

Thermal convection of a stratified fluid under the influence of gravity

Lectures 9, course on “Turbulence in Fluids”, Utrecht University

<http://www.staff.science.uu.nl/~delde102/TinF.html>

14 March 2022

Part 4: Non-linear scale-selection and “spectral transfer” in two-dimensional convection

The equations governing two-dimensional convection in spectral space

Transfer of energy to higher harmonics (smaller scales) by triad interactions

In the Lorenz model

In an extension of Lorenz’s model

Non-linear scale-selection

Conservation of kinetic energy and of enstrophy in 2D-flow

Spectral blocking and upscale spectral energy transfer in two-dimensional flow

Governing equations - two dimensional thermal convection in spectral space

(lecture 7)

Amplitudes: $\Theta_\gamma = i\Theta_\gamma^I \quad \Psi_\gamma = \Psi_\gamma^R$

Temperature:
$$\frac{d\Theta_\gamma}{dt} = \sum_{\alpha,\beta} I_{\gamma\alpha\beta} \Psi_\alpha \Theta_\beta + i\pi Ra a_x l_\gamma \Psi_\gamma - \pi^2 k_\gamma^2 \Theta_\gamma$$

Streamfunction:
$$\frac{d\Psi_\gamma}{dt} = \frac{1}{k_\gamma^2} \sum_{\alpha,\beta} k_\beta^2 I_{\gamma\alpha\beta} \Psi_\alpha \Psi_\beta + i\pi Pr \frac{a_x l_\gamma}{k_\gamma^2} \Theta_\gamma - \pi^2 Pr k_\gamma^2 \Psi_\gamma$$

Interaction-coefficient:
$$I_{\gamma\alpha\beta} \equiv \pi^2 a_x (n_\alpha l_\beta - n_\beta l_\alpha) \delta_{\gamma,\alpha+\beta}$$

$$\left. \begin{array}{l} \delta_{\gamma,\alpha+\beta} = 1 \text{ if } \alpha+\beta=\gamma \\ \text{else } \delta_{\gamma,\alpha+\beta} = 0 \end{array} \right\} \text{selection rule}$$

Flow of energy to higher harmonics in the Lorenz (1963) model of thermal convection

"Modes" included in the model:
(1,1),(0,2) and mirror images in l and n -axis

Triad interactions:
(1,1) and (-1,1) interact to contribute to (0,2)
(0,2) and (1,-1) interact to contribute to (1,1)

Because of symmetry relations, only equations needed for (1,1) & (0,2)

model equations:

$$\frac{d\Psi_{1,1}}{dt} \equiv \frac{dX}{dt} = \frac{a_x \text{Pr}}{\pi(a_x^2 + 1)} Y - \pi^2(a_x^2 + 1) \text{Pr} X$$

$$\frac{d\Theta_{1,1}}{dt} \equiv \frac{dY}{dt} = -2\pi^2 a_x XZ + \pi a_x \text{Ra} X - \pi^2(a_x^2 + 1) Y$$

$$\frac{d\Theta_{0,2}}{dt} \equiv \frac{dZ}{dt} = 4\pi^2 a_x XY - 4\pi^2 Z$$

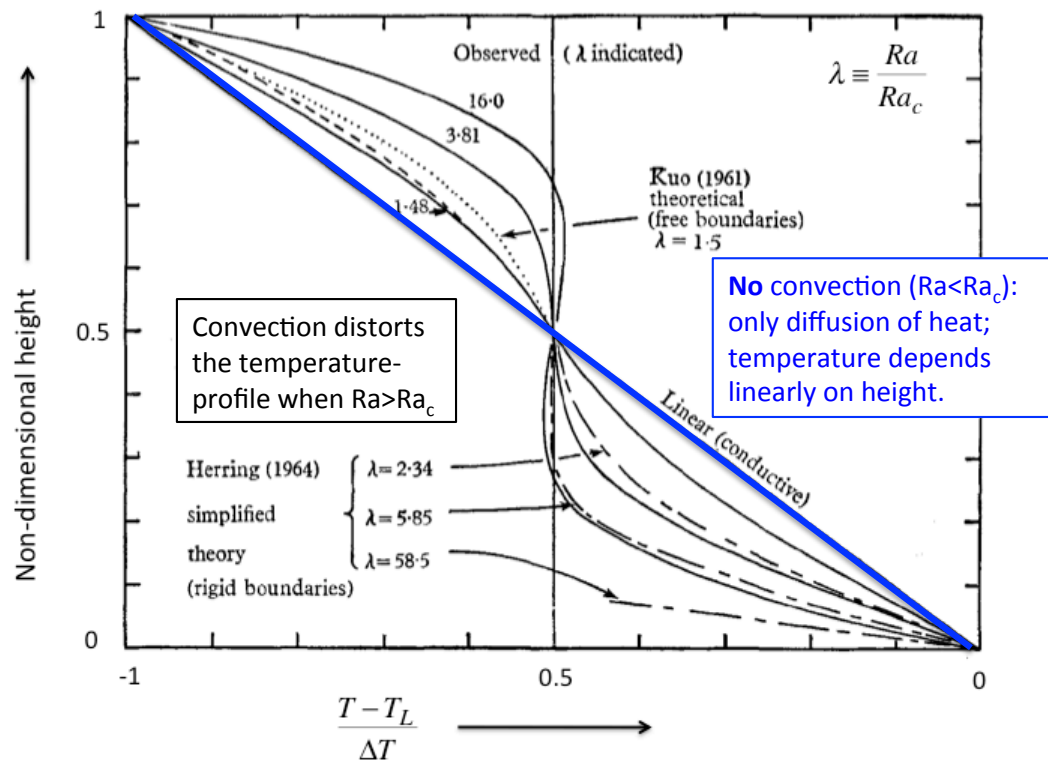
This term comes from the interaction of wave-vectors, (1,1) and (0,2)

This term comes from the interaction of wave-vectors, (1,1) and (-1,1)

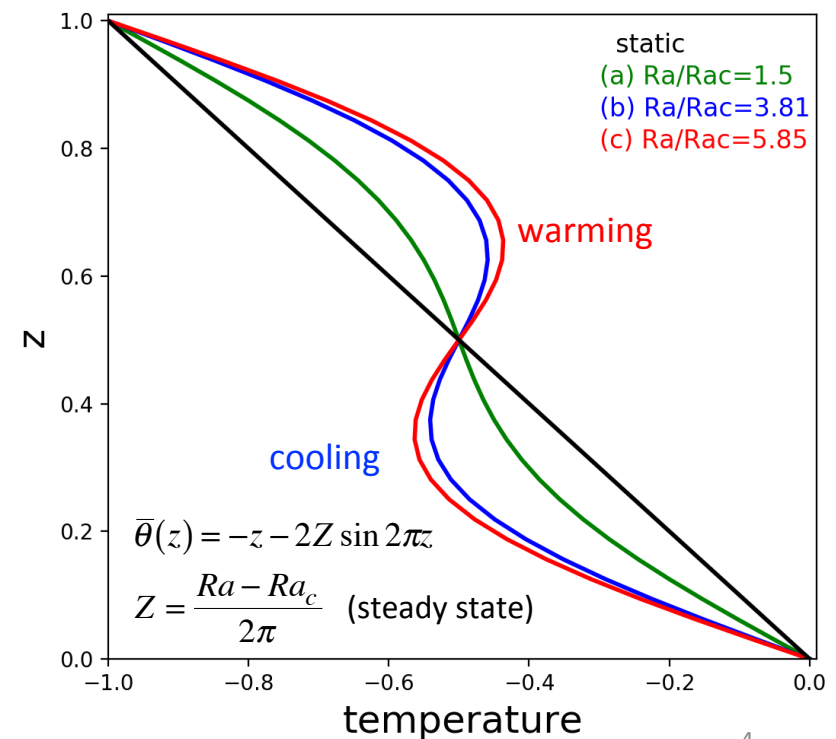
Non-linear interactions in Lorenz's (1963) model only describe the transfer of potential energy from the **"wave"** (1,1) to the **mean state** (0,2), because warm fluid goes up, while cold fluid goes down

Observed and modelled average temperature profile for different Rayleigh numbers

Observed (solid lines) (figure 4)



Modelled (Lorenz, 1963)



Convection cells have a well-defined scale

As if the spectrum is dominated by one Fourier mode

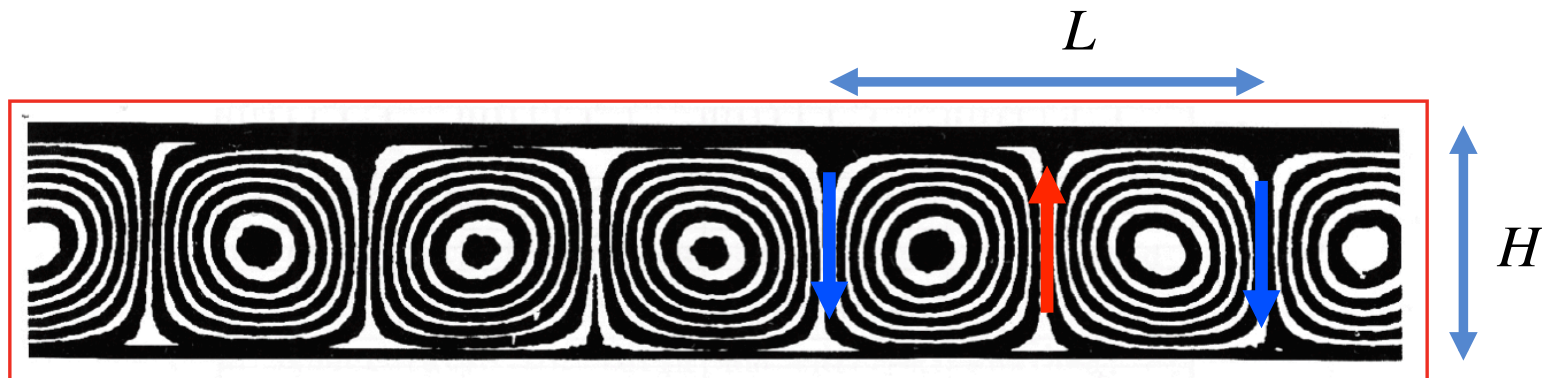


Figure 3. Side-view (differential interferograms) of two-dimensional convection cells in silicone oil in a rectangular box of relative dimensions 10:4:1, which is heated from below. Uniform heating produces convection “rolls” parallel to the shorter side (Van Dyke, 1982). The flow pattern is remarkable orderly!

Question of scale selection when $Ra \gg Ra_c$

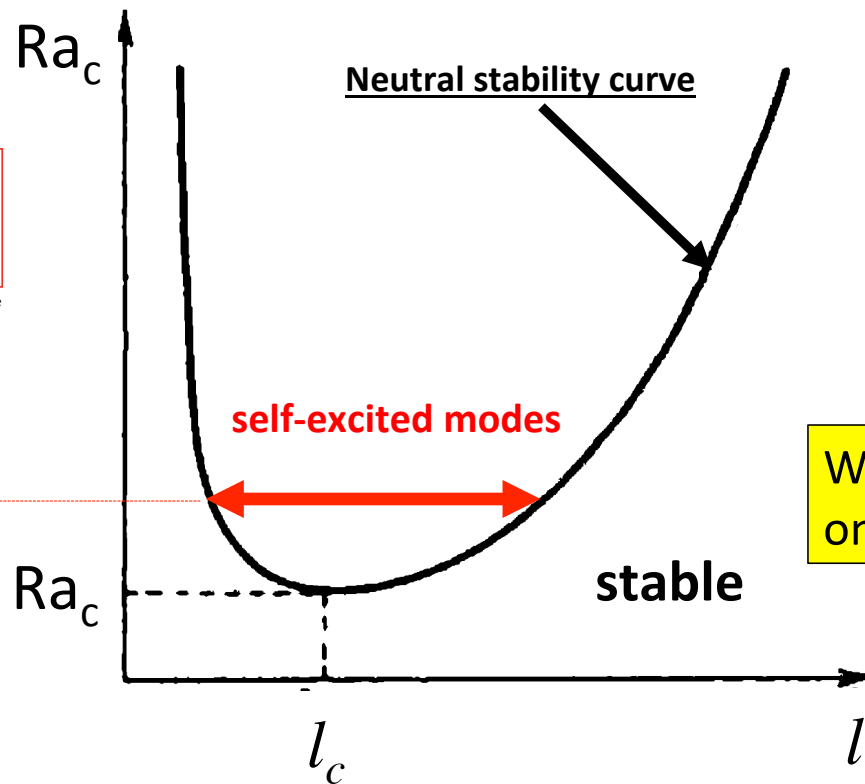
Previous slide:

Cells have a well-defined scale, as if the spectrum is dominated by one Fourier mode



Figure 3. Side-view (differential interferograms) of two-dimensional convection cells in silicone oil in a rectangular box of relative dimensions 10:4:1, which is heated from below. Uniform heating produces convection “rolls” parallel to the shorter side (Van Dyke, 1982). The flow pattern is remarkable orderly!

At $Ra > Ra_c$ more than one mode is unstable



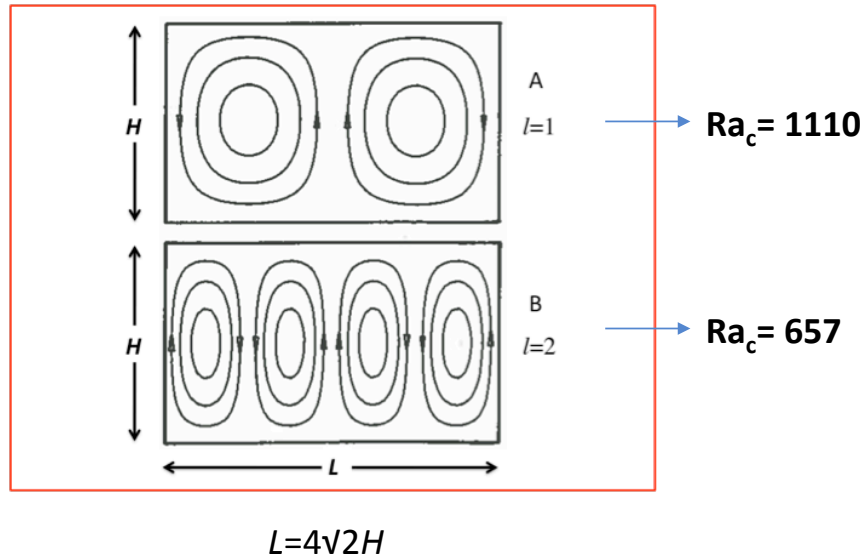
Why/how does one mode “win”?

Fig. 1. The neutral-stability curve for a layer of motionless fluid. The region of stable states is below the curve and the region of unstable states is above the curve.

An extension of the Lorenz model: the interaction of two “self-excited” modes

To study non-linear scale selection and energy transfer through the spectrum

Two self-excited modes: (1,1) and (2,1):



If $Ra > 1110$, both modes are “self-excited”

Triad interactions involving self-excited modes*:
(1,1) and (2,1) interact to contribute to (3,2)
(-1,1) and (2,1) interact to contribute to (1,2)

Additional “modes” needed to capture the interaction between (1,1) and (2,1):
(3,2), (1,2) and mirror images in l and n -axis

We construct a model, which includes the following wave-vectors (modes) in the streamfunction:
(1,1), (2,1), (1,2) and (3,2) and all mirror images in l and n -axis

We add modes (0,2) and (0,4) in the temperature

* Remember the “selection rule”: $\alpha + \beta = \gamma$

A 10-component model to study the interaction of two self-excited modes

$$\begin{aligned} A &\equiv \Psi_{j,1}^R; E \equiv \Theta_{j,1}^I; I \equiv \Theta_{0,2}^I; \\ B &\equiv \Psi_{j+1,1}^R; F \equiv \Theta_{j+1,1}^I; J \equiv \Theta_{0,4}^I; \\ C &\equiv \Psi_{1,2}^R; G \equiv \Theta_{1,2}^I; \\ D &\equiv \Psi_{2j+1,2}^R; H \equiv \Theta_{2j+1,2}^I. \end{aligned}$$

j is a positive integer

All triad interactions

γ	α	β
$(0,2)$	$(-j,1)$	$(j,1)$
$(0,2)$	$(-j-1,1)$	$(j+1,1)$
$(0,4)$	$(-1,2)$	$(1,2)$
$(0,4)$	$(-2j-1,2)$	$(2j+1,2)$
$(j,1)$	$(j+1,-1)$	$(-1,2)$
$(j,1)$	$(-j-1,-1)$	$(2j+1,2)$
$(j+1,1)$	$(j,-1)$	$(1,2)$
$(j+1,1)$	$(-j,1)$	$(2j+1,2)$
$(1,2)$	$(-j,1)$	$(j+1,1)$

These interactions (with $j=1$) are included in the Lorenz(63) model.

$(2j+1,1)$	$(j,1)$	$(j+1,1)$
$(j,1)$	$(j,-1)$	$(0,2)$
$(j+1,1)$	$(j+1,-1)$	$(0,2)$
$(1,2)$	$(1,-2)$	$(0,4)$
$(2j+1,2)$	$(2j+1,-2)$	$(0,4)$

Table 1: Non-zero interactions taken inside the truncation in the ten-coefficient model. The interaction of wave vectors, α and β , contributes to wave vector, γ .

From the equations on slide 2 and some careful book keeping we find the model equations on the next slide...

10-component model

26 non-linear terms !

$$\begin{aligned}
 \frac{dA}{dt} &= \frac{a_x j \text{Pr}}{\pi k_A^2} E - \pi^2 \text{Pr} k_A^2 A - (2j+1)\pi^2 a_x \frac{k_B^2 - k_C^2}{k_A^2} BC - \pi^2 a_x \frac{k_D^2 - k_B^2}{k_A^2} BD & (j,1) \\
 \frac{dB}{dt} &= \frac{a_x (j+1) \text{Pr}}{\pi k_B^2} F - \pi^2 \text{Pr} k_B^2 B - (2j+1)\pi^2 a_x \frac{k_C^2 - k_A^2}{k_B^2} AC + \pi^2 a_x \frac{k_D^2 - k_A^2}{k_B^2} AD & (j+1,1) \\
 \frac{dC}{dt} &= \frac{a_x j \text{Pr}}{\pi k_C^2} G - \pi^2 \text{Pr} k_C^2 C + (2j+1)\pi^2 a_x \frac{k_B^2 - k_A^2}{k_C^2} AB & (1,2) \\
 \frac{dD}{dt} &= \frac{a_x (2j+1) \text{Pr}}{\pi k_D^2} H - \pi^2 \text{Pr} k_D^2 D - \pi^2 a_x \frac{k_B^2 - k_A^2}{k_D^2} AB & (2j+1,2) \\
 \frac{dE}{dt} &= \pi a_x j \text{Ra} A - \pi^2 k_A^2 E - (2j+1)\pi^2 a_x BG - (2j+1)\pi^2 a_x CF - \pi^2 a_x BH - \pi^2 a_x DF - 2j\pi^2 a_x AI \\
 \frac{dF}{dt} &= \pi a_x (j+1) \text{Ra} B - \pi^2 k_B^2 F - (2j+1)\pi^2 a_x AG + (2j+1)\pi^2 a_x CE + \pi^2 a_x AH + \pi^2 a_x DE - 2(j+1)\pi^2 a_x BI \\
 \frac{dG}{dt} &= \pi a_x j \text{Ra} C - \pi^2 k_C^2 G + (2j+1)\pi^2 a_x AF + (2j+1)\pi^2 a_x BE - 4\pi^2 a_x CJ \\
 \frac{dH}{dt} &= \pi a_x (2j+1) \text{Ra} D - \pi^2 k_D^2 H - \pi^2 a_x AF + \pi^2 a_x BE - 4(2j+1)\pi^2 a_x DJ \\
 \frac{dI}{dt} &= -4\pi^2 I + 4j\pi^2 a_x AE + 4(j+1)\pi^2 a_x BF & (0,2) \\
 \frac{dJ}{dt} &= -16\pi^2 J + 8\pi^2 a_x CG + 8(2j+1)\pi^2 a_x DH , & (0,4)
 \end{aligned}$$

streamfunction

waves

temperature

(105)

Equations for spectral amplitudes of the stream function

wave-vector

(1,1)	$\frac{dA}{dt} = \frac{a_x j \text{Pr}}{\pi k_A^2} E - \pi^2 \text{Pr} k_A^2 A - (2j+1)\pi^2 a_x \frac{k_B^2 - k_C^2}{k_A^2} BC - \pi^2 a_x \frac{k_D^2 - k_B^2}{k_A^2} BD$	(j,1)
(2,1)	$\frac{dB}{dt} = \frac{a_x (j+1) \text{Pr}}{\pi k_B^2} F - \pi^2 \text{Pr} k_B^2 B - (2j+1)\pi^2 a_x \frac{k_C^2 - k_A^2}{k_B^2} AC + \pi^2 a_x \frac{k_D^2 - k_A^2}{k_B^2} AD$	(j+1,1)
(1,2)	$\frac{dC}{dt} = \frac{a_x j \text{Pr}}{\pi k_C^2} G - \pi^2 \text{Pr} k_C^2 C + (2j+1)\pi^2 a_x \frac{k_B^2 - k_A^2}{k_C^2} AB$	(1,2)
(3,2)	$\frac{dD}{dt} = \frac{a_x (2j+1) \text{Pr}}{\pi k_D^2} H - \pi^2 \text{Pr} k_D^2 D - \pi^2 a_x \frac{k_B^2 - k_A^2}{k_D^2} AB$	(2j+1,2)


Forcing


damping


non-linear interactions

streamfunction

What happens when Pr is very large?
What happens when Pr is very small?

Total wave numbers:

$$k_A^2 \equiv a_x^2 j^2 + 1; k_B^2 \equiv a_x^2 (j+1)^2 + 1; k_C^2 \equiv a_x^2 + 4; k_D^2 \equiv a_x^2 (2j+1)^2 + 4$$

Simplify *linear* terms in equations for stream function

$$\begin{aligned}
 (j,1) \quad \frac{dA}{dt} &= \lambda_A A + (2j+1)\pi^2 a_x \frac{k_C^2 - k_B^2}{k_A^2} BC - \pi^2 a_x \frac{k_D^2 - k_B^2}{k_A^2} BD \\
 (j+1,1) \quad \frac{dB}{dt} &= \lambda_B B - (2j+1)\pi^2 a_x \frac{k_C^2 - k_A^2}{k_B^2} AC + \pi^2 a_x \frac{k_D^2 - k_A^2}{k_B^2} AD \\
 (1,2) \quad \frac{dC}{dt} &= \lambda_C C + (2j+1)\pi^2 a_x \frac{k_B^2 - k_A^2}{k_C^2} AB \\
 (2j+1,2) \quad \frac{dD}{dt} &= \lambda_D D - \pi^2 a_x \frac{k_B^2 - k_A^2}{k_D^2} AB
 \end{aligned} \tag{108}$$

Forcing + damping

$\lambda_A, \lambda_B, \lambda_C$ and λ_D are **relaxation coefficients**
positive if the mode is **self-excited** (forcing > damping)
negative if the mode is **damped** (forcing < damping)

Simplify linear terms by “adiabatic elimination”

$$(j,1) \quad \frac{dA}{dt} = \lambda_A A + (2j+1)\pi^2 a_x \frac{k_C^2 - k_B^2}{k_A^2} BC - \pi^2 a_x \frac{k_D^2 - k_B^2}{k_A^2} BD$$

$\lambda_A > 0$ and $\lambda_B > 0$

$$(j+1,1) \quad \frac{dB}{dt} = \lambda_B B - (2j+1)\pi^2 a_x \frac{k_C^2 - k_A^2}{k_B^2} AC + \pi^2 a_x \frac{k_D^2 - k_A^2}{k_B^2} AD$$

$$(1,2) \quad \frac{dC}{dt} = \lambda_C C + (2j+1)\pi^2 a_x \frac{k_B^2 - k_A^2}{k_C^2} AB$$

$\lambda_C < 0$ and $\lambda_D < 0$

$$(2j+1,2) \quad \frac{dD}{dt} = \lambda_D D - \pi^2 a_x \frac{k_B^2 - k_A^2}{k_D^2} AB$$

time-derivatives of C and D are zero →

adiabatic elimination technique

$$C = -(2j+1)\pi^2 a_x \frac{k_B^2 - k_A^2}{\lambda_C k_C^2} AB .$$

and

$$D = \pi^2 a_x \frac{k_B^2 - k_A^2}{\lambda_D k_D^2} AB .$$

Forcing + damping

Damped modes (C and D) are “slaved” to the self-excited modes (A and B)

Simplify linear terms by “adiabatic elimination”

$$(j,1) \quad \frac{dA}{dt} = \lambda_A A + (2j+1)\pi^2 a_x \frac{k_C^2 - k_B^2}{k_A^2} BC - \pi^2 a_x \frac{k_D^2 - k_B^2}{k_A^2} BD$$

$$(j+1,1) \quad \frac{dB}{dt} = \lambda_B B - (2j+1)\pi^2 a_x \frac{k_C^2 - k_A^2}{k_B^2} AC + \pi^2 a_x \frac{k_D^2 - k_A^2}{k_B^2} AD$$

$$(1,2) \quad \frac{dC}{dt} = \lambda_C C + (2j+1)\pi^2 a_x \frac{k_B^2 - k_A^2}{k_C^2} AB$$

$$(2j+1,2) \quad \frac{dD}{dt} = \lambda_D D - \pi^2 a_x \frac{k_B^2 - k_A^2}{k_D^2} AB$$

Forcing + damping

$$k_A^2 < k_B^2 < k_C^2 < k_D^2$$

$$\frac{dA}{dt} = (\lambda_A + \mu_1 B^2) A$$

$$\frac{dB}{dt} = (\lambda_B + \mu_2 A^2) B$$

$$\mu_1 > 0 \text{ and } \mu_2 < 0$$

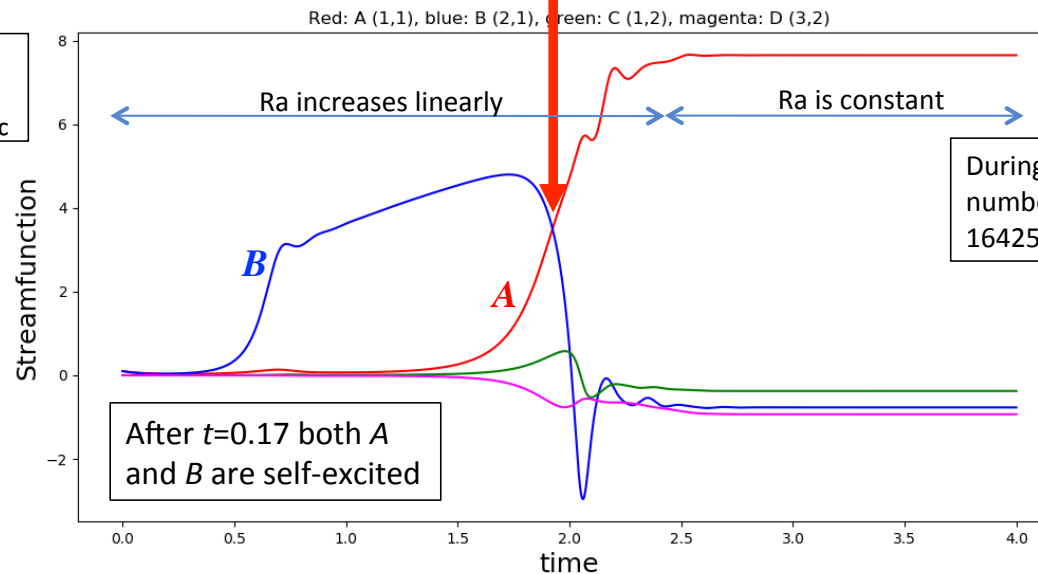
Above a certain value of A^2 , B is damped by A .

A numerical integration of the 10-component model (j=1)

Showing only the Fourier-amplitudes of the streamfunction, where **A** and **B** are the self-excited modes

B is “killed” by **A**

$\lambda_A > 0$ if $Ra > 657 = Ra_c$
 $\lambda_B > 0$ if $Ra > 1110 = 1.69Ra_c$



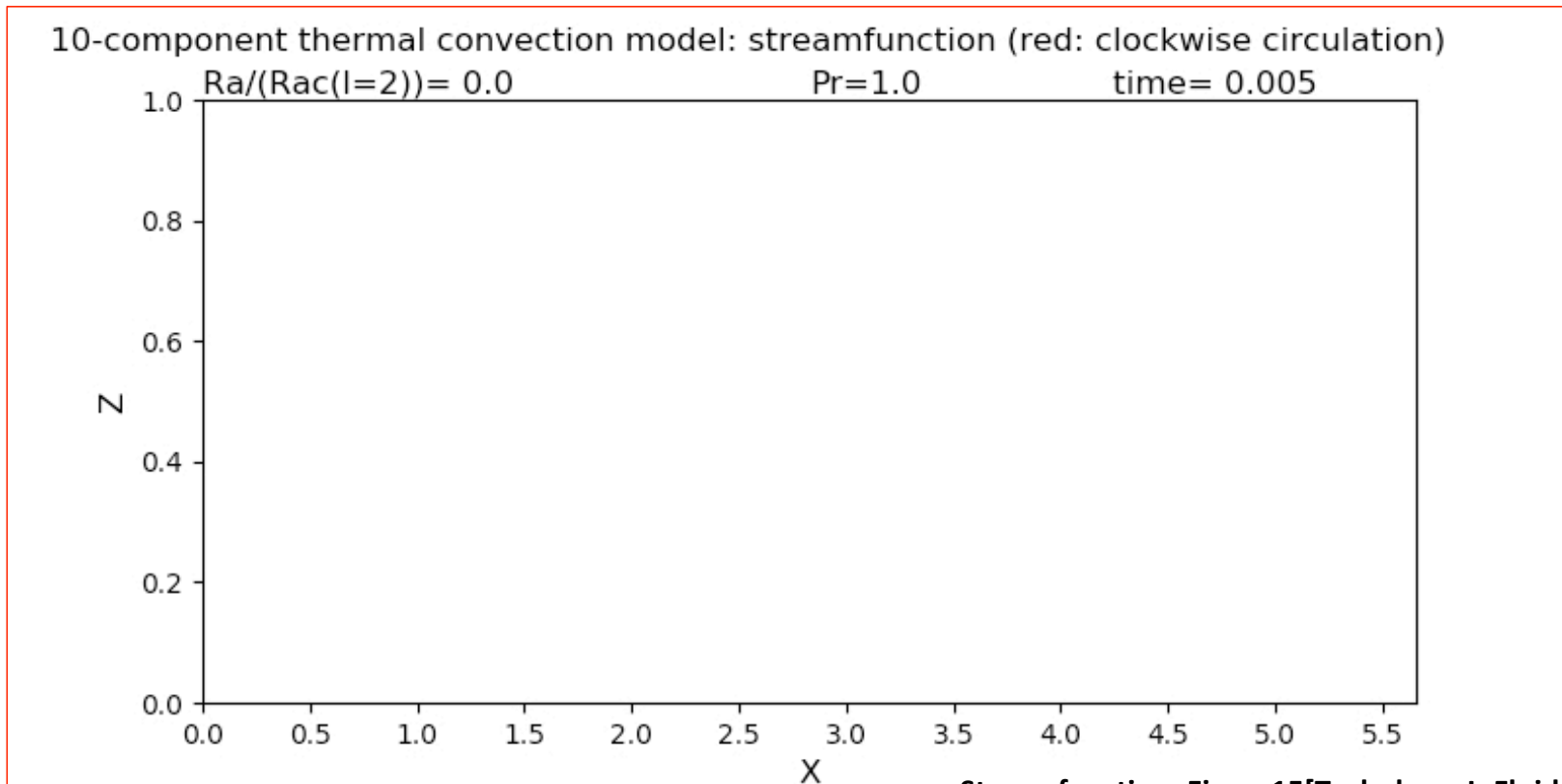
(units are non-dimensional)

Figure 14. **A** (blue), **B** (red), **C** (green) and **D** (magenta) of the ten-component model (eqs. 105), in a numerical integration with the state of rest as initial condition. The domain has an aspect ratio of $4\sqrt{2}$, so that the mode (i.e. (2,1)) has an aspect ratio of $2\sqrt{2}$. ($\alpha_x = 1/2\sqrt{2}$ and $Pr=1$). The mode with wave vector (2,1) is the fastest growing mode near the state of rest.

Animation of the integration: $\Psi(x,z)$

Transformation from spectral to physical space with:

$$\Psi(x,z) = -4A \cos(\pi a_x l x) \sin(\pi n z) - 4B \cos(2\pi a_x l x) \sin(\pi n z) - 4C \cos(\pi a_x l x) \sin(2\pi n z) - 4D \cos(3\pi a_x l x) \sin(2\pi n z)$$

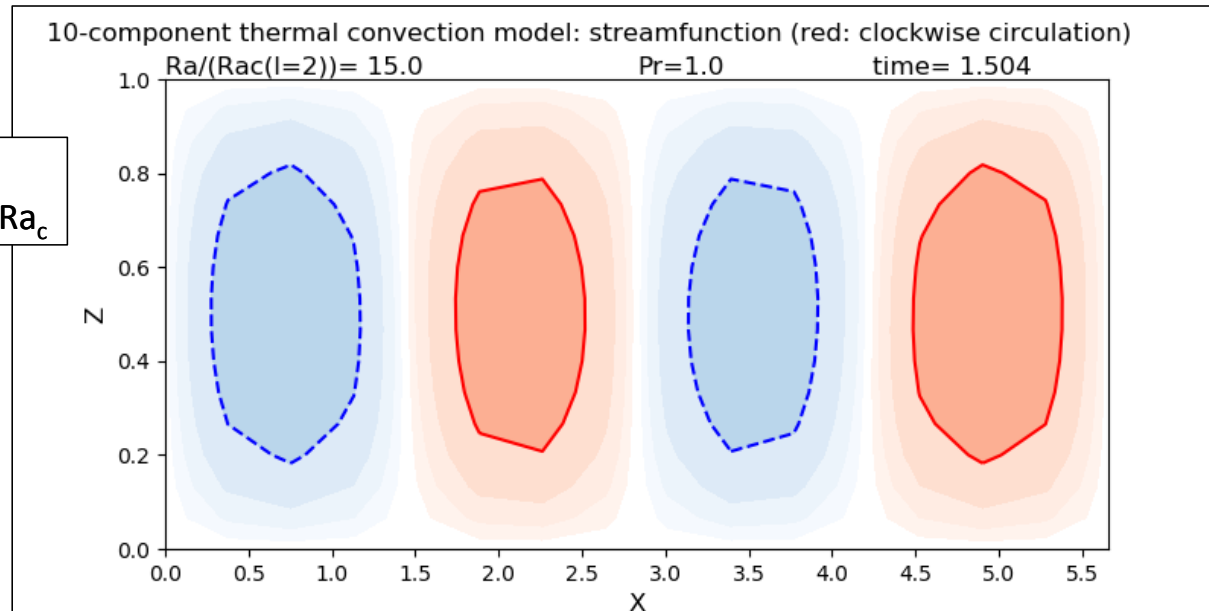


Streamfunction_Figure15[TurbulenceInFluids].mov

Low Ra: wave number 2 (B) grows exponentially, much faster than wave number 1

Ra = 15 Ra_c

$\lambda_A > 0$ if $Ra > 657 = Ra_c$
 $\lambda_B > 0$ if $Ra > 1110 = 1.69 Ra_c$

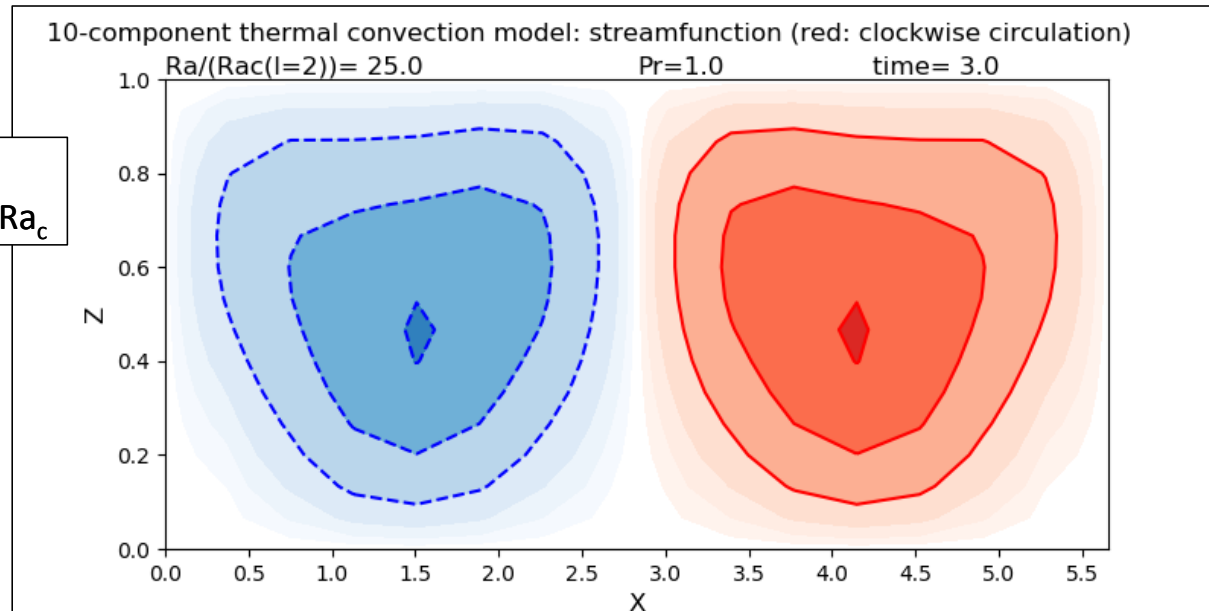


Large Ra: wave number 1 (A) grows at the cost of wave number 2

This is **non-linear scale-selection**

Ra = 25 Ra_c

$\lambda_A > 0$ if $Ra > 657 = Ra_c$
 $\lambda_B > 0$ if $Ra > 1110 = 1.69 Ra_c$

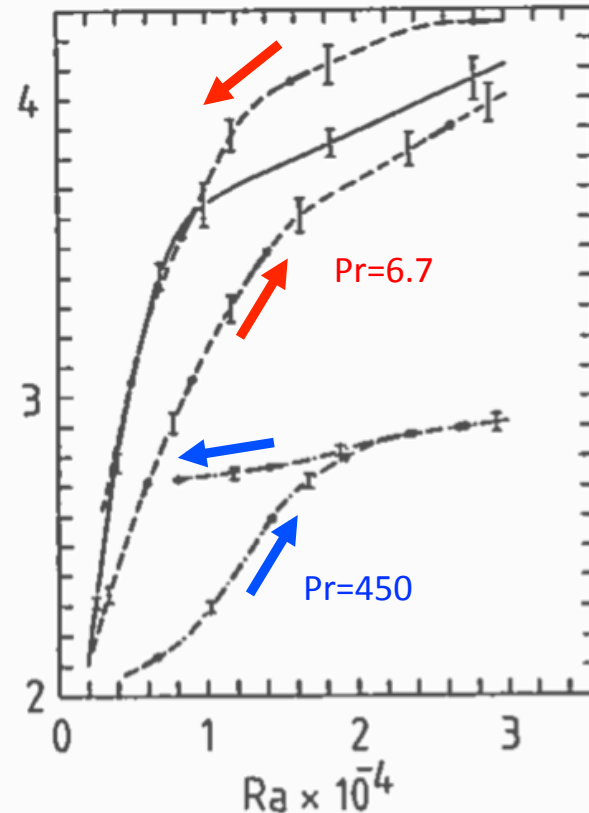


Observed increase aspect ratio of 2D convection cells

Aspect ratio of two-dimensional convection cells, with error limits, as measured in the laboratory by Willis et al. (1972) in an experiment in Ra increases and then decreases. Note the dependence of the state of the system on its history. This is a typical property of many non-linear systems, called **hysteresis**. (figure 16, lecture notes)

—, $Pr=0.71$;
 $\rightarrow$, $Pr=6.7$, Ra increasing;
 $\leftarrow$, $Pr=6.7$, Ra decreasing;
 $\rightarrow$, $Pr=450$, Ra increasing;
 $\leftarrow$, $Pr=450$, Ra decreasing.

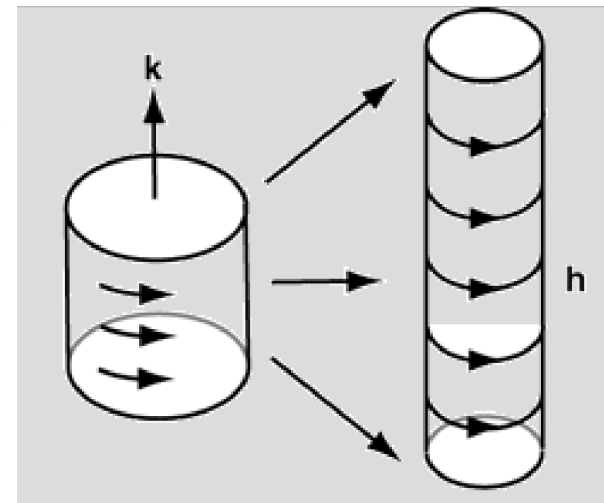
aspect ratio, L/H



Upscale spectral transfer is because total kinetic energy & total enstrophy are conserved

Vortex stretching is impossible in 2D-flow

Vorticity is conserved in two-dimensional (2D) flow



Total **enstrophy**, defined as one half of the mean square vorticity, is conserved

Total kinetic energy in terms of the streamfunction

Total kinetic energy: $K = \frac{1}{2} \int (\vec{v} \cdot \vec{v}) d\sigma = \frac{1}{2} \int (u^2 + w^2) d\sigma$

area-element in x - z space (see below eq. 69)

$$\nabla \psi \equiv \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial z} \right)$$

$$(\nabla \psi)^2 \equiv \nabla \psi \cdot \nabla \psi = \left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 = u^2 + w^2$$

$$\nabla(\psi \nabla \psi) = \psi \nabla^2 \psi + (\nabla \psi)^2 = \psi \nabla^2 \psi + u^2 + w^2 \longrightarrow \boxed{u^2 + w^2 = \nabla(\psi \nabla \psi) - \psi \nabla^2 \psi}$$

Total kinetic energy in terms of the stream function

Total kinetic energy: $K = \frac{1}{2} \int (\vec{v} \cdot \vec{v}) d\sigma = \frac{1}{2} \int (u^2 + w^2) d\sigma$

$$u^2 + w^2 = \nabla(\psi \nabla \psi) - \psi \nabla^2 \psi$$

$$K = \frac{1}{2} \int (u^2 + w^2) d\sigma = \frac{1}{2} \int (\nabla(\psi \nabla \psi) - \psi \nabla^2 \psi) d\sigma = \frac{1}{2} \int (\psi \nabla^2 \psi) d\sigma$$

Total enstrophy in terms of the stream function

Vorticity:

$$\vec{\omega} \equiv \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$$



$$\eta \equiv \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = - \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \equiv -\nabla^2 \psi$$

Total enstrophy:

$$E = \frac{1}{2} \int (\vec{\omega} \cdot \vec{\omega}) d\sigma = \frac{1}{2} \int \eta^2 d\sigma = \frac{1}{2} \int (\nabla^2 \psi)^2 d\sigma$$

two-dimensional flow

Total kinetic energy and total enstrophy in spectral space

$$K = \frac{1}{2} \int (\psi \nabla^2 \psi) d\sigma$$

From previous slide:

$$E = \frac{1}{2} \int (\nabla^2 \psi)^2 d\sigma$$

With $\psi = \sum_{\alpha} \Psi_{\alpha}(t) S_{\alpha}(x, z)$ $S_{\alpha}(x, z) = \exp\{i\pi(a_x l_{\alpha} x + n_{\alpha} z)\}$ $\int S_{\alpha} S_{\beta}^* d\sigma = \delta_{\alpha, \beta}$

$$K = \frac{1}{2} \sum_{\alpha} k_{\alpha}^2 \Psi_{\alpha}^2 \equiv \frac{1}{2} \sum_{\alpha} \varepsilon_{\alpha}$$

In spectral space:

$$E = \frac{1}{2} \sum_{\alpha} k_{\alpha}^4 \Psi_{\alpha}^2 = \frac{1}{2} \sum_{\alpha} k_{\alpha}^2 \varepsilon_{\alpha}$$

$$\varepsilon_{\alpha} \equiv k_{\alpha}^2 \Psi_{\alpha}^2$$

Next: we use these definitions to investigate the consequences of **conservation of K and E** .

2D flow

Centre of gravity of the kinetic energy spectrum

Suppose that all kinetic energy of the fluid flow is located in a restricted range of wave numbers, $0 < k < K$.

Due to nonlinear interactions some of this kinetic energy must flow to higher wave numbers ($k > K$), under the constraint,

$\sum_k \varepsilon_k$ is conserved. But, because entropy is conserved also, i.e. $\sum_k k^2 \varepsilon_k$ is conserved, more kinetic energy must flow to lower wave numbers within the interval, $0 < k < K$.

The centre of gravity of the kinetic energy spectrum must shift to lower wave numbers (larger scales)

This argument is due to G.K. Batchelor (1953)

2D-flow:

R. Fjortoft (1953):

Spectral “blocking” in 2-D flow

Consider an arbitrary triad interaction between α , β and γ .

The fact that both **K and E are conserved** is stated as follows

$$\frac{d}{dt}(\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma) = 0 \quad (1) \quad \frac{d}{dt}(k_\alpha^2 \varepsilon_\alpha + k_\beta^2 \varepsilon_\beta + k_\gamma^2 \varepsilon_\gamma) = 0 \quad (2)$$

Multiplying eq. 1 by k_α^2 and subtracting the result from eq. 2 we get

$$d\varepsilon_\gamma = -\frac{(k_\beta^2 - k_\alpha^2)}{(k_\gamma^2 - k_\alpha^2)} d\varepsilon_\beta; \text{ with } k_\gamma^2 > k_\beta^2 > k_\alpha^2 \longrightarrow d\varepsilon_\gamma \approx -d\varepsilon_\beta$$

Multiplying eq. 1 by k_γ^2 and subtracting the result from eq. 2 we get

$$d\varepsilon_\alpha = -\frac{(k_\gamma^2 - k_\beta^2)}{(k_\gamma^2 - k_\alpha^2)} d\varepsilon_\beta; \text{ with } k_\gamma^2 > k_\beta^2 > k_\alpha^2 \longrightarrow d\varepsilon_\alpha \approx -d\varepsilon_\beta$$

Energy flows simultaneously up and down the scale!

Conclusions

According to linear theory, many Fourier modes with different wavelengths grow when Ra exceeds the critical value for onset of convection.

Nevertheless, we observe convection cells with a well-defined wavelength, suggesting that not all modes grow exponentially, as predicted by linear theory.

An extension of Lorenz's model indicates that the larger scale Fourier mode can easily dominate the other modes in the nonlinear phase.

Spectral transfer to larger scales is a consequence of simultaneous conservation of total kinetic energy and total entropy by the non-linear (triad) interactions.

Next lecture (10) (part 5 of the lecture notes) is about pattern formation and presents a low-order model which is able to simulate 3D convection, allowing for solutions representing different plan-forms, such as 2D convection and hexagonal convection.

Tutorial: exercise 17 (numerical !)

(representative for an exam question)

* TenComponentConvectionModel[vrs1].py

(17)

For this exercise you are given the **python-code of the 10-component model of two-dimensional convection***, which integrates the twelve equations (105), and asked to install and run this code on your computer. Oscillations with a period in the range between 1 and 10 non-dimensional viscous time-units (H^2/ν) are observed in a layer of fluid with low Pr and Rayleigh numbers which are just super-critical to several times super-critical (Krishnamurti, 1973). The 10-component model also exhibits oscillations at low Pr ($=0.1$), with $\alpha_x=1/2\sqrt{2}$ and for Rayleigh number 1.5 times supercritical.

- (a)** Can you qualitatively foresee the existence of these oscillations from eqs. 112 and 113?
- (b)** Investigate these oscillations in the 10-component model (105). Are the oscillations qualitatively in accord with your predictions based on eqs. 112 and 113?
- (c)** Do you get oscillations with periods similar to periods observed by Krishnamurti?
- (d)** Is this solution “chaotic”? If so, why?
- (e)** Make a scatter plot (“map”) of the peak-values of A as a function of the previous peak-value of A . Compare this “map” with the Lorenz map (**figure 12**).
- (f)** Plot the time between successive peaks as a function of time half-way these peaks. What do you conclude from this?