

Chaos and predictability

Lecture 8, course on “Turbulence in Fluids”, Utrecht University, 2021

<http://www.staff.science.uu.nl/~delde102/TinF.htm>

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9 March 2022

Steady state convection in the Lorenz model

Chaotic solution when the system possesses no stable fixed point:
what does this imply?

- 1. Sensitive dependence on initial conditions**
- 2. Exponential divergence of nearby trajectories in phase space**
- 3. Volume contraction in phase space > (“strange” attractor)**
- 4. Lorenz map: order in chaos !**

Literature: Lorenz (1963) and Strogatz (1994), sections 9.2-9.4

Lorenz model equations:

A dissipative system

$$\frac{dX}{dt} = \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} Y - \pi^2(a_x^2 + 1) \text{Pr} X$$

$$\frac{dY}{dt} = -2\pi^2 a_x XZ + \pi a_x \text{Ra} X - \pi^2(a_x^2 + 1) Y$$

$$\frac{dZ}{dt} = 4\pi^2 a_x XY - 4\pi^2 Z$$

The **last term** on the right hand side of each equation represents **dissipation**. (the coefficient in front of these terms, including the minus sign, is negative).

Dissipation is due to molecular viscosity and molecular conduction of heat.

Steady state convection in the Lorenz model

Two steady state solutions, corresponding to finite amplitude convection, are

$$(X, Y, Z) = (X_0, Y_0, Z_0) \equiv \left(\frac{r}{\sqrt{2}\pi^2(a_x^2 + 1)}, \frac{\pi(a_x^2 + 1)r}{\sqrt{2}a_x}, \frac{r^2}{2\pi} \right)$$

and

$$(X, Y, Z) = (X_0, Y_0, Z_0) \equiv \left(\frac{-r}{\sqrt{2}\pi^2(a_x^2 + 1)}, \frac{-\pi(a_x^2 + 1)r}{\sqrt{2}a_x}, \frac{r^2}{2\pi} \right)$$

$$r \equiv \sqrt{(Ra - Ra_c)}$$

Counter-rotating convection cells.

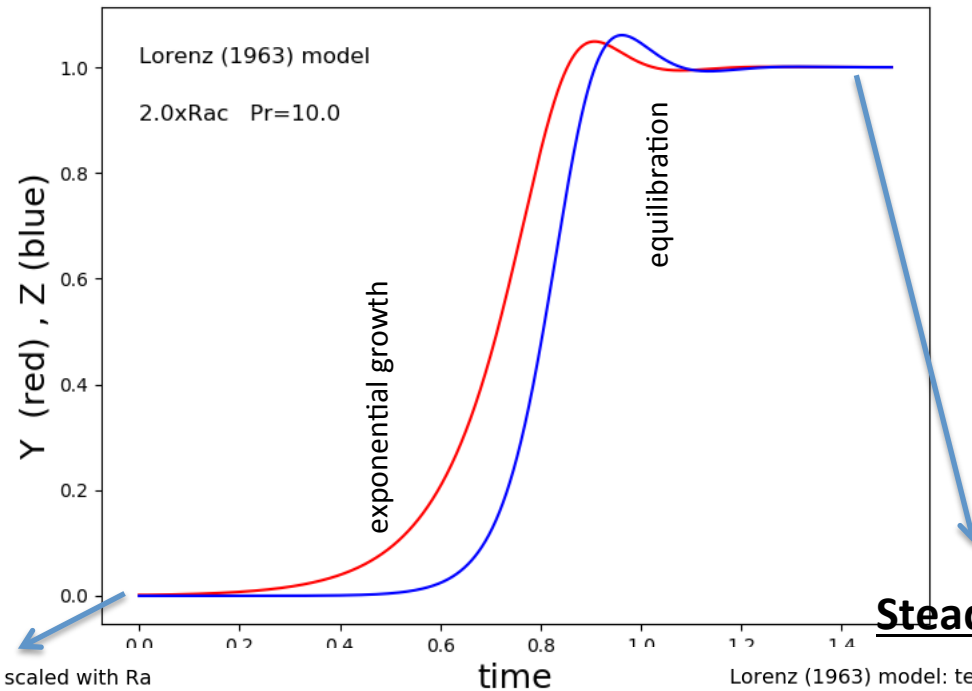
These solutions appear when $Ra > Ra_c$.

Lecture 7:

Numerical integration of the Lorenz equations

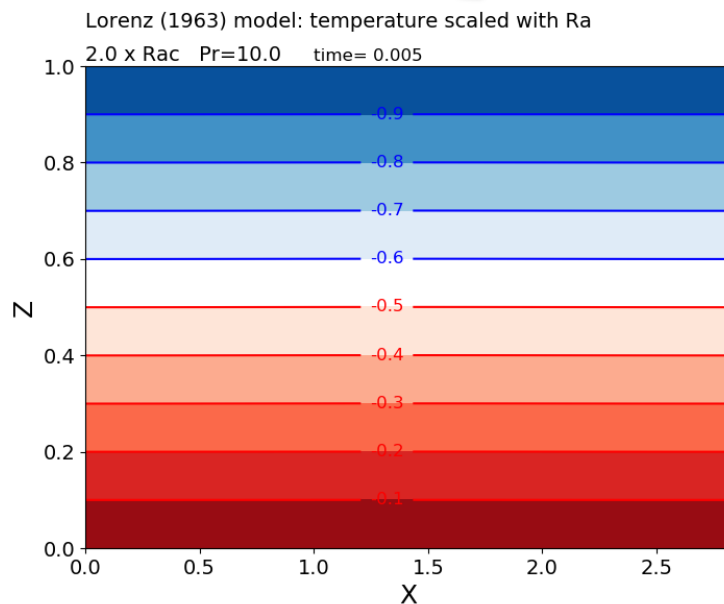
X and Z are scaled
with the steady state
value at $Ra = 2Ra_c$

$$Ra = 2Ra_c$$

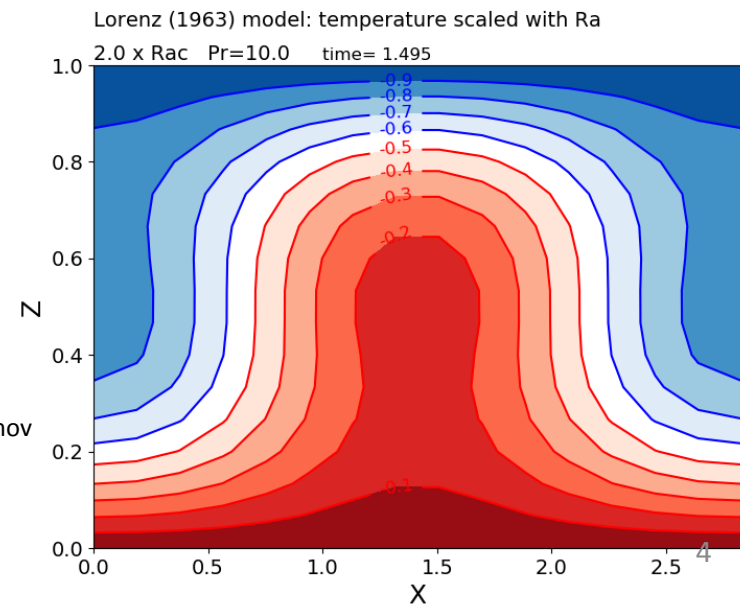


Initial state (rest):

Steady state convection



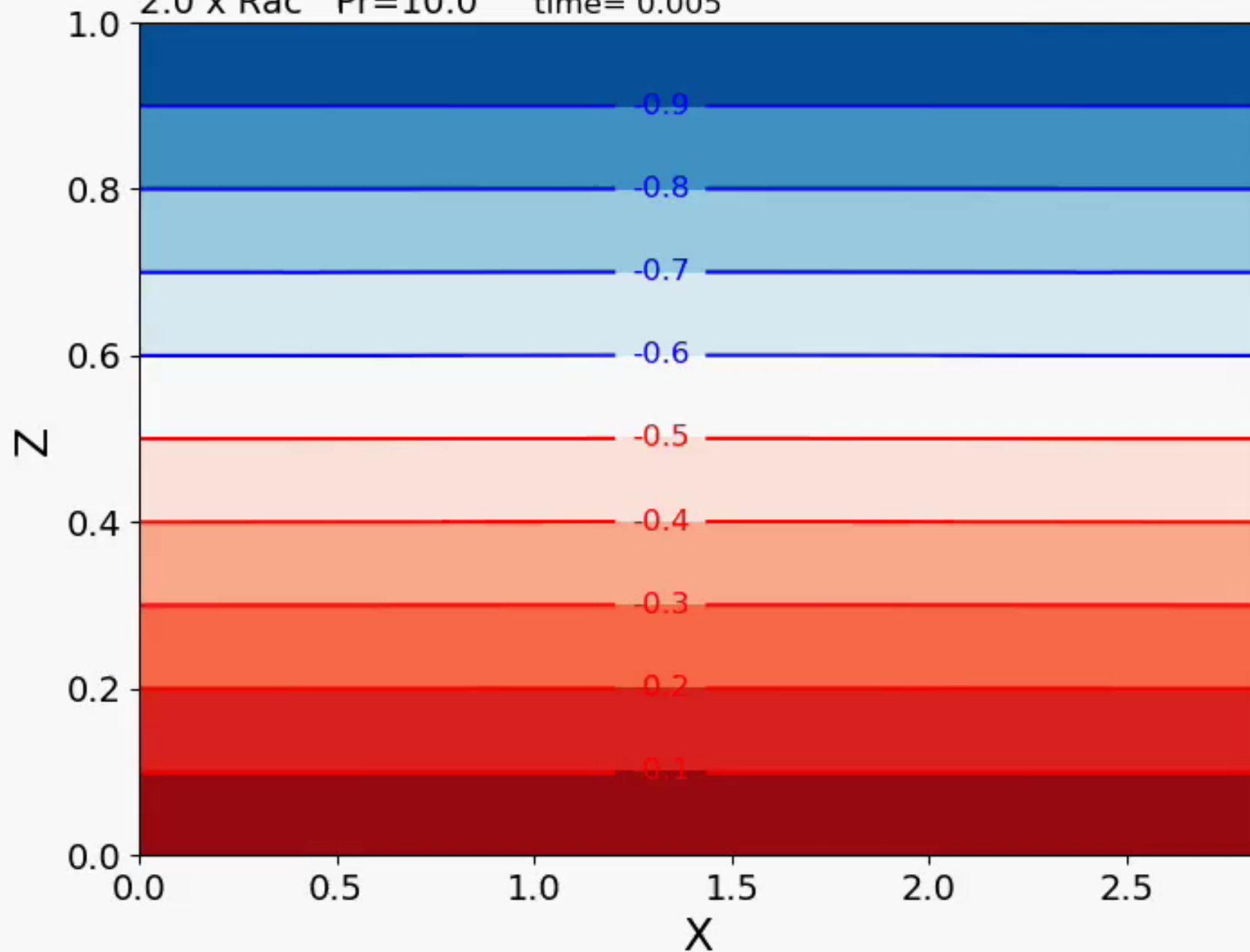
see animation on
blackboard:
LorenzModelTemp_2Rac.mov



Lorenz (1963) model: temperature scaled with Ra

Animation

2.0 x Rac Pr=10.0 time= 0.005



Homework assignment (will be graded)

Linear stability analysis of finite amplitude steady convection

(9)

Linearise the Lorenz equations (87-89) around the steady finite amplitude convective state and express this linearised system as in eqs. 91 and 92. Determine the linear stability of the finite amplitude steady states as a function of Ra and a_x , for $Pr=10$ (this is the value of Pr that was chosen by Lorenz in 1963) by determining the eigenvalues of the corresponding matrix, M . Draw a regime diagram, with Ra as the ordinate and a_x as the abscissa, indicating regions where finite amplitude convection is steady and stable and regions where finite amplitude is not steady and unstable. This can be done numerically. Determine the minimum value of Ra and the corresponding value of a_x for which all steady states are linearly unstable and for which the model only has time-dependent solutions (for $Pr=10$). **This exercise is given as homework.**

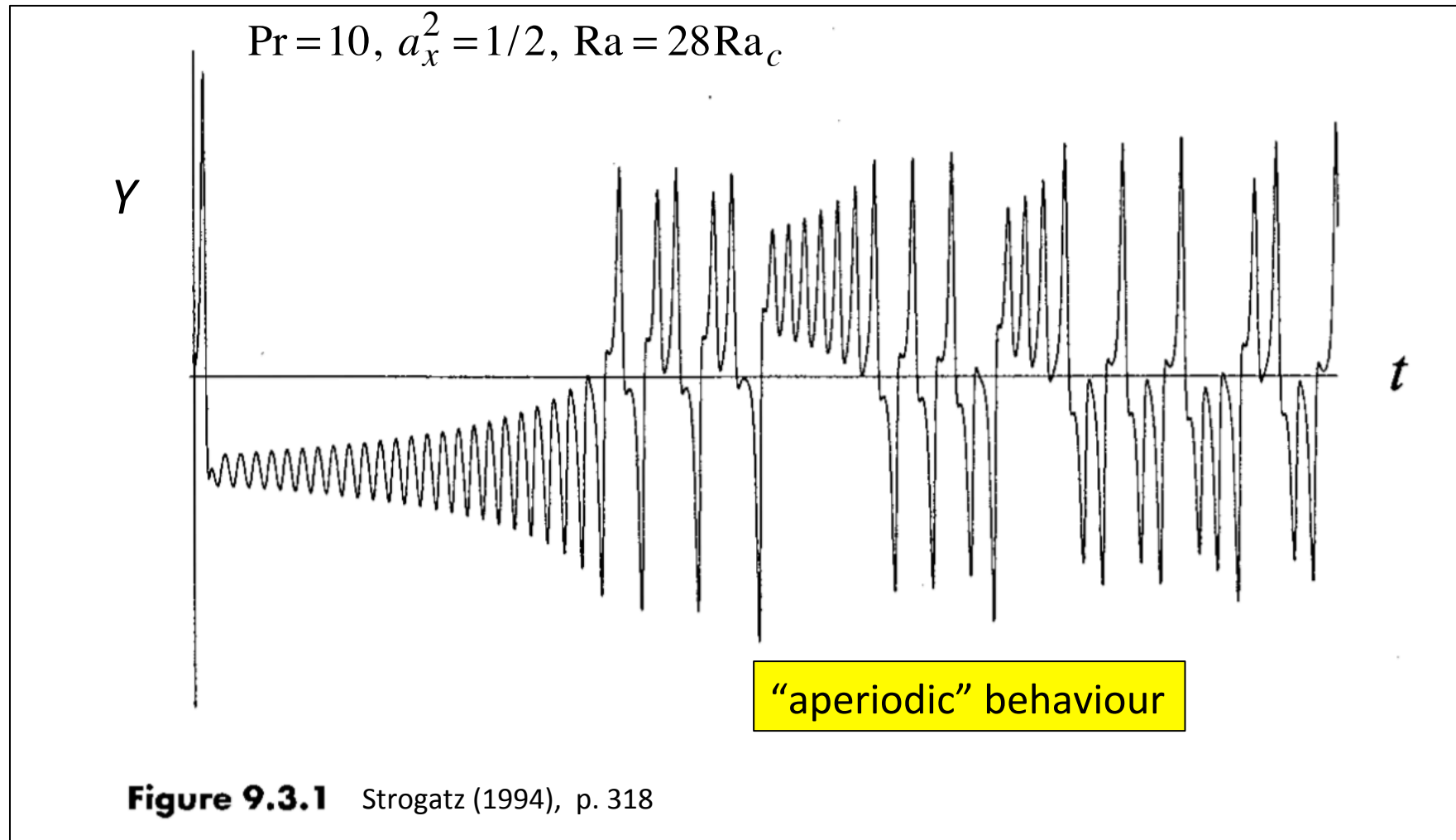
Hand in the answer no later than end of Wednesday, March 16, 2022.

HINT: If $Ra > 24.74 Ra_c$ (at $Pr=10$ and $a^2=1/2$), the finite amplitude steady convection is not stable to small perturbations.

What will happen $Ra > 24.74 Ra_c$?

Aperiodic numerical solution at larger Ra

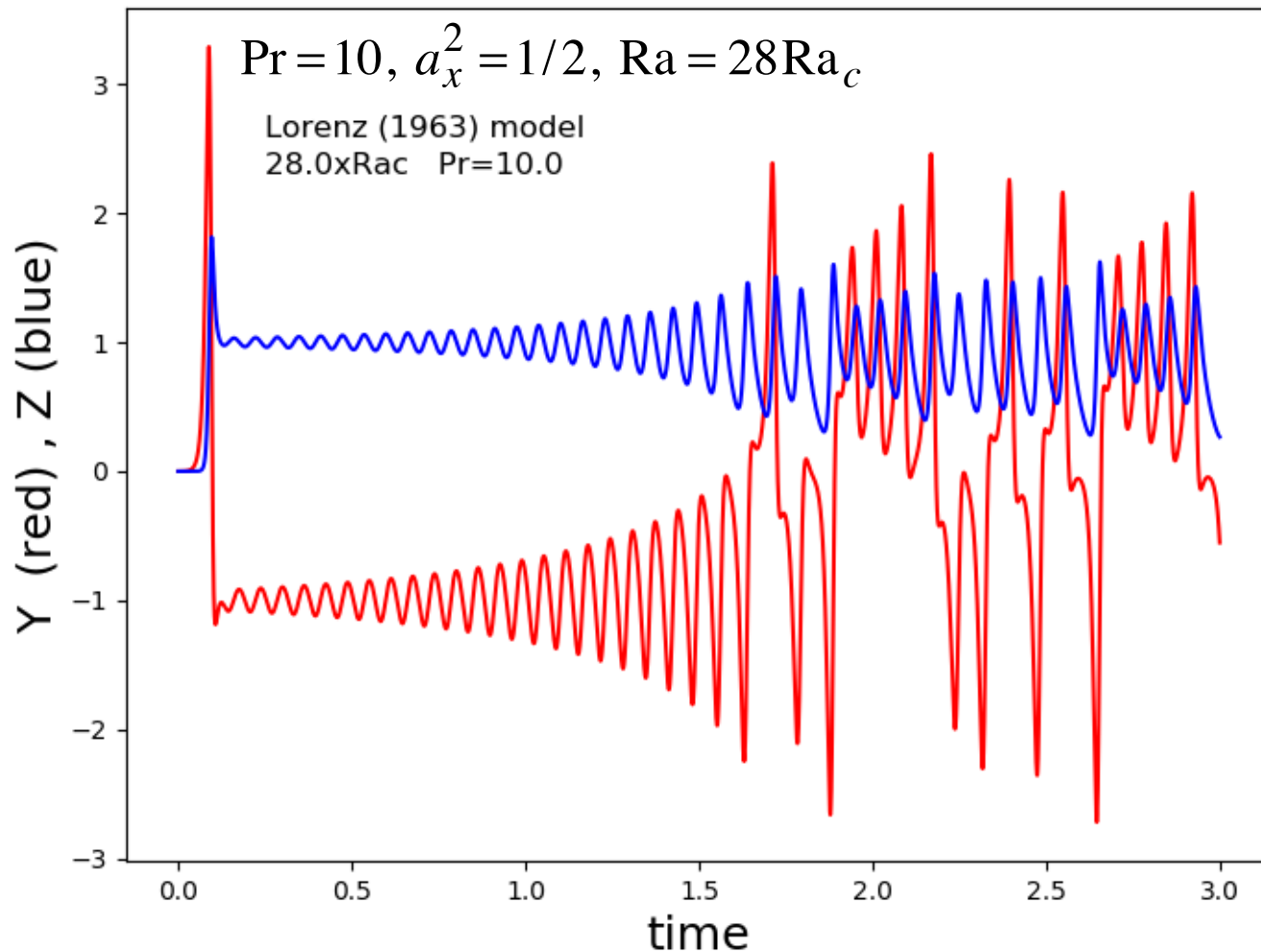
Lorenz (1963) integrated his model equations numerically. The result is shown below.



Aperiodic numerical solution

Integration done with Python-script (Runge-Kutta fourth order time stepping)

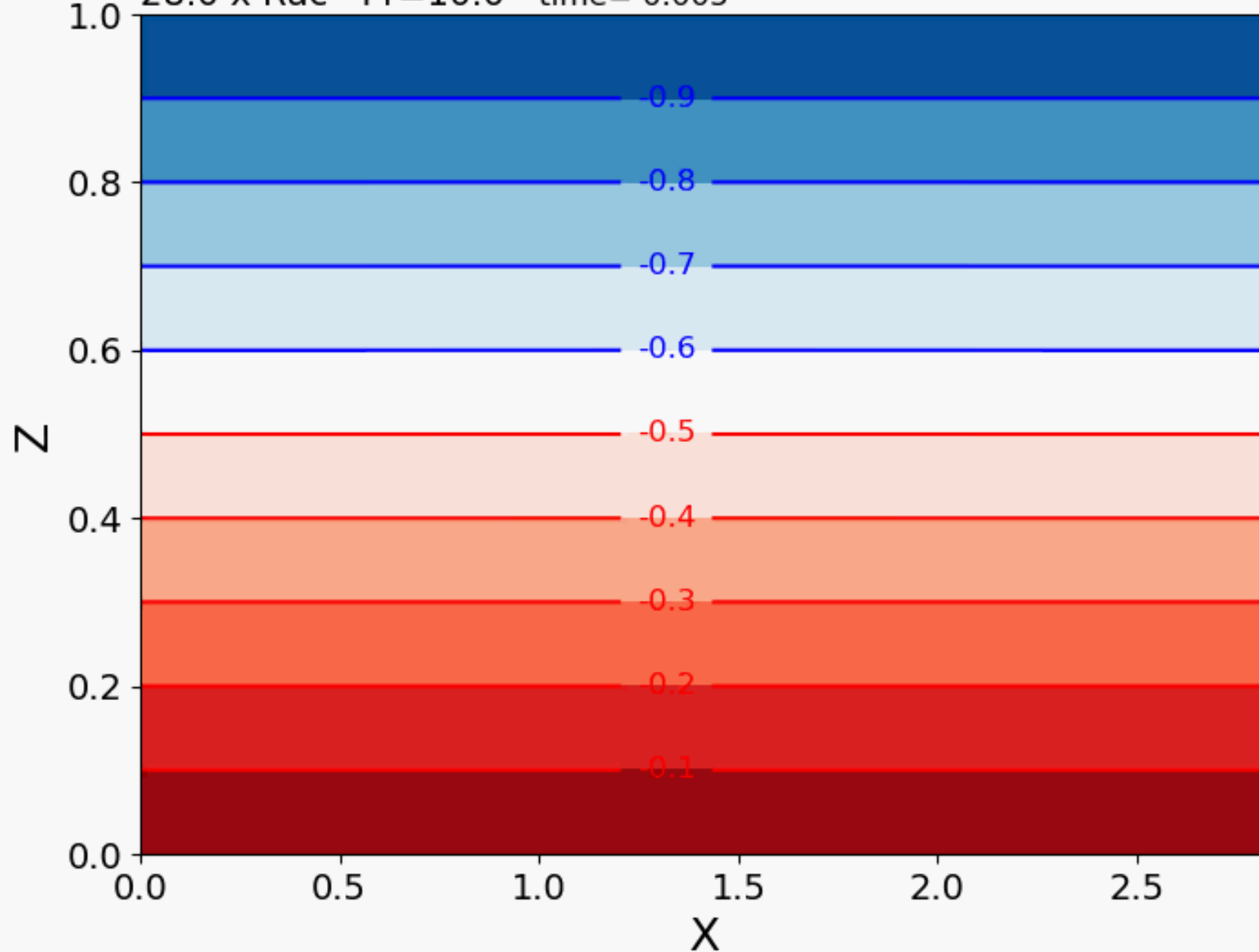
Compare this solution with the one on the previous slide (from the book by Strogatz)



Lorenz (1963) model: temperature scaled with Ra

Animation

28.0 x Rac Pr=10.0 time= 0.005



A “Strange Attractor” *

A wonderful structure in phase space

$$\text{Pr} = 10, a_x^2 = 1/2, \text{Ra} = 28\text{Ra}_c$$

Trajectory in phase space
projected onto the two-
dimensional x - z plane.

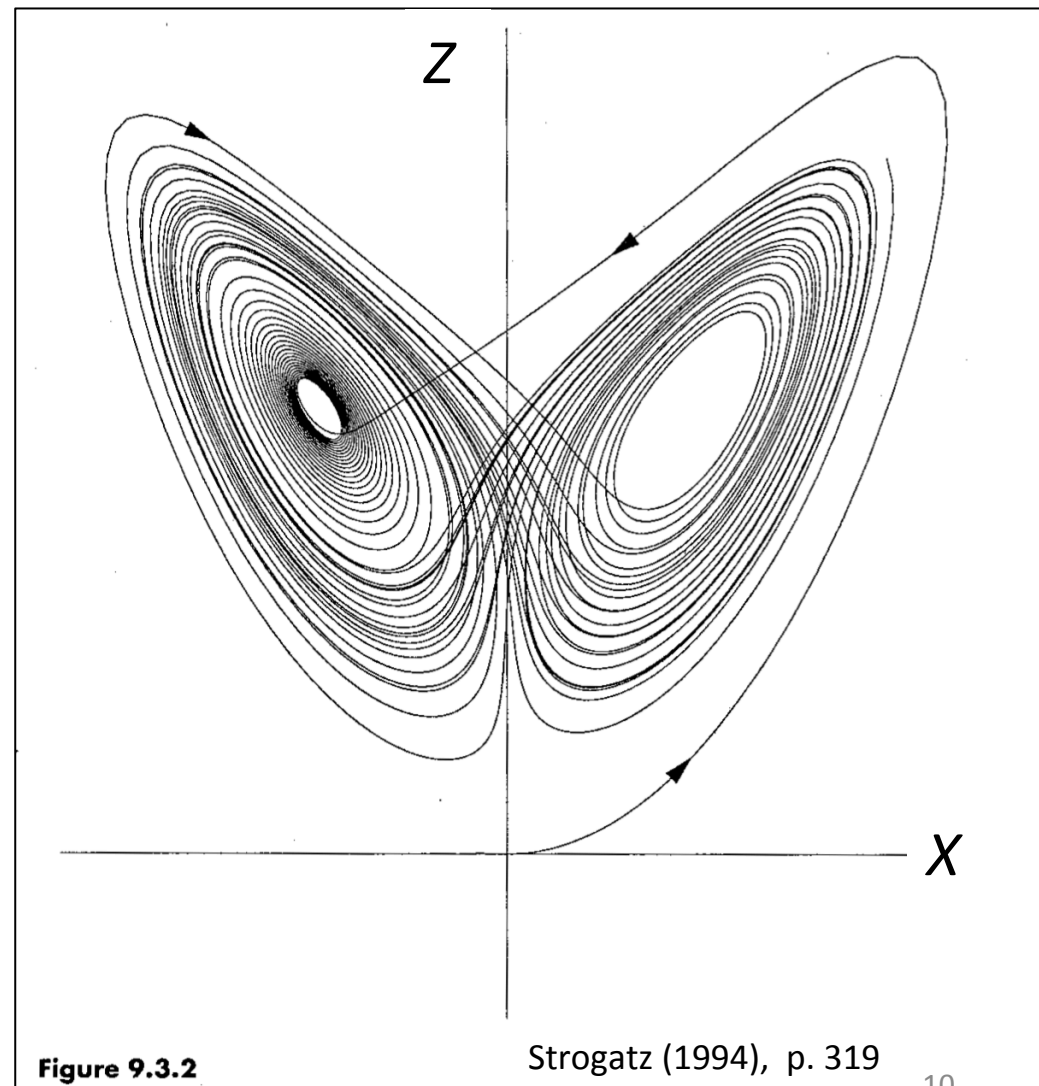
In 3-D, no self-intersections occur.

Looks like a set of **butterfly** wings.

Is this structure three dimensional?

Is this solution chaotic ?

* “Strange attractor” is a term coined in 1971 by the mathematicians, David Ruelle and Floris Takens, in a paper entitled: “On the Nature of Turbulence” (Lorenz was not cited).



Edward Lorenz stumbles upon the “strange attractor”

In the 1960's Edward Lorenz was working on a **statistical forecasting** project. He was performing numerical time-integrations of his simple non-linear model with his newly acquired computer. On one occasion he wanted to reexamine the results from an earlier simulation. Rather than rerun the simulation from the initial state, he decided to pick up computations partway into the original run by using the print-out from the earlier run as the starting point.

To his astonishment, the new simulation diverged significantly from the original. Eventually he realised that the initial values he used for the second simulation were rounded off from the initial run so that the initial values of his second run were slightly different. The minor differences at initialisation were magnified later in the run and led to very different states. Lorenz concluded that, if the real atmosphere evolved similarly to his numerical simulation, then very **long-range prediction would be impossible**.



Ed with his wife, Jane, in the White Mountains of New Hampshire.



Professor Edward Lorenz on 6 Nov 2007.

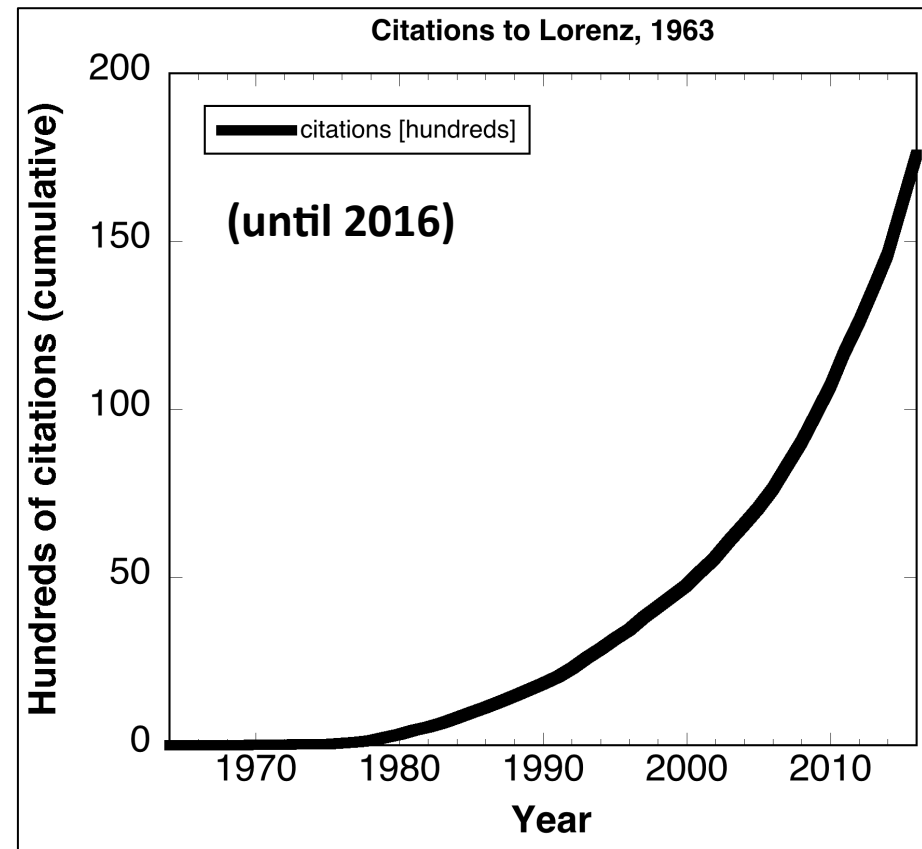
Intermezzo:

Huge but slow impact of Lorenz (1963)*

During the first 10 years after the year of publication there was very little interest in this paper, due to its somewhat inconvenient message, but after 1980 the impact of this paper “exploded”, now (1 March 2022) reaching 25197 citations.

With the present definition of “impact”, Lorenz’s 1963 paper had no “impact” !

* Lorenz, E.N., 1963: Deterministic non-periodic flow. **J.Atmos.Sci.** **20**. 130-141



Intermezzo:

Relative Impact of Lorenz (1963)

Lorenz (1963): 25197* citations (top 100)

Rayleigh (1916): 2613* citations

The paper mountain

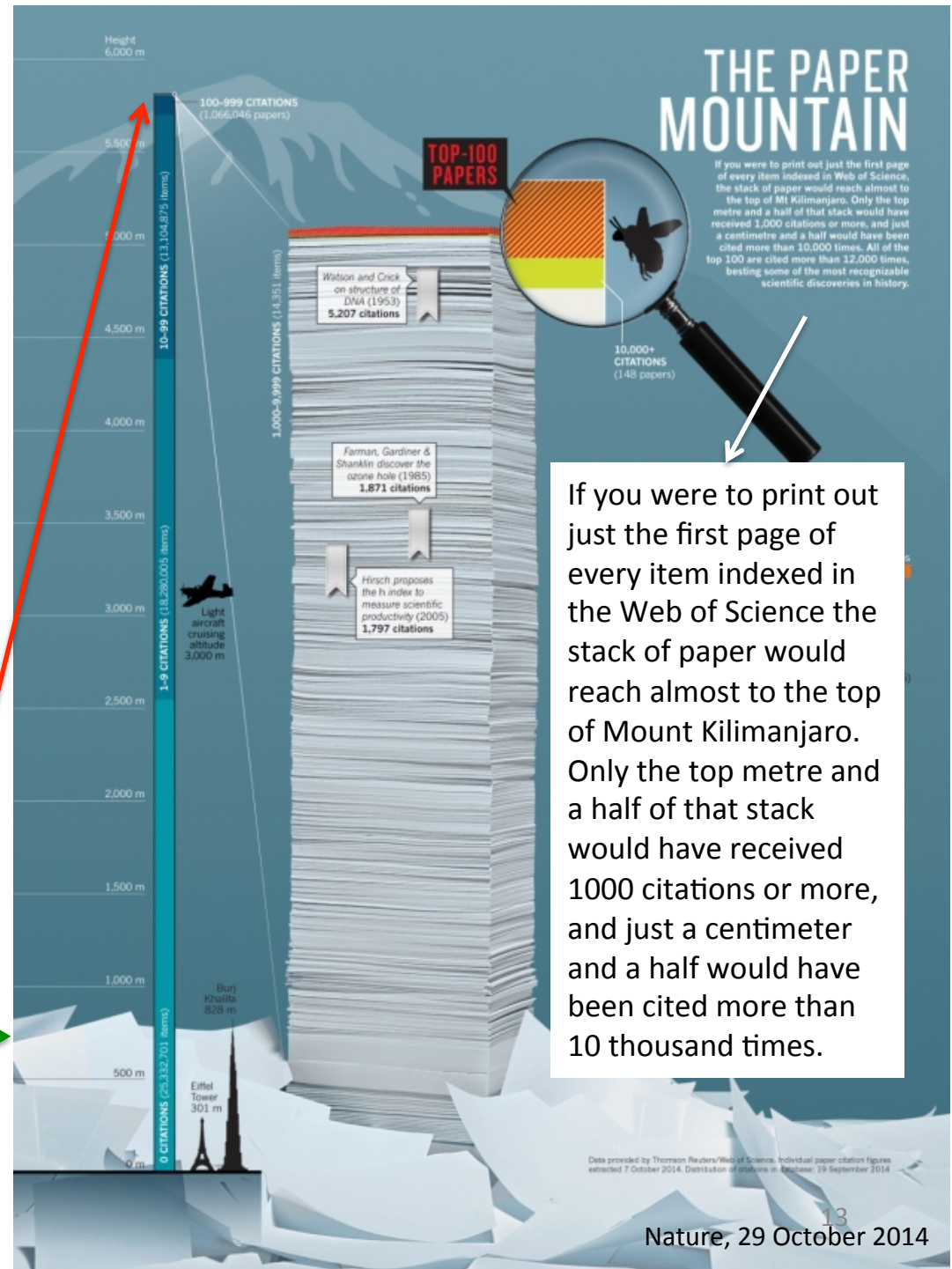
In 2014: millions of scientific papers!

1 million papers had 100-999 citations

13 million papers had 10-99 citations

18 million papers had 1-9 citations

25 million papers had 0 citations



* 1 March 2022

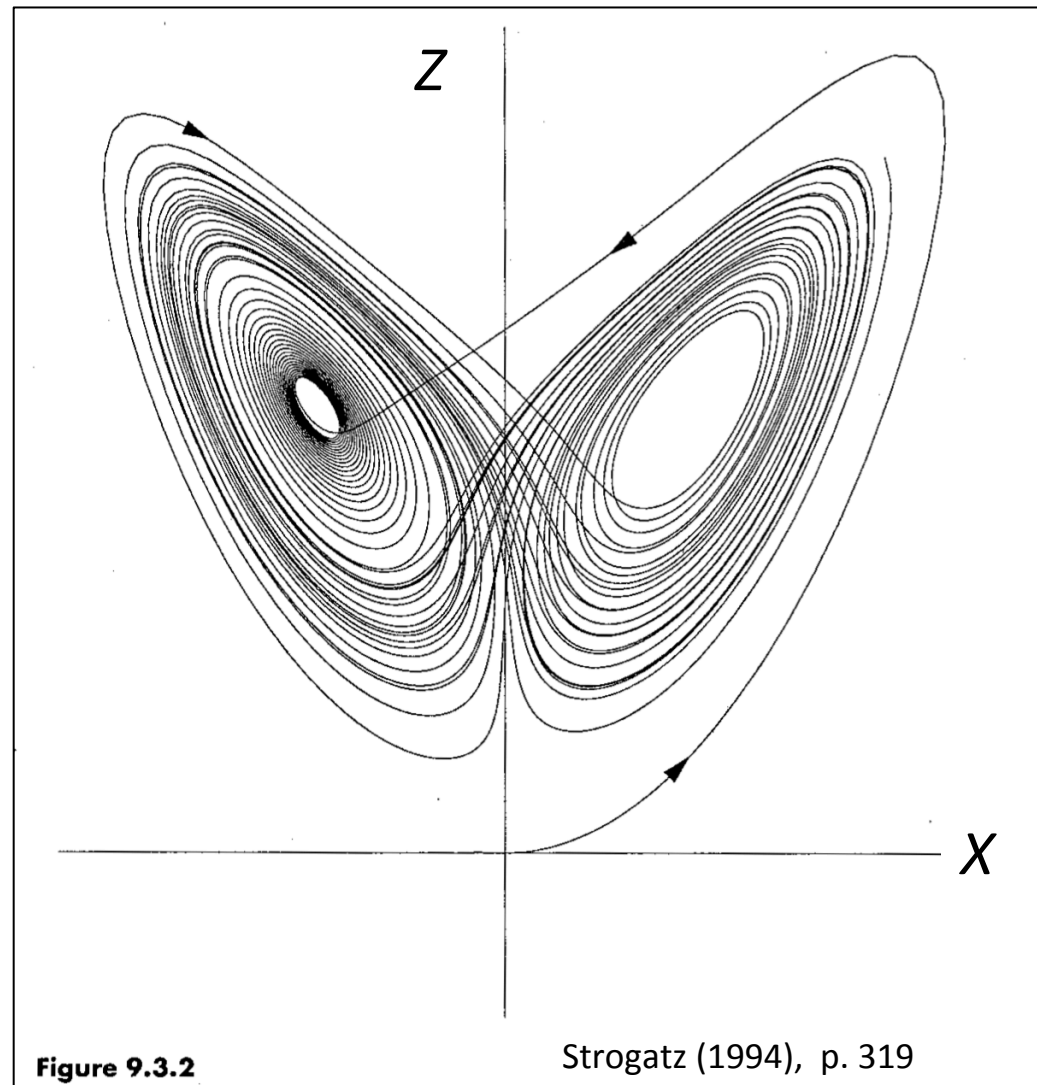
Nature, 29 October 2014

A “Strange Attractor*”

Is this chaos? Is this turbulence?

$$\text{Pr} = 10, a_x^2 = 1/2, \text{Ra} = 28\text{Ra}_c$$

Trajectory in phase space projected onto the two-dimensional x-z plane.



Ingredients of chaotic solution

There is no accepted definition of chaos.

But all agree on the following three **ingredients of chaos**.

Aperiodic behaviour means that the trajectories in phase space ***never*** settle down to fixed points, periodic orbits, or quasi-periodic orbits.

Deterministic means that the system has no randomly varying term or input parameters.

Sensitive dependence on initial conditions means that **nearby** trajectories in phase space separate exponentially fast, i.e. the system has a positive “Lyapunov exponent” (slide 17).

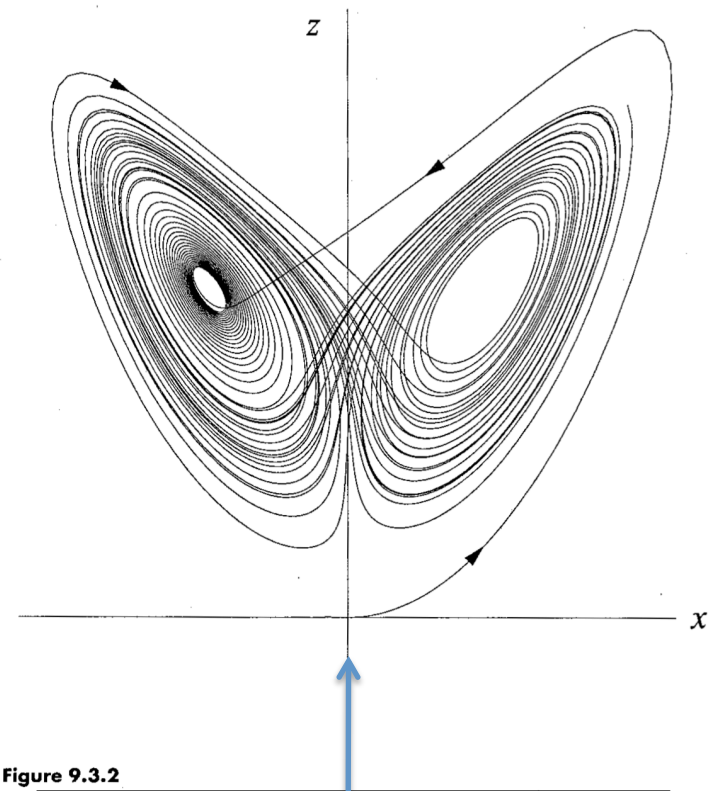


Figure 9.3.2

Solution of Lorenz model in chaotic regime projected onto x-z phase space.

Sensitive dependence on initial conditions

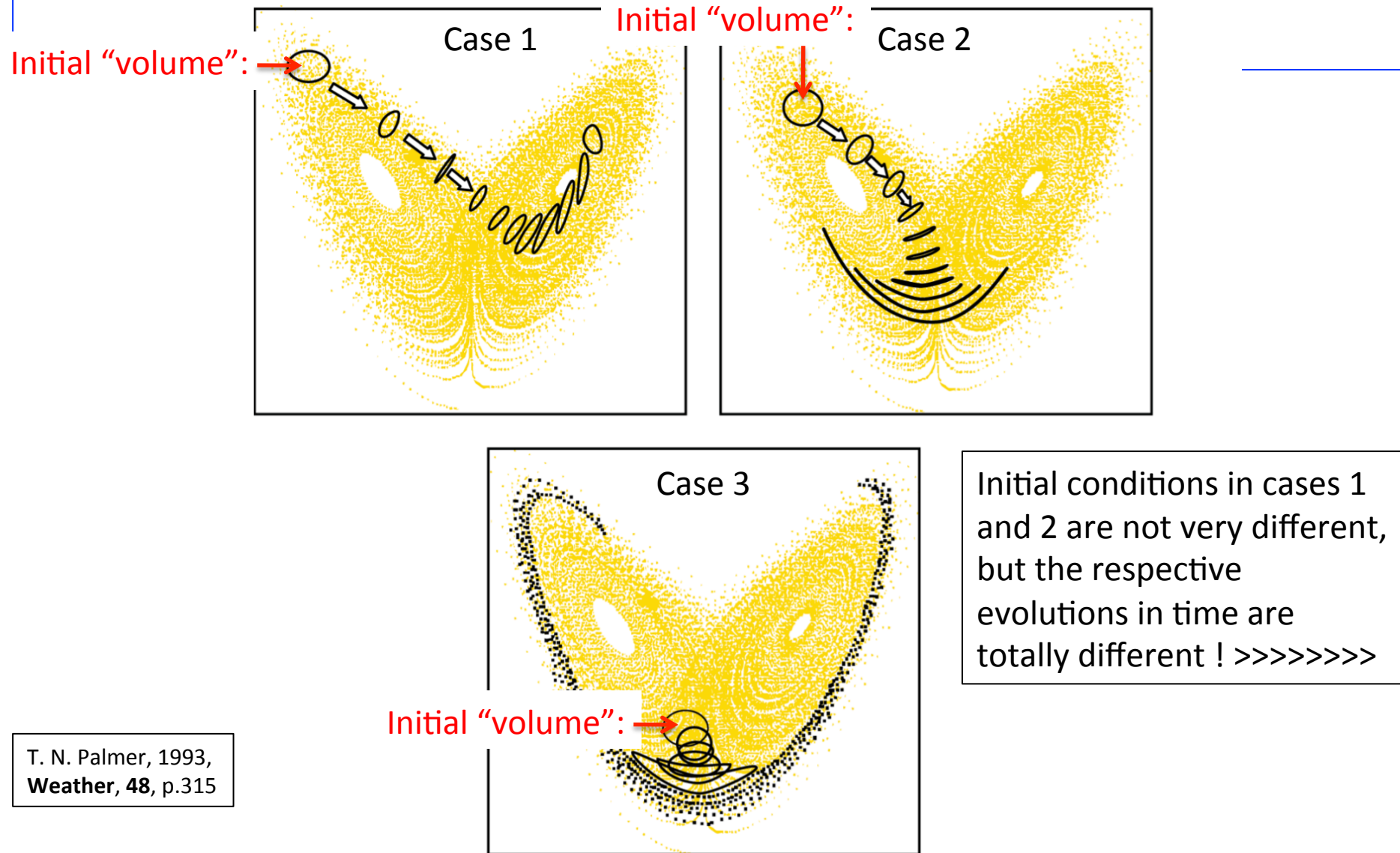


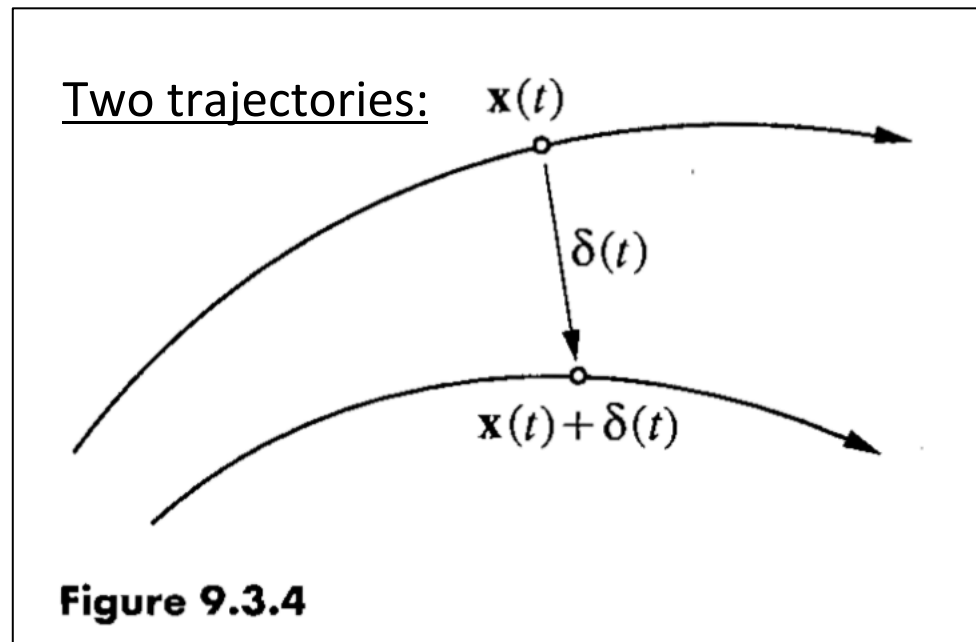
Figure 1. The scientific basis for ensemble forecasting illustrated by a Monte Carlo solution of the Liouville equation for the Lorenz (1963) system (2), showing that predictability is dependent on flow in a nonlinear system.

Sensitive dependence on initial conditions: separating trajectories in phase space

Nearby trajectories separate exponentially fast:

$$|\delta(t)| \approx |\delta_0| \exp(\lambda t)$$

“Lyapunov exponent”, λ



Strogatz (1994), p. 321

Lyapunov exponent is a measure of exponential divergence of nearby trajectories

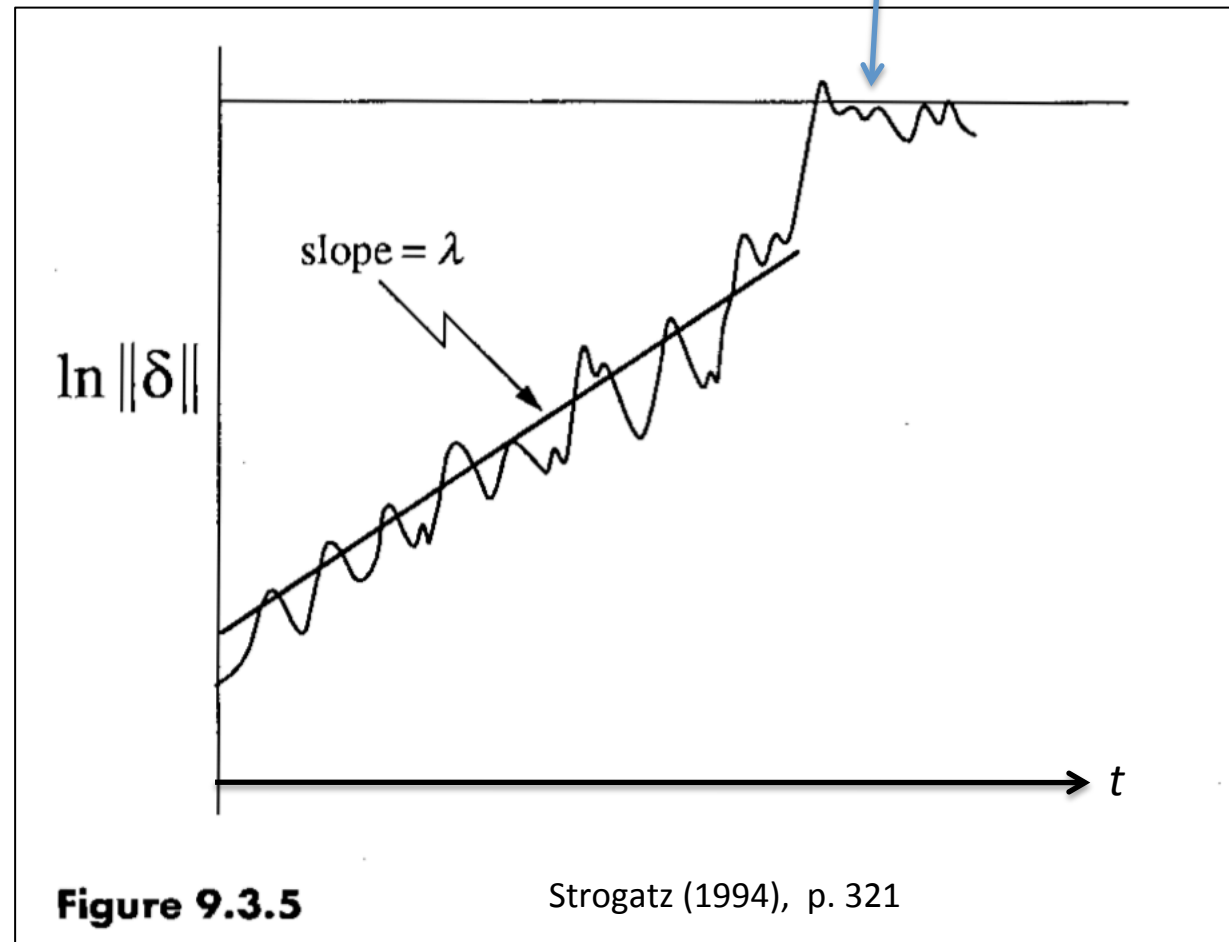
Nearby trajectories separate exponentially fast:

$$|\delta(t)| \approx |\delta_0| \exp(\lambda t)$$

“Lyapunov exponent”, λ

$$\ln|\delta(t)| \approx (\lambda t) + \ln|\delta_0|$$

(exercise 10d)



Time-horizon for prediction

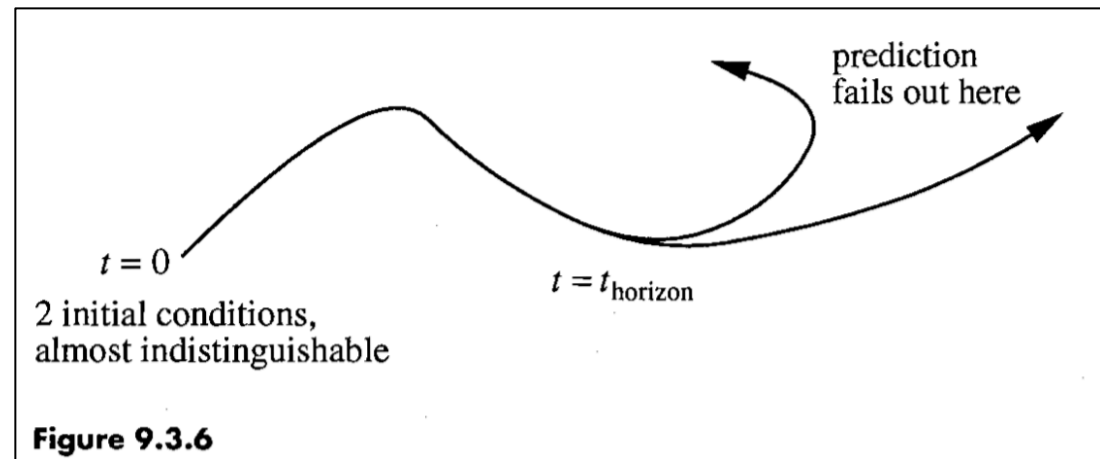
Suppose we measure the initial conditions of an experimental system with a very small error, $|\delta_0|$. After a time, t , the error, δ , is (see previous slide),

$$|\delta(t)| \approx |\delta_0| \exp(\lambda t) \quad \text{or} \quad \ln|\delta(t)| \approx (\lambda t) + \ln|\delta_0|$$

Let a be a measure of our tolerance. Our prediction is not accepted if $|\delta(t)| > a$
This occurs after a time in the order of

$$t_{\text{horizon}} \approx \frac{1}{\lambda} \ln \frac{a}{|\delta_0|}$$

Error doubling time = $\frac{\ln 2}{\lambda}$



Time-horizon : an example

Suppose **the initial error is $|\delta_0|=10^{-7}$** . For how long can we predict the state of the system if we **accept an error $a=10^{-3}$** ? Now, suppose that we improve our measurements a millionfold, such that **$|\delta_0|=10^{-13}$** . How much longer can we predict?

Answer to the first question:

$$t_{\text{horizon}} \approx \frac{1}{\lambda} \ln \frac{a}{|\delta_0|} \approx \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-7}} \approx \frac{4}{\lambda} \ln 10$$

Answer to the second question:

$$t_{\text{horizon}} \approx \frac{10}{\lambda} \ln 10$$

Therefore, after a **millionfold improvement in the initial uncertainty, we can predict only 2.5 times longer**

Lorenz (1963) system is “dissipative”

Volume contraction in phase space

Three-dimensional phase (X, Y, Z) is like a “fluid-parcel” which flows with a “velocity”,

$$\left(\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt} \right)$$

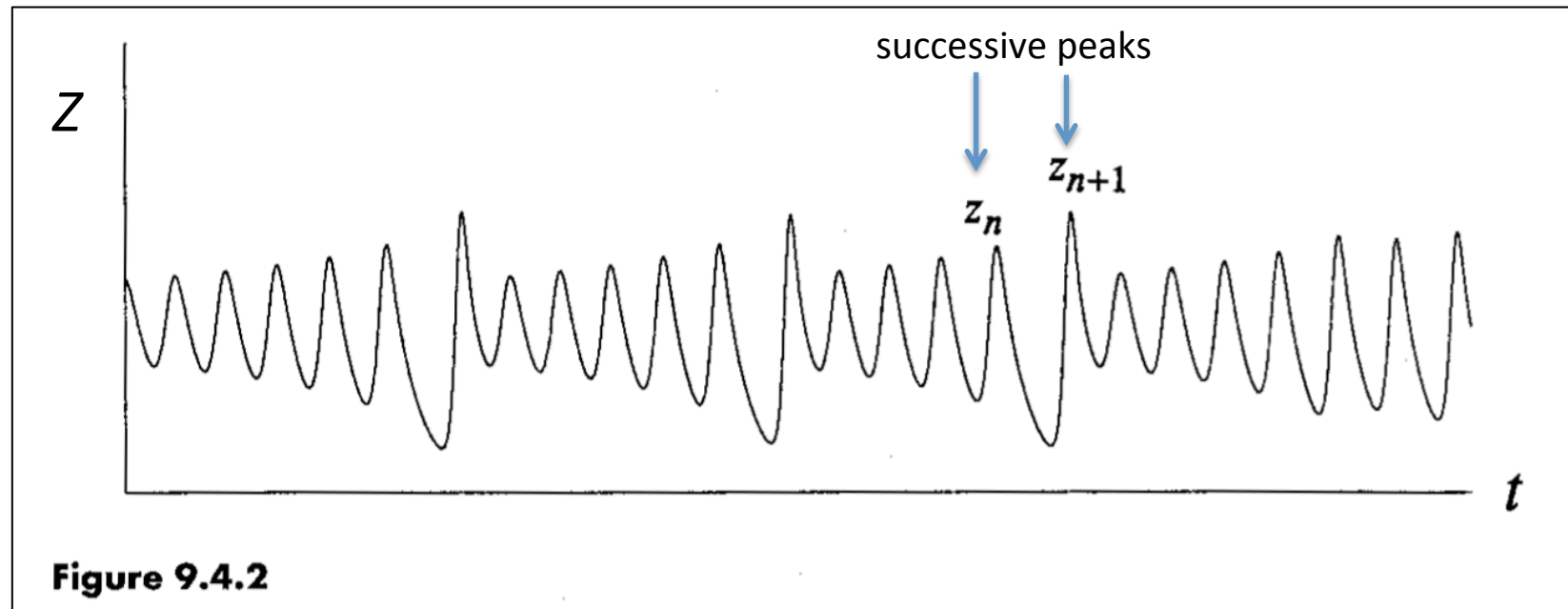
The change of the volume, $V=\Delta X\Delta Y\Delta Z$, of the “fluid-parcel” in phase space obeys

$$\frac{1}{V} \frac{dV}{dt} \approx \frac{\partial}{\partial X} \left(\frac{d\Delta X}{dt} \right) + \frac{\partial}{\partial Y} \left(\frac{d\Delta Y}{dt} \right) + \frac{\partial}{\partial Z} \left(\frac{d\Delta Z}{dt} \right) \approx -\pi^2(a^2+1)Pr - \pi^2(a^2+1) - 4\pi^2 < 0$$

All trajectories will ultimately become confined to a subspace in phase space **having zero volume** ! (page 135, Lorenz (1963)). Therefore, the **butterfly** is not three-dimensional. **Is it a two-dimensional surface?** If it is, then it is an extremely complicated surface in three-dimensional phase space !

Aperiodic solution: chaotic, but ... >>

An example of an aperiodic solution is shown in the figure below



Now: plot Z_{n+1} as function of Z_n . This is called a “discrete map” →

Discrete map: Lorenz map

The function, $Z_{n+1}=f(Z_n)$ is a **discrete map**.

The example here is called “**Lorenz Map**”.

There is almost no “thickness” to the “curve” represented by the dots.

Height of a peak can be predicted from height previous peak!

Unfortunately, the ***time between successive peaks*** in the graph on the previous slide is very erratic ([exercise 10c](#)) and therefore not predictable !

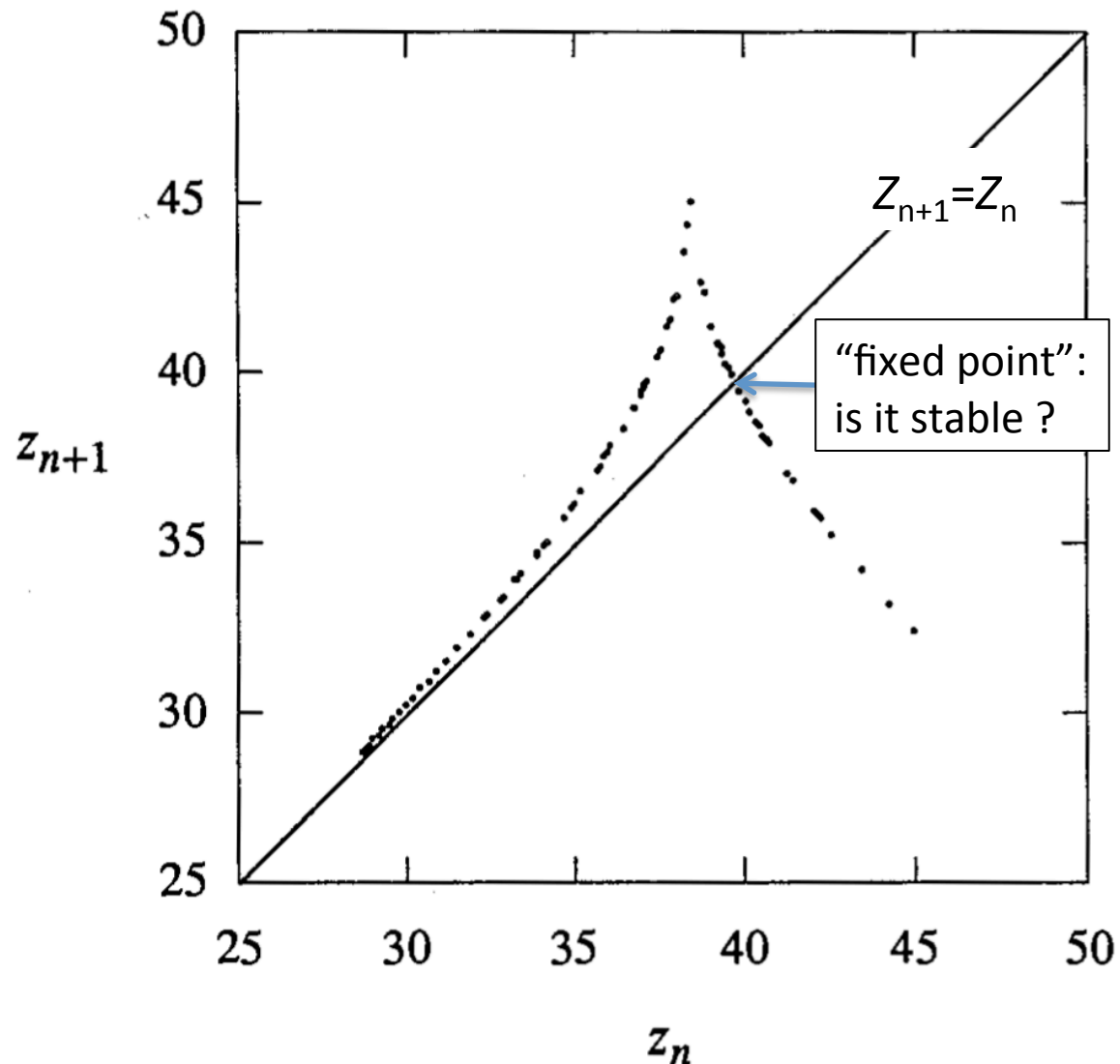


Figure 9.4.3

Is turbulence a manifestation of chaos?

At first, Edward Lorenz gave his 1963-paper the title, “*Deterministic Turbulence*”, but on the editor’s advice Lorenz replaced the term “*Turbulence*” by “*Nonperiodic Flow*”.

Is turbulence in fluids a manifestation of the mathematical chaos, as found in the low-order Lorenz (1963) model and many other such systems?

Does a more realistic model with more Fourier modes have chaotic solutions?

Can we use ideas derived from “chaos-theory” in an approach to the problem of predictability when the the system has **many degrees of freedom**?

Lorenz was interested in this question in relation to the question of predictability of weather and climate

Next week (lecture 9): the problem of scale – selection.

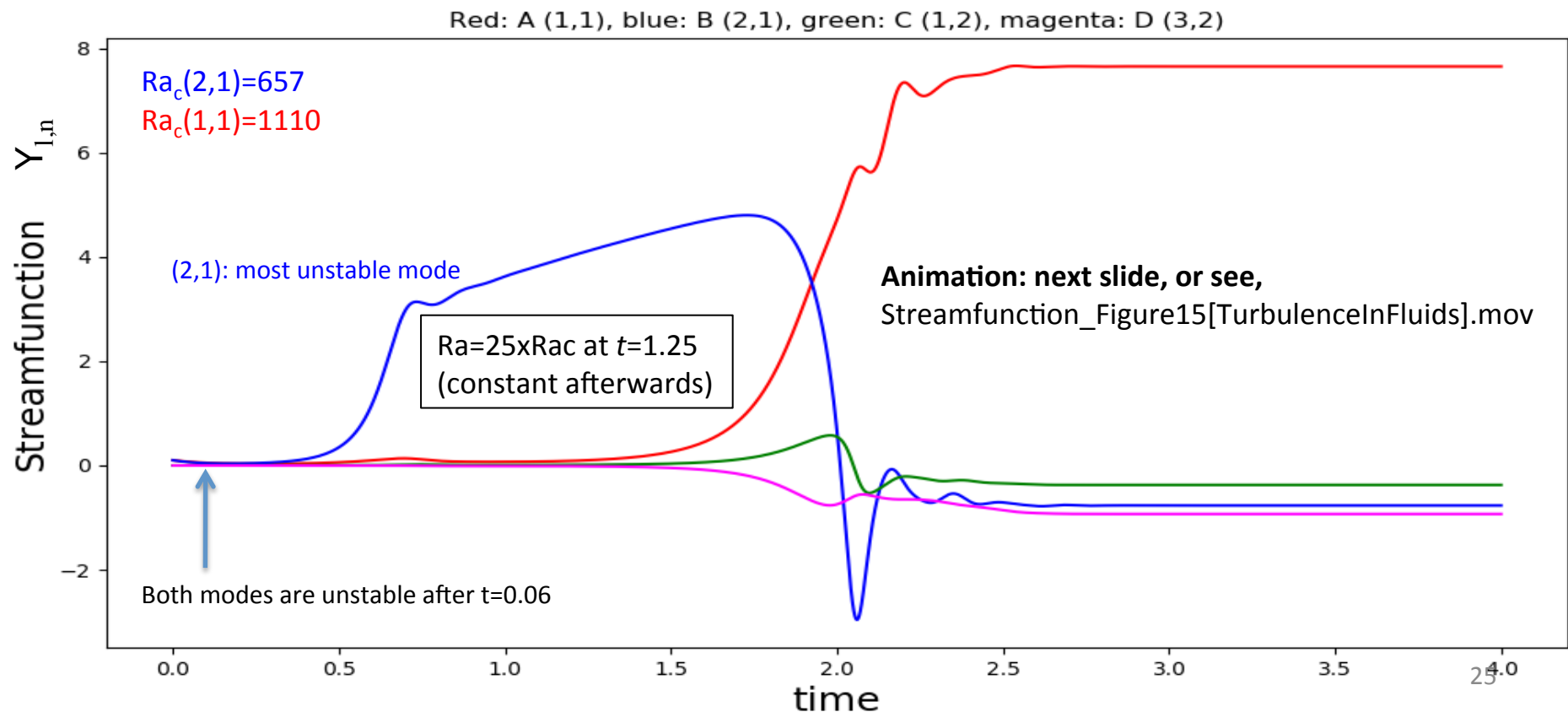
A more realistic model with more Fourier modes

Example

10-component model, including following wave numbers, $(l,n) = (1,1), (2,1), (1,2), (3,2), (0,2), (0,4)$.

Domain with aspect ratio: $4\sqrt{2}=5.66$

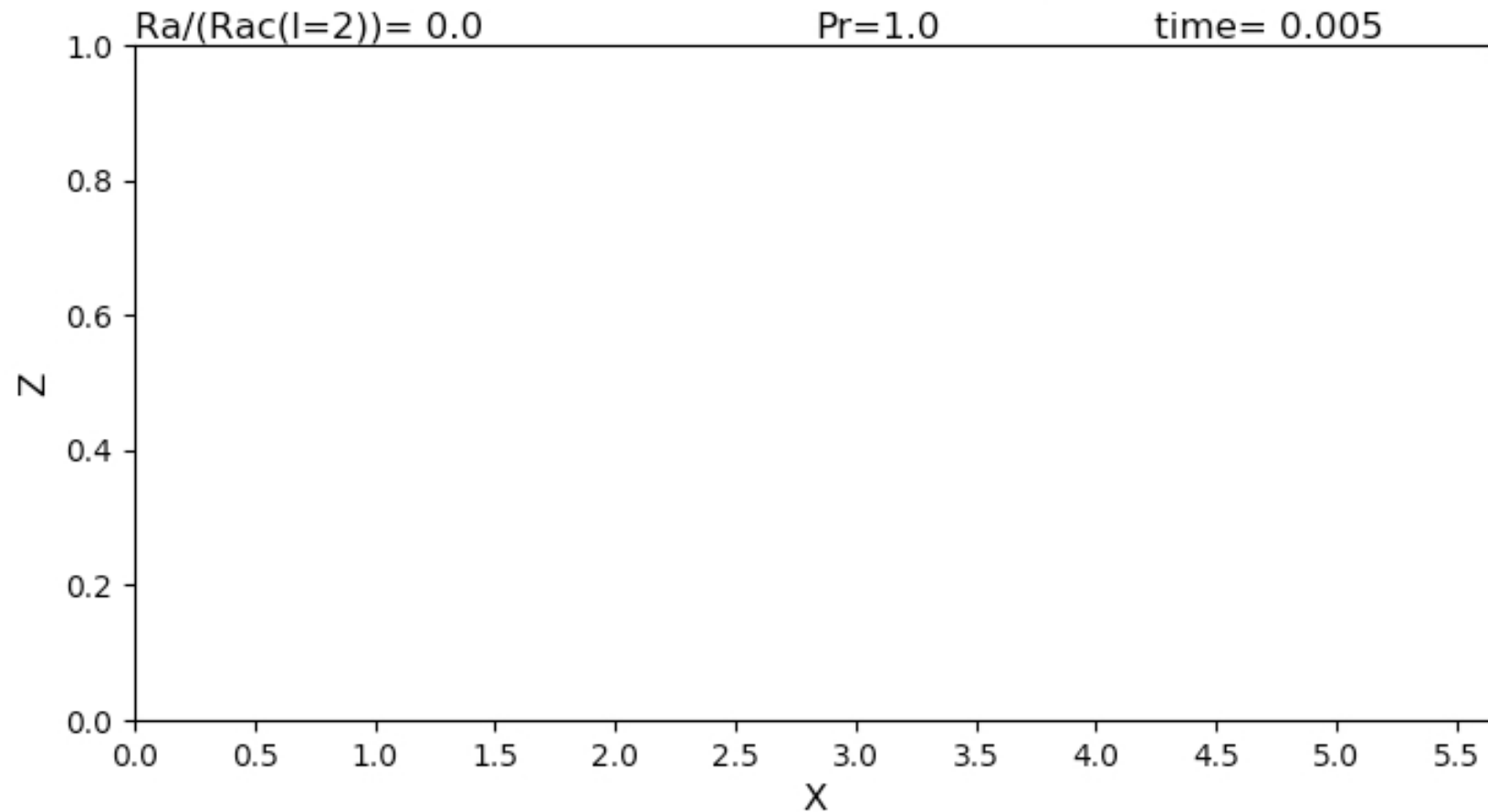
Below: a run in which **Ra is increased linearly** in time from 0 to $25 \times \text{Ra}_c(2,1)$ and is constant afterwards.



Animation of transfer of energy to a larger scale

File: Streamfunction_Figure15[TurbulenceInFluids].mov

10-component thermal convection model: streamfunction (red: clockwise circulation)



Tutorial: exercises 9 and 10

(9)

The total kinetic energy (K) and the total potential energy (P), associated with the perturbations (eq. 6) in two-dimensional convection (in which $v=0$) are, respectively,

$$K = \frac{1}{2} \int (u^2 + w^2) d\sigma \quad (95)$$

The integral is over the full domain

$$P = -\frac{1}{2} \text{Pr} \int \theta^2 d\sigma \quad (96)$$

Express K and P in spectral space and show that $K+P$ is conserved **by the non-linear terms** in the Lorenz model.

Hint: note that $(u^2 + w^2) = (\nabla \psi)^2 = \nabla(\psi \nabla \psi) - \psi \nabla^2 \psi$

(10)

For this exercise you need to know how to program a computer (in Mathematica, Python, in MATLAB, in C ...)

(a) Integrate the Lorenz (1963) equations in time numerically through at least 5 non-dimensional time units. Approximate the time-derivatives with the fourth order (RK-4) Runge-Kutta scheme (https://en.wikipedia.org/wiki/Runge-Kutta_methods). Do this for the exact same values of α_x , Ra , Pr , which Lorenz used to illustrate the chaotic solution of his model, i.e. $\alpha_x=1/(2^{1/2})$, $Ra=28Ra_c$ ($Ra_c=657$), $Pr=10$. The **initial condition** should lie close to steady state convection (insert a very small perturbation to this steady state). (continued on next slide)²⁷

(10) continued

(b) Locate local maxima of $Z(t)$, Z_n and plot Z_{n+1} as a function of Z_n , in other words reproduce the **Lorenz map** (see page 139 of Lorenz (1963), or p. 326-328 of Strogatz (1994)). According to the Lorenz map, the value of the next maximum (next peak in Z) is predictable.

(c) Now, plot the time between local maxima of $Z(t)$ as a function of time (e.g. the time half-way between two maxima). Discuss the plot.

(d) Perform a second integration with a slightly different initial condition, by weakly perturbing only one of the three variables. Plot a measure of the separation in phase space between the two solutions and estimate the associated **Lyapunov-exponent**, λ (see p. 320-323 of Strogatz (1994)). Do this for the variables, X , Y and Z separately.

(e) Compare the “climate” of the two solutions, i.e. the time average of X , Y , and Z . You may have to extend your integration in time by many more than 5 non-dimensional time units to get a “stable” answer. What conclusions can you draw? Now, compare the “climate” of the absolute values of the two solutions, i.e. the time average of $|X|$, $|Y|$, and $|Z|$. What do you conclude from this?

(f) Model climate can also be characterised by the **so-called “probability distribution function”** (“pdf”). A plot of the “pdf” as a function of X and Y gives the probability of finding the Lorenz model state vector in any point in X - Y plane of phase space. Plot this “pdf” and interpret your result in the light of **(e)**.

Lorenz' research on predictability

Looking for analogues:

Atmospheric Predictability as Revealed by Naturally Occurring Analogues

EDWARD N. LORENZ

Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge, Mass.¹

(Manuscript received 2 April 1969)

ABSTRACT

Two states of the atmosphere which are observed to resemble one another are termed *analogues*. Either state of a pair of analogues may be regarded as equal to the other state plus a small superposed "error." From the behavior of the atmosphere following each state, the growth rate of the error may be determined.

Five years of twice-daily height values of the 200-, 500-, and 850-mb surfaces at a grid of 1003 points over the Northern Hemisphere are procured. A weighted root-mean-square height difference is used as a measure of the difference between two states, or the error. For each pair of states occurring within one month of the same time of year, but in different years, the error is computed.

There are numerous mediocre analogues but no truly good ones. The smallest errors have an average doubling time of about 8 days. Larger errors grow less rapidly. Extrapolation with the aid of a quadratic hypothesis indicates that truly small errors would double in about 2.5 days. These rates may be compared with a 5-day doubling time previously deduced from dynamical considerations.

The possibility that the computed growth rate is spurious, and results only from having superposed the smaller errors on those particular states where errors grow most rapidly, is considered and rejected. The likelihood of encountering any truly good analogues by processing all existing upper-level data appears to be small.

J.Atmos.Sci., April 1969

Optimistic message:

Atmospheric states separated by 12 days resemble each other more than randomly selected states

Pessimistic message:

No truly good analogues.

The smallest errors have a doubling time of about 8 days.

Large errors grow less rapidly (why?).

On the Existence of Extended Range Predictability¹

EDWARD N. LORENZ

Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge, Mass. 02139

15 November 1972 and 9 January 1973

ABSTRACT

On the basis of five years of Northern Hemisphere isobaric height data, states of the atmosphere separated by 12 days or less are found on the average to resemble each other more closely than randomly selected states, even after adjustment for seasonal trend has been made. The existence of partial predictability of instantaneous weather patterns at least 12 days in advance is thereby confirmed.

Lorenz' research on predictability

Atmospheric predictability experiments with a large numerical model

By E. N. LORENZ,¹ *European Centre for Medium Range Weather Forecasts, Reading RG2 9AX, England*

ECMWF: founded in 1975

(Manuscript received January 28, 1982)

ABSTRACT

The instability of the atmosphere places an upper bound on the predictability of instantaneous weather patterns. The skill with which current operational forecasting procedures are observed to perform determines a lower bound. Estimates of both bounds are obtained by comparing the ECMWF operational forecast for each day of a 100-day sequence at one range with the operational forecast for the same day at another range, and with the analysis for that day. The estimated bounds are reasonably close together.

Predictions at least ten days ahead as skilful as predictions now made seven days ahead appear to be possible. Additional improvements at extended range may be realized if the one-day forecast is capable of being improved significantly.

Tellus, 34, 505-513 (1982)

Analysis ($j=0$) compared with 1-day forecast for the same day

Analysis of initial state should be improved.

Prediction for the same day, 0 days ahead and 10 days ahead

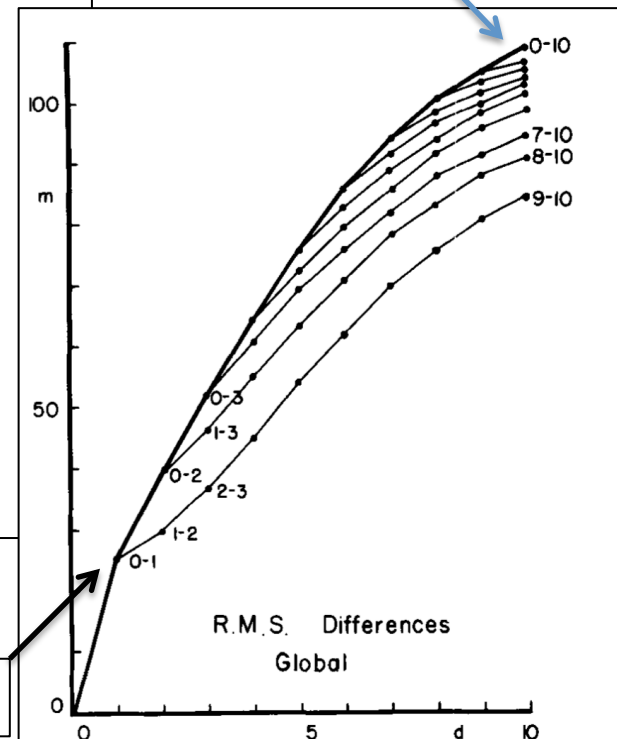


Fig. 1. Global root-mean-square 500-mb height differences E_{jk} in meters, between j -day and k -day forecasts made by the ECMWF operational model for the same day, for $j < k$, plotted against k . Values of (j, k) are shown beside some of the points. Heavy curve connects values of E_{0k} . Thin curves connect values of E_{jk} for constant $k - j$.

Lorenz' research on predictability led to *ensemble weather forecasting*

Lorenz (2007): “It has been a source of pleasure to me that the technique, known as “ensemble forecasting”, has within recent decades become standard practice.”

A typical ensemble of initial states may contain 100 members.

The ensemble of forecasts is commonly produced with the same model, but with reduced horizontal resolution.

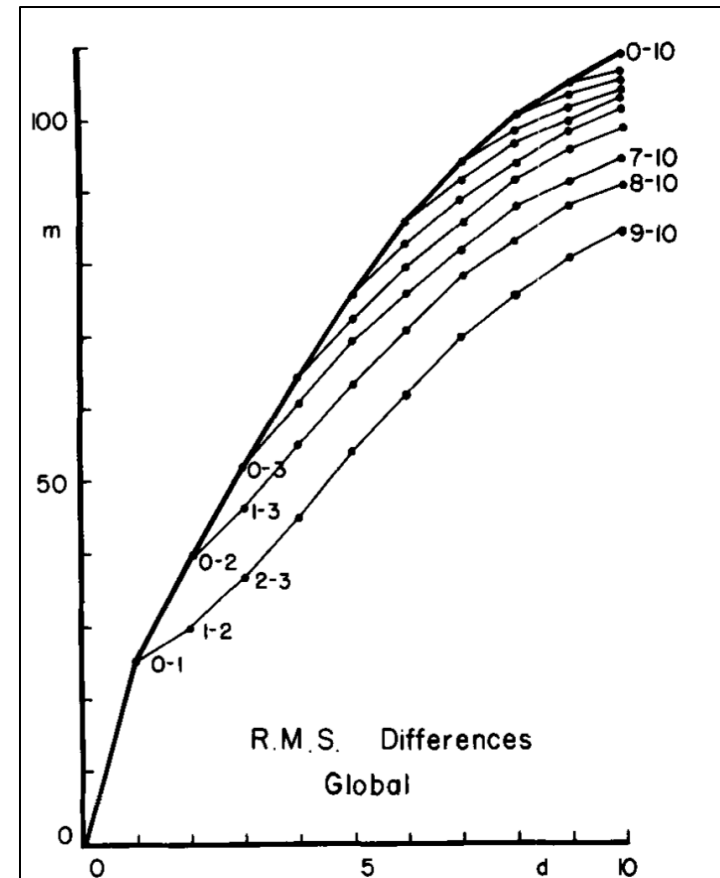
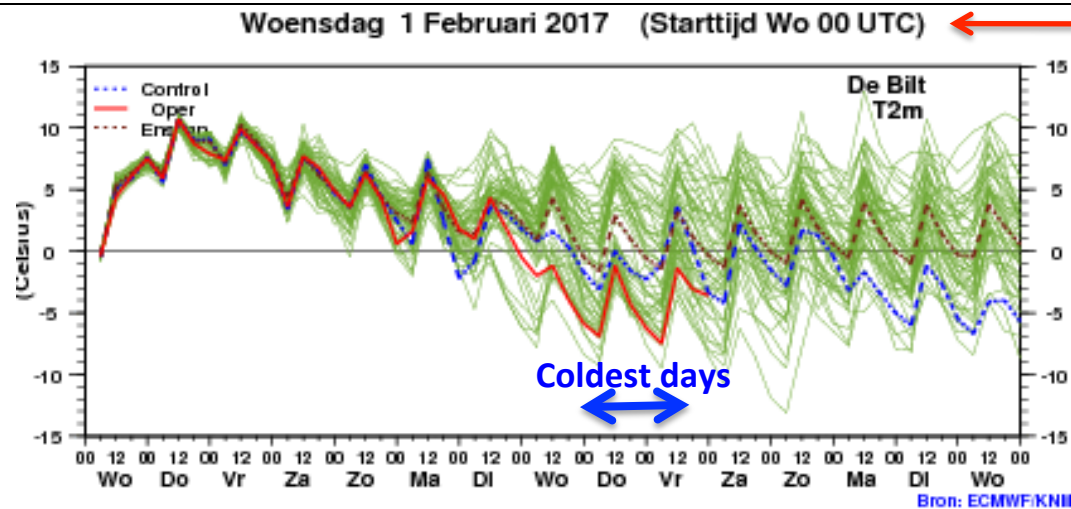


Fig. 1. Global root-mean-square 500-mb height differences E_{jk} in meters, between j -day and k -day forecasts made by the ECMWF operational model for the same day, for $j < k$, plotted against k . Values of (j, k) are shown beside some of the points. Heavy curve connects values of E_{0k} . Thin curves connect values of E_{jk} for constant $k-j$.

Ensemble prediction of temperature

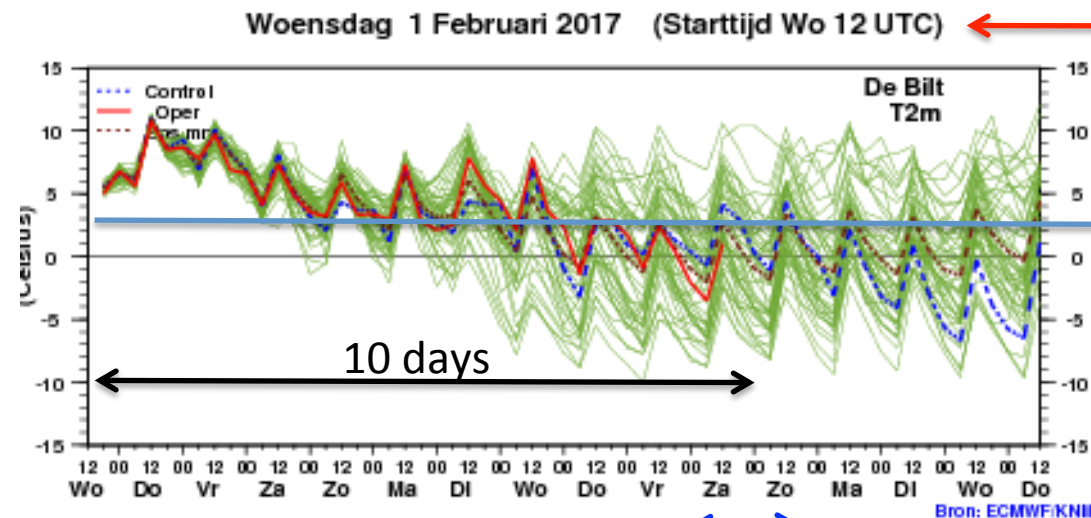
Temperature prediction for De Bilt, made by running a model in the order of 100 times with slightly different initial conditions.



Initialised at 00 UTC, 1 Feb. 2017

Uncertain prediction of a transition to colder weather after 3 Feb.

Did it occur? Yes

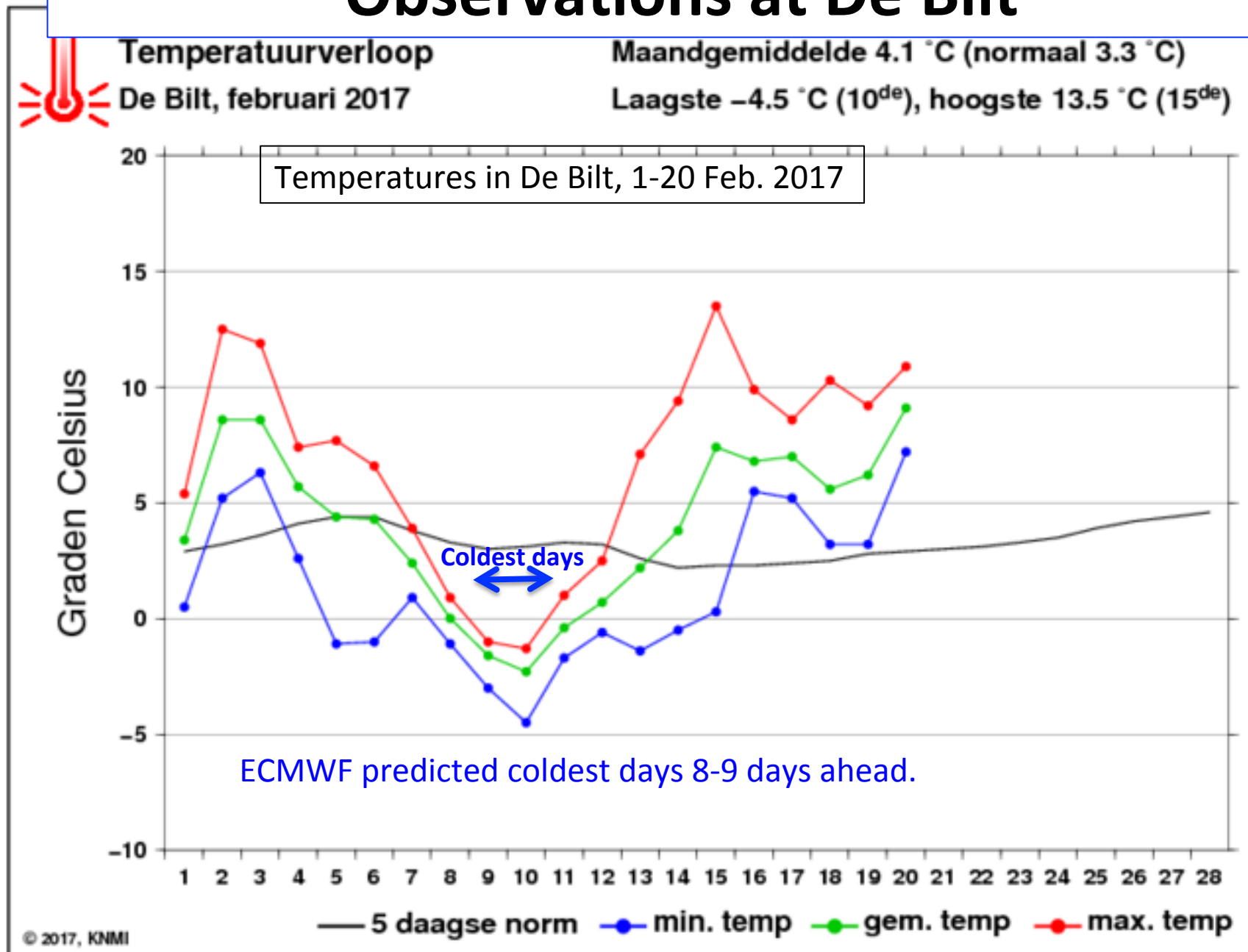


Initialised 12 hours later

Average February temperature (1980-2010)

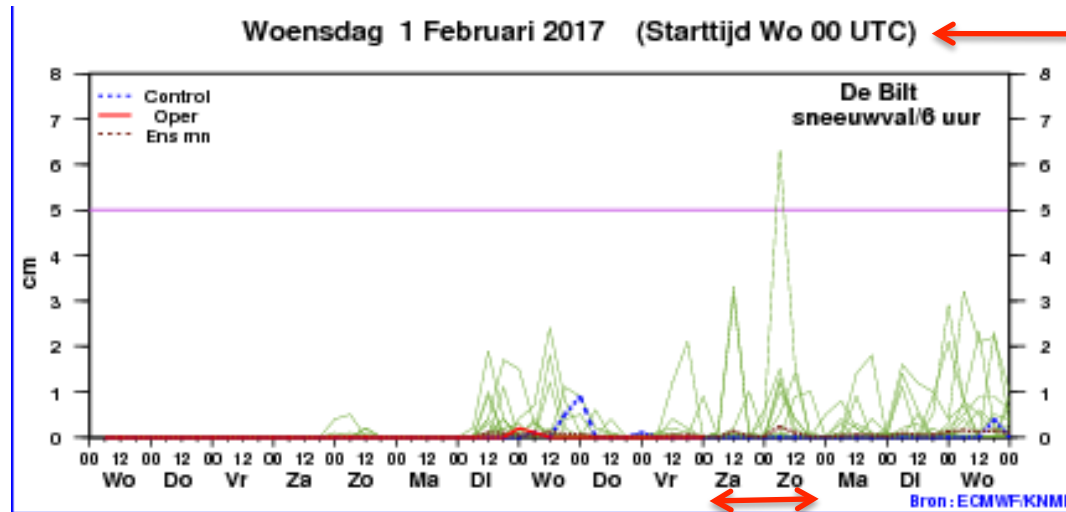
weekend of 11-12 Feb 2017: 10 cm of snow in De Bilt

Observations at De Bilt



Ensemble prediction of snow is more difficult

Snow-fall prediction (De Bilt)



Initialised at 00 UTC, 1 Feb. 2017

A few model runs predicted a little snow in the weekend of 11-12 Feb. 2017.

The view out of my window on Sunday morning

10 cm of snow on 12 Feb. 2017



Lorenz (2007) on the possibility of long range weather prediction

Optimistic message!

The first thing I would say is that current numerical prediction output are much better than I ever thought they would be at this time. I wasn't sure they would ever get as good as they are now—certainly not within my lifetime. So, this makes me think that they can become still better and makes me hopeful that we may actually get good forecasts a couple of weeks ahead some time. I still don't hold much hope for day-to-day forecasting a month ahead. Two weeks ahead doesn't seem unreasonable at all now even though we haven't reached that point. And . . . I gather that a lot of the improvement has been from the improvement in initial conditions,