Introduction

Toy-Climate-Models

Model 1, of ENSO, consisting of three time-dependent coupled nonlinear ordinary differential equations, is a simple example of a grid-point model having chaotic solutions.
Models 2, illustrating the tight control of climate by the biosphere, consists of two time-dependent coupled nonlinear ordinary differential equations plus seven diagnostic relations.
Models 3 is an energy balance model which illustrates the ice-albedo feedback. It is a very simple gridpoint model with very interesting non-linear behaviour

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More information about the toy models: http://www.staff.science.uu.nl/~delde102/SOAC.htm

Introduction

Toy-Climate-Models

1. The very simple model of ENSO (El Niño-Southern Oscillation) conveys the idea that ENSO is a chaotic oscillation.
2. The model of Climate biosphere interaction is known under the name, “Daisy World” and is an illustration of the Gaia-hypothesis.
3. This model called the “Budyko-Sellers model”. It illustrates hysteresis in the climate system and offering an explanation for the existence of ice-ages

The project consists of **writing a program that solves/integrates the model equations in time (to a steady state), and subsequently performing some interesting numerical experiments and an interesting analysis of the solutions which should come with a physical interpretation.**

Toy model 1

Model 1: A very simple model of El Niño

Based on:

Vallis, G.K, 1986: El Niño: A Chaotic Dynamical System? **Science**, 11 April 1986, 243-245.

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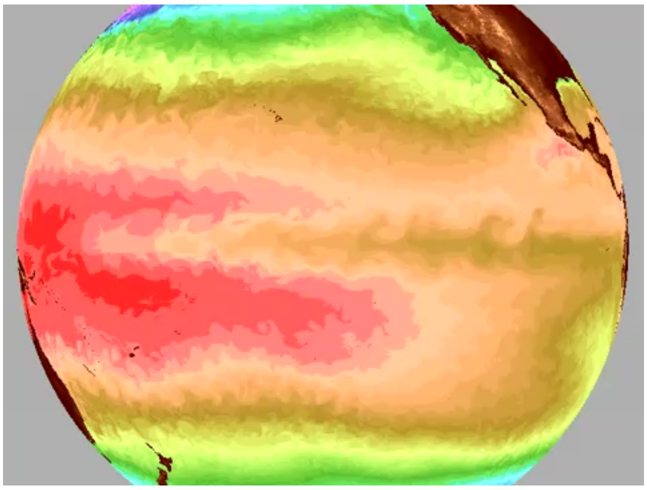
Toy model 1

El Niño: chaotic oscillation in Pacific Ocean temperature

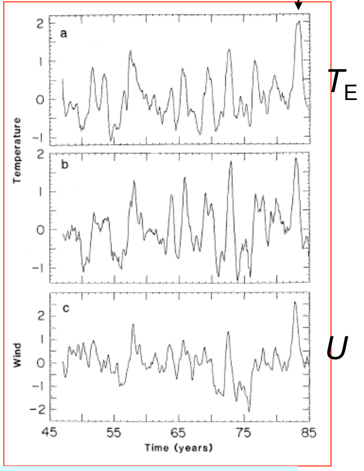


Toy model 1

El Niño-Southern Oscillation (ENSO)

El Niño-event: $T_E > T_W$, U>0

Fig. 1. (a and b) Time series for 1947 to 1985, of average sea-surface temperature anomaly (°C, annual cycle removed) for (a) the eastern equatorial Pacific (10°N to 10°S, 100°W to 80°W) and (b) the central equatorial Pacific (170°W to 120°W, 5°S to 5°N). (c) Time series of eastward surface wind anomaly (m sec⁻¹) in the central equatorial Pacific. Winds are combined averages over regions 150°E to 140°W, 0° to 10°N and 160°E to 160°W, 0° to 10°S. All series are of 5-month running means. The overall similarity of the fields is suggestive of large-scale controlling processes, and thus indicates that a low-order model can capture much of the essential dynamics. [Data courtesy of T. Barnett]



Taken from Vallis (1986)

A quasi-periodic phenomenon

Toy model 1

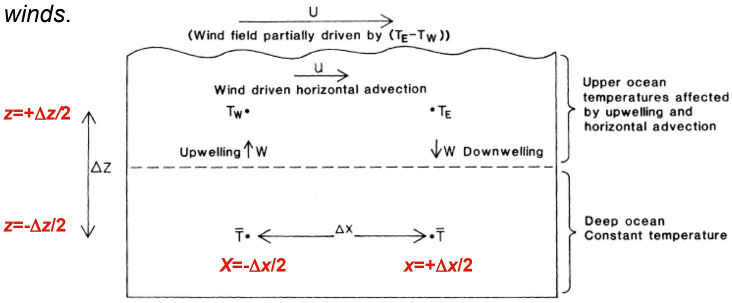
A conceptual model of El Niño

The **equatorial pacific ocean** is represented as a "**box**". The zonal current U at (x,z)= $(0,+1/2)$ in the upper layer of the ocean is governed by

$$\frac{dU}{dt} = B\frac{(T_E - T_W)}{\Delta x} - C(U - U^*)$$

First term r.h.s. represents pressure gradient term

B and C *are* constants and the steady state in the absence of an east-west temperature gradient, $U=U^*<0$ represents the effect of stress due to *steady tropical westward trade winds*.



If the temperature in the west in the upper layer of the ocean is greater than the temperature in the east in the upper layer of the ocean, that is, $T_W > T_E$, there is a **westward** surface wind, because of the convective tendency of air to rise (sink) over warm (cool) regions.

Fig. 2. Schema of model. Below the "thermocline" the temperature is constant, $\overline{T}$. The wind, driven in part by the temperature gradient $(T_e - T_w)$, generates a current which, by mass continuity, generates upwelling or downwelling as well as horizontal advection. Now if $\partial T/\partial t = -(uT)_x - (wT)_z$ and normal velocities are zero at domain boundaries, then $dT_w/dt = [0 - u(T_w + T_e)/2]/\Delta x + [w(\overline{T} + T_w)/2 - 0]/\Delta z = u(\overline{T} - T_e)/2\Delta x$, since $w/\Delta z = u/\Delta x$ by mass conservation. Addition of damping terms gives Eq. 2. Similarly for T_e.

Toy model 1

A low-order model of El Niño

Neglecting variations in the south-north direction and assuming incompressibility, we may approximate the continuity equation as follows.

$$U\Delta z = W\Delta x$$

Likewise, neglecting variations in the south-north direction, assuming incompressibility and adiabatic conditions, we may write the temperature equation in flux form as follows.

$$\frac{\partial T}{\partial t} = -\frac{\partial(uT)}{\partial x} - \frac{\partial(wT)}{\partial z}$$

Applying this equation to the gridpoints $(x,z)=(\pm\Delta x/2, \Delta z/2)$, using the boundary conditions, the continuity equation (see above) as well as linear interpolation of the temperature in both directions, we obtain the following two equations (see also the caption of the figure on the previous slide)

$$\frac{dT_W}{dt} = \frac{U}{2\Delta x}(\overline{T} - T_E) \qquad \frac{dT_E}{dt} = \frac{U}{2\Delta x}(T_W - \overline{T})$$

Toy model 1

A low-order model of El Niño

$$\frac{dT_W}{dt} = \frac{U}{2\Delta x}(\overline{T} - T_E) \qquad \frac{dT_E}{dt} = \frac{U}{2\Delta x}(T_W - \overline{T})$$

According to these two equations, the steady state solution in the absence of motion is undetermined. We, however, expect that the temperature of the ocean in rest will adjust to an externally (radiatively) determined temperature. In order to incorporate this effect we add a term to both equations such that they become:

$$\frac{dT_W}{dt} = \frac{U}{2\Delta x}(\overline{T} - T_E) - A(T_W - T^*) \qquad \frac{dT_E}{dt} = \frac{U}{2\Delta x}(T_W - \overline{T}) - A(T_E - T^*)$$

Where A is a constant and T^* is the temperature to which the ocean relaxes in the absence of motion. Without loss of generality we may measure the temperature with respect to the temperature of the deep ocean, i.e.

$$\overline{T} = 0$$

Toy model 1

The low-order model of El Niño

Thus, the three equations constituting the low order model of El Niño are

$$\frac{dT_W}{dt} = -\frac{UT_E}{2\Delta x} - A(T_W - T^*)$$
$$\frac{dT_E}{dt} = \frac{UT_W}{2\Delta x} - A(T_E - T^*)$$
$$\frac{dU}{dt} = B\frac{(T_E - T_W)}{\Delta x} - C(U - U^*)$$

This is a **non-linear dissipative dynamical system**; its solutions are bounded.

The first two equations are non-linear. For the case $U^*=0$ mathematical analysis is possible which gives some qualitative insight into the behaviour of the model (see the Vallis, 1986, p. 244). For B'<AC/T* (where $B'=B/(2\Delta x^2)$) only 1 steady state solution exists: U=0, $T_E=T_W=T^*$. If B increases, the system will bifurcate when B'=AC/T*. The existing solution becomes linearly unstable and two new stable solutions appear. All solutions become unstable when $B'>(4A+C)C^2/T^*(C-2A)$.

Toy model 1

Time integration

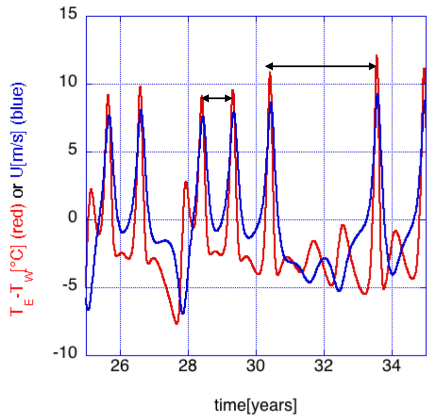
Using the Runge Kutta scheme we integrate this system of 3 non-linear ordinary differential equations numerically.

The parameter values chosen correspond to a frictional time scale C of 1/4 month^{-1}, a temperature decay time scale (A) of 1 year^{-1}, a mean current, U*, of -0.45 m/s, a basin size of 7500 km, $B=2/\Delta x$ and T* of 12°C. The value of B is approximately equivalent to supposing a current of 0.35 m/s can be generated in 1 month by a temperature difference of 2°C.

Toy model 1

Time integration: results

There is about 1 El Niño-event every 2 years, but the **period is not constant**. In between "events" T_E-T_W is about -2.5°C and U is about -1 m/s. This is typical of a **"chaotic"** system.



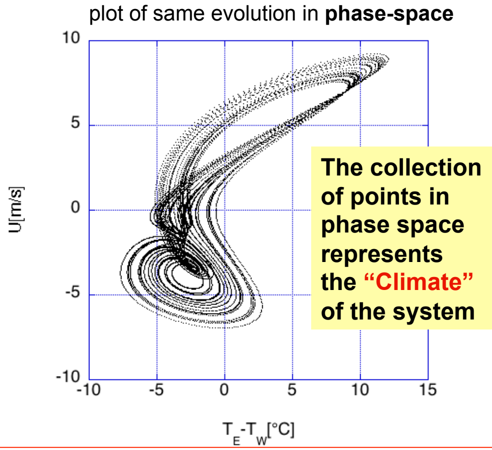
Investigate the behaviour of the model if there is a seasonal cycle in the U. Should there be a seasonal cycle in U*. If so, what period does it have?*

Toy model 1

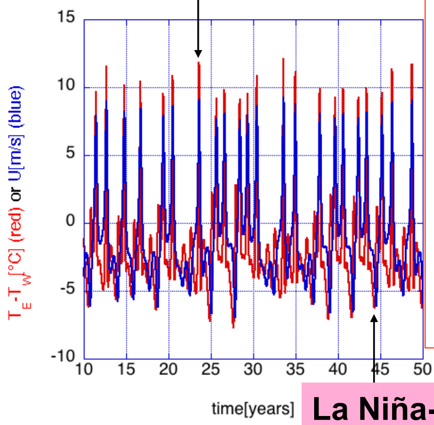
Time integration: results

"strange attractor"

plot of same evolution in **phase-space**



El Niño-event: T_E>T_W, U>0



La Niña-event: T_E<<T_W, U<<0

Toy model 1

Questions

Plot the probability density function (pdf) of the chaotic solution of the model (say: of T_E-T_W) and compare with the pdf of the Southern Oscillation Index

Apply Principal Component Analysis to the chaotic solution of the model

Compare this model with present day theoretical ideas about how El Niño works. What physical effects does it neglect?

Can you think of improvements to this model?

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References

Articles

Philander, S.G., 2003: Is El Niño Sporadic or Cyclic? **Ann.Rev.Earth&Planet.Sci**., 31, 579-594.

Dijkstra, H.A. and G.Burgers, 2005:Fluid Dynamics of El Niño variability. **Ann.Rev.Fluid Mech**., 34, 531-558.

Vallis, G.K, 1986: El Niño: A Chaotic Dynamical System? **Science**, 11 April 1986, 243-245.

Vallis, G.K, 1988: Conceptual Models of El Niño and the Southern Oscillation. **J.Geophys.Res, 93**, 13979-13991.

Websites

http://www.elnino.noaa.gov/
http://en.wikipedia.org/wiki/El_Niño–Southern_Oscillation
http://www.pmel.noaa.gov/tao/elnino/nino-home.html

Time integration

Using the Runge Kutta scheme we integrate this system of 3 non-linear ordinary differential equations numerically. The Pascal program is given on the right of this slide.

The parameter values chosen correspond to a frictional time scale C of 1/4 month^{-1}, a temperature decay time scale (A) of 1 year^{-1}, a mean current, U*, of -0.45 m/s, a basin size of 7500 km, B=2/Δx and T* of 12°C. The value of B is approximately equivalent to supposing a current of 0.35 m/s can be generated in 1 month by a temperature difference of 2°C.

```
program ElNino;

CONST
        dimi =3;{independent varables}
        dimj =4;{substeps in time-defferencing scheme}
        Dx=7500000.0;
        Tstar=12.0;
        Ustar=-0.45;
        localDir = '/Users/Shared/ResultsELNino/';

    type
        vector1 = array[1..dimi] of double;
        matrix1 = array[1..dimi,0..dimj] of double;

var
X:matrix1;
F:vector1;
time,dt,A,B,C,D:real;
nt,dimt,i,j:integer;
f1:text;

begin
{initilazation}
rewrite(f1,concat(localDir,'TimeEvolutionELNino'));
writeln(f1,'  time[years]    TE-TW[°C]     U[m/s]');
X[1,0]:=12.0;
X[2,0]:=12.0;
X[3,0]:=-0.45;
time:=0.0;
        A:=1/(365*24*3600);
        C:=0.25/(30*24*3600);
        B:=2.0/Dx;
        D:=1.0;
dt:=24*3600.0;
dimt:=20000;

for nt:=1 to dimt do begin
time:=time+dt;

for j:=1 to 4 do begin {substeps in R-K scheme}

for i:=1 to dimi do begin
if i=1 then F[i]:=-(D*X[3,j-1]*X[2,j-1]/(2*Dx))-(A*(X[1,j-1]-Tstar));
if i=2 then F[i]:=+(D*X[3,j-1]*X[1,j-1]/(2*Dx))-(A*(X[2,j-1]-Tstar));
if i=3 then F[i]:=B*(X[2,j-1]-X[1,j-1]/(Dx))-(C*(X[3,j-1]-Ustar));

if j=1 then X[i,j]:=X[i,0]+(F[i]*dt/2);
if j=2 then X[i,j]:=X[i,0]+(F[i]*dt/2);
if j=3 then X[i,j]:=X[i,0]+(F[i]*dt);
if j=4 then X[i,j]:=X[i,0]-(F[i]*dt/2);
end;{i}

end;{j}

{final step}
for i:=1 to dimi do begin
X[i,0]:=(X[i,1]+(2*X[i,2])+X[i,3]-X[i,4])/3;
end;{i}

if (time/(3600.0*24*365))>10 then begin
write(f1,(time/(3600.0*24*365)):12:2);
for i:=2 to dimi do begin
if i=2 then write(f1,X[i,0]-X[i-1,0]:12:2);
if i=dimi then write(f1,X[i,0]:12:2);
end;writeln(f1);
end;{if}

end;{nt, end time loop}

writeln('  time[years]          TE-TW[°C]           U[m/s]');
write((time/(3600.0*24*365)):12:2);
for i:=2 to dimi do begin
if i=2 then write(X[i,0]-X[i-1,0]:20:6);
if i=dimi then write(X[i,0]:20:6);
end;writeln;
writeln('computation terminated');
end.
```

Toy model 2

Model 2
A very simple model of Climate-biosphere interaction

Based on:

A.J.Watson and J.E.Lovelock, 1983: Biological homeostasis of the global environment: the parable of Daisyworld. **Tellus**, 35B, 284-289.

Supervisor: Aarnout van Delden
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http://www.staff.science.uu.nl/~delde102/SOAC.htm

Toy model 2

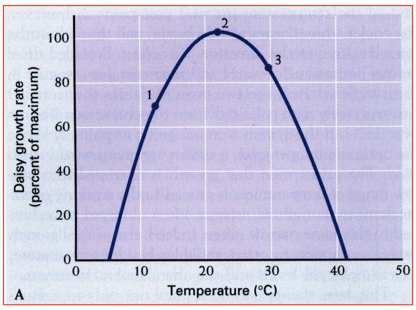
Daisy world

Illustration of the **Gaia hypothesis**, which suggests that life on Earth acts in such a way that earth's environment becomes adjusted to an optimum state for life's continuation. Evidence for this hypothesis is the low concentration of carbon dioxide.

Toy model 2

Daisy World

The Gaia hypothesis is illustrated through a computer model, called "Daisy World", of a cloudless planet in which the **environment** is defined by single variable: **temperature,** and the **biosphere** by a single species: **daisies**. Daisies grow best over a restricted range of temperatures, the growth rate peaking at 22.5°C and falling to zero below 5°C and above 40°C (see the **figure**).



Graedel & Crutzen (1995)

Toy model 2

Daisyworld: equations

Nine variables:

$A_w, A_b, A_g, \alpha_p, \beta_b, \beta_w, T_p, T_w, T_b$

Two time-dependent equations:

$$\frac{dA_w}{dt} = A_w\left(A_g\beta_w - \gamma\right)$$

$$\frac{dA_b}{dt} = A_b\left(A_g\beta_b - \gamma\right)$$

Seven diagnostic relations:

$$A_g = 1 - A_b - A_w$$

$$\beta_i = 1 - 4\frac{\left(T_{\text{opt}} - T_i\right)^2}{\left(T_{\max} - T_{\min}\right)^2}$$

$$\alpha_p = \alpha_g A_g + \alpha_b A_b + \alpha_w A_w$$

$$\left(T_i - T_p\right) = \frac{\frac{1}{4}S_0 L\left(\alpha_p - \alpha_i\right)}{b + \beta}$$

$$T_p = \frac{\frac{1}{4}S_0 L\left(1 - \alpha_p\right) - I_0}{b}$$

Parameters

$S_0, L, \alpha_g, \alpha_b, \alpha_w, \gamma, p, \beta, b, I_0$

$S_0 = 1366\ \text{W m}^{-2}$;
$\alpha_g = 0.25; \alpha_w = 0.75; \alpha_b = 0.15$;
$\gamma = 0.3\ \text{s}^{-1}; \beta = 16\ \text{W m}^{-2}\text{K}^{-1}$;
$b = 2.2\ \text{W m}^{-2}\text{K}^{-1}; I_0 = 220\ \text{W m}^{-2}$

Toy model 2

Questions

Calculate the equilibrium surface temperature as a function of the Solar luminosity.

Investigate the sensitivity of the solution to changes in the value of the parameters, b and β.

Give a physical interpretation of your results.

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References

Articles and books

A.J.Watson and J.E.Lovelock, 1983: Biological homeostasis of the global environment: the parable of Daisyworld. **Tellus**, 35B, 284-289.

James Lovelock, 1988: **The ages of Gaia**. Oxford University Press. 255 pp.

J.W. Kirchner, 2002: The Gaia hypothesis: fact, theory and wishful thinking, **Climatic Change**, 52, 391-408.

Wood, A.J. et al., 2008: Daisyworld: A review. **Reviews of Geophysics, 46**, RG1001, 23 pp., 2008, doi:10.1029/2006RG000217.

T.E. Graedel and P.J. Crutzen, 1995: **Atmosphere, Climate, and Change**. Scientific Americal Library. 196 pp.

Toy model 3

Model 3
Energy balance climate model

Based on:

M.I.Budyko, **Tellus**, 21 (1969), 610-619.
W.D. Sellers, **J.Appl.Meteor.**, 8 (1969), 392-400.

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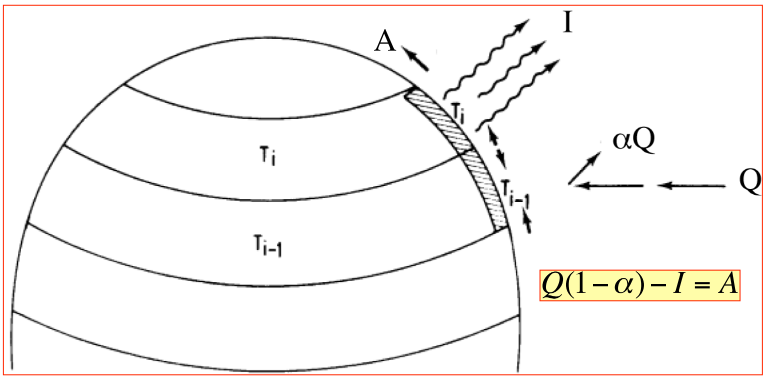
Toy model 3

One-dimensional (Budyko-Sellers) energy balance model (EBM)

A model of the latitude dependence of the surface temperature



Toy model 3

One-dimensional energy balance model

Aim is the estimation of the effect on the **temperature of the Earth's surface** of **changes in the planetary albedo** (due to changes in ice cover) and the Solar constant due to changes in activity of the Sun or due to changes in the earth's orbit as a **function of latitude, ϕ.**

For **mean annual conditions**, the equation for the vertically, **zonally averaged heat balance** of the earth-atmosphere system is:

$$Q(1-\alpha)=I+A$$

$Q(\phi)$=solar radiation coming to the outer boundary of the atmosphere
$\alpha(\phi,T)$=albedo;
$I(\phi,T)$=outgoing radiation
$A(\phi,T)$=loss (if A>0) of energy from a particular latitude belt as a result of the atmosphere and hydrosphere circulation, including heat redistribution of phase water transformations. For the earth-atmosphere system as a whole we have, A=0, giving the "**zero-dimensional EBM**".

Toy model 3

One-dimensional EBM"

Parametrization of solar radiation

$Q = \frac{S_0}{4} s(x)$ where $x = \sin\phi$

The distribution function $s(x)$ is defined as the **annual mean insolation at a particular latitude** divided by the global average insolation.
The global average of $s(x)$ is 1.

The annual mean insolation distribution function for current conditions can be approximated as

$s(x) = 1.0 - 0.477 P_2(x)$ where $P_2(x) = \frac{1}{2}(3x^2 - 1)$

Toy model 3

Specification of albedo

Introduce a **temperature dependent albedo**:

$$\alpha = \alpha_0 \text{ if } T \le T_0$$
$$\alpha = \alpha_0 + \frac{T - T_0}{T_1 - T_0}(\alpha_1 - \alpha_0) \text{ if } T_1 \ge T > T_0$$
$$\alpha = \alpha_1 \text{ if } T > T_1$$

$$\alpha_0 = 0.6 - 0.8;\ \alpha_1 = 0.1 - 0.3;$$
$$T_0 = -10°\text{C};\ T_1 = 0°\text{C};$$

Toy model 3

Dependence on temperature of Earth's surface, T_s

Parametrization of outgoing radiation

This depends on the radiation absorption-emission characteristics of the atmosphere. Therefore, not directly evident from simplified theory. An **empirical formula** has been suggested by **Budyko** (1969) and others:

$$I = I_0 + b(T - h\Gamma)$$

where I_0=205 W m^{-2} and b=2.23 W [m^2 °C]$^{-1}$

T is **sea-level** temperature **in °C(!)**, Γ is the lapse rate (6°C km^{-1}) and h the zonal mean height of Earth's surface

Toy model 3

$$Q(1-\alpha) - I = A$$

Parametrization of meridional flux divergence

To facilitate the solution of the above equation, the energy flux divergence within the atmosphere and oceans is **prescribed** as follows*.

$$A = \beta\left(T - T_p\right) \quad \text{with} \quad T_p = \frac{1}{2}\int_{-1}^{1} T\,\mathrm{d}x \qquad x \equiv \sin\phi$$

T_p is the planetary mean sea-level temperature;
β is a **relaxation coefficient**.
Another possibility is to describe the energy flux divergence by a **diffusive term** as follows**.

$$A = \frac{\partial}{\partial x}\left(1 - x^2\right) K_H \frac{\partial T}{\partial x}$$

K_H is diffusivity [W m^{-2} K^{-1}]

*after M.I.Budyko (Tellus, 21 (1969), 610-619

**after W.D. Sellers (J.Appl.Meteor., 8 (1969), 392-400

Toy model 3

Budyko model

Equilibrium: $Q(1-\alpha)-I=A$

With: $Q=\frac{S_0}{4}s(x)$

$I=I_0+b(T-h\Gamma)$

$A=\beta(T-T_P)$

gives

$$\beta(T-T_P)-bT=-\frac{S_0}{4}s(x)(1-\alpha)+I_0-bh\Gamma$$

This equation is solved as an initial-value problem by solving the following equation:

$$C\frac{dT}{dt}=Q(1-\alpha)-I-A$$

where C is a heat capacity and integrating the equation until a steady state is reached, i.e. l.h.s. is equal to zero.

Toy model 2

Sellers Model

The steady state solution of the one-dimensional EBM can be found by making the equation on the previous slide time-dependent, i.e.

$$C\frac{dT}{dt}=\frac{S_0}{4}s(x)(1-\alpha)-I_0+bh\Gamma-bT+\frac{\partial}{\partial x}\left(1-x^2\right)K_H\frac{\partial T}{\partial x}$$

(C is a heat capacity) and numerically integrating the equation in time with a suitable time-difference scheme* until the solution converges to a steady state⁺. The steady state will depend on the initial conditions.

+ l.h.s. equals zero

* For instance: the Matsuno scheme

Toy model 3

Questions

Investigate the global average temperature as a function the value of the Solar constant.

Investigate the solution as a function initial state

Plot the steady state OLR-TOA and the steady state absorbed Solar radiation as a function of latitude.

Introduce a more sophisticated parametrization of insolation. For instance you may introduce a seasonal cycle and investigate the solution in dependence on obliquity and eccentricity.

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