpha} \Psi_{\alpha} i\pi a_x l_{\beta} \Theta_{\beta} + i\pi a_x l_{\alpha} \Psi_{\alpha} i\pi n_{\beta} \Theta_{\beta}\} S_{\alpha} S_{\beta}$$

$$= \sum_{\alpha, \beta} \pi^2 a_x \{n_{\alpha} l_{\beta} - l_{\alpha} n_{\beta}\} \Psi_{\alpha} \Theta_{\beta} S_{\alpha} S_{\beta} \longrightarrow$$

Orthonormality is used to transform equations to “spectral space”

→ Non-linear terms = $\sum_{\alpha,\beta} \pi^2 a_x \{n_\alpha l_\beta - l_\alpha n_\beta\} \Psi_\alpha \Theta_\beta S_\alpha S_\beta$

Multiply by S_γ^* and integrate over the area $d\sigma$ and use the property of orthonormality:

$$\int S_\alpha S_\beta^* d\sigma = \delta_{\alpha,\beta}$$

Non-linear terms = $\sum_{\alpha,\beta} I_{\gamma\alpha\beta} \Psi_\alpha \Theta_\beta$

where: $I_{\gamma\alpha\beta} \equiv \pi^2 a_x (n_\alpha l_\beta - n_\beta l_\alpha) \delta_{\gamma,\alpha+\beta}$



Interaction coefficient



$$\left. \begin{array}{l} \delta_{\gamma,\alpha+\beta}=1 \text{ if } \alpha+\beta=\gamma \\ \text{else } \delta_{\gamma,\alpha+\beta}=0 \end{array} \right\} \text{selection rule}$$

Orthonormality is used to transform equations to “spectral space”

Non-linear terms = $\sum_{\alpha,\beta} \pi^2 a_x \{n_\alpha l_\beta - l_\alpha n_\beta\} \Psi_\alpha \Theta_\beta S_\alpha S_\beta$

Also multiply linear terms by S_γ^* and integrate over the volume $d\sigma$ and use

$$\int S_\alpha S_\beta^* d\sigma = \delta_{\alpha,\beta}$$

The spectral form of the temperature equation becomes

$$\frac{d\Theta_\gamma}{dt} = \sum_{\alpha,\beta} I_{\gamma\alpha\beta} \Psi_\alpha \Theta_\beta + i\pi R a a_x l_\gamma \Psi_\gamma - \pi^2 k_\gamma^2 \Theta_\gamma$$



All interaction coefficients are zero unless the selection rule (in vector form),

$$\alpha + \beta = \gamma$$

is satisfied

Spectral form of the equations

Previous slide:
$$\frac{d\Theta_\gamma}{dt} = \sum_{\alpha,\beta} I_{\gamma\alpha\beta} \Psi_\alpha \Theta_\beta + i\pi Ra a_x l_\gamma \Psi_\gamma - \pi^2 k_\gamma^2 \Theta_\gamma$$

The spectral form of the equation for the stream-function is similar:

$$\frac{d\Psi_\gamma}{dt} = \frac{1}{k_\gamma^2} \sum_{\alpha,\beta} k_\beta^2 I_{\gamma\alpha\beta} \Psi_\alpha \Psi_\beta + i\pi Pr \frac{a_x l_\gamma}{k_\gamma^2} \Theta_\gamma - \pi^2 Pr k_\gamma^2 \Psi_\gamma$$

The **Fourier coefficients are complex**. Therefore: $\Theta_\gamma = \Theta_\gamma^R + i\Theta_\gamma^I$

Use of the boundary conditions

Stress-free, perfectly conducting bottom and top boundary condition:

$$w = \frac{\partial \psi}{\partial x} = 0; \quad \frac{\partial u}{\partial z} = -\frac{\partial^2 \psi}{\partial z^2} = 0; \quad \theta = \frac{\partial^2 \theta}{\partial z^2} = 0 \quad \text{at } z = 0 \text{ and } z = 1$$

Temperature is real. Therefore, $\Theta_\gamma = \Theta_{-\gamma}^*$

$$\text{or } \Theta_{l,n}^R = \Theta_{-l,-n}^R \quad \text{and} \quad \Theta_{l,n}^I = -\Theta_{-l,-n}^I$$

The boundary conditions at the top and bottom imply a **sine-dependence in the vertical**:

$$\Theta_{l,n}^R = -\Theta_{l,-n}^R \quad \text{and} \quad \Theta_{l,n}^I = -\Theta_{l,-n}^I$$

Let us impose a **cosine-dependence of the temperature in the horizontal direction**:

$$\Theta_{l,n}^R = \Theta_{-l,n}^R \quad \text{and} \quad \Theta_{l,n}^I = \Theta_{-l,n}^I$$

These relations imply that $\Theta_{l,n}^R = 0$ (see the appendix to this lecture)

Temperature and vertical velocity in phase in the horizontal direction



Therefore, streamfunction is out of phase with the temperature

Because $\Theta_{l,n}^R = 0$ this implies that $\Psi_{l,n}^I = 0$

Symmetries in wave-number space

We now have (previous slide): $\Theta_{l,n}^I = -\Theta_{-l,-n}^I$

$$\Theta_{l,n}^I = -\Theta_{l,-n}^I$$

$$\Theta_{l,n}^I = \Theta_{-l,n}^I$$

Therefore, we need only to explicitly formulate equations for the time-evolution of amplitudes belonging to **one quadrant in wave-number space**: e.g. the quadrant for which $l \geq 0$ and $n > 0$

The spectral energy cascade in thermal convection

Suppose that one so-called fundamental mode is **self-excited** (linearly unstable), i.e. has a positive linear growth rate. Let this be $(l,n)=(1,1)$. The initial effect of the non-linear interactions is to distort the horizontal mean temperature profile, making the temperature field **more isothermal in the mid-region and enhancing the unstable temperature gradients near the upper and lower boundaries**, thereby increasing the vertical heat transport. This is achieved through **the interaction of $(1,1)$ and $(-1,1)$, which creates $(0,2)$ in the temperature field, but not in the stream function field** (**check this!**). In the second instance the original symmetric cellular stream function pattern is distorted due to the interaction between $(\pm 1,1)$ and $(0,2)$, which creates $(\pm 1,3)$. After that the interaction between $(1,3)$ and $(1,-1)$ will create $(2,2)$, and so on. The energy flows up the spectrum to modes with wavelengths $1/2, 1/3, 1/4 \dots$ of the wavelength of the fundamental self-excited (linearly unstable) mode. If this fundamental mode has **even parity**, i.e. $(l+n)$ is an even number, the selection rule leads to the following property of this cascade of energy. The wave vectors, which are excited through the non-linear interactions, have **even parity**. This is the reason why odd-parity modes are usually neglected in studies on cellular convection. The idea behind these studies is that convection starts at the critical Rayleigh number in the form of a fundamental mode, $(l,n)=(1,1)$ of even parity and proceeds to create higher harmonics of even-parity through the nonlinear cascade effect, described above, to reach the finite amplitude equilibrium state.

Observed horizontally averaged temperature in a convecting fluid between horizontal plates

Horizontally averaged steady state temperature profiles at various Rayleigh numbers for seawater with a salinity 3.5 %. T_L is the temperature of the lower boundary; ΔT is the temperature difference between upper and lower boundary. The height is scaled with H (the depth of the fluid layer).

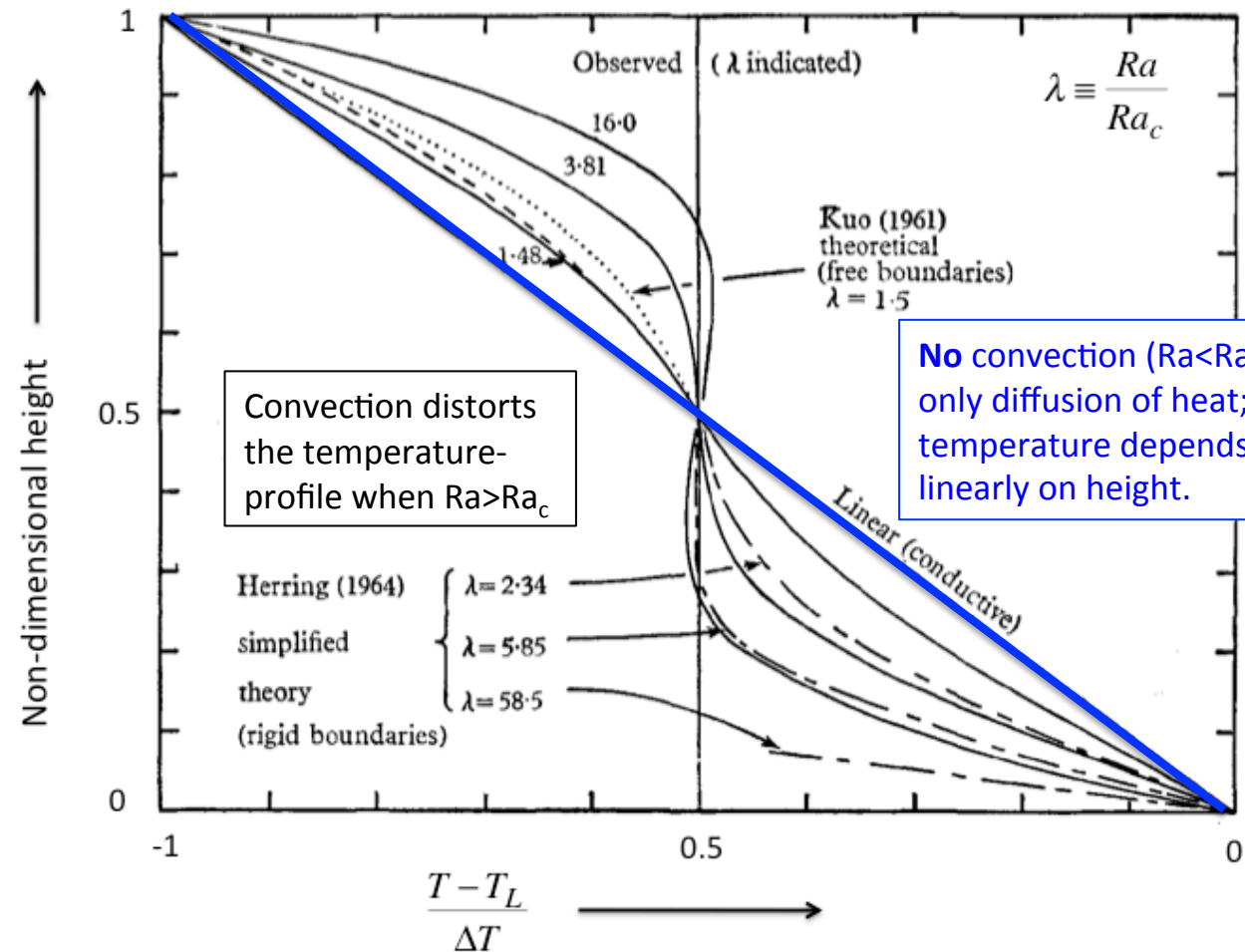
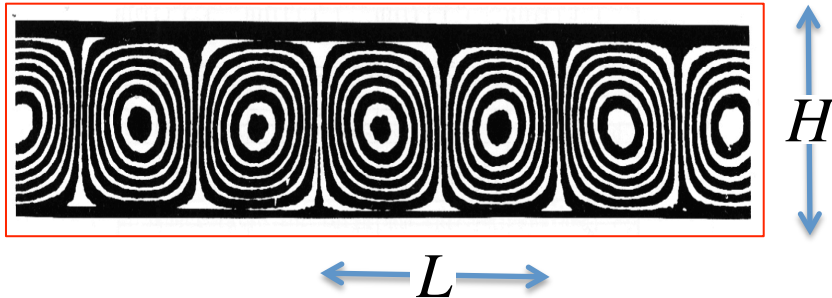


Figure 4 (lecture notes)

Observed horizontally averaged temperature in a convecting fluid between horizontal plates

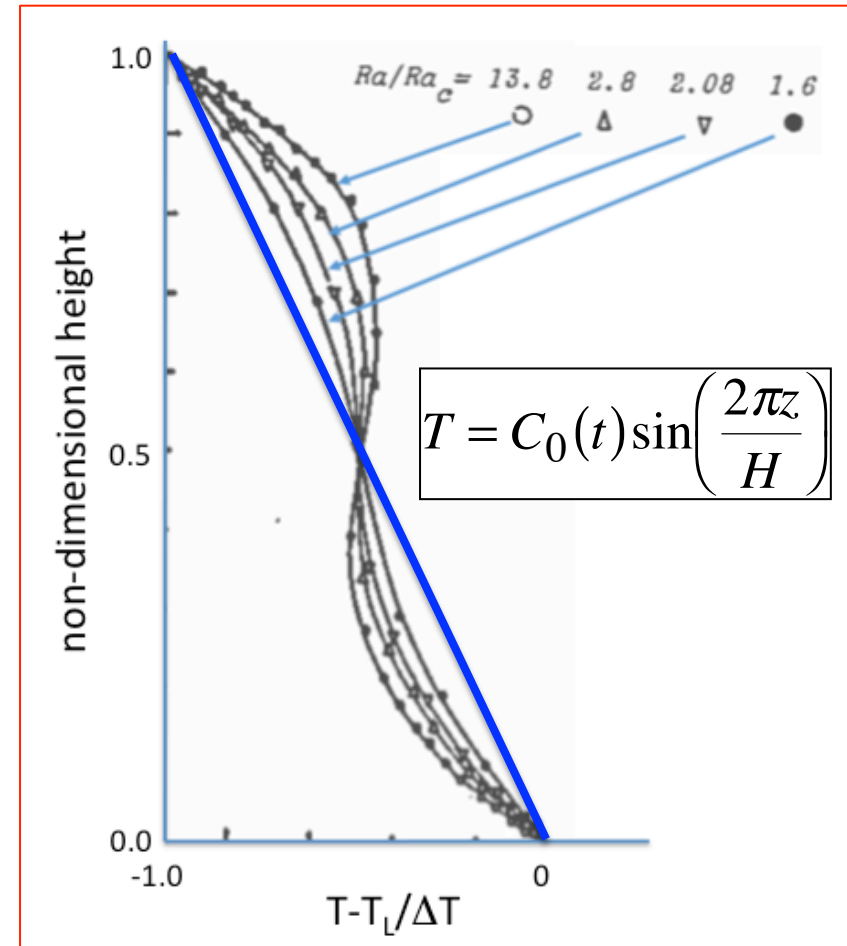


Lowest order mathematical description:

In view of the observations, shown in the figures on this slide, a reasonable fit to the temperature distribution in a fluid layer filled with two-dimensional convection cells is

$$T'(x,z,t) = C_0(t) \sin\left(\frac{2\pi z}{H}\right) + C_1(t) \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi z}{H}\right)$$

In other words, the temperature perturbation can be represented as a sum of **two Fourier modes with time-dependent amplitudes.**



Lorenz (1963) model of thermal convection

The lowest-order model of finite amplitude convection that can be constructed is one in which the Fourier series,

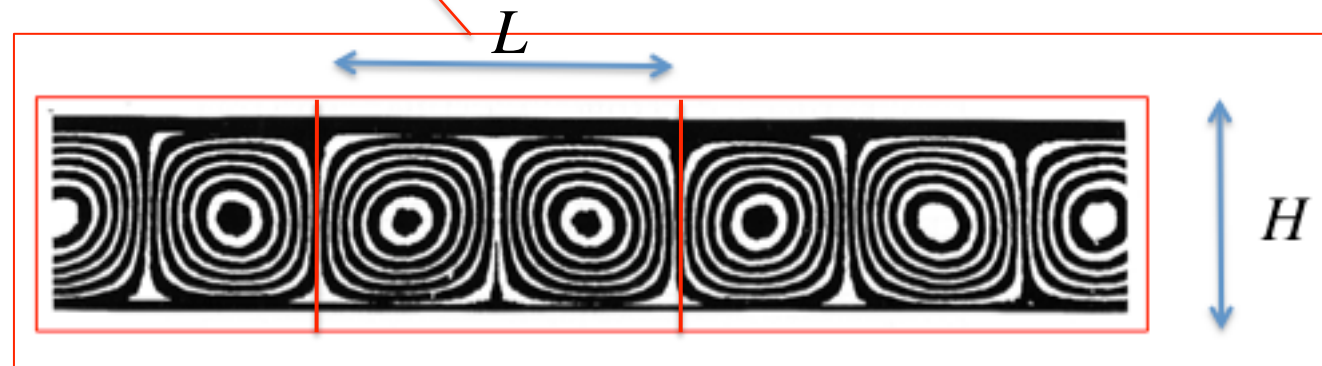
$$\theta(x, z, t) = \sum_{\alpha} \Theta_{\alpha}(t) S_{\alpha}(x, z)$$

is truncated to include only two term:

$$\theta(x, z, t) = \Theta_{0,2}(t) \sin\left(\frac{2\pi z}{H}\right) + \Theta_{1,1}(t) \exp(i\pi a_x x) \sin\left(\frac{\pi z}{H}\right)$$

This is the Lorenz (1963) model.

$$a_x \equiv 2H/L$$



The Lorenz (1963) model of thermal convection

We study the first basic non-linear interaction in isolation. The temperature field is expanded into the two Fourier components, $(l,n)=(1,1)$ and $(l,n)=(0,2)$, while the streamfunction field is expanded into only one Fourier component, $(l,n)=(1,1)$. In other words, we take into account only the following wave vectors, $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, $(0,2)$, $(0,-2)$ in the expansion,

$$\sum_{\alpha} \begin{bmatrix} \Psi_{\alpha} \\ \Theta_{\alpha} \end{bmatrix} S_{\alpha}(x,z)$$

All interactions with wave vectors outside this truncation are neglected.

In view of the symmetry relations, we need to formulate equations for the time-evolution of the amplitudes, corresponding to only three amplitudes, i.e.

$$\Psi_{1,1}^R, \Theta_{1,1}^I \text{ and } \Theta_{0,2}^I.$$

Why is $\Psi_{0,2} = 0$? (see the appendix to this lecture)

The Lorenz (1963) model of thermal convection

The only non-linear interactions involved are the following: (1,1) and (-1,1) interact to contribute to (0,2) and as a feedback (1,-1) and (0,2) interact to contribute to (1,1).

If all interactions with wave vectors outside the truncation are neglected, **the non-linear terms are energy conserving** (exercise 9). The resulting equations for $\Psi_{1,1}^R(X)$, $\Theta_{1,1}^I(Y)$ and $\Theta_{0,2}^I(Z)$ are

$$\frac{dX}{dt} = \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} Y - \pi^2(a_x^2 + 1) \text{Pr} X$$

$$\frac{dY}{dt} = -2\pi^2 a_x XZ + \pi a_x \text{Ra} X - \pi^2(a_x^2 + 1) Y$$

$$\frac{dZ}{dt} = 4\pi^2 a_x XY - 4\pi^2 Z$$

These are the so-called “**Lorenz (1963) equations**” (next slide)

$$\text{Ra} \equiv \frac{g\alpha\Gamma H^4}{\kappa\nu}$$

$$\text{Pr} \equiv \frac{\nu}{\kappa}$$

$$a_x \equiv 2H/L$$

The Lorenz (1963) model of thermal convection

The only non-linear interactions involved are the following: (1,1) and (-1,1) interact to contribute to (0,2) and (1,-1) and (0,2) interact to contribute to (1,1) (**the feedback**).

If all interactions with wave vectors outside the truncation are neglected, **the non-linear terms are energy conserving** (exercise 9). The resulting equations are

$$\frac{dX}{dt} = \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} Y - \pi^2(a_x^2 + 1) \text{Pr} X$$

$$\frac{dY}{dt} = -2\pi^2 a_x XZ + \pi a_x \text{Ra} X - \pi^2(a_x^2 + 1) Y$$

$$\frac{dZ}{dt} = 4\pi^2 a_x XY - 4\pi^2 Z$$

Original Lorenz (1963)-equations:

$$X' = -\sigma X + \sigma Y, \quad (25)$$

$$Y' = -XZ + rX - Y, \quad (26)$$

$$Z' = XY - bZ. \quad (27)$$

These equations differ from the **original Lorenz (1963) equations**.

This is because **Lorenz's non-dimensional units are different**, e.g. his time unit is $\pi^2(a_x^2 + 1) \frac{\kappa}{H^2}$

The Lorenz (1963) model of thermal convection

$$\frac{dX}{dt} = \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} Y - \pi^2(a_x^2 + 1) \text{Pr} X$$

$$\frac{dY}{dt} = -2\pi^2 a_x XZ + \pi a_x \text{Ra} X - \pi^2(a_x^2 + 1) Y$$

$$\frac{dZ}{dt} = 4\pi^2 a_x XY - 4\pi^2 Z$$

rest



Three **steady state solutions** (or “**fixed points**”). One fixed point is $\vec{A} \equiv (X, Y, Z) = (0, 0, 0)$

The **stability** of this fixed point is investigated by linearising the set of three equations around this steady state of rest, which can then be written compactly as

$$\frac{d}{dt} \vec{A} = M \vec{A} \quad \text{where} \quad M \equiv \begin{pmatrix} -\pi^2(a_x^2 + 1) \text{Pr} & \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} & 0 \\ \pi a_x \text{Ra} & -\pi^2(a_x^2 + 1) & 0 \\ 0 & 0 & -4\pi^2 \end{pmatrix}$$

The Lorenz (1963) model of thermal convection stability of the rest state

Previous slide:

$$\boxed{\frac{d}{dt} \vec{A} = M \vec{A}} \quad \text{where} \quad M \equiv \begin{pmatrix} -\pi^2(a_x^2 + 1) \text{Pr} & \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} & 0 \\ \pi a_x \text{Ra} & -\pi^2(a^2 + 1) & 0 \\ 0 & 0 & -4\pi^2 \end{pmatrix}$$

A positive eigenvalue of the matrix, M , implies instability. This is the case if

$$\text{Ra} > \frac{\pi^4 (a_x^2 + 1)^3}{a_x^2}$$

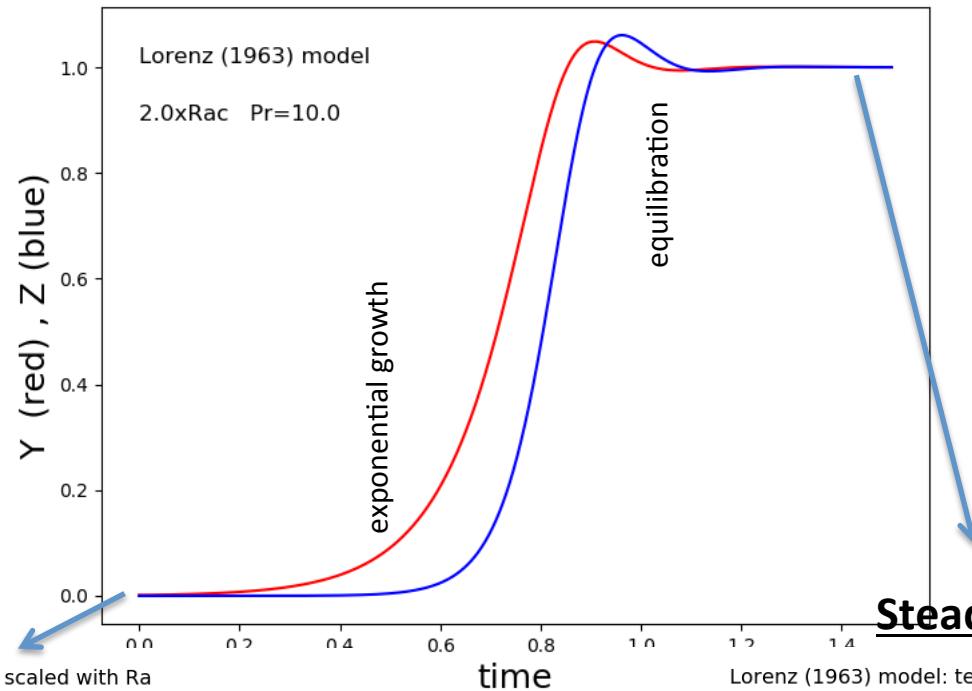
↓
Check this by assuming solutions of the form,
 $\boxed{X = X_0 \exp(\lambda t)}$

Critical Rayleigh number:

$$\text{Ra}_{\text{crit}} = \frac{\pi^4 (a_x^2 + 1)^3}{a_x^2} \longrightarrow \text{Minimum value when } a_x = \frac{1}{\sqrt{2}} \quad \text{or} \quad \frac{L}{H} = 2\sqrt{2}$$

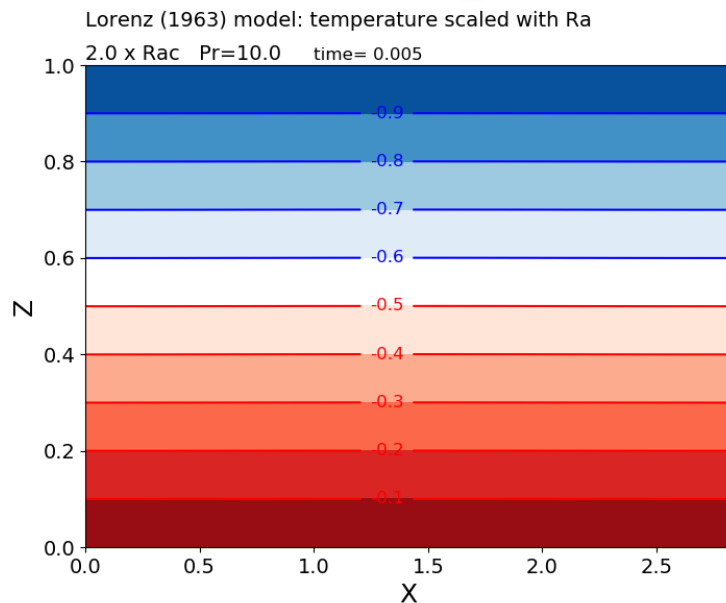
Numerical integration of the Lorenz equations

X and Z are scaled
with Ra

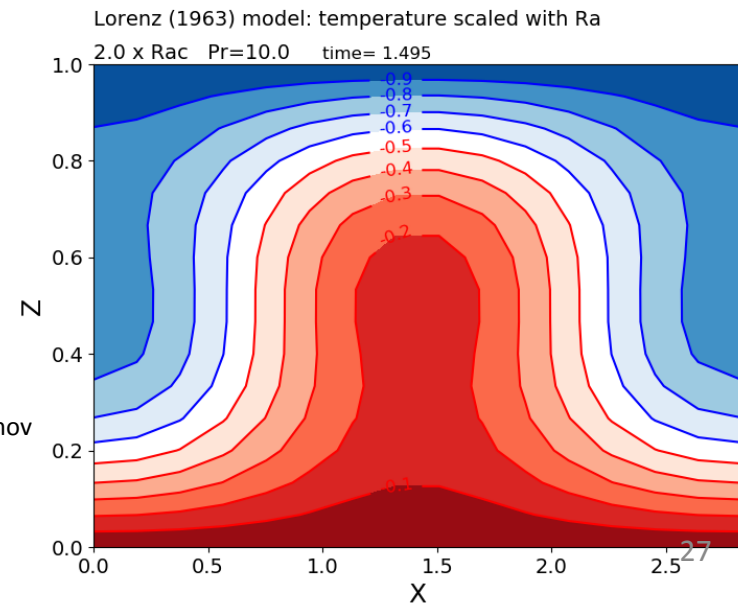


Initial state (rest):

Steady state convection



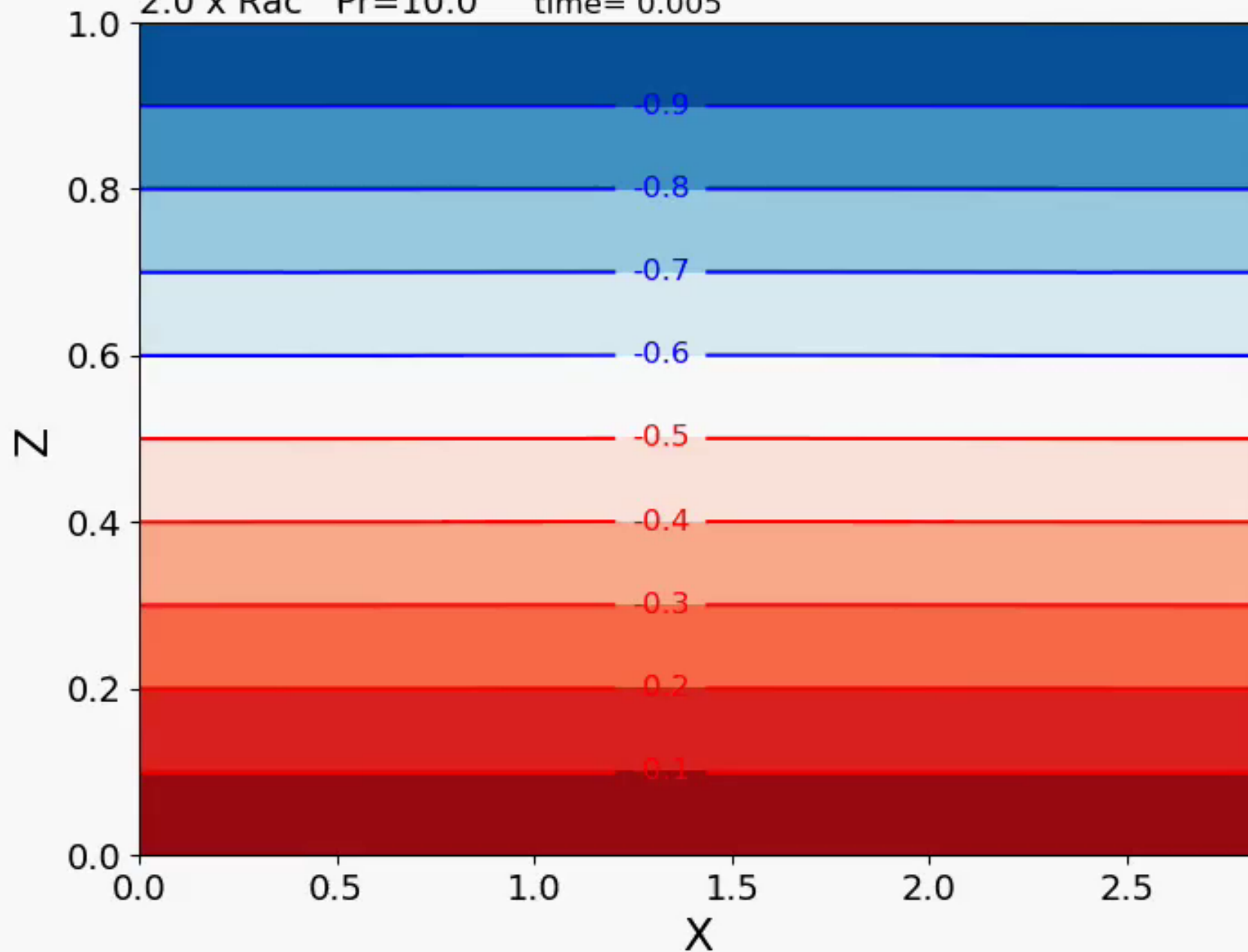
see animation on
blackboard:
LorenzModelTemp_2Ra.mov



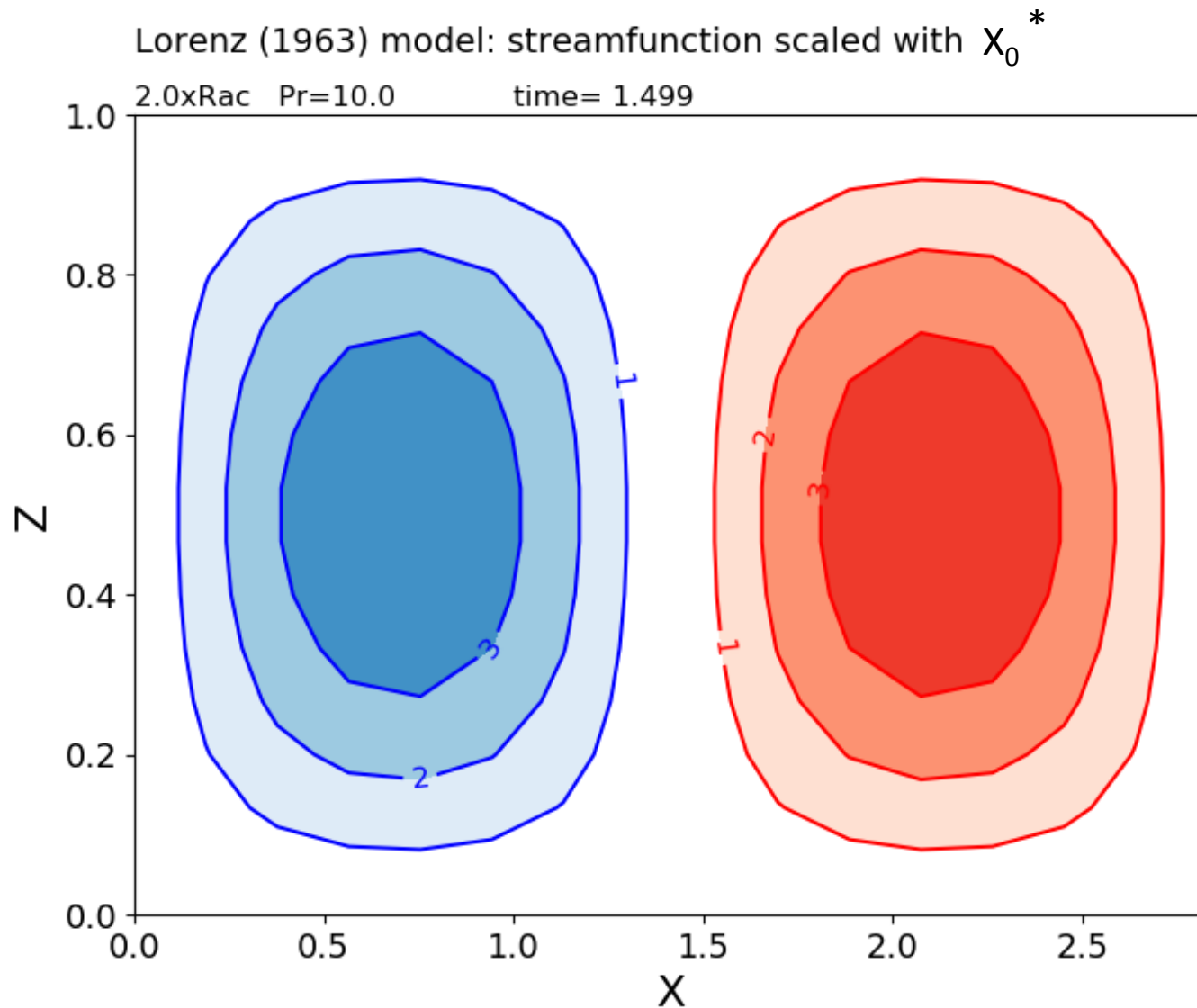
Lorenz (1963) model: temperature scaled with Ra

Animation

2.0 x Rac Pr=10.0 time= 0.005

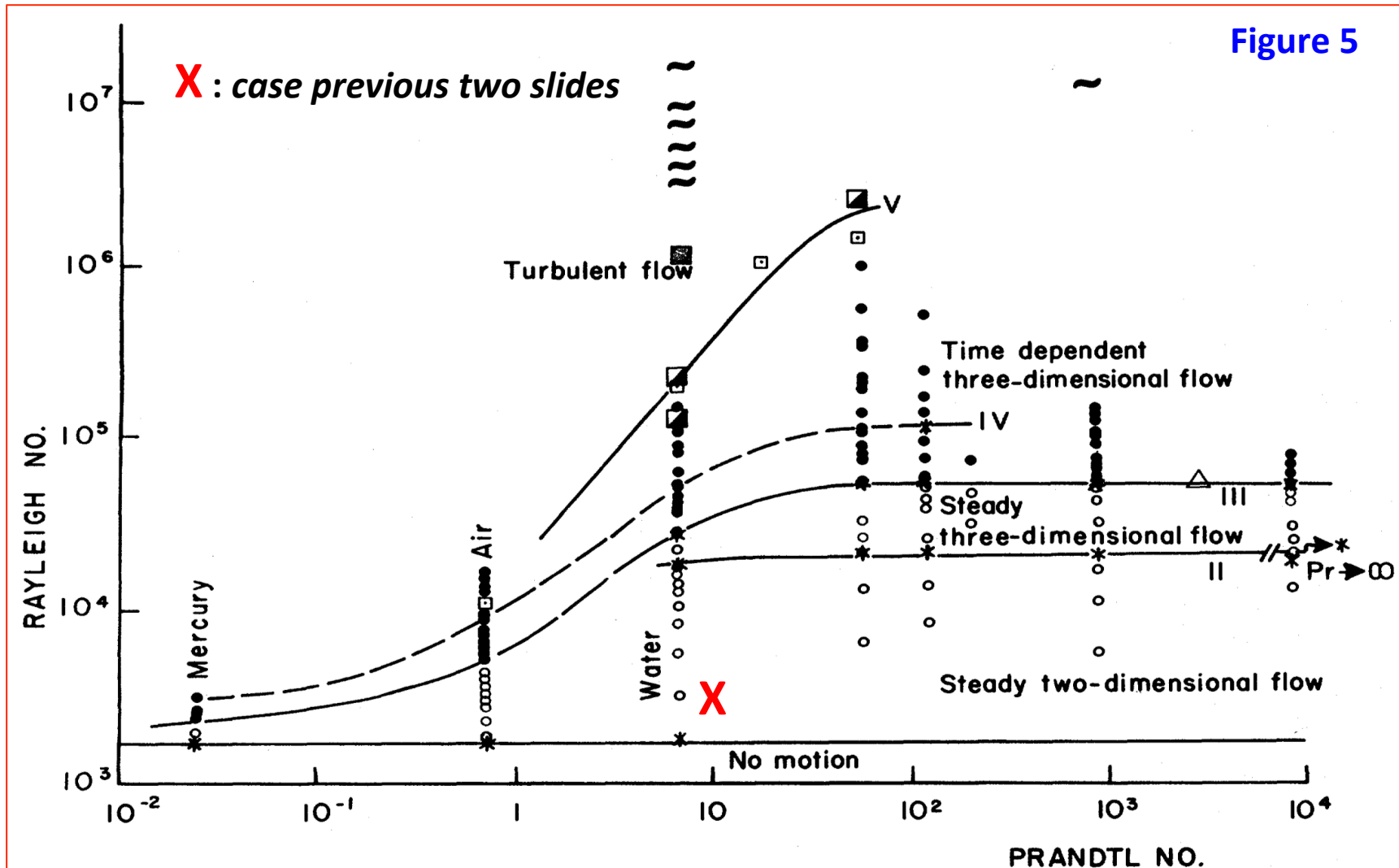


Stream function pattern in Lorenz model



* X_0 is the steady steady state value of X :
$$X_0 = \frac{\sqrt{Ra - Ra_c}}{\sqrt{2}\pi^2(a_x^2 + 1)}$$

Figure 5



Regime diagram of convection in fluids with different Prandtl numbers as a function of Rayleigh number for convection between rigid boundaries (not stress-free), distilled from **laboratory experiments**. Two-dimensional convection is observed at low Ra . As the Rayleigh number is increased, three-dimensional cells appear. If Ra is increased further, steady state regular cellular flow disappears and is replaced by a random array of transient plumes (see exercise 5).

Tutorial: exercises 5 to 8

(5)

Figure 5 (previous slide) shows a regime diagram of laboratory convection as a function of Ra and Pr , demonstrating that time dependent flow emerges when Ra is sufficiently large. Time-dependent turbulent flow emerges for lower Ra when Pr is small than when Pr is large. Which non-linear terms are responsible for this “early” onset of time-dependent flow at low Pr : those in eq. 57 (momentum advection) or those in eq. 58 (temperature advection)? Is the Lorenz model more applicable to low Pr convection or to high Pr convection?

(6)

The Lorenz model has two other steady state solutions (next to the state of rest), corresponding to finite amplitude convection. Derive expressions for X , Y , and Z in these steady states. Write these expressions in terms of r and a_x (hint: the steady state values of X , Y , and Z do not depend on Pr).



$$r \equiv \sqrt{(Ra - Ra_c)}$$

(7)

Show that the solution of the Lorenz model in phase space (i.e. as a function of X , Y and Z) for large t will either collapse to a point, to a line or to a two-dimensional surface (e.g. a torus) for all values of Ra , Pr and a_x . This means that the volume in phase space, spanned by the collection of trajectories corresponding to different initial states, will, as each point in phase space is displaced in accordance with eqs. 87-89, shrink to zero for large t . What is the associated e -folding time? In the words of Lorenz (1963) (p.135), *all trajectories ultimately become confined to a specific subspace having zero volume*. If the trajectories do not lie on a closed line or a torus in phase space, this subspace is referred as a “**strange attractor**”, a term which was introduced by the French mathematician, David Ruelle, and the Dutch mathematician, Floris Takens, in 1971 (see Ruelle (2006) and section 9.3 in Strogatz (1994)).

(8)

Derive expression for the growth-rate of small perturbations to the state of rest, similar to the expression in eq. 39, but now in the non-dimensional units of the Lorenz model (eqs. 87-89). You will find that perturbations grow at a rate, which depends on the Rayleigh number, Ra , and the aspect ratio (or a_x). Make a contour plot of the growth-rate as a function of Ra (vertical axis) and a_x (horizontal axis). HINT: the growth-rate corresponds to the positive eigenvalue of the matrix, M , in eq. 92.

Homework

(9)

Linearise the Lorenz equations (87-89) around the steady finite amplitude convective state and express this linearised system as in eqs. 91 and 92. Determine the linear stability of the finite amplitude steady states as a function of Ra and a_x , for $Pr=10$ (this is the value of Pr that was chosen by Lorenz in 1963) by determining the eigenvalues of the corresponding matrix, M . Draw a regime diagram, with Ra as the ordinate and a_x as the abscissa, indicating regions where finite amplitude convection is steady and stable and regions where finite amplitude is not steady and unstable. This can be done numerically. Determine the minimum value of Ra and the corresponding value of a_x for which all steady states are linearly unstable and for which the model only has time-dependent solutions (for $Pr=10$). **This exercise is given as homework.**

Hand in the answer on Wednesday, March 16, 2022.

Next lecture (8)

Steady state convection in the three-component Lorenz model: physical interpretation

Transition to chaos

Chaotic solution: what does this mean?

Exponential divergence of nearby trajectories (**Lyapunov-exponent**)
(*application: ensemble weather prediction*)

Predictability in chaos: **Lorenz map**

Is turbulence a manifestation of chaos?

Does a more realistic model of convection have chaotic solutions?

Order in a non-linear system: **scale-selection** in a 10-component spectral model of convection (the slaving principle)

Lorenz' research on predictability of weather

Tutorial: exercises 9 and 10

(9)

The total kinetic energy (K) and the total potential energy (P), associated with the perturbations (eq. 6) in two-dimensional convection (in which $v=0$) are, respectively,

$$K = \frac{1}{2} \int (u^2 + w^2) d\sigma \quad (95)$$

$$P = -\frac{1}{2} \text{Pr} \int \theta^2 d\sigma \quad (96)$$

Express K and P in spectral space and show that $K+P$ is conserved **by the non-linear terms in the Lorenz model.**

Hint: note that $(u^2 + w^2) = (\nabla \psi)^2 = \nabla(\psi \nabla \psi) - \psi \nabla^2 \psi$

(10)

For this exercise you need to know how to program a computer (in Mathematica, Python, in MATLAB, in C ...)

(a) Integrate the Lorenz (1963) equations (87-89) in time numerically through at least 8 non-dimensional time units. Approximate the time-derivatives with the fourth order (RK-4) Runge-Kutta scheme (https://en.wikipedia.org/wiki/Runge-Kutta_methods). Do this for the exact same values of σ , Ra , Pr , which Lorenz used to illustrate the chaotic solution of his model, i.e. $\sigma=1/(2^{1/2})$, $Ra=28Ra_c$ ($Ra_c=657$), $Pr=10$. The **initial condition** should lie close to steady state convection. Insert a very small perturbation to this steady state, i.e. such that $X = X_0 + 0.1$

(b) Locate local maxima of $Z(t)$, Z_n and plot Z_{n+1} as a function of Z_n , in other words reproduce the **Lorenz map** (see page 139 of Lorenz (1963), or p. 326-328 of Strogatz (1994)). According to the Lorenz map, the value of the next maximum (next peak in Z) is predictable.

(c) Now, plot the time between local maxima of $Z(t)$ as a function of time (e.g. the time half-way between two maxima). Discuss the plot.

(d) Perform a second integration with a slightly different initial condition, by weakly perturbing only one of the three variables. Plot a measure of the separation in phase space between the two solutions and estimate the associated Lyapunov-exponent, λ (see p. 320-323 of Strogatz (1994)).

(10) CONTINUED

(e) Compare the “climate” of the two solutions, i.e. the time average of X , Y , and Z . You may have to extend your integration in time by many more than 5 non-dimensional time units to get a “stable” answer. What conclusions can you draw? Now, compare the “climate” of the absolute values of the two solutions, i.e. the time average of $|X|$, $|Y|$, and $|Z|$. What do you conclude from this?

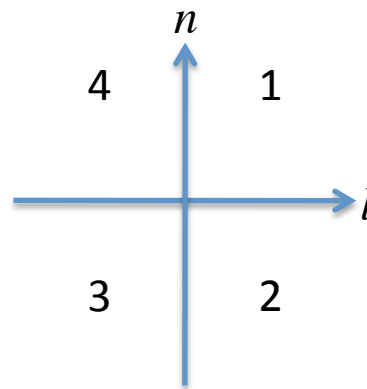
(f) Model climate can also be characterised by the **so-called “probability distribution function”** (“pdf”). A plot of the “pdf” as a function of X and Y gives the probability of finding the Lorenz model state vector in any point in X - Y plane of phase space. Plot this “pdf” and interpret your result in the light of **(e)**.

Use of the boundary conditions

Why is $\Theta_{l,n}^R = 0$?

We have (slide 14): $\Theta_{l,n}^R = \Theta_{-l,-n}^R$ (1) $\Theta_{l,n}^R = -\Theta_{l,-n}^R$ (2) $\Theta_{l,n}^R = \Theta_{-l,n}^R$ (3)

Wave number space has four quadrants, which are numbered as shown below.



Suppose that the Fourier coefficient $\Theta_{l,n}$ is **positive in quadrant 1**. According to (3) $\Theta_{l,n}$ is positive in quadrant 4, which means, according to (2), that it should be **negative in quadrant 3**, which is **not consistent with (1)**, except if $\Theta_{l,n}=0$.

Book keeping of the triad interactions

The Lorenz (1963) model: the only interactions that are “allowed” are listed in the table

α	β	γ
$(-1,1)$	$(1,1)$	$(0,2)$
$(1,1)$	$(-1,1)$	$(0,2)$

The **temperature equation** including only the nonlinear term on the r.h.s. is

$$\frac{d\Theta_\gamma}{dt} = \sum_{\alpha,\beta} I_{\gamma\alpha\beta} \Psi_\alpha \Theta_\beta$$

Interaction coefficient:
$$I_{\gamma\alpha\beta} \equiv \pi^2 a_x (n_\alpha l_\beta - n_\beta l_\alpha) \delta_{\gamma,\alpha+\beta}$$

The first triad interaction in the table gives $I_{\gamma\alpha\beta} \equiv 2\pi^2 a_x$ $\alpha=(-1,1)$ and $\beta=(1,1)$

The second triad interaction in the table gives $I_{\gamma\alpha\beta} \equiv -2\pi^2 a_x$ $\alpha=(1,1)$ and $\beta=(-1,1)$

$$\begin{aligned} \text{This makes } \sum_{\alpha,\beta} I_{\gamma\alpha\beta} \Psi_\alpha \Theta_\beta &= 2\pi^2 a_x \Psi_{-1,1}^R i \Theta_{1,1}^I - 2\pi^2 a_x \Psi_{1,1}^R i \Theta_{-1,1}^I \\ &= 2\pi^2 a_x \Psi_{1,1}^R i \Theta_{1,1}^I + 2\pi^2 a_x \Psi_{1,1}^R i \Theta_{1,1}^I = 4\pi^2 i a_x \Psi_{1,1}^R i \Theta_{1,1}^I \end{aligned}$$

Book keeping of the triad interactions

The Lorenz (1963) model: the only interactions that are “allowed” are listed in the table

α	β	γ
$(-1,1)$	$(1,1)$	$(0,2)$
$(1,1)$	$(-1,1)$	$(0,2)$

The **streamfunction equation** including only the nonlinear term on the r.h.s. is

$$\frac{d\Psi_\gamma}{dt} = \frac{1}{k_\gamma^2} \sum_{\alpha,\beta} k_\beta^2 I_{\gamma\alpha\beta} \Psi_\alpha \Psi_\beta$$

Interaction coefficient:
$$I_{\gamma\alpha\beta} \equiv \pi^2 a_x (n_\alpha l_\beta - n_\beta l_\alpha) \delta_{\gamma,\alpha+\beta}$$

The first triad interaction in the table gives $I_{\gamma\alpha\beta} \equiv 2\pi^2 a_x$ $\alpha=(-1,1)$ and $\beta=(1,1)$

The second triad interaction in the table gives $I_{\gamma\alpha\beta} \equiv -2\pi^2 a_x$ $\alpha=(1,1)$ and $\beta=(-1,1)$

This makes
$$\sum_{\alpha,\beta} k_\beta^2 I_{\gamma\alpha\beta} \Psi_\alpha \Psi_\beta = 2\pi^2 a_x k_{1,1}^2 \Psi_{-1,1}^R \Psi_{1,1}^R - 2\pi^2 a_x k_{1,1}^2 \Psi_{1,1}^R \Psi_{-1,1}^R = 0$$

Remember: $k_\beta^2 = a_x^2 l_\beta^2 + n_\beta^2$