

Dynamic Meteorology: lecture 9

14/12/2022 (Wednesday, 13:15)

Sections 1.23-1.24

Pressure and potential temperature as a vertical coordinate

Hadley and Ferrel circulations (section 1.24)

Continuity equation in isentropic coordinates

Eddy-driven meridional circulation of mass (section 1.26)

Introduction to project 5 (problem 1.24)

Tutorial after this lecture (9)

Weather forecast for Sunday 18 December 2022 →

Project 5 (problem 1.24)

hand in hypothesis on or before Monday 20 December 2021

Problem 1.18

Range of "normal values" 18/12/1992-2021

TX: 6.05-8.15 C

TN: -0.6-4.0 C

SP: 0.0-40.5 %

RH: 0.025-1.55 mm

average precipitation 2.0

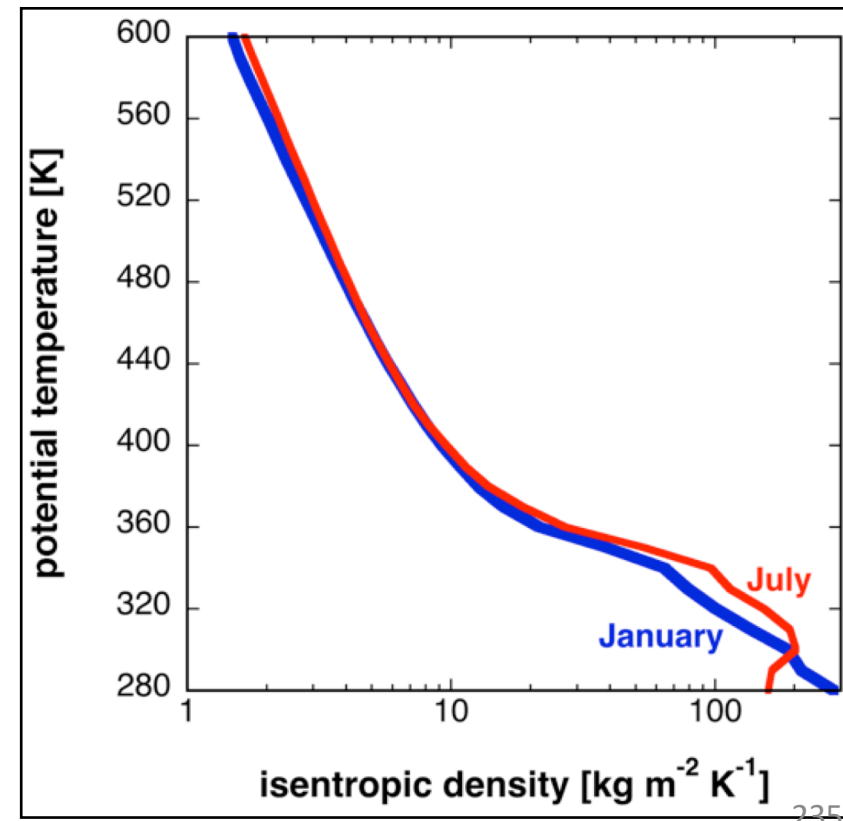
Pressure or potential temperature as a new **vertical coordinate**

Hydrostatic relation: transform from height to pressure as a vertical coordinate:

$$\frac{\partial p}{\partial z} = -\rho g$$

Hydrostatic relation + empirical fact that isentropic density, σ , is positive (**figure on the right**) is used to transform from height to potential temperature as a vertical coordinate:

$$\sigma \equiv -\frac{1}{g} \frac{\partial p}{\partial \theta} = \rho \frac{\partial z}{\partial \theta}$$



Equation of motion (z -coordinate)

x -component of the equation of motion:

$$\frac{du}{dt} = -\theta \frac{\partial \Pi}{\partial x} + fv + \frac{uv \tan \phi}{a}$$

y -component of the equation of motion:

$$\frac{dv}{dt} = -\theta \frac{\partial \Pi}{\partial y} + fu - \frac{u^2 \tan \phi}{a}$$

z -component of the equation of motion:

$$0 = -\theta \frac{\partial \Pi}{\partial z} - g$$

(hydrostatic balance)

Transformation to a different vertical coordinate (p or θ instead of z) has implications for the terms containing derivatives, which are underlined in the equations above.

Pressure as the vertical coordinate: Material time-derivative

Pressure-coordinate:

$$u(x, y, p, t) : \quad du = \frac{\partial u}{\partial t} dt + \left[\frac{\partial u}{\partial x} \right]_p dx + \left[\frac{\partial u}{\partial y} \right]_p dy + \frac{\partial u}{\partial p} dp$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial t} + u \left[\frac{\partial u}{\partial x} \right]_p + v \left[\frac{\partial u}{\partial y} \right]_p + \frac{dp}{dt} \frac{\partial u}{\partial p}}$$

↑
“vertical velocity”:

$\omega \equiv \frac{dp}{dt}$

↑
“omega”

Potential temperature as the vertical coordinate: Material time-derivative

theta-coordinate:

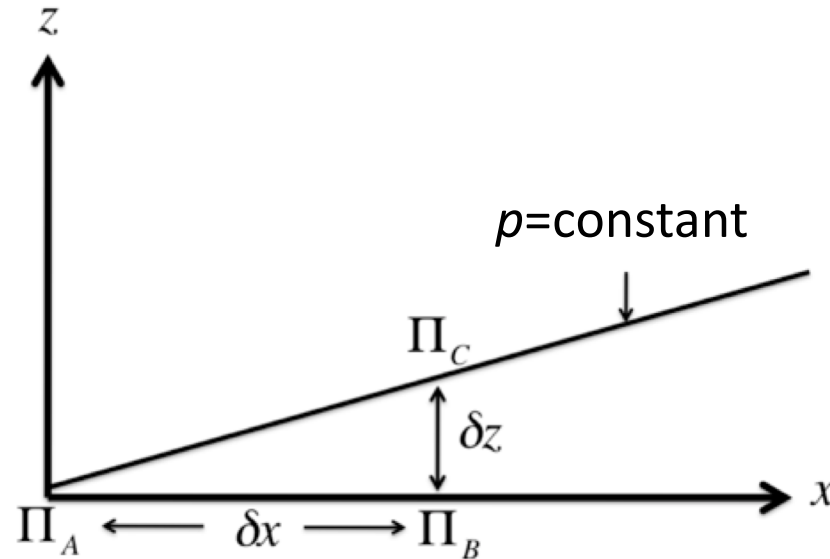
$$u(x,y,\theta,t) : du = \frac{\partial u}{\partial t} dt + \left[\frac{\partial u}{\partial x} \right]_{\theta} dx + \left[\frac{\partial u}{\partial y} \right]_{\theta} dy + \frac{\partial u}{\partial \theta} d\theta$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial t} + u \left[\frac{\partial u}{\partial x} \right]_{\theta} + v \left[\frac{\partial u}{\partial y} \right]_{\theta} + \frac{d\theta}{dt} \frac{\partial u}{\partial \theta}}$$

“vertical velocity”: $\frac{d\theta}{dt}$

“cross-isentropic flow”

Pressure gradient term and geopotential

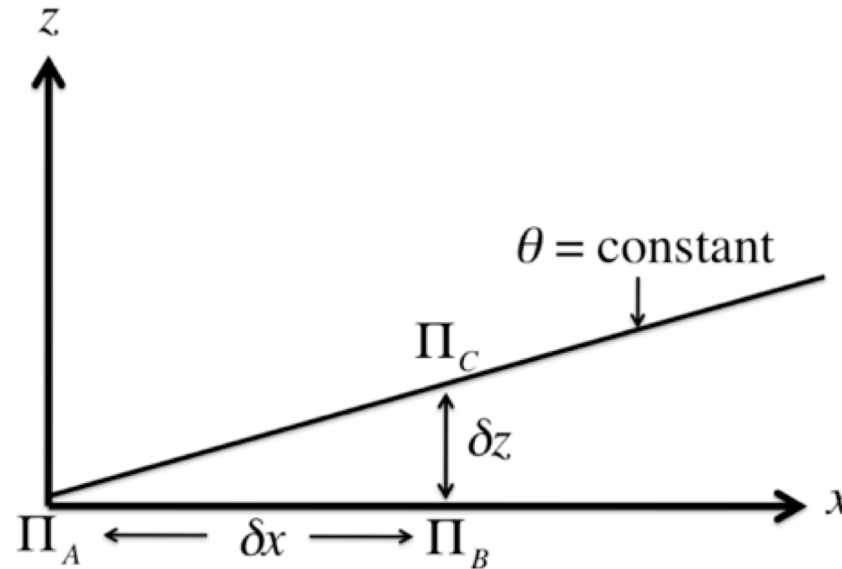


$$0 = \frac{\Pi_B - \Pi_A}{\delta x} + \frac{\Pi_C - \Pi_B}{\delta z} \frac{\delta z}{\delta x} \quad \text{and} \quad \boxed{\frac{\partial \Pi}{\partial z} = -\frac{g}{\theta}}$$

$$\rightarrow \theta \left(\frac{\partial \Pi}{\partial x} \right)_z = g \left(\frac{\partial z}{\partial x} \right)_p = \left(\frac{\partial \Phi}{\partial x} \right)_p \quad \text{Geopotential: } \Phi = gz \quad g \text{ is constant}$$

In isobaric coordinates the “pressure gradient term” is linear!!!

Pressure gradient term and isentropic streamfunction



$$\frac{\Pi_C - \Pi_A}{\delta x} = \frac{\Pi_B - \Pi_A}{\delta x} + \frac{\Pi_C - \Pi_B}{\delta z} \frac{\delta z}{\delta x}$$

$$\text{or } \left(\frac{\partial \Pi}{\partial x} \right)_{y\theta t} = \left(\frac{\partial \Pi}{\partial x} \right)_{yz t} + \left(\frac{\partial \Pi}{\partial z} \right)_{xy t} \left(\frac{\partial z}{\partial x} \right)_{y\theta t}$$

$$\boxed{\frac{\partial \Pi}{\partial z} = -\frac{g}{\theta}}$$



$$\theta \left(\frac{\partial \Pi}{\partial x} \right)_{yz t} = \theta \left(\frac{\partial \Pi}{\partial x} \right)_{y\theta t} + g \left(\frac{\partial z}{\partial x} \right)_{y\theta t} \equiv \left(\frac{\partial \Psi}{\partial x} \right)_{y\theta t}$$

$$\Psi \equiv \theta \Pi + g z \rightarrow$$

Pressure gradient term and isentropic streamfunction

Previous slide:
$$\theta \left(\frac{\partial \Pi}{\partial x} \right)_{yzt} = \theta \left(\frac{\partial \Pi}{\partial x} \right)_{y\theta t} + g \left(\frac{\partial z}{\partial x} \right)_{y\theta t} \equiv \left(\frac{\partial \Psi}{\partial x} \right)_{y\theta t}$$

Isentropic stream function: $\Psi \equiv \theta \Pi + gz$

or: $\Psi = c_p T + gz$

In isentropic coordinates the “pressure gradient term” is linear!!!

Hydrostatic equation in θ -coordinate

Isentropic stream function:

$$\Psi \equiv \theta \Pi + gz$$



$$\frac{\partial \Psi}{\partial \theta} \equiv \theta \frac{\partial \Pi}{\partial \theta} + \Pi + g \frac{\partial z}{\partial \theta}$$

$$\text{with } \frac{\partial \Pi}{\partial z} = -\frac{g}{\theta}$$



Hydrostatic equation:

$$\frac{\partial \Psi}{\partial \theta} \equiv \Pi$$

Equation of motion in two coordinate systems

Pressure as vertical coordinate:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \left[\frac{\partial u}{\partial x} \right]_p + v \left[\frac{\partial u}{\partial y} \right]_p + \omega \frac{\partial u}{\partial p} = - \left[\frac{\partial \Phi}{\partial x} \right]_p + fv + \frac{uv \tan \phi}{a}$$

$$\Phi = gz$$

↑
geopotential

Potential temperature as vertical coordinate:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \left[\frac{\partial u}{\partial x} \right]_{\theta} + v \left[\frac{\partial u}{\partial y} \right]_{\theta} + \frac{d\theta}{dt} \frac{\partial u}{\partial \theta} = - \left[\frac{\partial \Psi}{\partial x} \right]_{\theta} + fv + \frac{uv \tan \phi}{a}$$

$$\Psi \equiv \theta \Pi + gz$$

↑
isentropic streamfunction

Continuity equation in pressure coordinate

Define an element of air between two surfaces of equal pressure

The mass, δm , is equal to $\delta m = -\frac{\delta x \delta y \delta p}{g}$ (using hydrostatic equation)

Conservation of mass: $\frac{d}{dt}(\delta m) = -\frac{d}{dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$

Continuity equation in pressure coordinate

Define an element of air between two surfaces of equal pressure

The mass, δm , is equal to $\delta m = -\frac{\delta x \delta y \delta p}{g}$

Conservation of mass: $\frac{d}{dt}(\delta m) = -\frac{d}{dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$

Multiply this equation by $\frac{1}{\delta m}$ $\rightarrow$ $\frac{1}{\delta x} \frac{d\delta x}{dt} + \frac{1}{\delta y} \frac{d\delta y}{dt} + \frac{1}{\delta p} \frac{d\delta p}{dt} = 0$

$$\frac{1}{\delta x} \frac{d\delta x}{dt} \approx \frac{\delta(u)}{\delta x} \approx \frac{\partial u}{\partial x}$$

Therefore:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0$$

Why is this equation remarkable ?

Zonal average continuity equation

Previous slide:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0$$

Take the zonal average of this equation:

$$\left(\left[\frac{\partial u}{\partial x} \right] + \left[\frac{\partial v}{\partial y} \right] \right)_p + \left[\frac{\partial \omega}{\partial p} \right] = 0$$

Square brackets indicate zonal average, e.g.:

$$\left[\frac{\partial u}{\partial x} \right] \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial x} d\lambda = 0$$

→

$$\left(\left[\frac{\partial v}{\partial y} \right] \right)_p + \left[\frac{\partial \omega}{\partial p} \right] = 0$$

Zonal average continuity equation

Previous slide:

$$\left(\left[\frac{\partial v}{\partial y} \right] \right)_p + \left[\frac{\partial \omega}{\partial p} \right] = 0$$

We satisfy this equation if : $[v] \equiv \frac{\partial[\psi]}{\partial p}$ and $[\omega] \equiv -\frac{\partial[\psi]}{\partial y}$

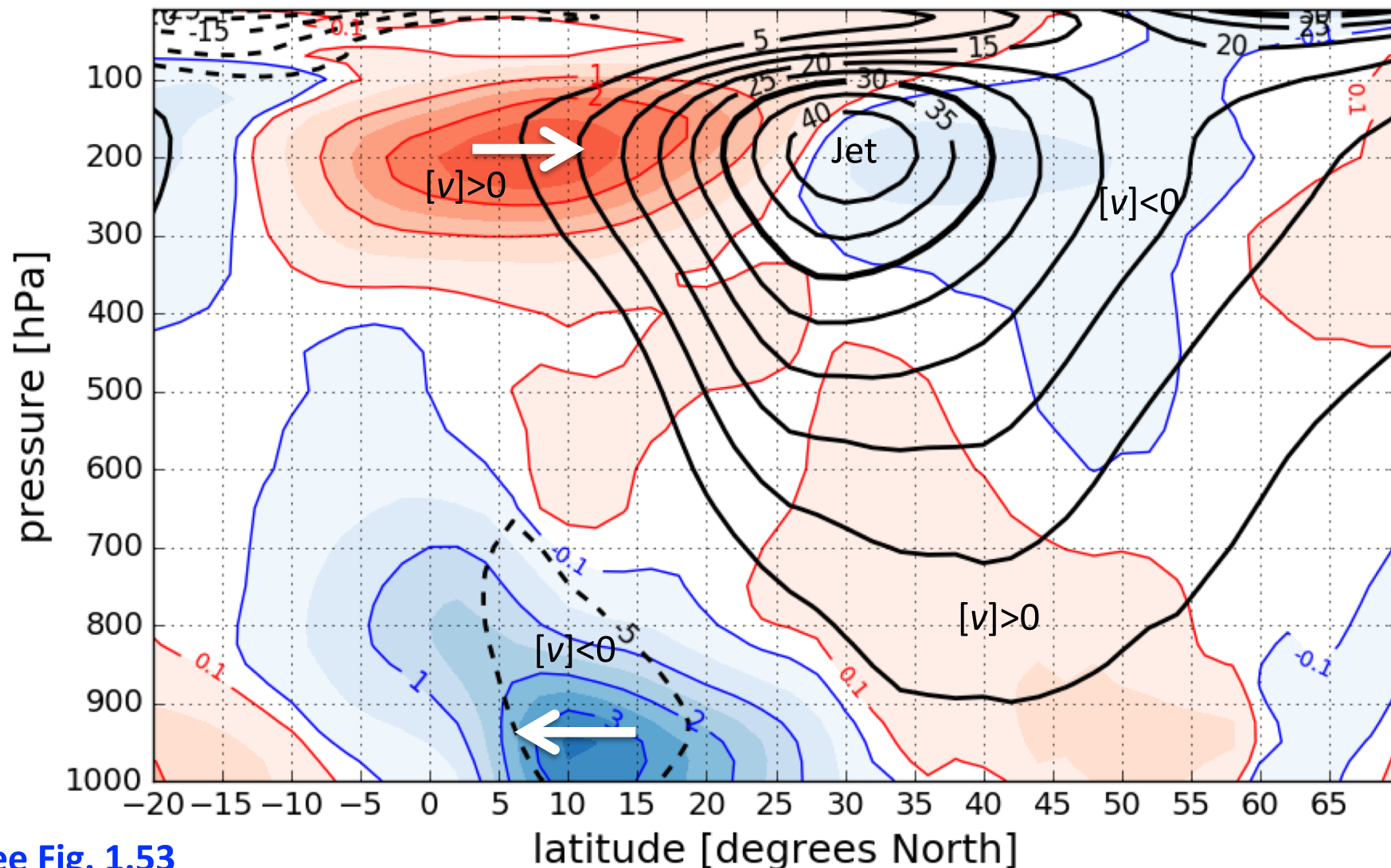
With $\psi(p=0) = 0 \longrightarrow$

$$\psi(y, p) = \int_0^p [v] dp$$

“streamfunction”

Hadley circulation: horizontal motion ($[v]$)

Average (time and zonal) horizontal velocity components $[u]$ (black contours) and $[v]$ (red: northward; blue:southward) in January 1979-2017 (ERA-Interim). Labels in units of m s^{-1} .



Hadley circulation: “Stokes” streamfunction

Slide 247:

$$\psi(y, p) = \int_0^p [v] dp$$

Slightly different definition:
(taking curvature earth into account)

$$\Psi_m(y, p) = \frac{2\pi a \cos \phi}{g} \int_0^p [v] dp$$

(“Stokes” streamfunction: units are kg s^{-1})

Ensemble average mass streamfunction in January 1979-2017. (labels in units of 10^9 kg s^{-1})

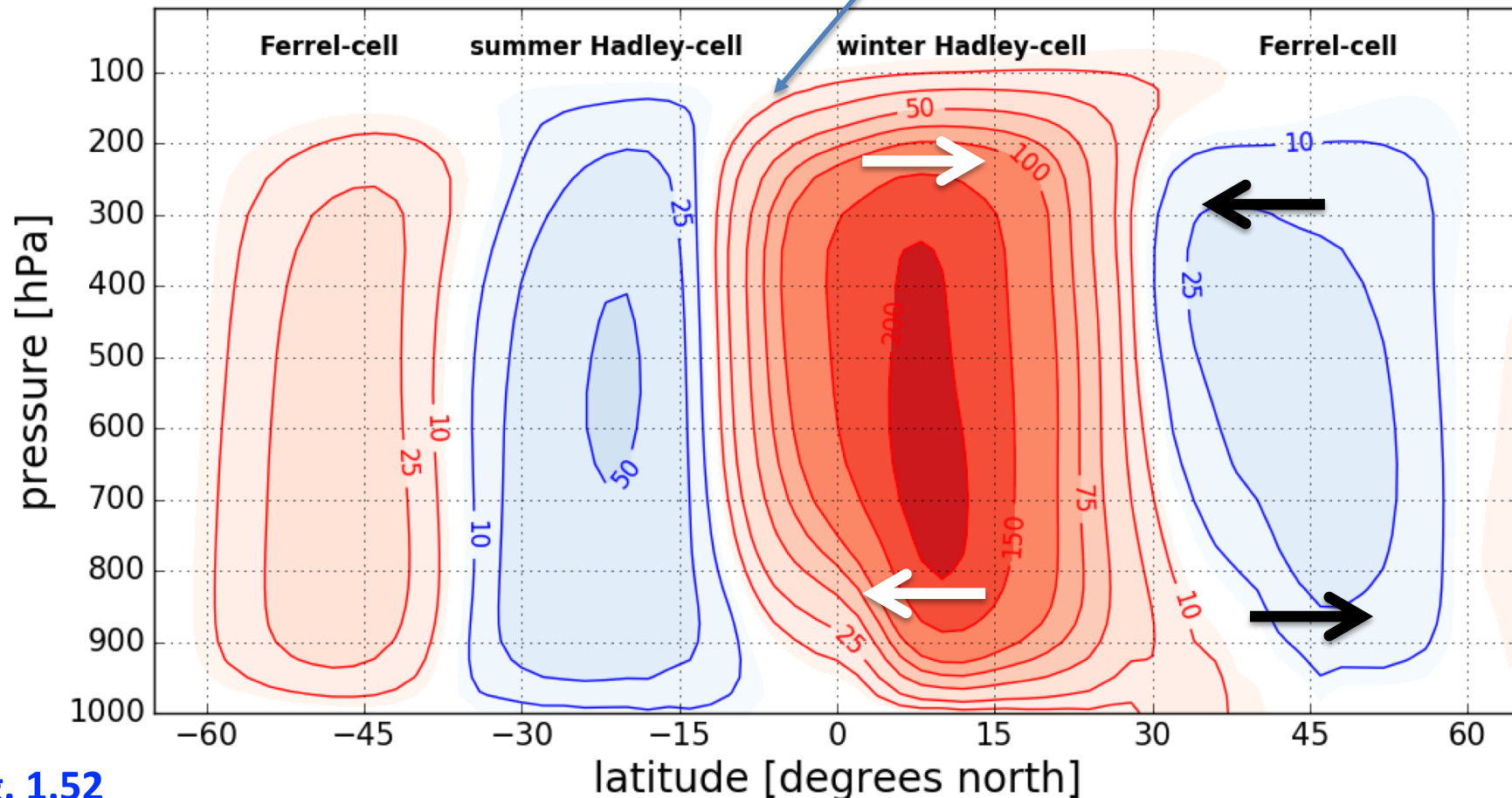
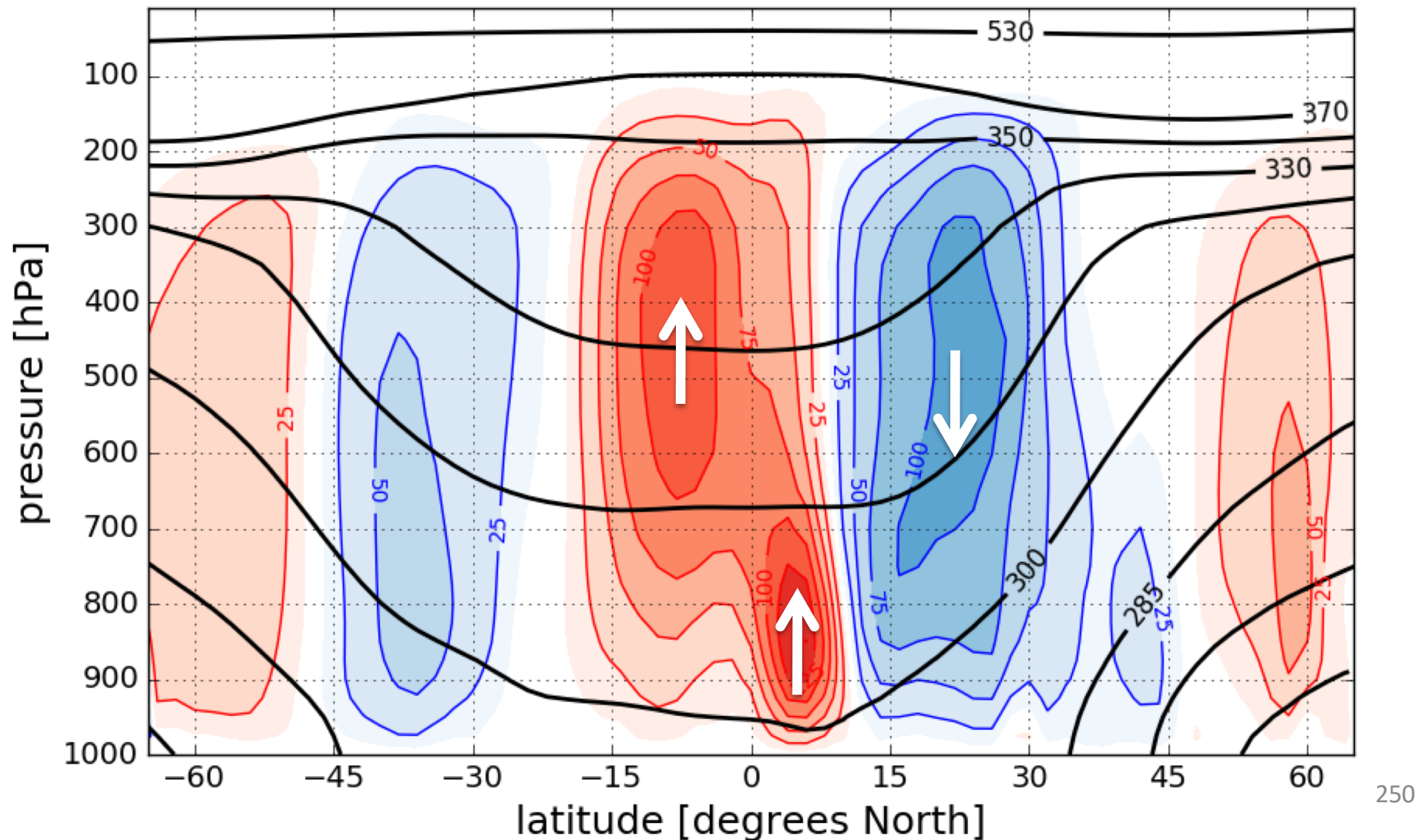


Fig. 1.52

Hadley circulation: vertical motion ($[\omega]$)

Average (time and zonal) vertical velocity $[\omega]$ (red: up; blue: down) and potential temperature (black contours) in January 1979-2017 (ERA-Interim). Labels indicate the absolute value of $[\omega]$ in units of Pa hr^{-1} and potential temperature in K.



Continuity equation

Define an element of air between two surfaces of equal θ

The mass, Δm , is equal to $\Delta m = \sigma \Delta x \Delta y \Delta \theta$

Isentropic density: $\sigma = -\frac{1}{g} \frac{\partial p}{\partial \theta}$ [kg m⁻² K⁻¹]

$$\frac{d\Delta m}{dt} = 0$$

Derivation is
analogous to
derivation in p -
coordinates
(this lecture)

$$\frac{\partial \sigma}{\partial t} = - \left(\frac{\partial \sigma u}{\partial x} \right)_{\theta} - \left(\frac{\partial \sigma v}{\partial y} \right)_{\theta} - \frac{\partial}{\partial \theta} \left(\sigma \frac{d\theta}{dt} \right) \quad (1.200)$$

$$\frac{\partial \sigma}{\partial t} = -\vec{\nabla} \cdot \vec{I}$$

adiabatic

diabatic

Mass flux:

$$\vec{I} \equiv \left(\sigma u, \sigma v, \sigma \frac{d\theta}{dt} \right) = (\sigma u, \sigma v, 0) + \left(0, 0, \sigma \frac{d\theta}{dt} \right)$$

Mass flux: isentropic and cross-isentropic

Continuity equation:
$$\frac{\partial \sigma}{\partial t} = - \left(\frac{\partial \sigma u}{\partial x} \right)_\theta - \left(\frac{\partial \sigma v}{\partial y} \right)_\theta - \frac{\partial}{\partial \theta} \left(\sigma \frac{d\theta}{dt} \right) \quad (1.200) \quad \sigma = -\frac{1}{g} \frac{\partial p}{\partial \theta} \text{ [kg m}^{-2}\text{K}^{-1}\text{]}$$

Same equation:
$$\frac{\partial \sigma}{\partial t} = -\vec{\nabla} \cdot \vec{I}$$

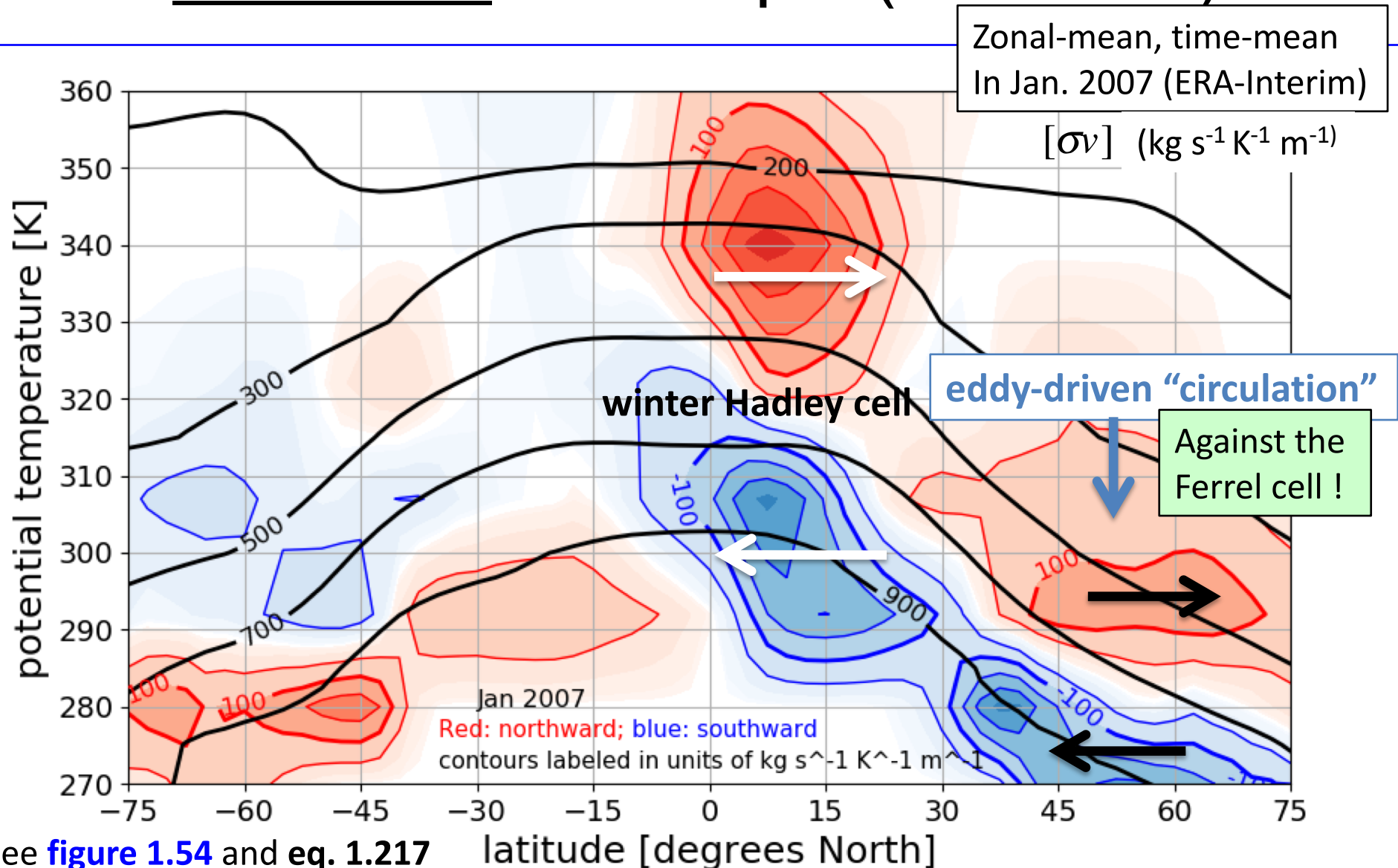
Mass flux:
$$\vec{I} \equiv \left(\sigma u, \sigma v, \sigma \frac{d\theta}{dt} \right) = (\sigma u, \sigma v, 0) + \left(0, 0, \sigma \frac{d\theta}{dt} \right)$$

Isentropic mass flux

Cross-isentropic mass flux

Next slide: meridional isentropic mass flux

Mass flux: isentropic (adiabatic)



See [figure 1.54](#) and [eq. 1.217](#) latitude [degrees North]

Time mean, zonal mean cross-isentropic mass flux in **January 2007** as a function of latitude and potential temperature. Labels are indicated in units of kg s^{-1} per Kelvin per metre in longitude. **Red: northward**; **blue: southward**. Also shown is zonal-mean pressure (black contours; labels in units of hPa).

Isentropic and cross-isentropic mass fluxes

$$\frac{\partial \sigma}{\partial t} = - \left(\frac{\partial \sigma u}{\partial x} \right)_\theta - \left(\frac{\partial \sigma v}{\partial y} \right)_\theta - \frac{\partial}{\partial \theta} \left(\sigma \frac{d\theta}{dt} \right) \quad (1.200) \quad \sigma = -\frac{1}{g} \frac{\partial p}{\partial \theta} \text{ [kg m}^{-2}\text{K}^{-1}\text{]}$$

$$\frac{\partial \sigma}{\partial t} = -\vec{\nabla} \cdot \vec{I}$$

Mass flux:

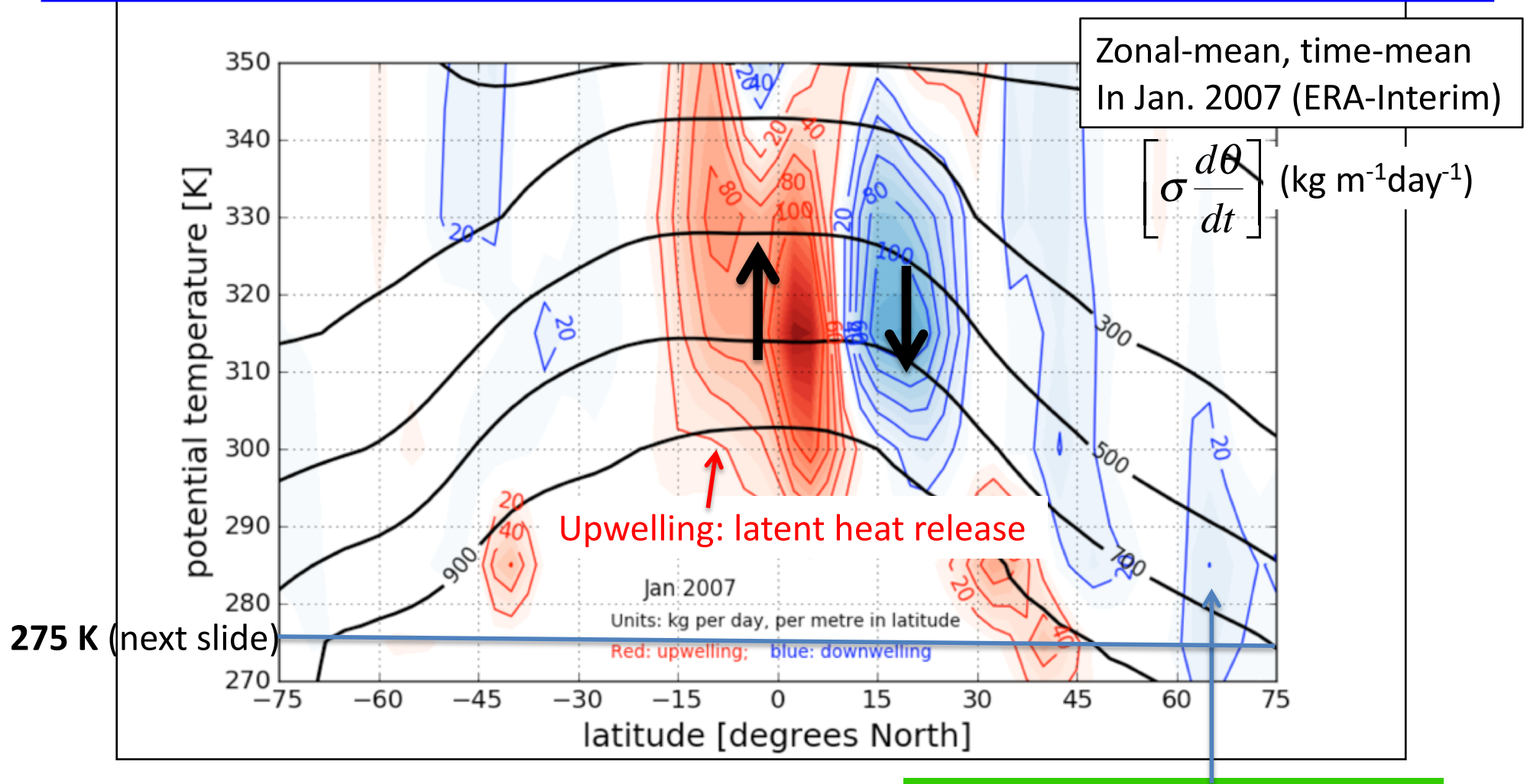
$$\vec{I} \equiv \left(\sigma u, \sigma v, \sigma \frac{d\theta}{dt} \right) = (\sigma u, \sigma v, 0) + \left(0, 0, \sigma \frac{d\theta}{dt} \right)$$

Isentropic mass flux

Cross-isentropic mass flux

Next slide: Cross-isentropic mass flux

Mass flux: cross-isentropic



What is causing the cooling?

FIGURE 1.60. Time mean, zonal mean cross-isentropic mass flux in January 2007 as a function of latitude and potential temperature. Labels are indicated in units of kg day^{-1} per metre in latitude. **Red: upwelling; blue: downwelling.** Also shown is zonal mean pressure (black contours; labels in units of hPa).

Cross-isentropic up- and downwelling at 275 K

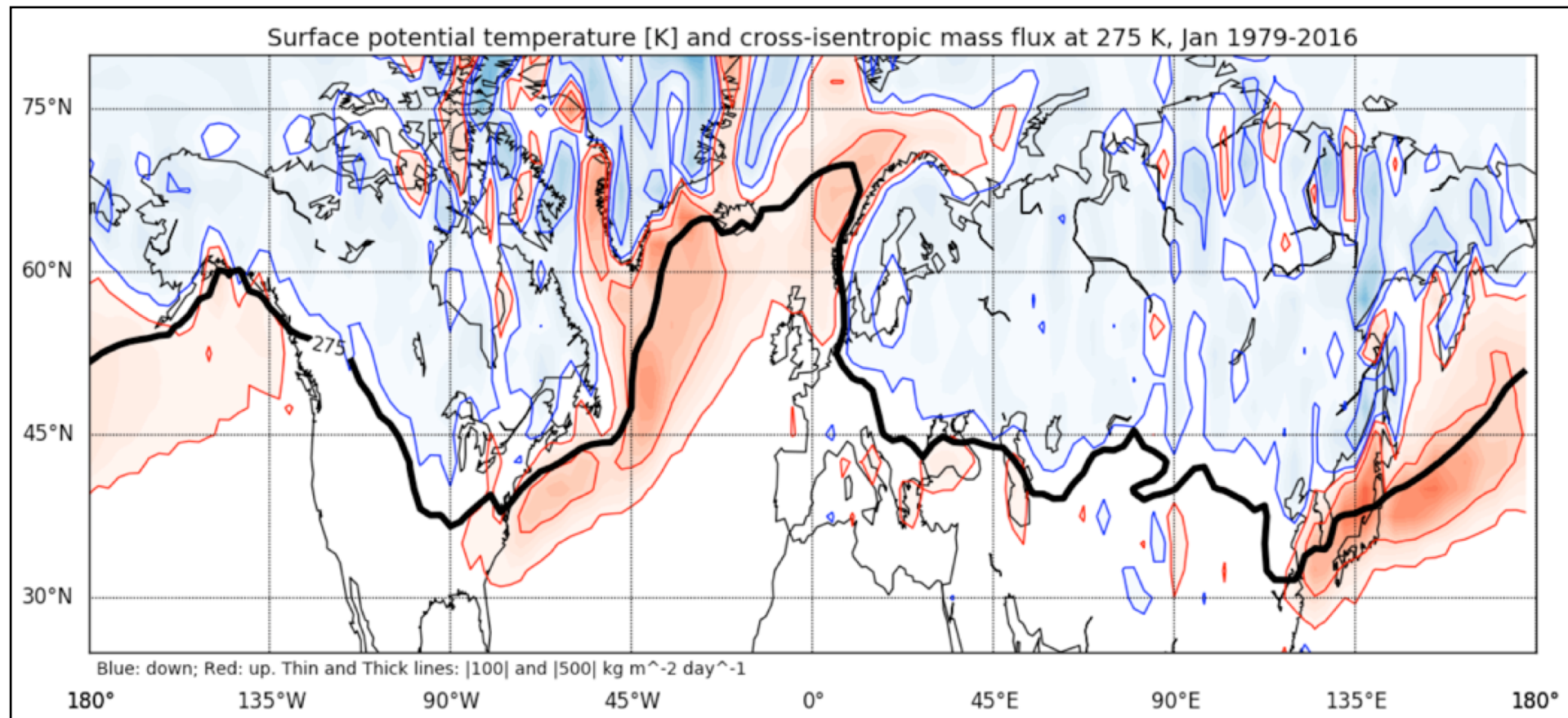
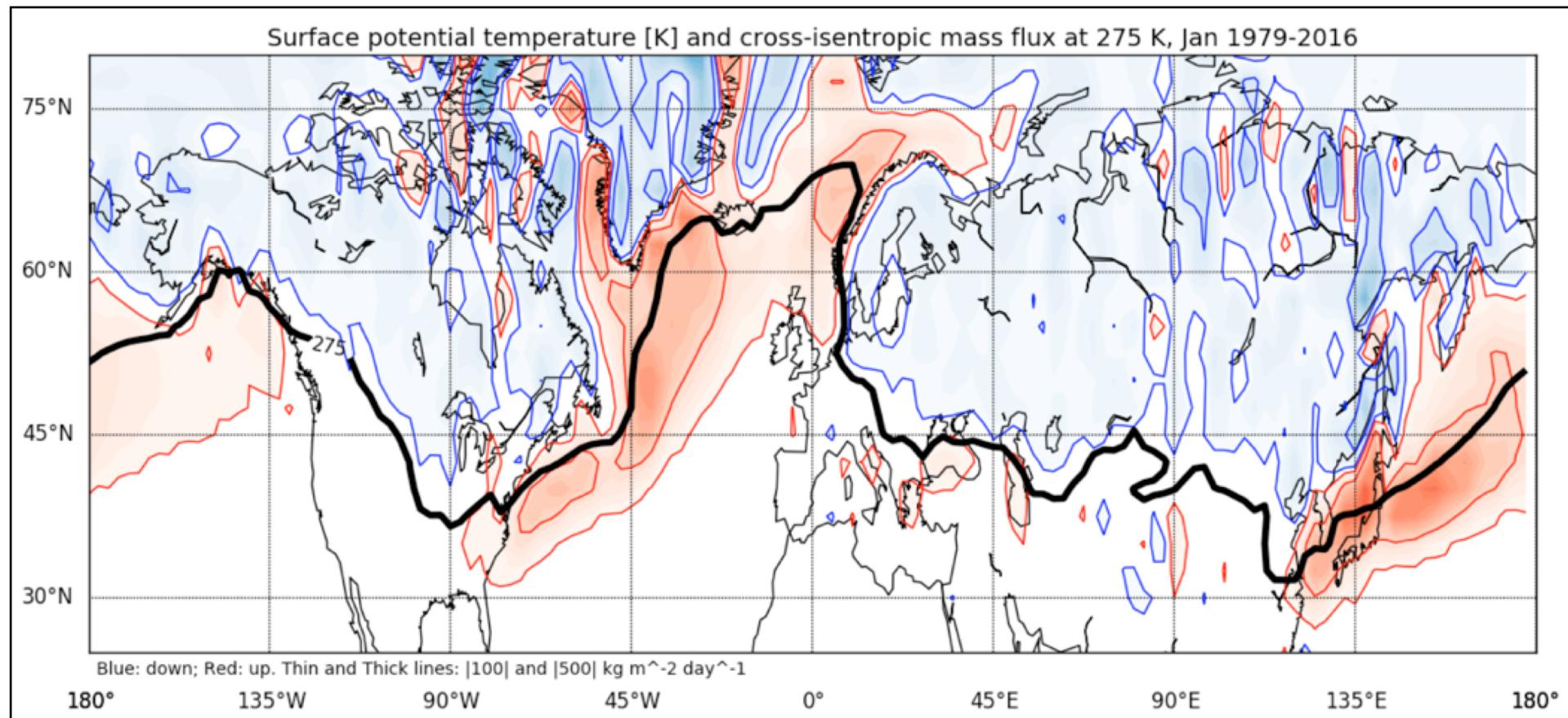


FIGURE 1.62. Ensemble January average cross-isentropic mass flux in the Northern Hemisphere at $\theta=275$ K for the years 1979-2016. Red and blue contours correspond to $|100|$ and $|500|$ kg m⁻² day⁻¹. Red shading: **upwelling** ($d\theta/dt > 0$) greater than 100 kg m⁻² day⁻¹. Blue shading **downwelling** ($d\theta/dt < 0$) greater than 100 kg m⁻² day⁻¹. The **thick black line** indicates the ensemble average position of the 275 K isentrope at the earth's surface in January (1979-2016). Based on the ERA-Interim re-analysis (<http://apps.ecmwf.int/datasets/>).

Cross-isentropic up- and downwelling at 275 K



Interpretation: in winter air enters the Lower Underworld over the cold snow-covered continents and exits the Lower Underworld over the warm oceans during outbreaks of cold continental air, due to the *remote effect* of high emission of radiation from the relatively warm sea surface below.

Next:

Dynamic Meteorology: lecture 10

16/12/2022 (Friday, 09:00), KBG 208

Sections 1.25-1.27

(Potential) vorticity equation in isentropic coordinates (section 1.25)

Vorticity equation in flux-form (impermeability theorem)

Potential vorticity mixing (section 1.26)

Zonal mean- and eddy fluxes of mass and vorticity (section 1.26)

Eddy-driven (“Brewer-Dobson”) meridional circulation of mass (section 1.26)

Tutorial after this lecture (9)

Weather forecast for Sunday 18 December 2022

Project 5 (problem 1.24)

hand in hypothesis on or before Monday 19 December 2022

Problem 1.18

Weather Forecast 18 December 2022

Range of "normal values" 18/12/1992-2021

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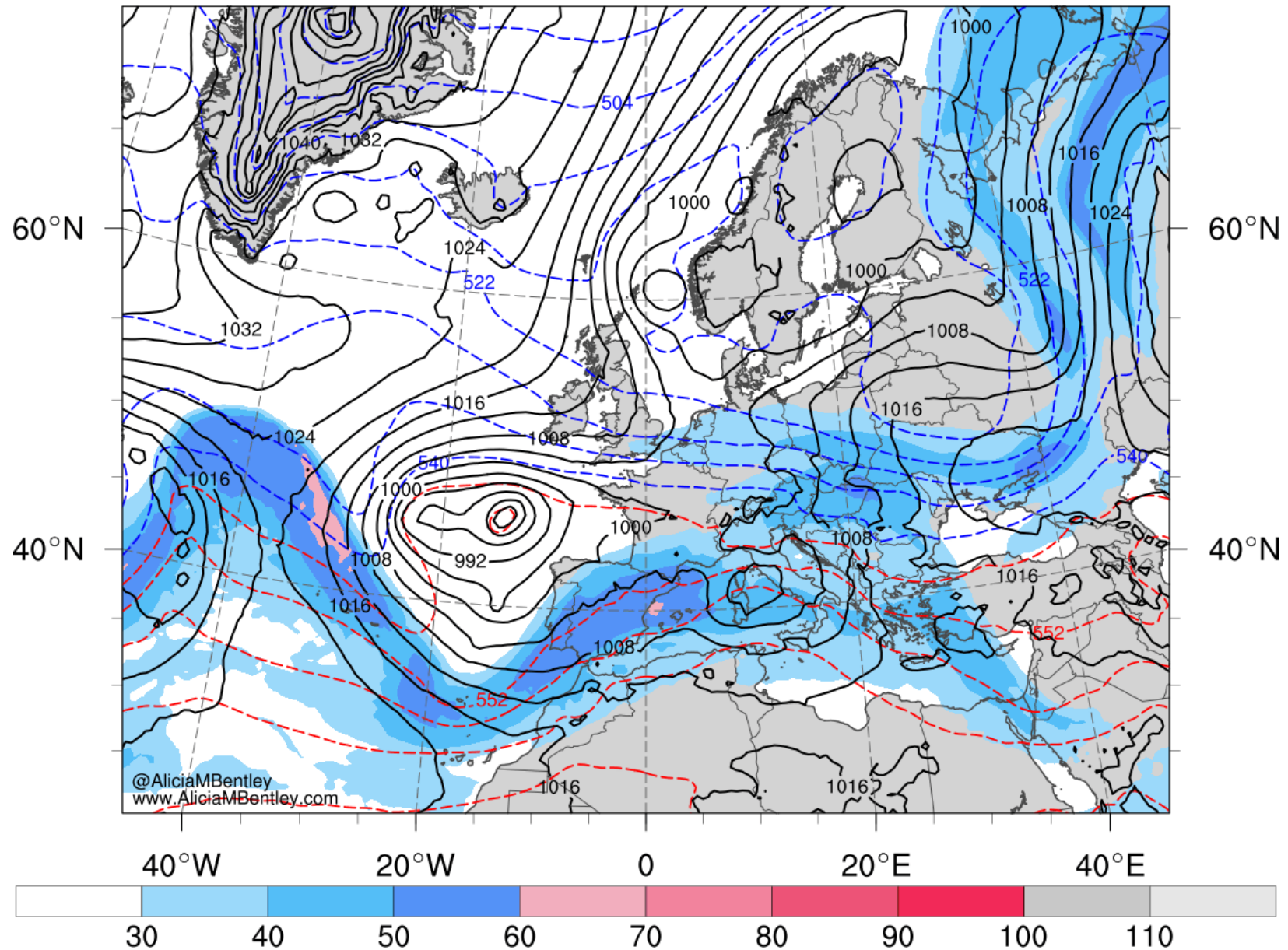
[Link to Alicia Bentley's web page](#)

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(https://www.atmos.albany.edu/student/abentley/realtime/standard.php?domain=europe&variable=mslp_jet)

MSLP (black, hPa), 1000-500-hPa thickness (red/blue, dam), 250-hPa wind speed (shaded, m/s)
Initialized: 0000 UTC 14 Dec 22 | Forecast hour: 12 | Valid: 1200 UTC 14 Dec 2022



MSLP (black, hPa), 1000-500-hPa thickness (red/blue, dam), 250-hPa wind speed (shaded, m/s)
Initialized: 0000 UTC 14 Dec 22 | Forecast hour: 108 | Valid: 1200 UTC 18 Dec 2022

