A lecture on the advection equation

Third year bachelor course, "Turbulence in Fluids" (NS376B)
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This lecture treats the advection equation, which expresses conservation of momentum of an incompressible fluid parcel. Turbulence in fluids is due to the non-linearity of the advection equation. To study non-linear effects in fluid flow we should really start by considering the full 3-dimensional Navier-Stokes equations with some relevant boundary conditions. Except for some rather special cases, this would very quickly lead us to a mathematical problem with such a complexity that it would be difficult to handle. Therefore, we instead start here with the most simple equation, which contains the advective nonlinearity in its most rudimentary form: the one-dimensional advection equation. We will discuss some particular properties of this equation, which are characteristic for advection of fluids. It is difficult, if not impossible, to relate the one-dimensional advection equation to any specific fluid flow situation, but the presence of the advection term gives the equation properties, which are related to the properties of real fluids.

1. General properties of the one-dimensional advection equation

The advection equation in one dimension states that the velocity, $u(x,t)$, of a fluid particle is conserved following the particle motion (x is distance and t is time). Without external forces, this equation is

$$\frac{du}{dt} = 0,$$

or

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 . \qquad (1)$$

The second term in this equation, the so-called "**advection term**", is non-linear.

We impose the following **boundary conditions**.

$$u(0,t) = u(L,t) = 0.$$

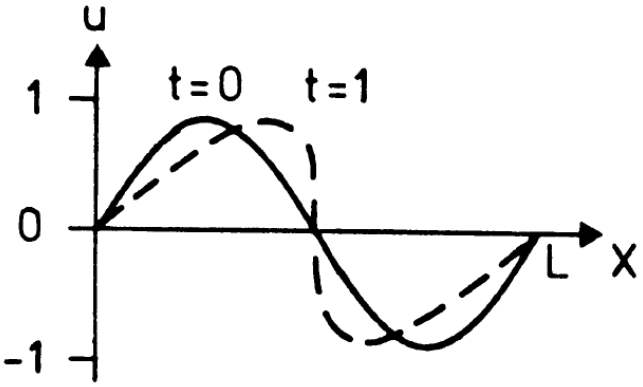

FIGURE 1a. Initial distribution of u (t=0, full line) and distribution of u just at the onset of "breaking" (t=1, dashed line) (L=2π and u_0=1). Figure modified from Platzman (1964).

An initial condition, which obeys these boundary conditions is

$$u(x,0) = u_0 \sin\left(\frac{2\pi x}{L}\right) .$$

This initial state, which is shown in **figure 1a**, may be thought of as a wave disturbance in a fluid flow.

We now investigate how the advective process changes the shape of this wave. One way of doing this is to use the **method of characteristics**, i.e. to determine curves in the x-t plane, which fluid particles follow. For eq. 1 these curves are straight lines, where the slope, dx/dt=u, is given by the initial value of u.

Some typical characteristics of eq. 1 (i.e. lines of constant u in the x-t plane) are shown in **figure 1b**. After some time the characteristics from different initial points will intersect and thus we have a multi-valued velocity at certain points. Also shown in **figure 1b** is the envelope of the "cusp-region", inside which the solution is triple-valued (three characteristics through each point). This behaviour is unphysical, but we interpret it in terms of a "**wave-breaking**" in the following way. Since each value of u is propagated along x with speed u, it follows that the wave crest (u>0) is propagated forward (toward larger values of x) and the trough (u<0) is propagated backwards. So, the slope, S=$\partial u/\partial x$, of the wave profile becomes steeper where S is negative initially and flatter where S is positive initially.

This breaking process may be examined quantitatively by computation of the change of slope along a characteristic:

$$\frac{dS}{dt} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) .$$

After changing the order of differentiation in the first term on the right hand side and using (1) we obtain

$$\frac{dS}{dt} = -S^2 .$$

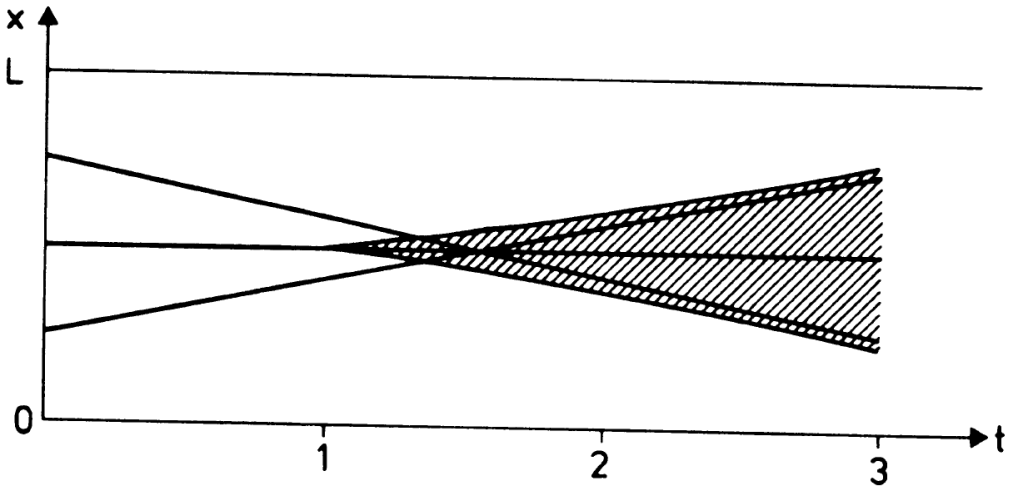

FIGURE 1b. Characteristics of eq. (1) in the t-x plane, assuming that $L=2\pi$ and $u_0=1$. Inside the hatched, cusp-shaped region the solution is multi-valued. Figure modified from Platzman (1964).

Integration of this equation, i.e.

$$\int_{S_0}^{S} -S^{-2} dS = \int_{0}^{t} dt \ ,$$

where S_0 is the value of S at t=0, yields

$$S = \frac{1}{t + S_0^{-1}} \ .$$

If S_0<0, i.e. if the slope is negative at t=0, it will become increasingly negative until at a critical time, t_c, breaking occurs. After this time the solution is no longer meaningful.

Physically this is not a satisfying model, because of the discontinuity forming at the breaking point. However, if we incorporate a simple dissipation term (a linear damping) into eq. 1 as follows,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\varepsilon u \ , \qquad (2)$$

where ε is a constant positive damping coefficient, the slope equation becomes

$$\frac{dS}{dt} = -S^2 - \varepsilon S \ . \qquad (3)$$

If we substitute y=1/S, we get

$$\frac{dy}{dt} = \varepsilon y + 1 \ ,$$

which has the solution,

$$y = C^* \exp(\varepsilon t) - \frac{1}{\varepsilon},$$

where C^* is an integration constant. Thus,

$$S = \frac{\varepsilon}{C \exp(\varepsilon t) - 1}$$

where C is a constant related to the initial slope, S_0 as follows:

$$C = \frac{\varepsilon}{S_0} + 1 .$$

Breaking occurs if S goes to infinity, or if

$$S_0 = \frac{\varepsilon}{\exp(-\varepsilon t) - 1} .$$

Since $-1<(\exp(-\varepsilon t)-1)<0$, breaking only occurs if $-\infty<S_0<-\varepsilon$. Therefore, not all initial slopes will lead to breaking. This is illustrated in **figure 2**. Three solutions for three different initial conditions are drawn. The value of the parameter, ε, determines for which initial slope breaking will occur.

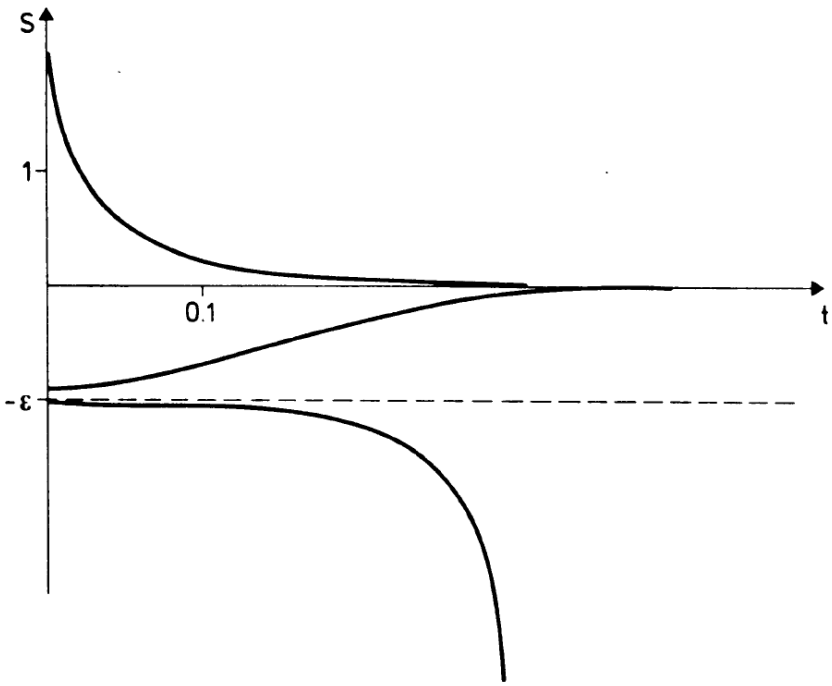

FIGURE 2. Time evolution of the slope for some different initial values in the presence of dissipation.

This behaviour is typical of the solution of many nonlinear equations or models. From eq. 2 we see that there are two terms, which govern the time evolution of u. The dissipation term on the right-hand side of (2) will try to damp the initial value towards zero, while the advective term will force an increase of the velocity gradient, which will ultimately lead to "breaking". Which of these two competing effects wins depends on the initial slope relative to the value of ε. The slope equation can be written as

$$\frac{dS}{dt} = -S(S+\varepsilon) \ . \qquad (4)$$

We see that, if $S<-\varepsilon$, dS/dt will be negative and an initially negative slope will become increasingly negative, i.e. the wave will break.

If $S<-\varepsilon$, dS/d$t>0$ for $S<0$ and dS/d$t<0$ for $S>0$. This will give a solution, which asymptotically approaches $S=0$ as t goes to infinity. We also see from (4) that both $S=0$ and $S=-\varepsilon$ are steady solutions of the slope equation. A slight perturbation around $S=0$ will eventually lead to S returning to zero. Therefore, the steady state $S=0$ is stable to small perturbations. The steady state $S=-\varepsilon$, on the other hand, is unstable to small perturbations, as any small deviation from it will either lead to the steady state, $S=0$ or $S \rightarrow -\infty$.

The method of determining the steady states and then analysing the stability of each steady state to small perturbations will be used extensively in this course. In this case we see that we have two steady states, but only one of them is stable to small perturbations. Solutions starting closing to the unstable steady state will either reach to stable steady state asymptotically, or they will be negatively infinite. Solutions starting sufficiently close to the stable steady state will always approach this steady state asymptotically. This steady state, therefore, is most interesting.

Finally, it should be mentioned that the dissipation term in (2) can be replaced by a "diffusion" term. This results in the so-called "**Burgers equation**":

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\nu\frac{\partial^2 u}{\partial x^2} \ , \qquad (5)$$

where ν represents a constant diffusion coefficient. Equation (5) can be solved analytically (see Platzman, 1964). The solution of (5) does not have any discontinuities, in contrast to the solution of (2). Therefore, different descriptions of dissipation may lead to solutions with very different qualitative behaviour. In Burger's equation diffusion most strongly damps the solutions with large gradients, so that breaking will not occur.

2. The spectral method

When examining the nonlinear properties of the advection equation in the previous section, we assumed an initial condition in the form of a sine wave. Any initial condition satisfying the boundary conditions,

$$u(0,t) = u(L,t) = 0 \qquad (6)$$

may of course be constructed by the addition of sine waves with different wavelengths in the form of Fourier series. This also applies to the solution for later times (t>0), as long as this solution is single-valued. When breaking occurs, the solution has an infinite spatial derivative at one point. Under these circumstances a Fourier series expansion of the solution is not possible, nor valid. It may, nevertheless, be instructive to express the solution of the relatively simple one-dimensional advection equation in terms of a Fourier series, and investigate the solution before breaking occurs, in particular because this method will be applied to more complicated physical problems later in this course.

Let us introduces **Fourier basis functions** as

$$S_\alpha(x) \equiv \exp\left(2\pi i l_\alpha x / L\right) . \tag{7}$$

The **wave number**, l_α, may have any integer value (positive and negative). The velocity, u, can be expressed as a sum over all possible wave numbers, l_α, as follows

$$u(x,t) \equiv \sum_\alpha U_\alpha(t) S_\alpha(x) \tag{8}$$

Here, $\alpha \equiv l_\alpha$. The symbol, α, is used to indicate the **wave number vector**, corresponding to the wave number, l_α. In a one-dimensional problem (one spatial dimension), like the present one, there is no need to distinguish between α and l_α. However, in a problem with more than one spatial dimension this distinction is necessary. If, for example, we have a problem with two spatial coordinates, x and z, the wave vector, $\alpha \equiv (l_\alpha, n_\alpha)$, where the wave number in the x-direction is l_α, the wave number in the z-direction is n_α. We will encounter such a problem when we derive the Lorenz (1963) model of thermal convection, later in this course.

Orthonormality of the basis functions, S, is expressed as

$$\frac{1}{L}\int_0^L S_\alpha S_\beta^* dx = \delta_{\alpha,\beta} , \tag{9}$$

where δ is the **Kronecker delta function** and the asterisk indicates a **complex conjugate** (see the appendix to this lecture).

The advection equation (1), which is repeated for convenience here,

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} ,$$

is transformed by first substituting the Fourier expansion (8), yielding

$$\sum_\alpha \frac{dU_\alpha}{dt} S_\alpha \equiv -\left\{\sum_\alpha U_\alpha S_\alpha\right\}\left\{\sum_\beta \left(2\pi i l_\beta / L\right) U_\beta S_\beta\right\}$$

or

$$\sum_{\alpha}\frac{dU_\alpha}{dt}S_\alpha \equiv -\sum_{\alpha,\beta}\left(2\pi i l_\beta / L\right)U_\alpha U_\beta S_\alpha S_\beta \qquad (10)$$

where $\sum_{\alpha,\beta}$ is a double sum over all wave vectors, α and β, or, in other words, over all integer combinations (positive and negative) of l_α and l_β.

We may now separate this equation into a set of nonlinearly coupled ordinary differential equations by making use of the orthogonal properties of the basis functions, as follows.

Multiply the equation by S_γ^* and integrate the resulting equation (each term in the equation) over the distance, $0 \le x \le L$. This yields

$$\frac{1}{L}\int_0^L\left\{\sum_{\alpha}\frac{dU_\alpha}{dt}S_\alpha S_\gamma^*\right\}dx \equiv -\frac{1}{L}\int_0^L\left\{\sum_{\alpha,\beta}\left(2\pi i l_\beta / L\right)U_\alpha U_\beta S_\alpha S_\beta S_\gamma^*\right\}dx \ .$$

The integral on the left hand side is non-zero only if $\alpha = \gamma$. The integral on the right hand side is non-zero only if $\alpha + \beta = \gamma$ (because $S_\alpha S_\beta = S_{\alpha+\beta}$). We are, in fact, projecting eq. (10) on one basis function, S_γ.

This procedure yields a non-linear first-order ordinary equation, describing the time rate of change of the amplitude or Fourier coefficient, U_γ, as follows.

$$\frac{dU_\gamma}{dt} = -\sum_{\alpha,\beta}\left(2\pi i l_\beta / L\right)U_\alpha U_\beta \delta_{\gamma,\alpha+\beta} \ . \qquad (11)$$

The Fourier-coefficient, U_γ, is expressed as a sum of a real part (superscript, R) and an imaginary part (superscript, I) as follows:

$$U_\gamma = U_\gamma^R + iU_\gamma^I . \qquad (12)$$

The velocity, u, is a **real physical variable**, which means that (**exercise 1**)

$$U_\gamma = U_{-\gamma}^* \ , \qquad (13a)$$

or

$$U_\gamma^R = U_{-\gamma}^R ;\ U_\gamma^I = -U_{-\gamma}^I \qquad (13b)$$

(see the appendix to this lecture). The **boundary conditions**, u=0 at x=0 and at x=L (eq. 6), implying a sine-dependence, are fulfilled if

$$U_\gamma^R = -U_{-\gamma}^R ;\ U_\gamma^I = -U_{-\gamma}^I . \qquad (14)$$

Conditions (13b) and (14) imply that

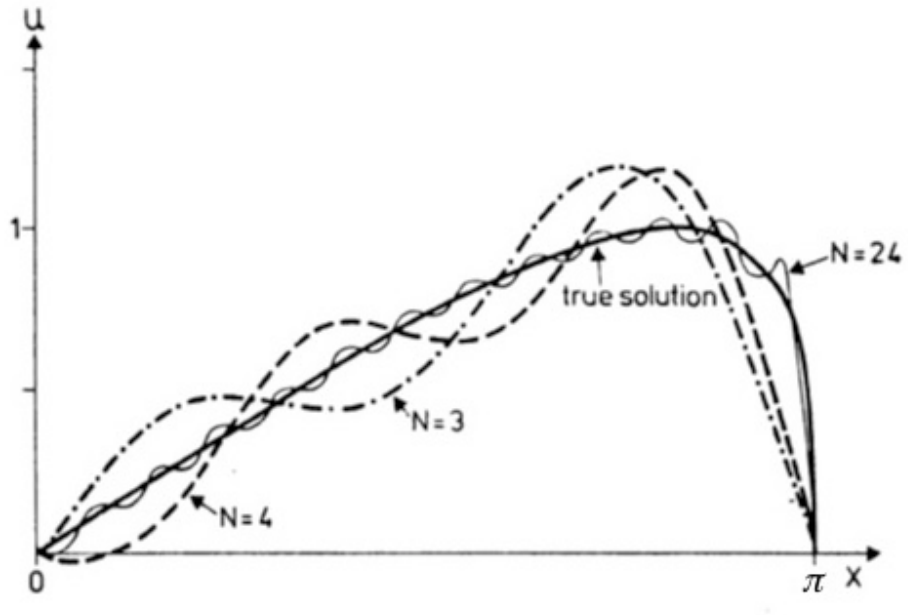

Figure 3. A wave at the onset of breaking (full line) and its approximate form given by a limited number of spectral components, N (L=2π).

$$U_\gamma^R = 0 \text{ and } U_0^I = 0 \,, \tag{15}$$

so that, according to the definition (12)

$$U_\gamma = iU_\gamma^I \tag{16}$$

and eq. 11 becomes

$$\frac{dU_\gamma^I}{dt} = \sum_{\alpha,\beta} \left(2\pi l_\beta / L\right) U_\alpha^I U_\beta^I \delta_{\gamma,\alpha+\beta} \,, \tag{17a}$$

or

$$\frac{dU_\gamma^I}{dt} = \sum_{\alpha,\beta} I_{\gamma\alpha\beta} U_\alpha^I U_\beta^I, \tag{17b}$$

where the **"interaction coefficient"**,

$$I_{\gamma\alpha\beta} \equiv \left(2\pi l_\beta / L\right) \delta_{\gamma,\alpha+\beta} \tag{18}$$

describes the interaction between different spectral components, i.e. the energy transfer from one spectral component to the other. In the example shown in **figure 1a**, energy is transferred from wave number 1 to higher wave numbers, or smaller scales,

as time passes. When the wave reaches the point of breaking, energy will have spread over the full spectrum of wave numbers. This is another way of saying that the wave, just before the onset of breaking, can only be described accurately if we take many Fourier components into account, as is illustrated in **figure 3**. However, the problem is tractable only if we take a finite number of wave numbers into account, i.e. if we truncate the Fourier series (8). The next section describes a severely truncated approximation to the advection equation. This approximation is called a "**low-order model**".

3. A low order model

To investigate schematically how the nonlinear energy transfer between spectral components takes place, we analyse a very severely truncated spectral representation of the advection equation, which takes into account only wave numbers, l=1 and l=2. We should also take into account wave numbers, l=-1 and l=-2. In view of eq. 18 and the **selection rule**,

$$\gamma = \alpha + \beta \ , \qquad (20)$$

the only possible interactions are listed in **Table 1**.

γ	α	β
1	*2*	*-1*
1	*-1*	*2*
2	*1*	*1*

Table 1. Wave number combinations giving non-zero contributions to the interaction coefficient (eq. 18) in the lowest order nonlinear truncation of the advection equation, which takes only wave numbers 1 and 2 inside the truncation.

Interactions with wave numbers outside the truncation are neglected. The low order model, derived form eq. 17a, consists of the following two coupled nonlinear equations.

$$\frac{dX}{dt} = -CXY \ ; \qquad (20a)$$

$$\frac{dY}{dt} = CX^2 \ , \qquad (20b)$$

with

$$X \equiv U_1^I ; \ Y \equiv U_2^I \text{ and } C \equiv \frac{2\pi}{L} \ . \qquad (21)$$

If we multiply eq. 20a by X and multiply eq. 20b by Y and subsequently add the resulting equations we obtain

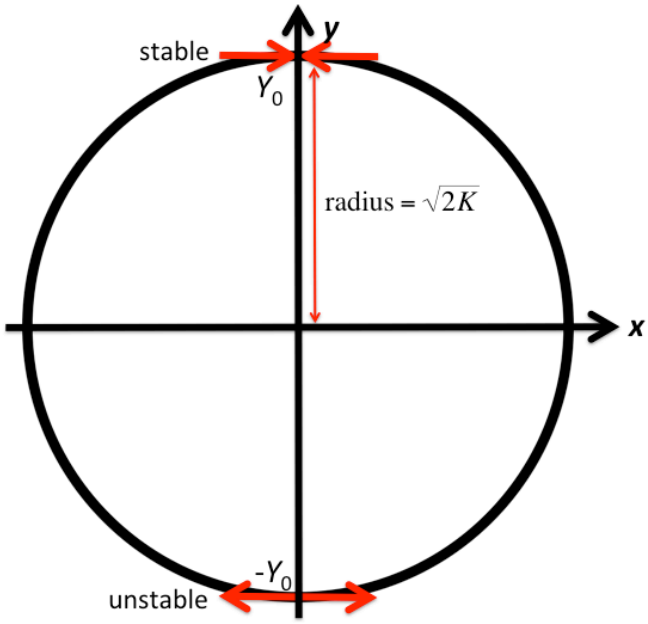

Figure 4. Steady states (red dots), at $(0, Y_0)$ and at $(0, -Y_0)$, and their stability properties, indicated by arrows, of the two-component conservative low order model (eqs. 20a,b).

$$X\frac{dX}{dt}+Y\frac{dY}{dt}=0,$$

or alternatively,

$$\frac{d}{dt}\left[\frac{1}{2}\left(X^2+Y^2\right)\right]\equiv\frac{dK}{dt}=0\ . \tag{22}$$

This shows that total kinetic energy, K, is conserved. Kinetic energy is redistributed between the two components by the nonlinear interactions. In this model the **kinetic energy spectrum**, K_γ, is restricted to two wave numbers. Eq. 22 indicates that the solution of the low-order model (20a,b) lies on a circle in **state-space** or **phase-space** (**figure 4**). The radius of this circle is equal to $2K$. Since K is conserved, K is determined by the initial condition at t=0, i.e.

$$K=\frac{1}{2}\left(X(0)^2+Y(0)^2\right). \tag{23}$$

The **steady state solution** are defined as values of X and Y for which

$$\frac{dX}{dt}=0 \text{ and } \frac{dY}{dt}=0 \tag{24}$$

We thus find the system

$$X_0=0 \text{ and } X_0Y_0=0. \tag{24}$$

The subscript '0' stands for steady state. Together with eq. 23, we find that the steady state points in phase-space are $(0,Y_0)$ and $(0,-Y_0)$. These points are encircled in **figure 4**. In both steady state solutions all kinetic energy is in the smallest scale of motion.

Multiple steady states are characteristic for nonlinear systems. The nonlinear system (20a,b) will ultimately settle into one of the two steady states only if the steady state is stable to small perturbations. The stability of a steady state can be examined by perturbing the steady state as follows

$$X = (X_0 + \delta X), \tag{25a}$$
$$Y = (Y_0 + \delta Y). \tag{25a}$$

Inserting these definitions into the governing equations (20a,b) yields

$$\frac{d}{dt}(X_0 + \delta X) = -C(X_0 + \delta X)Y(Y_0 + \delta Y), \tag{26a}$$
$$\frac{d}{dt}(Y_0 + \delta Y) = C(X_0 + \delta X)^2. \tag{26b}$$

Taking into account that (X_0,Y_0) represents the steady state,

$$\frac{d\delta X}{dt} = -C(X_0\delta Y + Y_0\delta X + \delta X\delta Y) \approx -C(X_0\delta Y + Y_0\delta X) = -CY_0\delta X. \tag{27a}$$
$$\frac{d\delta Y}{dt} = C(2X_0\delta X + (\delta X)^2) \approx 2CX_0\delta X = 0. \tag{27b}$$

We have neglected products of (small) perturbations. Since $C>0$, linear stability of the steady state is determined by the sign of Y_0. The steady state is stable if $Y_0>0$ and unstable if $Y_0<0$ (**figure 4**). However, beware that a perturbation in the Y-direction (i.e. δY) of the steady state, which lies on the Y-axis, is always neutrally stable (refer to eq. 27b). Why is this?

This lecture is intended to introduce the **spectral method** and **linear stability analysis** and also to illustrate some typical properties of non-linear systems, in particular of fluid advection, such as **multiple equilibrium states** and **transfer of energy between scales of motion**. We will return to these topics later in this course.

4. Exercises

Exercise 1.

(a) If we express the velocity, u, as a Fourier series as follows

$$u(x,t) \equiv \sum_{\gamma} U_{\gamma}(t) S_{\gamma}(x)$$

with

$$S_{\gamma}(x) \equiv \exp\left(2\pi i l_{\gamma} x / L\right)$$

and U_{γ} a complex Fourier coefficient or time-dependent amplitude, then show that u is a real physical variable if (eq. 13a)

$$U_{\gamma} = U_{-\gamma}^{*},$$

where the asterisk indicates a complex conjugate (**see appendix**).

(b) What are the values of the Fourier coefficients, U_{γ}^{R} and U_{γ}^{I}, at t=0 if

$$u(x,0) = u_0 \sin\left(\frac{2\pi x}{L}\right),$$

with $u_0 = 1$ m/s?

Exercise 2.

Derive a low-order model, based on the Fourier transform of the one-dimensional advection equation eq. 2, which is repeated here:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\varepsilon u \ , \qquad (2)$$

taking the wave numbers, 1 and 2 into account. The interactions, satisfying the selection rule (eq.19), are listed in **table 1**. The term on the right hand side of (2) is damping term if ε>0. Repeat the analysis of section (3) for this low order model. This analysis consists of the following steps.

(a) Is kinetic energy conserved?

(b) Determine the steady states.

(c) Determine the linear stability of the steady states.

Exercise 3.

Derive a low-order model, based on eq. 17, which is the Fourier transform of the one-dimensional advection equation eq. 1, taking wave numbers, 1, 2 and 3 into account (inside the truncation of the Fourier series approximation of u). The interactions, satisfying the selection rule (eq.19), are listed in **table 2**. Show that kinetic energy is conserved by this three-component low order model

γ	α	β
1	*2*	*-1*
1	*-1*	*2*
1	*3*	*-2*
1	*-2*	*3*
2	*1*	*1*
2	*3*	*-1*
2	*-1*	*3*
3	*1*	*2*
3	*2*	*1*

Table 2. Wave number combinations giving non-zero contributions to the interaction coefficient (eq. 18) in a low order nonlinear truncation of the advection equation, which takes wave numbers 1, 2 and 3 inside the truncation.

Exercise 4.

The two-component low-order model of section 3 is not realistic. More wave numbers are needed inside the truncation of the Fourier series of u to obtain a better approximation to the correct solution before breaking occurs. Write a program (e.g. in Python) that integrates the Fourier transform of eq. 2 in time, taking a prescribed number of wave numbers inside the truncation of the Fourier series approximation of u. The time-derivative can be approximated numerically with a simple **Euler forward time stepping scheme** (provided that the time step is small enough) or, better, with a **Runge-Kutta scheme** [1]. Integrate the equations until just before the point of "wave-breaking" (if u_0=1 m/s and $L=2\pi$ m, breaking occurs when $t>1$ s). How many wave numbers do you need inside the truncation to reproduce the solution as illustrated in **figure 1**? Check kinetic energy conservation. Plot and discuss the evolution in time of the kinetic energy spectrum (the kinetic energy as a function wave number and time).

[1] https://en.wikipedia.org/wiki/Runge–Kutta_methods

5. References

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6. Appendix

The **Kronecker delta function** is defined as

$\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$;
$\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$.

The **complex conjugate** of

$$S_\gamma \equiv \exp\left(il_\gamma x\right) = \cos\left(il_\gamma x\right) + i\sin\left(il_\gamma x\right)$$

is

$$S_\gamma^* \equiv \exp\left(-il_\gamma x\right) = \cos\left(-il_\gamma x\right) + i\sin\left(-il_\gamma x\right) = \cos\left(il_\gamma x\right) - i\sin\left(il_\gamma x\right) .$$