

09/12/2022 (Friday, 09:00)

Dynamical Meteorology: lecture 8

Sections 1.21 and 1.22

Circulation and vorticity

Large-scale vortices in the atmosphere (polar vortex and tropical cyclone)

Potential vorticity and isentropic density

The tropopause (lapse rate, dynamical tropopause, cold point, isentropic)

The Underworld, the Middleworld and the Overworld

Tutorial after this lecture (8)

Project 3 (hand in today: 9/12/22)

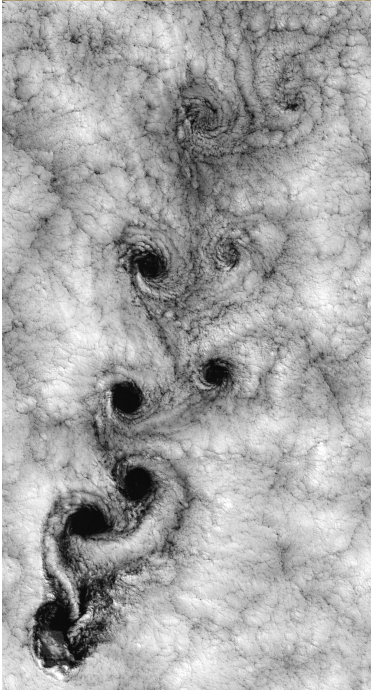
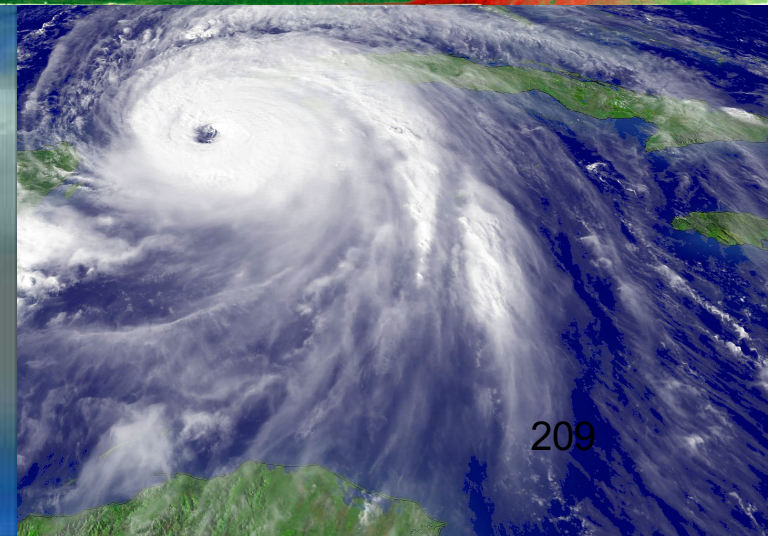
Problem 1.18

(a.j.vandelden@uu.nl)

(<http://www.uu.nl/~nvdelden/dynmeteorology.htm>)

Section 1.22: measures of rotation

Circulation, vorticity and potential vorticity



Circulation

Circulation, Γ , is the line integral around a closed curve C of fluid velocity:

$$\Gamma \equiv \oint_C \vec{v} \cdot d\vec{l}$$

With **Stokes theorem** this becomes an area integral of vorticity:

$$\oint_C \vec{v} \cdot d\vec{l} = \iint_S (\vec{\nabla} \times \vec{v}) \cdot \hat{n} dS = \iint_S \vec{\omega} \cdot \hat{n} dS$$



Area enclosed by curve, C



unit vector normal to surface, S



vorticity vector

vorticity = circulation per unit area

Planetary vorticity

Planetary vorticity is the circulation due to Earth's rotation divided by area spanned by latitude circle.

velocity due to Earth's rotation = $\Omega a \cos \phi$



Radius of the earth

$$\text{circulation} = \Gamma \equiv \oint_C \vec{v} \cdot d\vec{l} = 2\pi\Omega a^2 \cos^2 \phi$$

$$\text{planetary vorticity} = \frac{\Gamma}{\pi a^2 \cos^2 \phi} = 2\Omega$$

The planetary vorticity vector is perpendicular to the plane of rotation!

Local vertical component of planetary vorticity: $2\Omega \sin \phi \equiv f$

Absolute vorticity = planetary vorticity + relative vorticity

Forced and free vortex

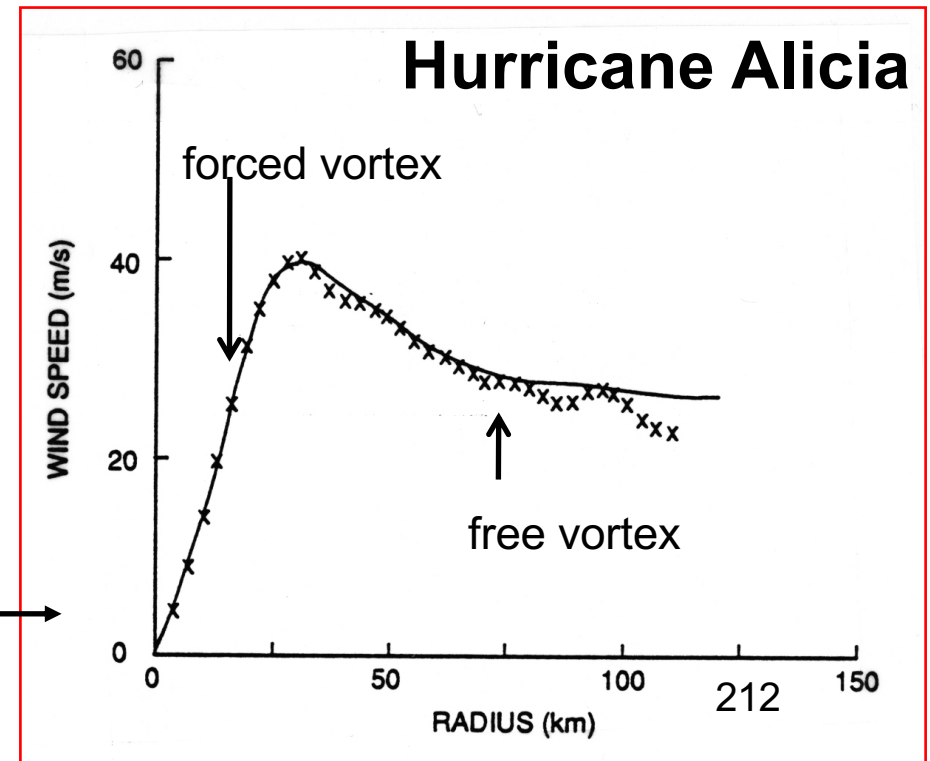
Swirling flow (tangential velocity, v_θ) in the centre of an axisymmetric vortex can be approximated as a function of r (radius) as

$$v_\theta \equiv \omega r \longrightarrow \text{solid rotation if } \omega = \text{constant}$$

For larger r the swirling frequently flow obeys something like

$$v_\theta \equiv \frac{\Gamma}{2\pi r} \longrightarrow \text{"free vortex"}$$

Figure 1.49. Time mean axisymmetric swirling (tangential) wind velocity in **hurricane Alicia** (solid line) and the "gradient wind" (crosses) in the time interval 12-18 UTC, 17 August 1983 (H.E. Willoughby, **J.Atmos.Sci.**, 47, 265-247)



Circulation in a hurricane

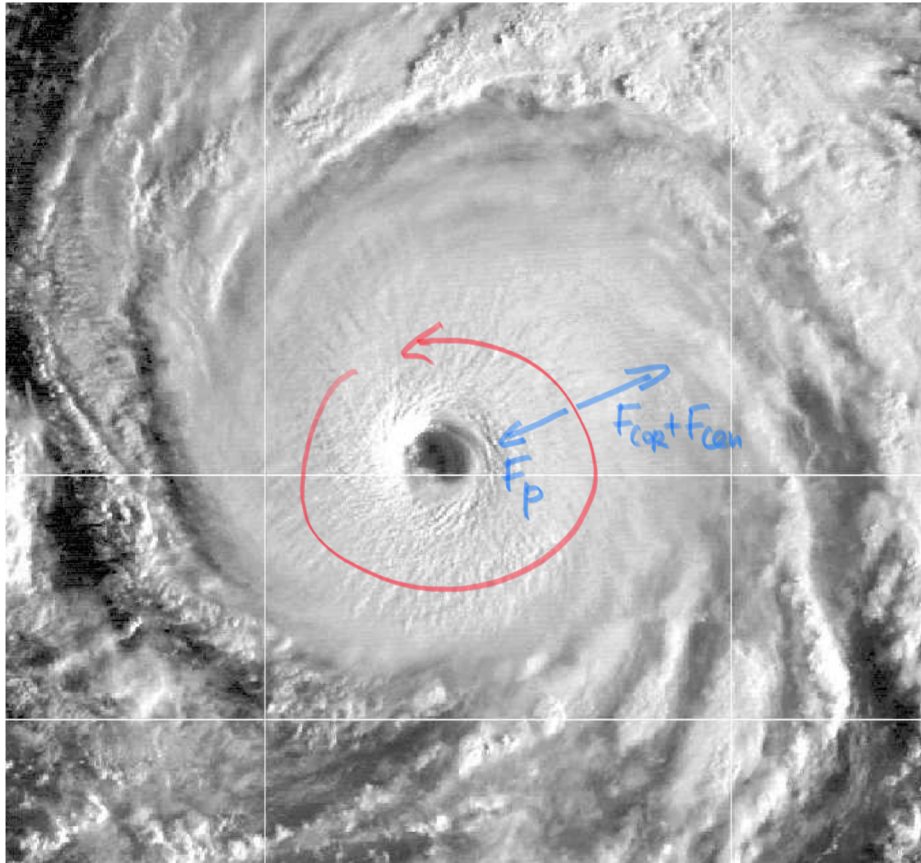
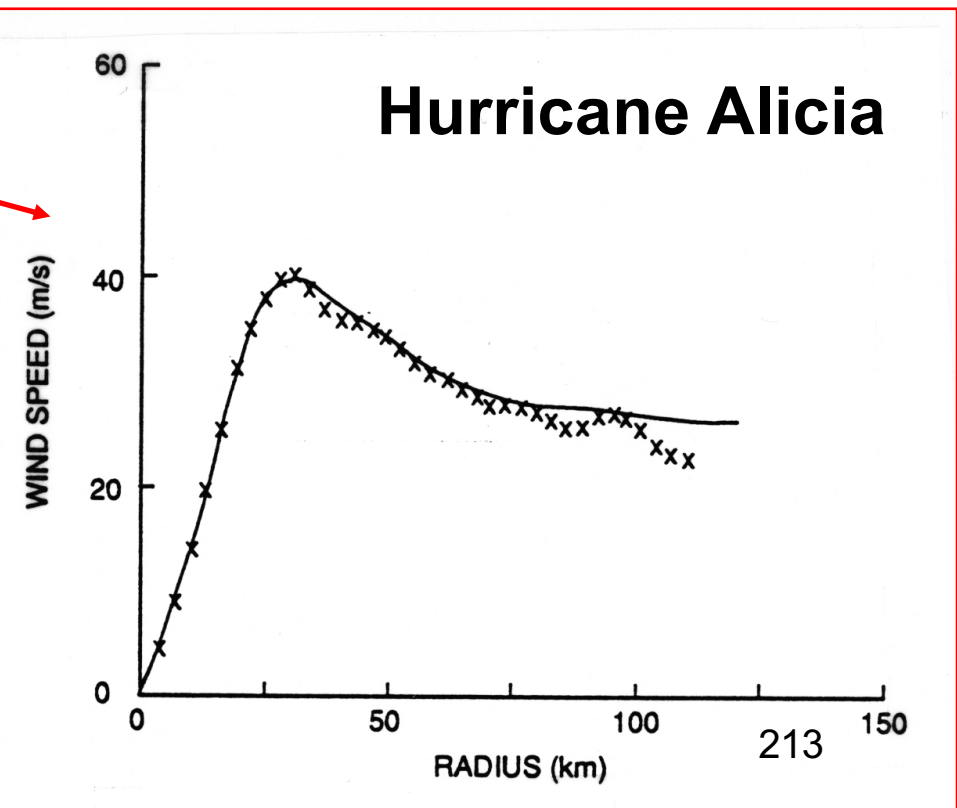


Figure 1.49. Time mean axisymmetric swirling (tangential) wind velocity in hurricane Alicia (solid line) and the “gradient wind” (crosses) in the time interval 12-18 UTC, 17 August 1983 (H.E. Willoughby, *J.Atmos.Sci.*, 47, 265-247)

Estimate the average relative vorticity inside the *radius of maximum wind* in hurricane Alicia (problem 1.18).

Gradient wind balance: $F_p = F_{cor} + F_{cen}$



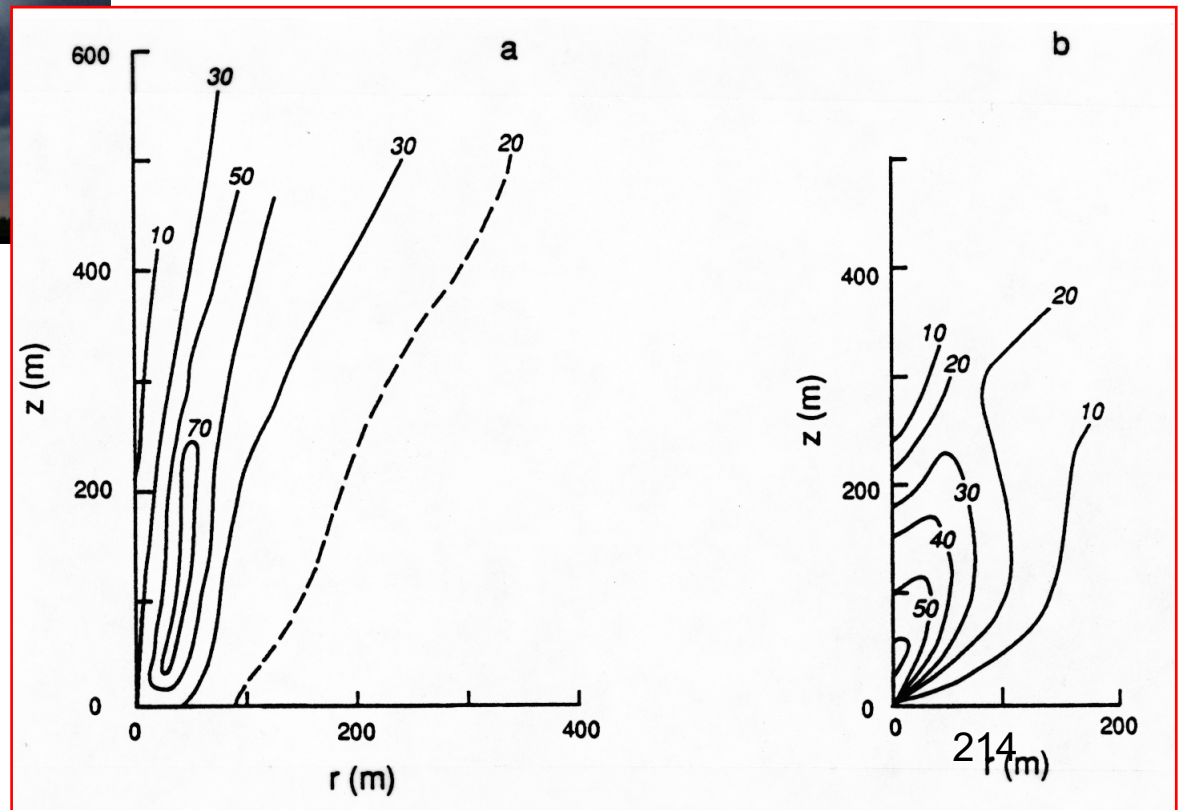
Vorticity in a tornado



Estimate the average relative vorticity inside the *radius of maximum wind* in “Hoeker’s tornado” (see below)

Hoeker's photogrammetric analysis of the 1957 Dallas tornado showing (a) tangential and (b) vertical velocities in m s^{-1}

(from Kessler, E., ed. 1985: **Thunderstorm Morphology and Dynamics**. Second edition. University of Oklahoma Press, Norman, 411 pp.)



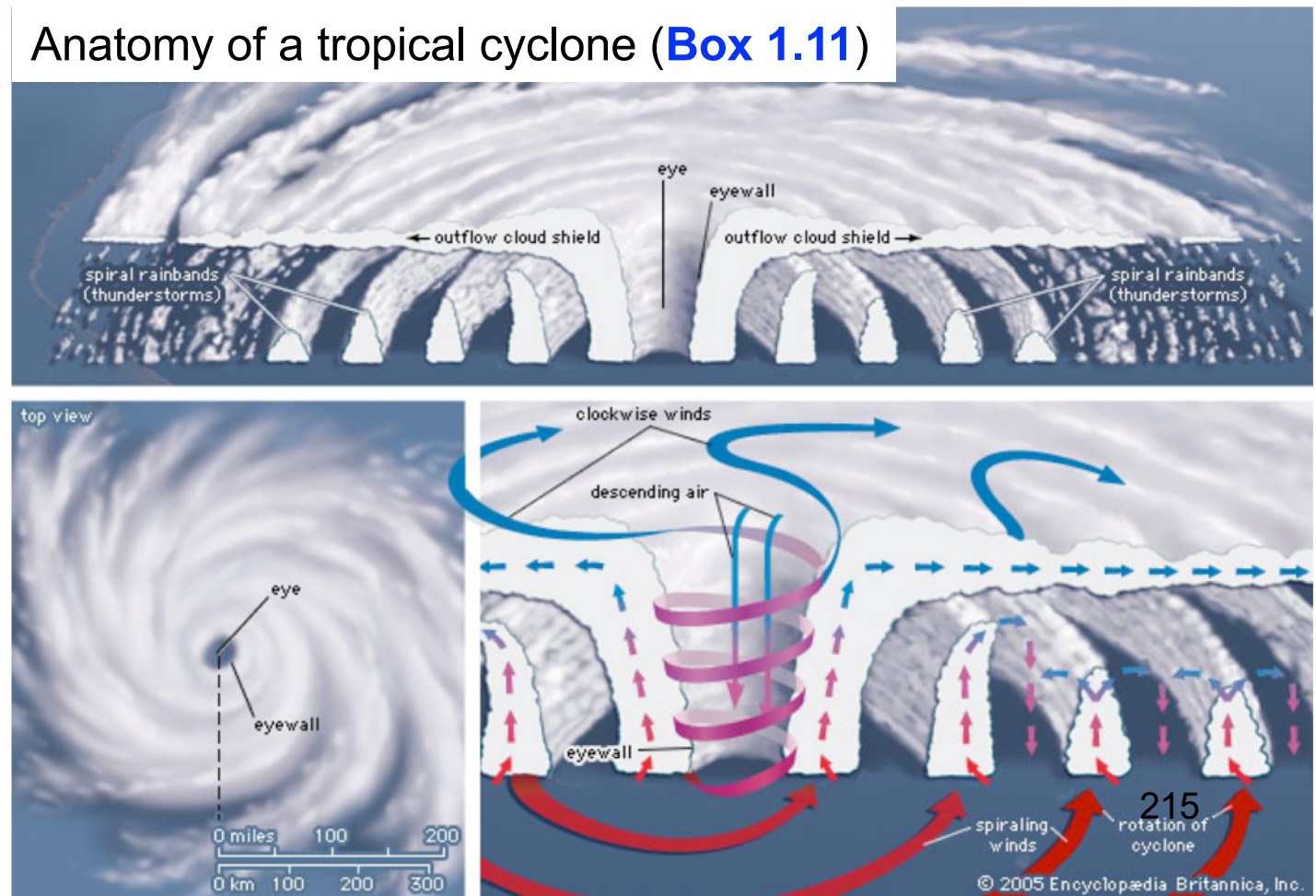
Absolute circulation equation

Absolute circulation: $\Gamma_a \equiv \oint_C \vec{v}_a \cdot d\vec{l}$ ↖ integral around a closed loop

Problem 1.18:

Use circulation to analyse rotation of the flow in a tropical cyclone

Anatomy of a tropical cyclone (Box 1.11)



Absolute circulation equation

Absolute circulation: $\Gamma_a \equiv \oint_C \vec{v}_a \cdot d\vec{l}$

$\uparrow$
closed curve line integral

Evolution of absolute circulation

Absolute circulation: $\Gamma_a \equiv \oint_C \vec{v}_a \cdot d\vec{l}$

$\uparrow$
 closed curve line integral

Time derivative: $\frac{d_a \Gamma_a}{dt} = \frac{d_a}{dt} \oint \vec{v}_a \cdot d\vec{l} = \oint \frac{d_a}{dt} (\vec{v}_a \cdot d\vec{l})$

Evolution of absolute circulation

Absolute circulation: $\Gamma_a \equiv \oint_C \vec{v}_a \cdot d\vec{l}$

$\uparrow$
 closed curve line integral

Time derivative: $\frac{d_a \Gamma_a}{dt} = \frac{d_a}{dt} \oint \vec{v}_a \cdot d\vec{l} = \oint \frac{d_a}{dt} (\vec{v}_a \cdot d\vec{l}) \rightarrow$

Chain rule: $\frac{d_a}{dt} (\vec{v}_a \cdot d\vec{l}) = \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l} + \vec{v}_a \cdot \frac{d_a (d\vec{l})}{dt} = \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l} + \vec{v}_a \cdot d\vec{v}_a = \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l} + \frac{1}{2} d(\vec{v}_a \cdot \vec{v}_a)$

$$\rightarrow \frac{d_a \Gamma_a}{dt} = \oint \frac{d_a}{dt} (\vec{v}_a \cdot d\vec{l}) = \oint \left\{ \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l} + \frac{1}{2} d(\vec{v}_a \cdot \vec{v}_a) \right\} = \oint \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l}$$

Circulation equation on an isentropic surface

Previous slide:

$$\frac{d_a \Gamma_a}{dt} = \oint \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l}$$

Equation of motion in absolute reference frame:

$$\frac{d_a \vec{v}_a}{dt} = -\theta \vec{\nabla} \Pi - g \hat{k}$$

$$\frac{d_a \Gamma_a}{dt} = -\oint (\theta \vec{\nabla} \Pi + g \hat{k}) \cdot d\vec{l} = -\oint \theta \vec{\nabla} \Pi \cdot d\vec{l}$$

Circulation equation on an isentropic surface

Previous slide:

$$\frac{d_a \Gamma_a}{dt} = \oint \frac{d_a \vec{v}_a}{dt} \cdot d\vec{l}$$

Equation of motion in absolute reference frame:

$$\frac{d_a \vec{v}_a}{dt} = -\theta \vec{\nabla} \Pi - g \hat{k}$$

$$\frac{d_a \Gamma_a}{dt} = -\oint (\theta \vec{\nabla} \Pi + g \hat{k}) \cdot d\vec{l} = -\oint \theta \vec{\nabla} \Pi \cdot d\vec{l}$$

Integrate on an **isentropic surface**:

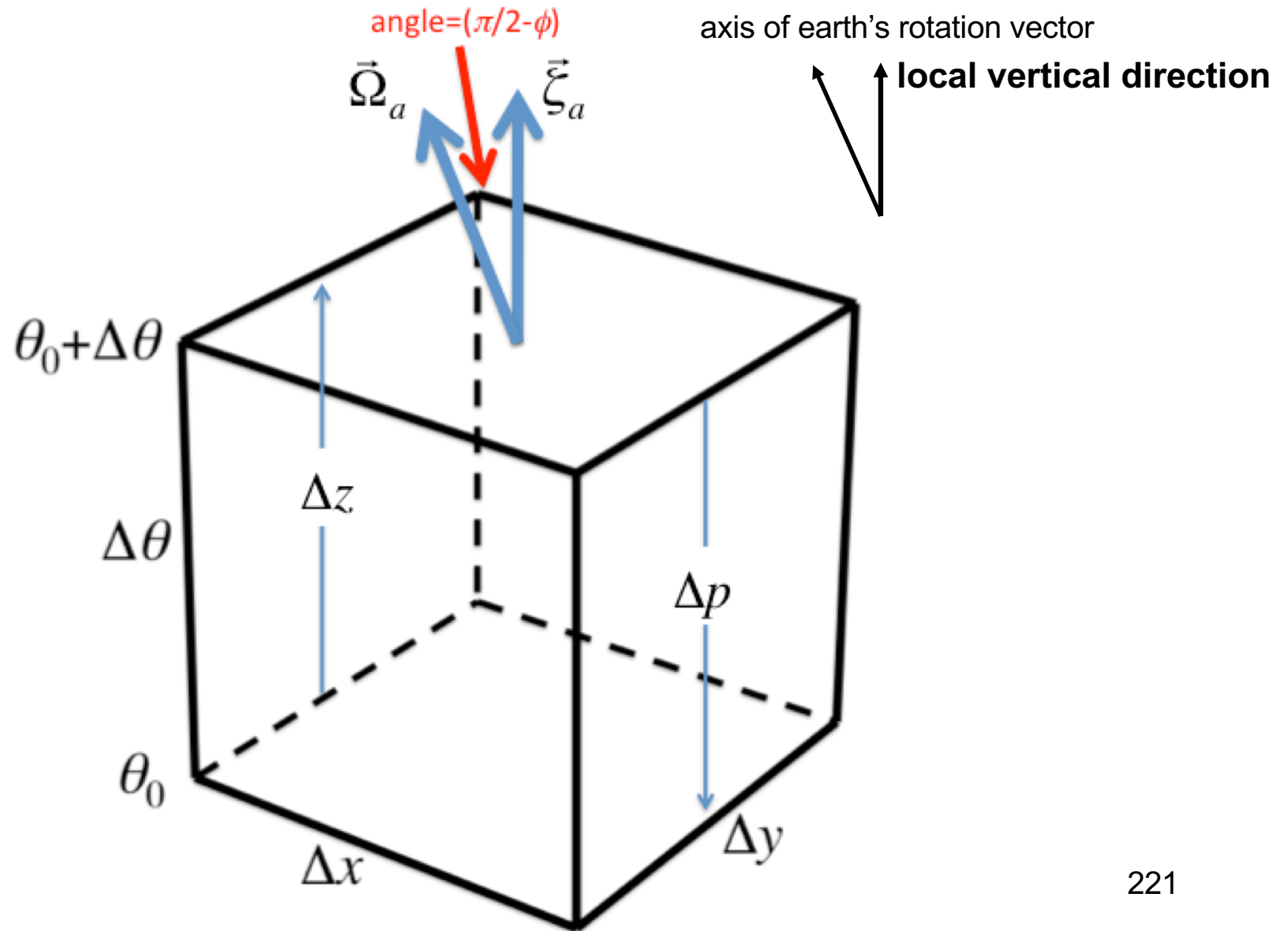
$$\frac{d_a \Gamma_a}{dt} = -\theta \oint \vec{\nabla} \Pi \cdot d\vec{l} = 0 \rightarrow$$

→

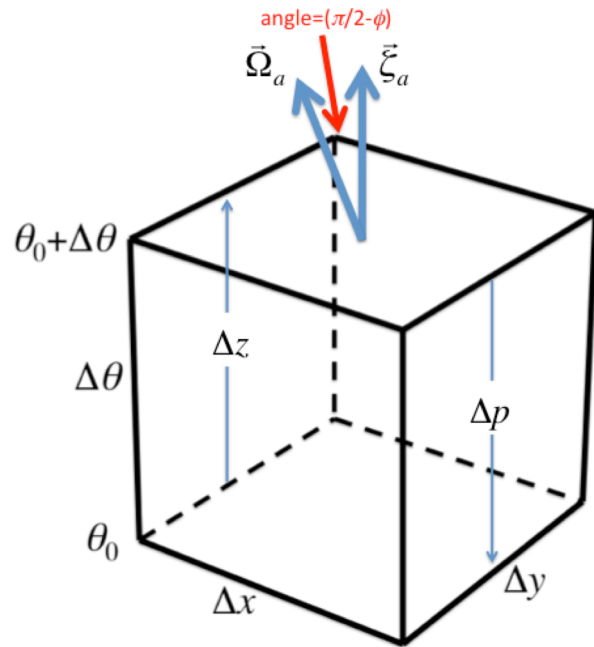
Absolute circulation on an isentropic surface is conserved

Apply these ideas to

a cubic fluid (air) parcel between isentropes



isentropic density



Mass: $m = \rho \Delta x \Delta y \Delta z = -\frac{\Delta p}{g} \Delta x \Delta y$

$$= -\frac{\Delta p}{g \Delta \theta} \Delta \theta \Delta x \Delta y \equiv \sigma \Delta \theta \Delta x \Delta y$$

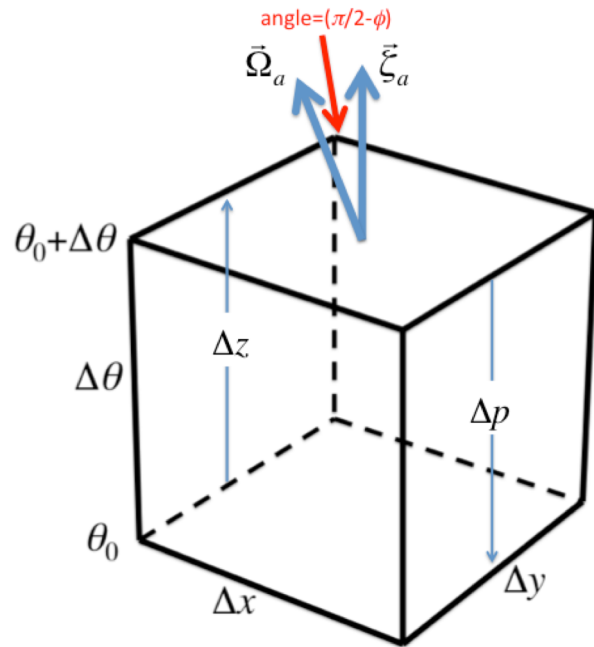
“volume”

Isentropic density:

$$\sigma \equiv -\frac{1}{g} \frac{\Delta p}{\Delta \theta}$$

(density in isentropic coordinates)

conservation of circulation/mass



Mass: $m = \rho \Delta x \Delta y \Delta z = -\frac{\Delta p}{g} \Delta x \Delta y$

$$= -\frac{\Delta p}{g \Delta \theta} \Delta \theta \Delta x \Delta y \equiv \sigma \Delta \theta \Delta x \Delta y$$

“volume”

Isentropic density:

$$\sigma \equiv -\frac{1}{g} \frac{\Delta p}{\Delta \theta}$$

(density in isentropic coordinates)

Conservation of mass: $\frac{d_a m}{dt} = 0$

Conservation of circulation: $\frac{d_a \bar{\Gamma}_a}{dt} = 0$

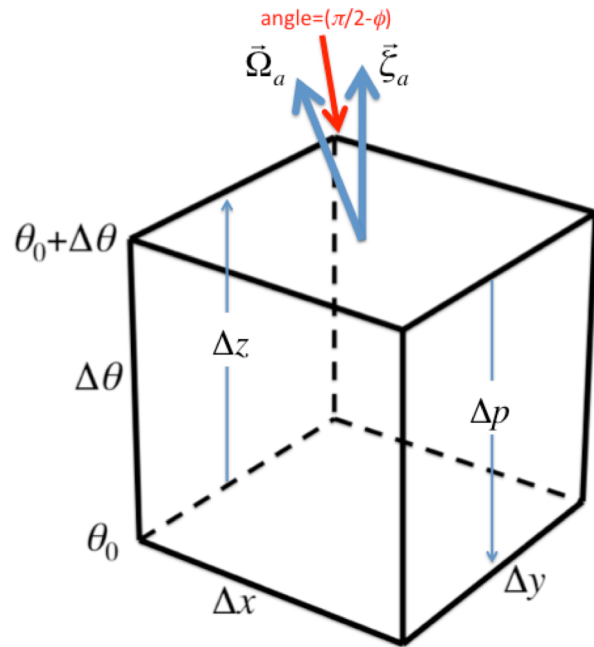
$$\left. \begin{array}{l} \frac{d_a m}{dt} = 0 \\ \frac{d_a \bar{\Gamma}_a}{dt} = 0 \end{array} \right\} \frac{d_a \left(\frac{\bar{\Gamma}_a}{m} \right)}{dt} = 0 \quad \#$$

$\bar{\Gamma}_a$: vertical average of isentropic circulation

However, note that also:

$$\frac{d_a}{dt} (m \bar{\Gamma}_a) = 0$$

Conservation of potential vorticity



Previous slide:

$$\frac{d_a}{dt} \left(\frac{\bar{\Gamma}_a}{m} \right) = \frac{d_a}{dt} \left(\frac{\bar{\zeta}_a \Delta x \Delta y}{\sigma \Delta x \Delta y \Delta \theta} \right) = \frac{d_a}{dt} \left(\frac{\bar{\zeta}_a}{\sigma \Delta \theta} \right) = 0$$

Previous slide:

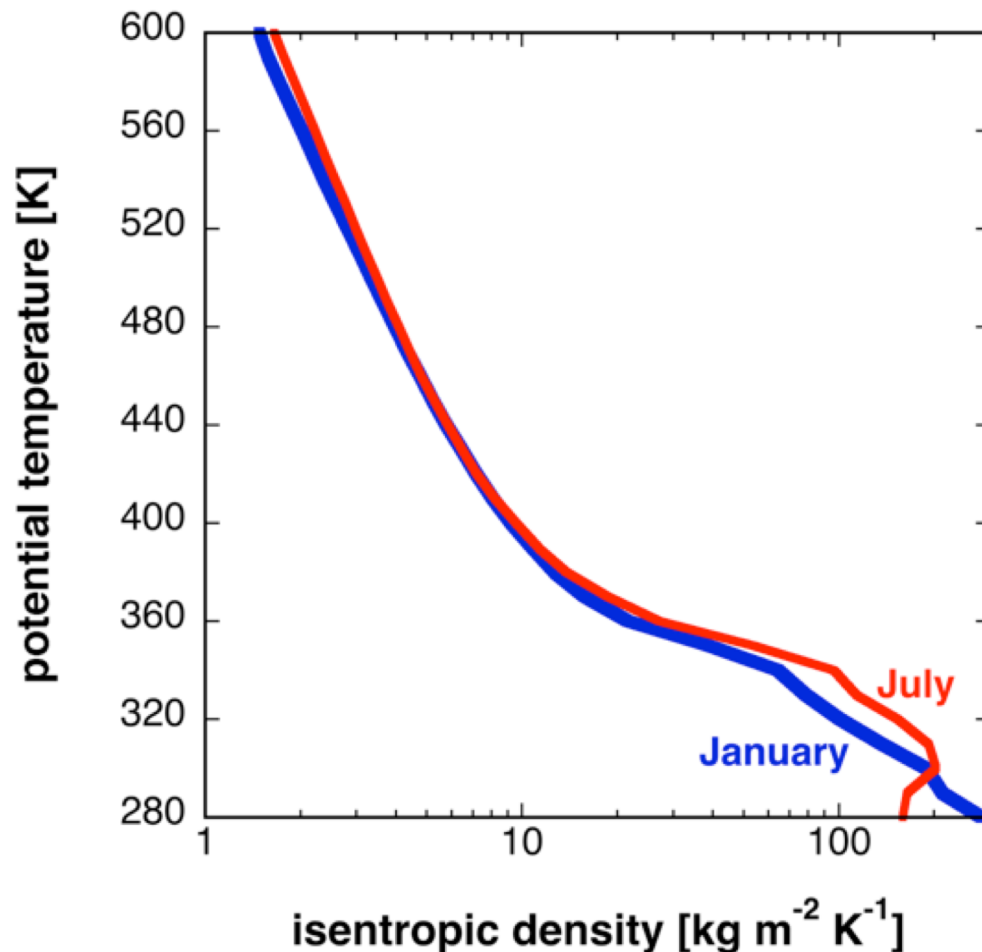
$$\sigma \equiv -\frac{1}{g} \frac{\Delta p}{\Delta \theta} = -\frac{1}{g} \frac{\partial p}{\partial \theta}$$

$\Delta\theta$ is constant
(adiabatic)

$$\frac{d_a}{dt} \left(\frac{\bar{\zeta}_a}{\sigma} \right) = 0$$

“potential vorticity”, $Z \equiv \frac{\bar{\zeta}_a}{\sigma}$ is conserved

Distribution of isentropic density



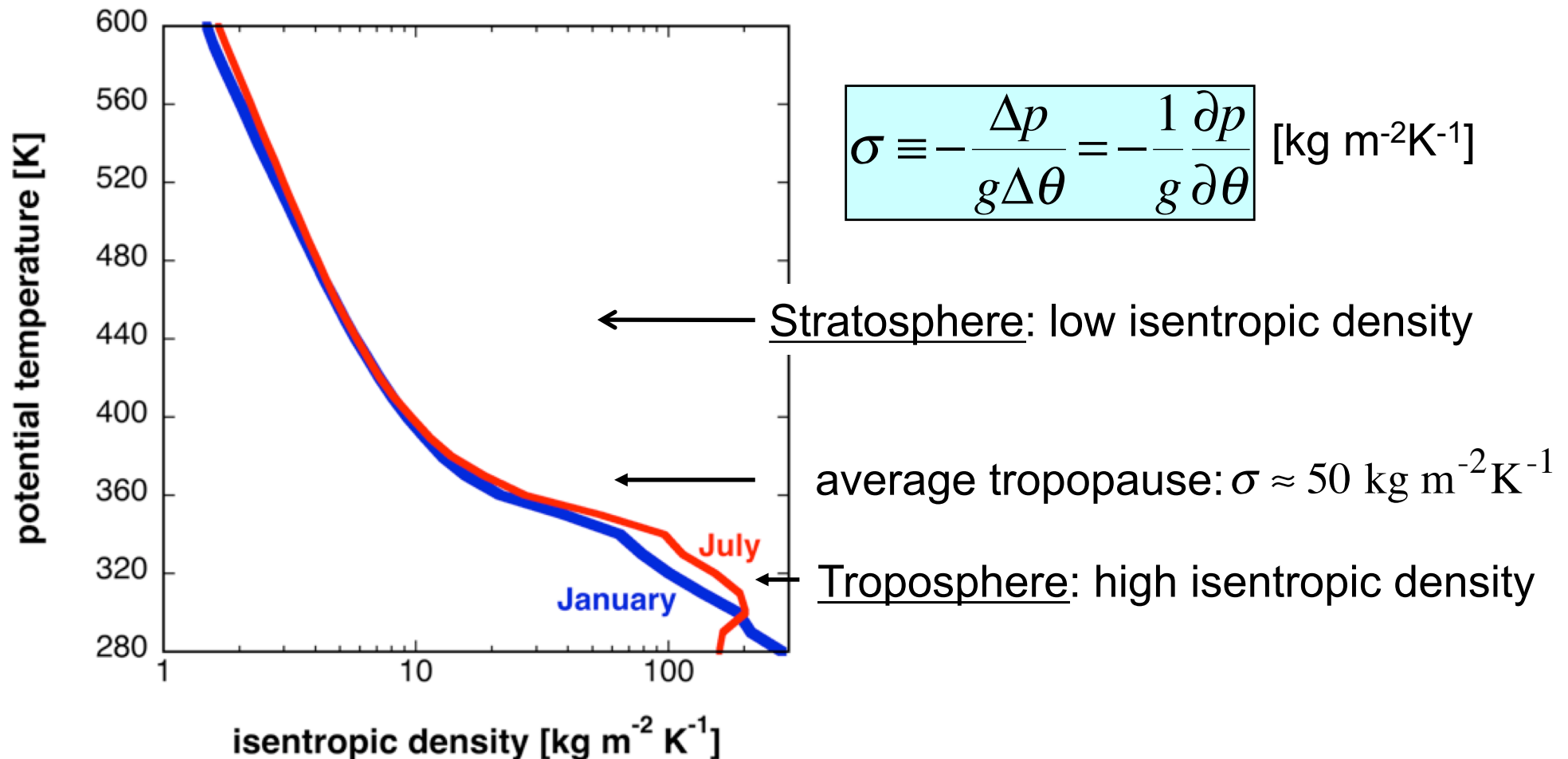
$$\sigma \equiv -\frac{\Delta p}{g\Delta\theta} = -\frac{1}{g}\frac{\partial p}{\partial\theta} \quad [\text{kg m}^{-2}\text{K}^{-1}]$$

Boxes 1.1-1.4:

Isentropic density is principally determined by radiative flux divergence (see [figure 1.46](#), page 118)

Monthly average isentropic density in the northern hemisphere (area weighted average between 10° N and the North Pole) in **July** (thin red curve) and in **January** (thicker blue curve). ($\theta=350$ K and $\theta=600$ K correspond, respectively, to $p=200$ hPa and $p=20$ hPa)

Distribution of isentropic density



$$\sigma \equiv -\frac{\Delta p}{g \Delta \theta} = -\frac{1}{g} \frac{\partial p}{\partial \theta} \quad [\text{kg m}^{-2} \text{K}^{-1}]$$

Monthly average isentropic density in the northern hemisphere (area weighted average between 10° N and the North Pole) in **July** (thin red curve) and in **January** (thicker blue curve).
 ($\theta=350 \text{ K}$ and $\theta=600 \text{ K}$ correspond, respectively, to $p=200 \text{ hPa}$ and $p=20 \text{ hPa}$)

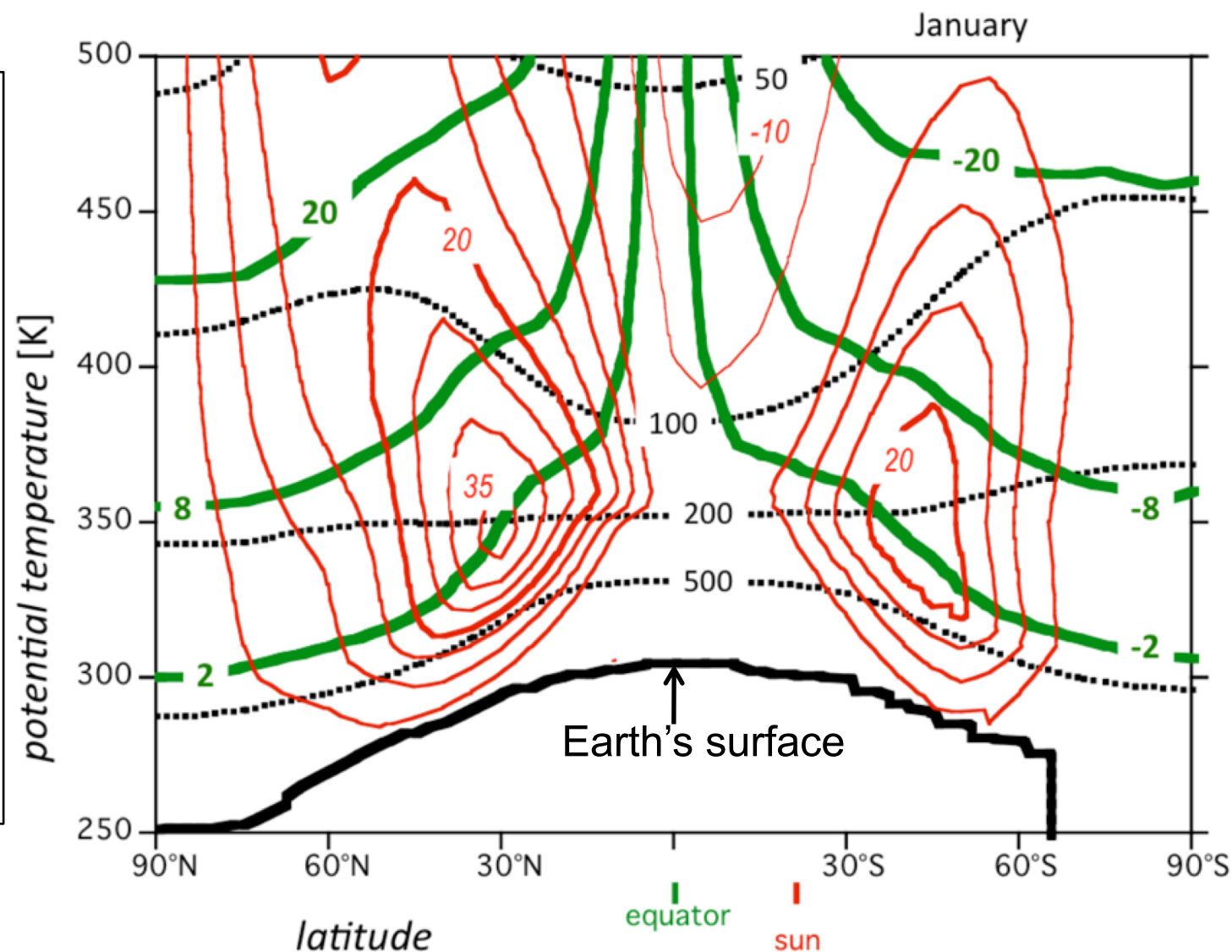
Zonal mean potential vorticity

Unit of potential vorticity (PVU): $1 \text{ PVU} = 10^{-6} \text{ m}^2 \text{K}^1 \text{kg}^{-1} \text{s}^{-1}$

Figure 1.47

January-mean zonal-mean **isentropic potential vorticity** (equation below) (**green** lines, labeled in PVU, where $1 \text{ PVU} = 10^{-6} \text{ K m}^2 \text{kg}^{-1} \text{s}^{-1}$), of **pressure** (dashed lines, labeled in hPa) and of **zonal wind** (**red** lines, labeled in m/s, interval: 5 m/s) as a function of potential temperature and latitude

$$Z \equiv \frac{\bar{\zeta}_a}{\sigma}$$



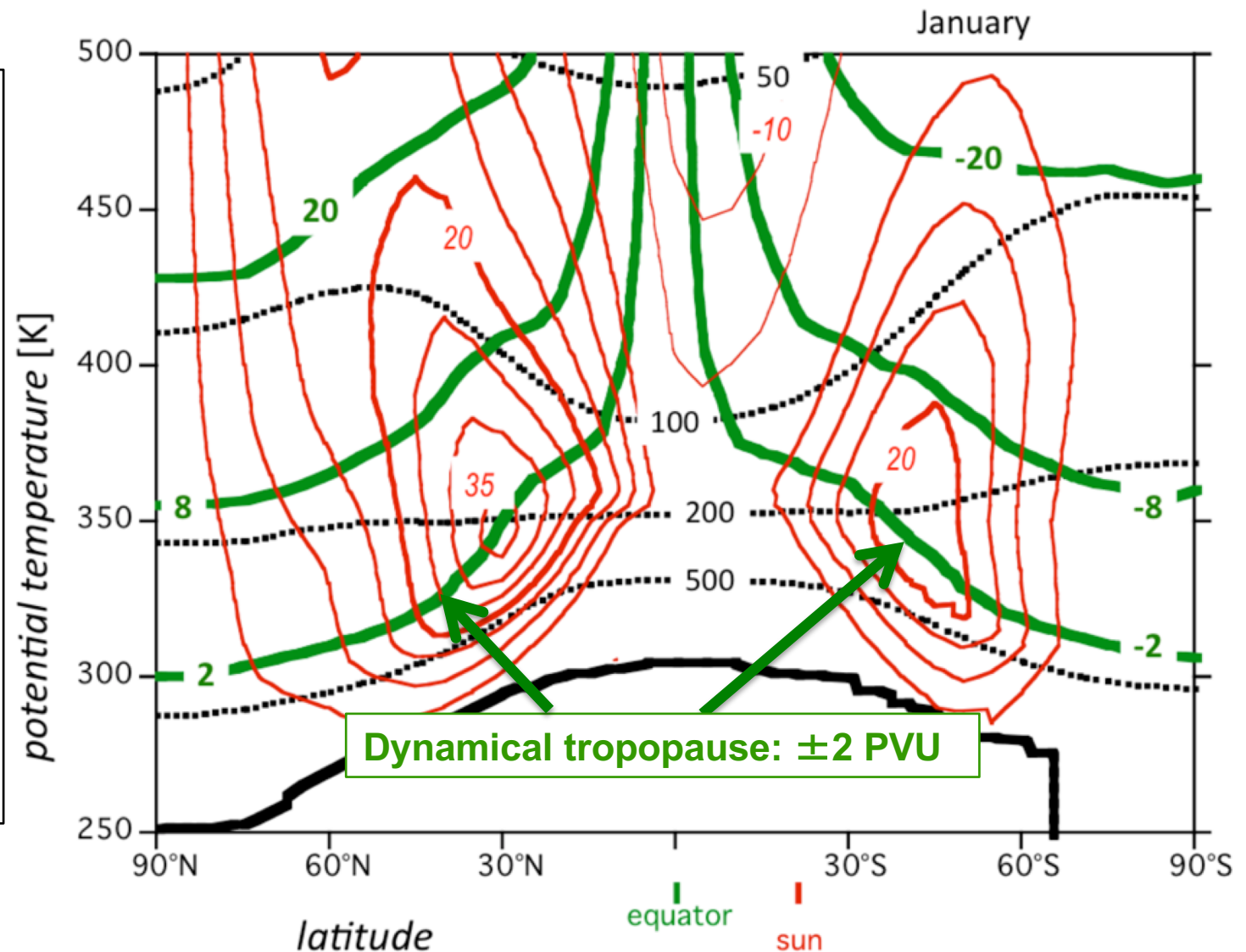
Zonal mean potential vorticity

Unit of potential vorticity (PVU): 1 **PVU** = $10^{-6} \text{ m}^2 \text{ K}^1 \text{ kg}^{-1} \text{ s}^{-1}$]

Figure 1.47

January-mean zonal-mean **isentropic potential vorticity** (equation below) (**green** lines, labeled in PVU, where 1 PVU = $10^{-6} \text{ K m}^2 \text{ kg}^{-1} \text{ s}^{-1}$), of **pressure** (dashed lines, labeled in hPa) and of **zonal wind** (**red** lines, labeled in m/s, interval: 5 m/s) as a function of potential temperature and latitude

$$Z \equiv \frac{\bar{\zeta}_a}{\sigma}$$



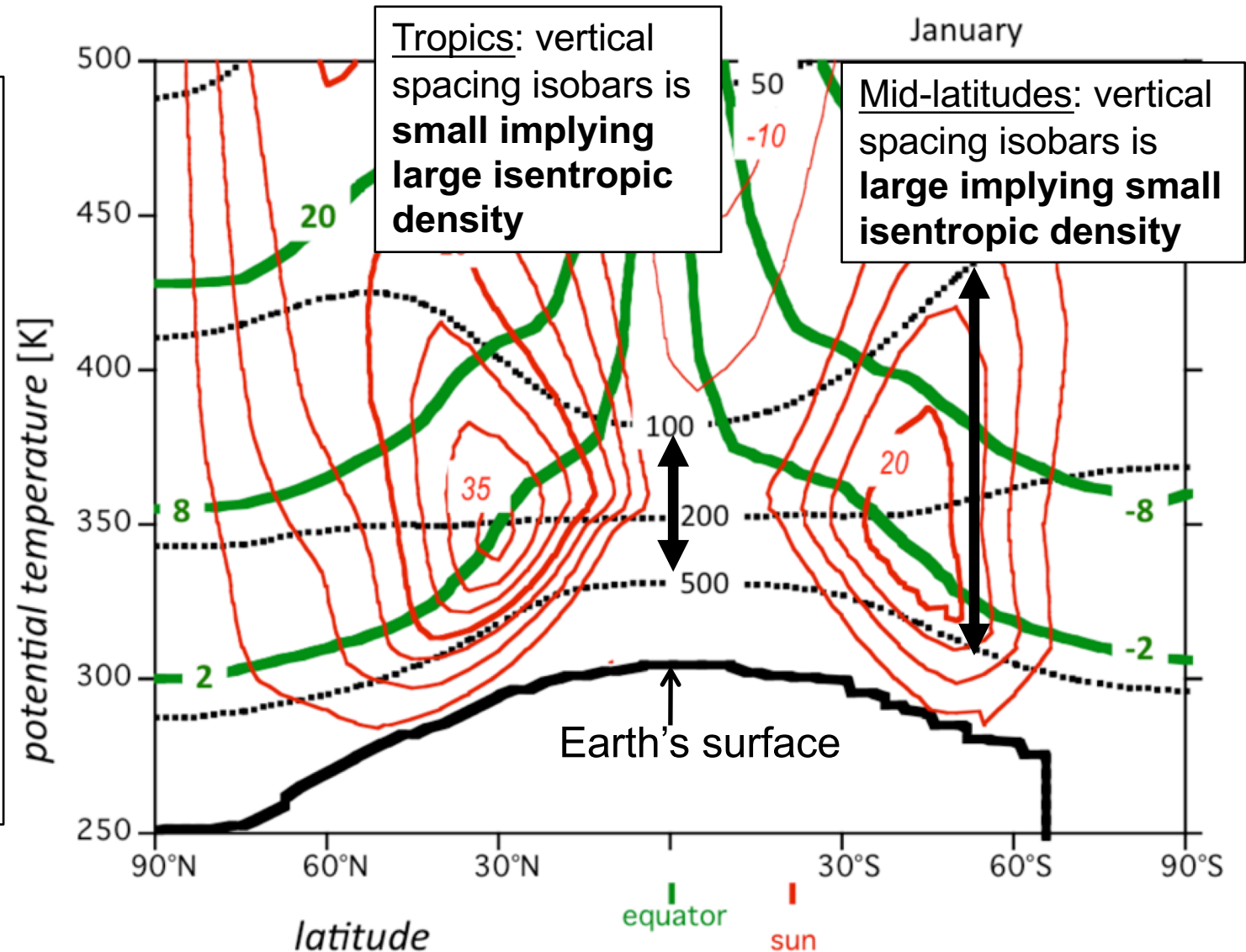
Zonal mean potential vorticity

Unit of potential vorticity (PVU): 1 **PVU** = $10^{-6} \text{ m}^2 \text{ K}^1 \text{ kg}^{-1} \text{ s}^{-1}$]

Figure 1.47

January-mean zonal-mean **isentropic potential vorticity** (equation below) (**green** lines, labeled in PVU, where 1 PVU = $10^{-6} \text{ K m}^2 \text{ kg}^{-1} \text{ s}^{-1}$), of **pressure** (dashed lines, labeled in hPa) and of **zonal wind** (**red** lines, labeled in m/s, interval: 5 m/s) as a function of potential temperature and latitude

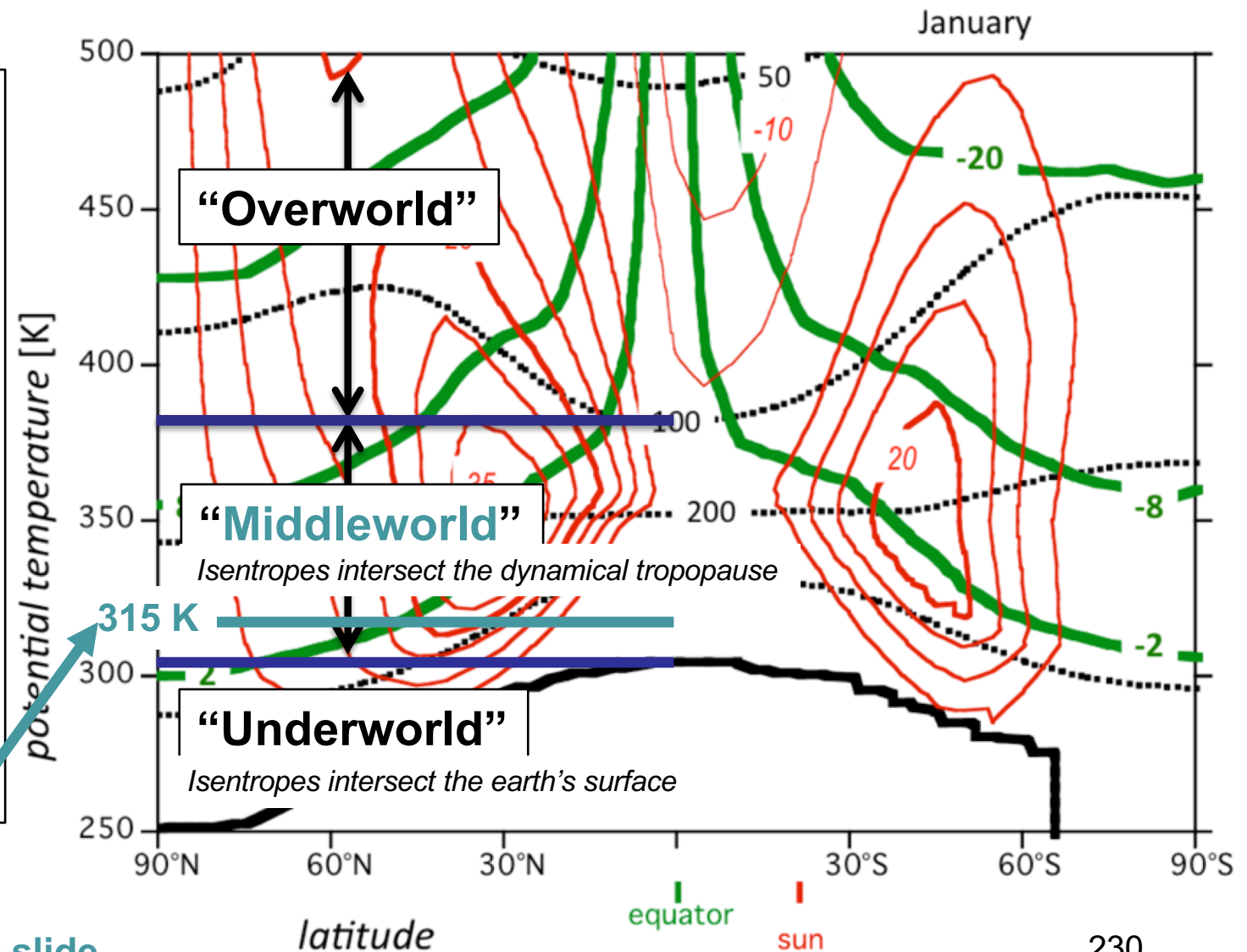
$$Z \equiv \frac{\bar{\zeta}_a}{\sigma}$$



Atmospheric “Worlds”

Figure 1.47

January-mean zonal-mean **isentropic potential vorticity** (eq. 1.128a) (**green** lines, labeled in PVU, where $1 \text{ PVU} = 10^{-6} \text{ K m}^2 \text{ kg}^{-1} \text{ s}^{-1}$), of **pressure** (dashed lines, labeled in hPa) and of zonal-mean **zonal wind velocity** (**red** lines, labeled in m/s, interval: 5 m/s) as a function of potential temperature and latitude

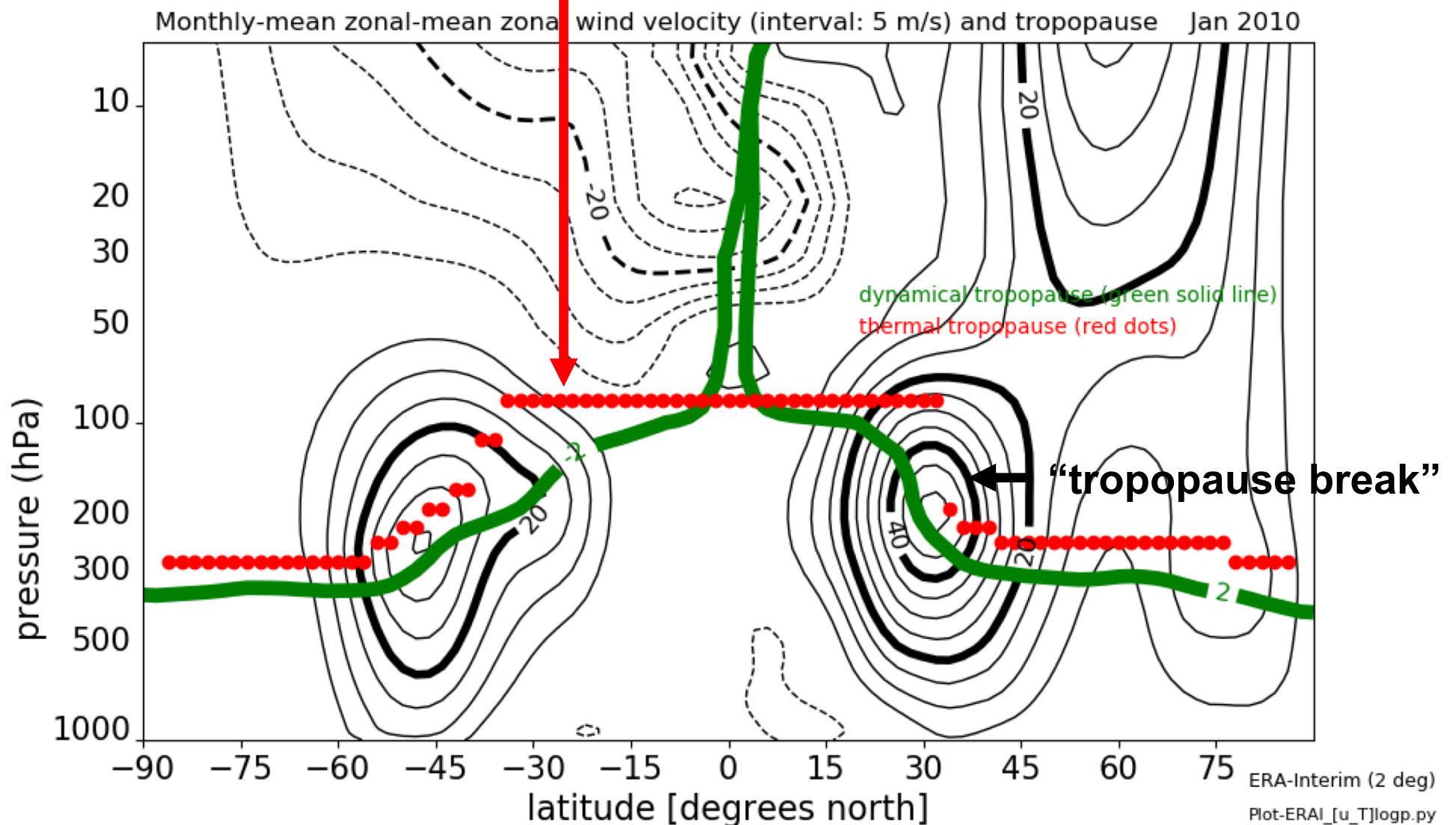


animation on next slide

Thermal and dynamical tropopause

Thermal tropopause: point above which $\partial T/\partial z > -2$ K/km

Dynamical tropopause: point at which $Z=2$ PVU

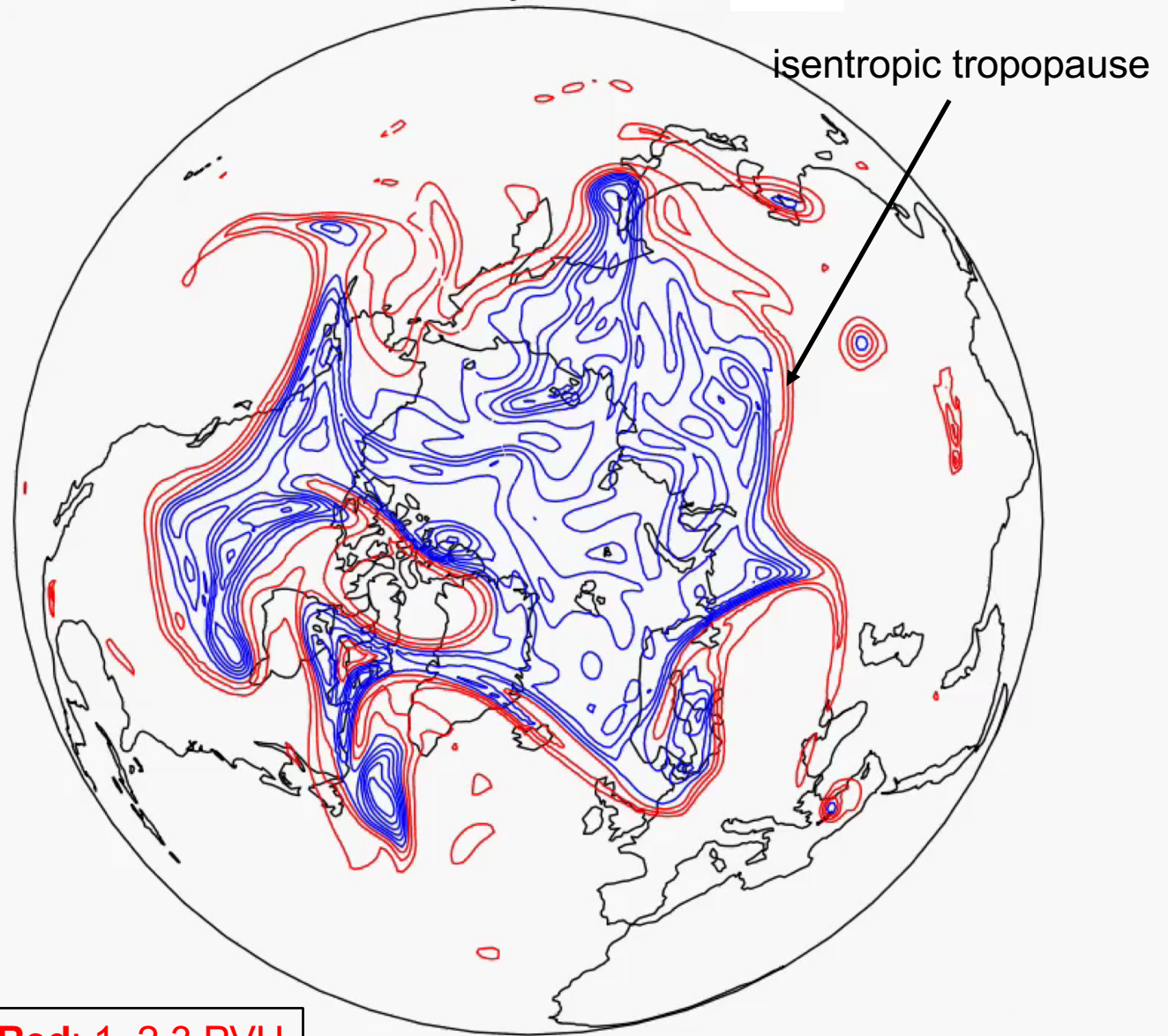


Potential vorticity at $\theta=315$ K and the isentropic tropopause

PV at 315 K on day 0 after 1 Nov. 2006

High PV at high latitudes is due to the low isentropic density, which, in turn, is due to the existence of an “Underworld” (a cold air mass) at these latitudes.

The convoluted PV-distribution in middle latitudes is due to “**breaking planetary (Rossby) waves**”, which **propagate along a PV-gradient**, sometimes referred to as a “Rossby wave-guide”.



Red: 1, 2 3 PVU
Blue: ≥ 4 PVU

PV315K_1nov2006-28feb2007 (Converted).mov
SurfZoneSeason.py

ERA-Interim re-analysis

Next:

Dynamic Meteorology: lecture 9

14/12/2022 (Wednesday, 13:15)

Sections 1.22-1.24

Pressure and potential temperature as a vertical coordinate (section 1.23)
Hadley, Ferrel and Brewer-Dobson circulations (section 1.24)
Cross-isentropic mass transport
Introduction to project 5 !

Tutorial after this lecture (8) (Friday 9 December 2022):
Project 3 (hand in today: end of 09/12/22)
Problem 1.18 (page 128)